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CYCLIC PERMUTATIONS IN DETERMINING CROSSING NUMBERS

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Abstract

The crossing number of a graph G is the minimum number of edge crossings over all drawings of G in the plane. Recently, the crossing numbers of join products of two graphs have been studied. In the paper, we extend know results concerning crossing numbers of join products of small graphs with discrete graphs. The crossing number of the join product $G^* + D_n$ for the disconnected graph G^* consisting of five vertices and of three edges incident with the same vertex is given. Up to now, the crossing numbers of $G + D_n$ were done only for connected graphs G. In the paper also the crossing numbers of $G^* + P_n$ and $G^* + C_n$ are given. The paper concludes by giving the crossing numbers of the graphs $H + D_n$, $H + P_n$, and $H + C_n$ for four different graphs H with $|E(H)| \leq |V(H)|$. The methods used in the paper are new. They are based on combinatorial properties of cyclic permutations.

Keywords: graph, drawing, crossing number, join product, cyclic permutation.

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1. Introduction

The *crossing number* of a simple graph G, denoted $\mathrm{cr}(G)$, with vertex set $V(G)$ and edge set $E(G)$ is defined as the minimum possible number of edge crossings in a drawing of G in the plane. A drawing with the minimum number of crossings (an *optimal* drawing) must be a *good* drawing; that is, each two edges have at most one point in common, which is either a common end-vertex or a crossing.

Moreover, no three edges cross in a point. The investigation on the crossing number of graphs is a classical and very difficult problem. Garey and Johnson [1] proved that determining $\mathrm{cr}(G)$ is NP-complete.

Over the past decade, some results concerning crossing numbers of join products of two graphs have been obtained. The purpose of this article is to extend the known results concerning this topic. The join product of two graphs G_i and G_j, denoted $G_i + G_j$, is obtained from vertex-disjoint copies of G_i and G_j by adding all edges between $V(G_i)$ and $V(G_j)$. For $|V(G_i)| = m$ and $|V(G_j)| = n$, the edge set of $G_i + G_j$ is the union of disjoint edge sets of the graphs G_i, G_j, and the complete bipartite graph $K_{m,n}$. Let P_n and C_n be the *path* and the *cycle* of n vertices, respectively, and let D_n denote the *discrete graph* (sometimes called *empty graph*) on n vertices. Using Kleitman's result [9], the crossing numbers for join of two paths, join of two cycles, and for join of path and cycle were studied in [3]. Moreover, the exact values for crossing numbers of $G + D_n$ and $G + P_n$ for all graphs G of order at most four are given in [5]. The crossing numbers of the graphs $G + P_n$ and $G + C_n$ are also known for very few graphs G of order five and six; see [4, 6], and [7]. In all these cases, the graph G is connected and contains at least one cycle.

The aim of the paper is to give the crossing number of the join product $G^* + D_n$ for the disconnected graph G^* consisting of five vertices and of three edges incident with the same vertex. The methods used in the paper are new. They are based on combinatorial properties of cyclic permutations. The similar methods were partially used earlier in the papers [2, 10]. We were unable to determine the crossing number of the join product $G^* + D_n$ using the methods used in [4, 6], and [7]. In Section 6 we refer the graph H on five vertices and six edges for which the crossing number of $H + D_n$ was obtained using previous methods; see also [6]. Nevertheless, $\mathrm{cr}(H + D_n) = \mathrm{cr}(G^* + D_n)$.

Let D be a good drawing of the graph G. We denote the number of crossings in D by $\mathrm{cr}_D(G)$. For a subgraph G_i of the graph G, let $D(G_i)$ be the subdrawing of G_i induced by D. For edge-disjoint subgraphs G_i and G_j of G, we denote by $\mathrm{cr}_D(G_i, G_j)$ the number of crossings of edges in G_i and edges in G_j, and by $\mathrm{cr}_D(G_i)$ the number of crossings among edges of G_i in D. It is easy to see that for any three edge-disjoint subgraphs G_i, G_j, and G_k of the graph G the following equations hold

$$\mathrm{cr}_D(G_i \cup G_j) = \mathrm{cr}_D(G_i) + \mathrm{cr}_D(G_j) + \mathrm{cr}_D(G_i, G_j),$$
$$\mathrm{cr}_D(G_i \cup G_j, G_k) = \mathrm{cr}_D(G_i, G_k) + \mathrm{cr}_D(G_j, G_k).$$

In the paper, some proofs are based on Kleitman's result on crossing numbers of complete bipartite graphs. More precisely, he proved that

$$\mathrm{cr}(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor, \quad \text{if} \quad m \le 6\,. \tag{1}$$

The paper is organized as follows. In section 2 we discuss all possible good drawings of the graph $G^* + D_n$ in which, for some vertices $t_i \in V(D_n)$ of degree five, no edge incident with t_i crosses G^*. For such vertices, all possible rotations of incident edges are summarized and the corresponding cyclic permutations of five elements are characterized. In Section 3 we determine the smallest necessary number of crossings among edges of a subgraph isomorphic with $K_{5,2}$ in a drawing of $G^* + D_n$ in which no edge of $K_{5,2}$ crosses G^*. Table 2 summarizes the minimal values of necessary crossings among the edges in such $K_{5,2}$ depending on the vertex rotations of both vertices of degree five. In the next section we prove several lemmas that are used in the proof of the main result. This result, namely the crossing number of the graph $G^* + D_n$, is presented in Section 5. In Section 6, based on the main result, the crossing numbers of $G^* + P_n$ and $G^* + C_n$ are given. The paper concludes by giving the crossing numbers of $H_i + D_n$, $H_i + P_n$, and $H_i + C_n$ for four different graphs H_i with $|E(H_i)| \leq |V(H_i)|$.

2. The Graph $G^* + D_n$ and Its Drawings

Consider the graph G^* of order five with one isolated vertex and one vertex of degree three. Of course, it forces that others three vertices are of degree one. Let us denote by v_1 the vertex of degree three in the graph G^*. Let v_2, v_3, and v_4 be the vertices of degree one, and let v_5 be the isolated vertex in G^*. The graph $G^* + D_n$ consists of one copy of the graph G^* and n vertices $t_1, t_2, \ldots, t_n$, where any vertex t_i, $i = 1, 2, \ldots, n$, is adjacent with every vertex of G^*. In Figure 1 there are two drawings of the graph $G^* + D_n$ with $4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor$ crossings. The subgraph $K_{5,n}$ is drawn in the same way as in [12] with $4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$ crossings and the edge v_1v_3 of G^* not belonging to $K_{5,n}$ is crossed by $\left\lfloor \frac{n}{2} \right\rfloor$ edges of $K_{5,n}$. Hence, $\mathrm{cr}(G^* + D_n) \leq 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor$.

Let T^i, $1 \leq i \leq n$, denote the subgraph induced by the five edges incident with the vertex t_i. Then

$$G^* + D_n = G^* \cup K_{5,n} = G^* \cup \left(\bigcup_{i=1}^{n} T^i \right).$$

Two vertices t_i and t_j of $G^* + D_n$ are *antipodal* in a drawing of $G^* + D_n$ if the subdrawing of $T^i \cup T^j$ has no crossings. A drawing is *antipode-free* if it has no antipodal vertices.

In a good drawing D of $G^* + D_n$, the *rotation* $\mathrm{rot}_D(t_i)$ of a vertex t_i is the cyclic permutation that records the (cyclic) counter-clockwise order in which the edges leave t_i. We use the notation (12345) if the counter-clockwise order the edges incident with the vertex t_i is t_iv_1, t_iv_2, t_iv_3, t_iv_4, and t_iv_5. We emphasize that a rotation is a cyclic permutation; that is, (12345), (23451), (34512), (45123),

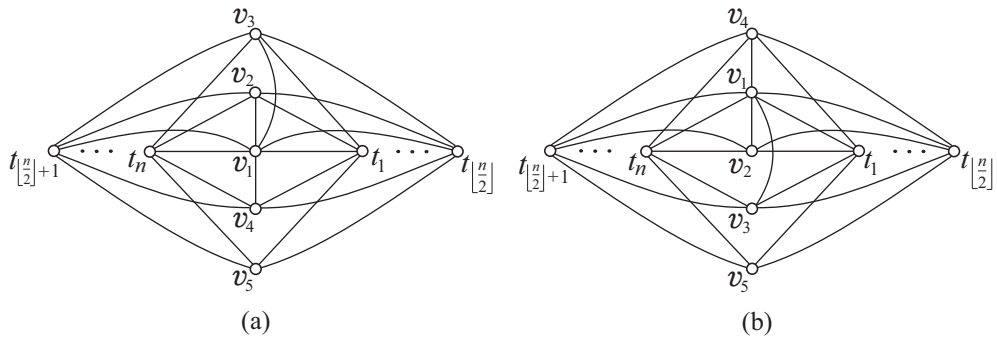

Figure 1. Two drawings of the graph $G^* + D_n$.

and (51234) denote the same rotation. Thus, $5!/5 = 24$ different $\mathrm{rot}_D(t_i)$ can appear in a drawing of the graph $G^* + D_n$; see Table 1.

In the subdrawing of G^* induced by D, the rotation $\mathrm{rot}_D(v_1)$ of the vertex v_1 is defined analogously. Since v_1 is adjacent with only three vertices of G^*, there are only two possible rotations of the vertex v_1 represented by the cyclic permutations (234) and (243).

As the complete bipartite graph $K_{5,n}$ is a subgraph of $G^* + D_n$, let us discuss some properties of crossings among edges of its subgraph $K_{5,2}$. Assume, in general, D is a good drawing of the graph $K_{m,n}$ with the vertices $t_1, t_2, \ldots, t_n$ of degree m. The rotation $\mathrm{rot}_D(t_i)$, $i = 1, 2, \ldots, n$, is defined in the same way as above, i.e., as the cyclic permutation of m elements. Let $K_{m,2}$ be the subgraph of $K_{m,n}$ with the vertices t_i and t_j of degree m. Similarly as in the graph $G^* + D_n$, we can use the symbol $\mathrm{cr}_D(T^i, T^j)$ for the number of crossings between the edges incident with t_i and the edges incident with t_j. Woodall [11] proved that if both vertices t_i and t_j have the same rotation in D, then $\mathrm{cr}_D(T^i, T^j) \geq \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor$. It is easy to see that $\mathrm{cr}_D(T^i, T^j) = 0$ only if $\mathrm{rot}_D(t_j)$ is inverse to $\mathrm{rot}_D(t_i)$.

Let $Q(\mathrm{rot}_D(t_i), \mathrm{rot}_D(t_j))$ denote the minimum number of interchanges of adjacent elements of $\mathrm{rot}_D(t_i)$ required to produce the inverse cyclic permutation of $\mathrm{rot}_D(t_j)$ or, equivalently, from $\mathrm{rot}_D(t_j)$ to the inverse of $\mathrm{rot}_D(t_i)$. Woodall proved that

$$\mathrm{cr}_D(T^i, T^j) \geq Q(\mathrm{rot}_D(t_i), \mathrm{rot}_D(t_j)) \tag{2}$$

and that

$$\mathrm{cr}_D(T^i, T^j) \equiv Q(\mathrm{rot}_D(t_i), \mathrm{rot}_D(t_j)) (\mathrm{mod}\, 2) \qquad \text{if } m \text{ is odd.} \tag{3}$$

This implies that, in a good drawing D of the graph $G^* + D_n$, $\mathrm{cr}_D(T^i, T^j) = 0$ only if $\mathrm{rot}_D(t_i)$ is inverse to $\mathrm{rot}_D(t_j)$ and $\mathrm{cr}_D(T^i, T^j) \geq 4$ if $\mathrm{rot}_D(t_i) = \mathrm{rot}_D(t_j)$. Moreover, $\mathrm{cr}_D(T^i, T^j) = Q(\mathrm{rot}_D(t_i), \mathrm{rot}_D(t_j)) + 2k$ for some nonnegative integer k.

In a good drawing D of the graph $G^* + D_n$, we separate the subgraphs T^i, $i = 1, 2, \ldots, n$, of $G^* + D_n$ into two subsets. Let us denote by R_0 the set of subgraphs T^i, $i \in \{1, 2, \ldots, n\}$, for which $\mathrm{cr}_D(G^*, T^i) = 0$. Every other subgraph T^i crosses G^* at least once in D.

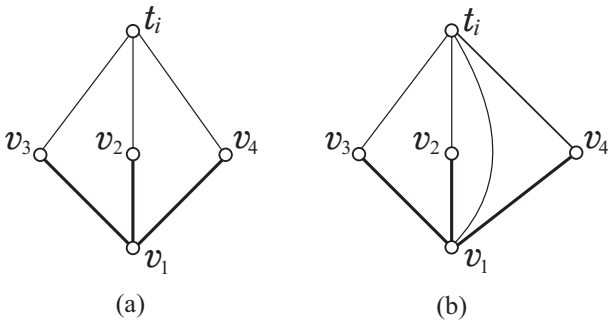

Figure 2. Two subdrawings of F^i.

For $T^i \in R_0$, let F^i denote the subgraph $G^* \cup T^i$, $i \in \{1, 2, \ldots, n\}$, of $G^* + D_n$ and let $D(F^i)$ be its subdrawing induced by D. Our aim is to list all possible rotations $\mathrm{rot}_D(t_i)$ which can appear in D if the edges of T^i do not cross the edges of G^*. Let us start with the subdrawing of F^i induced by the edges incident with the vertices v_2, v_3, and v_4 shown in Figure 2(a). By symmetry, we may assume that the rotation $\mathrm{rot}_D(v_1)$ is represented by the cyclic permutation (234) in this drawing as identical considerations may be applied if $\mathrm{rot}_D(v_1)$ is represented by (243). The rotation $\mathrm{rot}_D(t_i)$ of the vertex t_i is (243) in this subdrawing, which is inverse to $\mathrm{rot}_D(v_1)$. In $D(F^i)$, the edge t_iv_1 divides one of three quadrangular regions of the subdrawing in Figure 2(a). In Figure 2(b) there is the subdrawing of F^i in which the edge t_iv_1 divides the region with the vertices v_2 and v_4 on its boundary. Hence, $\mathrm{rot}_D(t_i) = (1432)$ in this case. If the edge t_iv_1 divides the region with the vertices v_2 and v_3 or the region with the vertices v_3 and v_4 on the boundary, $\mathrm{rot}_D(t_i) = (1243)$ and $\mathrm{rot}_D(t_i) = (1324)$, respectively. Every of these three subdrawings of $F^i \setminus v_5$ produces four drawings of F^i depending on in which region the vertex v_5 is placed. Thus, to obtain all allowed rotations $\mathrm{rot}_D(t_i)$ of $D(F^i)$, in each of the cyclic permutations (1432), (1243), and (1324) we simply add the number 5 into all four positions between the numbers 1, 2, 3, and 4. Hence, there are twelve possible cyclic permutations representing $\mathrm{rot}_D(t_i)$ for which $\mathrm{cr}_D(G^*, T^i) = 0$.

In the rest of the paper, each cyclic permutation will be represented by the permutation with 1 in the first position. These twelve permutations under our consideration are denoted by A_r and B_r, $r = 1, 2, \ldots, 6$; see the left side of Table 1. The permutation is of type A or B if the vertex v_5 is placed in the triangular or in the quadrangular region in the subdrawing $D(F^i \setminus v_5)$, respectively. For

example, from the drawing in Figure 2(b) one can obtain the configurations A_1, A_2, B_1, and B_6. Let us divide the permutations A_r and B_r, $r = 1, 2, \ldots, 6$, into subsets

$$M_1^A = \{A_1, A_3, A_5\}, \qquad M_1^B = \{B_1, B_3, B_5\},$$
$$M_2^A = \{A_2, A_4, A_6\}, \qquad M_2^B = \{B_2, B_4, B_6\},$$

and let

$$M = M_1 \cup M_2, \quad \text{where} \quad M_1 = M_1^A \cup M_1^B \quad \text{and} \quad M_2 = M_2^A \cup M_2^B.$$

In a fixed drawing of the graph $G^* + D_n$, some permutations from M may not occur. For a drawing D of the graph $G^* + D_n$, we denote by M_D the subset of M containing only permutations which represent rotations of the vertices t_i for which $\mathrm{cr}_D(G^*, T^i) = 0$. We denote by $M_{1_D}^A$ and $M_{2_D}^A$ the set of all permutations of type A that exist in the drawing D belonging to the sets M_1 and M_2, respectively. Similarly are used notations $M_{1_D}^B$ and $M_{2_D}^B$ for the sets of permutations of type B. For a subgraph $F^i = G^* \cup T^i$, we say that F^i has configuration X_r, denoted $\mathrm{conf}(F^i) = X_r$, if $\mathrm{rot}_D(t_i) = X_r$ for $X \in \{A, B\}$ and $r \in \{1, 2, \ldots, 6\}$.

M_1	M_2	$\overline{M}_1$	$\overline{M}_2$
$A_1 : (15432)$	$A_2 : (14325)$	$\overline{A}_1 : (12345)$	$\overline{A}_2 : (15234)$
$A_3 : (15324)$	$A_4 : (13245)$	$\overline{A}_3 : (14235)$	$\overline{A}_4 : (15423)$
$A_5 : (15243)$	$A_6 : (12435)$	$\overline{A}_5 : (13425)$	$\overline{A}_6 : (15342)$
$B_1 : (14532)$	$B_2 : (13254)$	$\overline{B}_1 : (12354)$	$\overline{B}_2 : (14523)$
$B_3 : (13524)$	$B_4 : (12453)$	$\overline{B}_3 : (14253)$	$\overline{B}_4 : (13542)$
$B_5 : (12543)$	$B_6 : (14352)$	$\overline{B}_5 : (13452)$	$\overline{B}_6 : (12534)$

Table 1. All cyclic permutations of 5 elements.

For each $X \in \{A, B\}$ and $r \in \{1, 2, \ldots, 6\}$, let $\overline{X}_r$ denote the inverse permutation to the permutation X_r. In the right side of Table 1, all twelve inverse permutations $\overline{X}_r$ are divided into two sets

$$\overline{M}_1 = \{\overline{A}_1, \overline{A}_3, \overline{A}_5, \overline{B}_1, \overline{B}_3, \overline{B}_5\} \quad \text{and} \quad \overline{M}_2 = \{\overline{A}_2, \overline{A}_4, \overline{A}_6, \overline{B}_2, \overline{B}_4, \overline{B}_6\},$$

and let $\overline{M} = \overline{M}_1 \cup \overline{M}_2$. In a similar way as above, we use the notations $\overline{M}_1^A$, $\overline{M}_1^B$, $\overline{M}_2^A$, and $\overline{M}_2^B$ such that $\overline{M}_1 = \overline{M}_1^A \cup \overline{M}_1^B$ and $\overline{M}_2 = \overline{M}_2^A \cup \overline{M}_2^B$.

We remark that if T^i does not cross the edges of G^*, then $\mathrm{rot}_D(t_i)$ must contain the elements 2, 3, and 4 in such a way that the omission of the elements 1 and 5 induces the cyclic sub-permutation (243). Let us define the functions

$$\pi_1 : \{2, 3, 4\} \to \{2, 3, 4\}, \quad \text{with} \quad \pi_1(2) = 4, \quad \pi_1(3) = 2, \quad \text{and} \quad \pi_1(4) = 3,$$

and

$$\pi_2 : \{2,4\} \to \{2,4\}, \quad \text{with} \quad \pi_2(2) = 4, \quad \text{and} \quad \pi_2(4) = 2.$$

Let $\Pi_1 : M \cup \overline{M} \to M \cup \overline{M}$, be the function obtained by applying π_1 on the corresponding elements of the permutations in $M \cup \overline{M}$. Let $\Pi_2 : M \cup \overline{M} \to M \cup \overline{M}$, be the function obtained by applying π_2. Thus, for $X \in \{A, B, \overline{A}, \overline{B}\}$,

$$\Pi_1(X_1) = X_3, \quad \Pi_1(X_3) = X_5, \quad \Pi_1(X_5) = X_1,$$

$$\Pi_1(X_2) = X_4, \quad \Pi_1(X_4) = X_6, \quad \Pi_1(X_6) = X_2,$$

and

$$\begin{aligned}
&\Pi_2(A_1) = \overline{A}_2, \quad \Pi_2(A_2) = \overline{A}_1, \quad \Pi_2(\overline{A}_1) = A_2, \quad \Pi_2(\overline{A}_2) = A_1,\\
&\Pi_2(A_3) = \overline{A}_6, \quad \Pi_2(A_6) = \overline{A}_3, \quad \Pi_2(\overline{A}_3) = A_6, \quad \Pi_2(\overline{A}_6) = A_3,\\
&\Pi_2(A_5) = \overline{A}_4, \quad \Pi_2(A_4) = \overline{A}_5, \quad \Pi_2(\overline{A}_5) = A_4, \quad \Pi_2(\overline{A}_4) = A_5,\\
&\Pi_2(B_1) = \overline{B}_6, \quad \Pi_2(B_6) = \overline{B}_1, \quad \Pi_2(\overline{B}_1) = B_6, \quad \Pi_2(\overline{B}_6) = B_1,\\
&\Pi_2(B_3) = \overline{B}_4, \quad \Pi_2(B_4) = \overline{B}_3, \quad \Pi_2(\overline{B}_3) = B_4, \quad \Pi_2(\overline{B}_4) = B_3,\\
&\Pi_2(B_5) = \overline{B}_2, \quad \Pi_2(B_2) = \overline{B}_5, \quad \Pi_2(\overline{B}_5) = B_2, \quad \Pi_2(\overline{B}_2) = B_5.
\end{aligned}$$

3. Necessary Crossings Between T^i and T^j

If two different subgraphs F^i and F^j with configurations from M cross in a drawing of $G^* + D_n$, then only the edges of T^i cross the edges of T^j. We will deal with the minimum numbers of crossings between two different subgraphs F^i and F^j depending on their configurations. Let D be a good drawing of the graph $G^* + D_n$, and let X, Y be configurations from M_D. We shortly denote by $\mathrm{cr}_D(X,Y)$ the number of crossings in D between T^i and T^j for different $T^i, T^j \in R_0$ such that F^i, F^j have configurations X, Y, respectively. Finally, let $\mathrm{cr}(X,Y) = \min\{\mathrm{cr}_D(X,Y)\}$ over all pairs X and Y from M among all good drawings of the graph $G^* + D_n$. Our aim is to establish $\mathrm{cr}(X,Y)$ for all pairs $X, Y \in M$, i.e., the minimum number of crossings between two different subgraphs F^i and F^j with configurations X and Y over all good drawing of the graph $G^* + D_n$.

Let D be any good drawing of the graph $G^* + D_n$. For some $i \in \{1, 2, \ldots, n\}$, assume that the subdrawing of F^i induced by D has configuration A_1. This unique drawing of F^i contains four regions. Let us denote these four regions by ω_2, $\omega_{2,3}$, $\omega_{3,4}$, and ω_4, depending on which of v_2, v_3, and v_4 are located on the boundary of the corresponding region. Without loss of generality we may assume that the unbounded region is $\omega_{3,4}$ since our considerations do not rely on which vertices are on the bounded region; see Figure 3(a). Now, let us count the

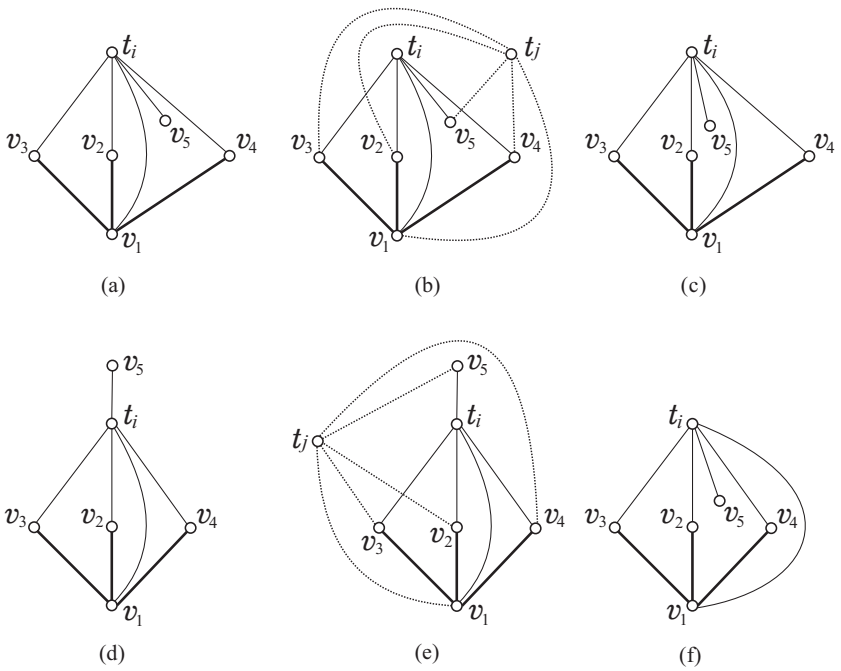

Figure 3. The discussed subdrawings of F^i and $F^i \cup T^j$.

minimum necessary number of crossings between F^i and F^j, $i \neq j$, depending on the configuration of F^j. For each region, at most three vertices of G^* can be adjacent with the vertex t_j of F^j without crossings. Thus, the edges of T^j must cross the edges of T^i at least twice.

Let us first list the configurations of F^j which can cross F^i only twice. As $\mathrm{rot}_D(t_i) = (15432)$, by (2), the permutation which represents $\mathrm{conf}(F^j)$ must be obtained from $\overline{A}_1 = (12345)$ by at most two interchanges of adjacent elements. Since $\mathrm{conf}(F^j) \in M$ and $\mathrm{rot}_D(t_j)$ contains the cyclic sub-permutation (243), the adjacent elements 2 and 3 or the adjacent elements 3 and 4 must be changed. Thus, after this step the permutations $A_4 = (13245)$ and $A_6 = (12435)$ can be obtained. Hence, $Q(\mathrm{rot}_D(t_i), \mathrm{rot}_D(t_j)) = 1$ if $\mathrm{conf}(F^j) \in \{A_4, A_6\}$. But, $\mathrm{cr}_D(T^i, T^j) \neq 1$ and by (3), $\mathrm{cr}(A_1, X) \geq 3$ for $X \in \{A_4, A_6\}$. In the second step, the possible changes are 1–3, 1–5, and 4–5 in the permutation (13245) and 1–2, 1–5, and 3–5 in (12435). It is easy to verify that $Q(\mathrm{rot}_D(t_i), \mathrm{rot}_D(t_j)) = 2$ only if $\mathrm{conf}(F^j) \in \{A_3, A_5, B_2, B_4, B_6\}$.

Now we show that $\mathrm{cr}(A_1, X) = 2$ only for $X = B_2$ and $X = B_6$. The vertex t_j must be placed in one of the regions $\omega_{3,4}$ and ω_4 with three vertices of G^* adjacent with t_j without crossings. Assume that t_j is placed in the region $\omega_{3,4}$, i.e., $t_j \in \omega_{3,4}$. If T^j crosses T^i only twice, the unique possibility is that the edge $t_j v_2$ crosses the edge $t_i v_3$ and the edge $t_j v_5$ crosses the edge $t_i v_4$; see

Figure 3(b). So, $\mathrm{rot}_D(t_j) = (13254)$ and $\mathrm{conf}(F^j) = B_2$. If $t_j \in \omega_4$, the edge t_jv_2 must cross the edge t_iv_1 and the edge t_jv_3 must cross the edge t_iv_4. In this case, $\mathrm{rot}_D(t_j) = (14352)$ and $\mathrm{conf}(F^j) = B_6$. We remark that t_j cannot be placed in the region $\omega_{2,3}$, because the edge t_jv_5 must cross at least two edges of T^i in this case. Thus, $\mathrm{cr}(A_1, B_2) = \mathrm{cr}(A_1, B_6) = 2$ and all other subgraphs F^j with configurations in M different from B_2 and B_6 cross F^i at least three times. But, it implies from (2) and (3) that if $Q(\mathrm{rot}_D(t_i), \mathrm{rot}_D(t_j)) = 2$ and $\mathrm{cr}_D(T^i, T^j) \neq 2$, then $\mathrm{cr}_D(T^i, T^j) \geq 4$. Thus, $\mathrm{cr}(A_1, A_3) \geq 4$, $\mathrm{cr}(A_1, A_5) \geq 4$, and $\mathrm{cr}(A_1, B_4) \geq 4$. Clearly, also $\mathrm{cr}(A_1, A_1) \geq 4$ and all subgraphs F^j with configurations A_2, A_4, A_6, B_1, B_3, and B_5 cross F^i at least three times; see the first row in Table 2. We remark that it is possible to show that also $\mathrm{cr}(A_1, A_4) > 3$ and $\mathrm{cr}(A_1, A_6) > 3$, but we do not need higher values in our proofs.

Assume that $\mathrm{conf}(F^i) = A_3$. The subdrawing of F^i induced by D can be obtained from the drawing in Figure 3(a) in such a way that the vertices v_2, v_3, and v_4 are replaced by the vertices v_4, v_2, and v_3, respectively. Hence, $\mathrm{rot}_D(t_i) = A_3 = (15324)$ is obtained from $A_1 = (15432)$ using transformation π_1. But, in this case, π_1 must be applied on all $\mathrm{rot}_D(t_j)$ with $T^j \in R_0$. Thus, the function Π_1 transforms the previous configurations B_2 and B_6 with $\mathrm{cr}(A_1, B_2) = \mathrm{cr}(A_1, B_6) = 2$ to B_4 and B_2, respectively. Hence, we have $\mathrm{cr}(A_3, B_4) = \mathrm{cr}(A_3, B_2) = 2$. Similarly, Π_1 transforms A_1, A_3, A_5, and B_4 to A_3, A_5, A_1, and B_6, respectively. This implies that the number 4 appears four times in the corresponding columns of the third row in Table 2. In the remaining columns there is the number 3. In the same way, by transformation π_1, the configuration A_5 of F^i is obtained from the drawing which represent the configuration A_3. Now, using transformation Π_1, $\mathrm{cr}(A_5, B_4) = \mathrm{cr}(A_5, B_6) = 2$ and $\mathrm{cr}(A_5, A_j) \geq 4$ for $j = 1, 3, 5$, as well as $\mathrm{cr}(A_5, B_2) \geq 4$; see the fifth row in Table 2.

Let us focus on the configuration A_2 of F^i. If the positions of the vertices v_2 and v_4 in the drawing of A_2 in Figure 3(c) are exchanged, the drawing of $\overline{A}_1$ is obtained. Hence, $\overline{A}_1 = (12345)$ is obtained from $A_2 = (14325)$ using transformation π_2. It is obvious that $\mathrm{cr}(X, Y) = \mathrm{cr}(\overline{X}, \overline{Y})$ for all $X, Y \in M$. This implies that the lower bounds of $\mathrm{cr}(\overline{A}_1, \overline{X})$ are known for all $\overline{X} \in \overline{M}$. As $\Pi_2(\overline{A}_1) = A_2$, applying the function Π_2 on the elements of $\overline{M}$ the lower bounds of $\mathrm{cr}(A_2, X)$ for all $X \in M$ can be obtained. Concretely, the results $\mathrm{cr}(A_2, B_5) = 2$ and $\mathrm{cr}(A_2, B_1) = 2$ are obtained from $\mathrm{cr}(\overline{A}_1, \overline{B}_2) = 2$ and $\mathrm{cr}(\overline{A}_1, \overline{B}_6) = 2$, respectively, using $\Pi_2(\overline{B}_2) = B_5$ and $\Pi_2(\overline{B}_6) = B_1$. Similarly, as Π_2 transforms $\overline{A}_3$, $\overline{A}_5$, and $\overline{B}_4$ to A_6, A_4, and B_3, respectively, $\mathrm{cr}(A_2, A_6) \geq 4$, $\mathrm{cr}(A_2, A_4) \geq 4$, and $\mathrm{cr}(A_2, B_3) \geq 4$. Of course, $\mathrm{cr}(A_2, A_2) \geq 4$ also applies. For the rest six configurations in M, the number 3 is obtained; see the second row in Table 2. Similarly as above, using Π_1, the second row can be transformed to the fourth row and the fourth row to the sixth. As $\mathrm{cr}(X, Y) = \mathrm{cr}(Y, X)$ for $X, Y \in M$, also the values in the first six columns in Table 2 are known. It is easy to see that, for the

completeness of Table 2, only the values of $\mathrm{cr}(B_i, B_j)$ for some $i \in \{1,2,3,4,5,6\}$ and $j = 1,2,3,4,5,6$ are needed.

$-$	A_1	A_2	A_3	A_4	A_5	A_6	B_1	B_2	B_3	B_4	B_5	B_6
A_1	4	3	4	3	4	3	3	2	3	4	3	2
A_2	3	4	3	4	3	4	2	3	4	3	2	3
A_3	4	3	4	3	4	3	3	2	3	2	3	4
A_4	3	4	3	4	3	4	2	3	2	3	4	3
A_5	4	3	4	3	4	3	3	4	3	2	3	2
A_6	3	4	3	4	3	4	4	3	2	3	2	3
B_1	3	2	3	2	3	4	4	1	2	3	2	3
B_2	2	3	2	3	4	3	1	4	3	2	3	2
B_3	3	4	3	2	3	2	2	3	4	1	2	3
B_4	4	3	2	3	2	3	3	2	1	4	3	2
B_5	3	2	3	4	3	2	2	3	2	3	4	1
B_6	2	3	4	3	2	3	3	2	3	2	1	4

Table 2. The necessary numbers of crossings between T^i and T^j for the configurations of F^i and F^j.

Assume that a subdrawing of F^i induced by D has configuration $B_1 = (14532)$. This unique subdrawing contains four regions ω_2, ω_4, $\omega_{2,3}$, and $\omega_{3,4}$. Without loss of generality, let $\omega_{3,4}$ be the unbounded region; see Figure 3(d). If $T^j \in R_0$ crosses the edges of T^i once, then, by (2), $Q(\mathrm{rot}_D(t_i), \mathrm{rot}_D(t_j)) = 1$ and $\mathrm{rot}_D(t_j)$ must be obtained from $\overline{B}_1 = (12354)$ by only one exchange of adjacent elements. As $\mathrm{rot}_D(t_j)$ must contain the cyclic sub-permutation (243), only the adjacent elements 2 and 3 are exchanged in $\overline{B}_1 = (12354)$ and the configuration $B_2 = (13254)$ with $\mathrm{cr}(B_1, B_2) = 1$ is obtained. Figure 3(e) shows such subdrawing of $G^* \cup T^i \cup T^j$ with only one crossing between T^i and T^j. Since all four other interchanges of adjacent elements in $\overline{B}_1 = (12354)$ produce the cyclic permutations not containing the cyclic sub-permutation (243), all configurations $X \in M$ with $\mathrm{cr}(B_1, X) = 2$ are these which we can obtain from $B_2 = (13254)$ by only one exchange of adjacent elements, concretely 1-3, 2-5, 1-4, and 4-5. This forces that only the configurations $B_5 = (12543)$, $B_3 = (13524)$, $A_2 = (14325)$, and $A_4 = (13245)$ are these for which the corresponding T^j can cross the edges of T^i twice. Hence, we add the result $\mathrm{cr}(B_1, B_3) = \mathrm{cr}(B_1, B_5) = 2$ to the row corresponding to B_1 in Table 2. Since $\mathrm{cr}(B_1, B_1) = 4$, only the results for $\mathrm{cr}(B_1, B_4)$ and $\mathrm{cr}(B_1, B_6)$ are unknown. But we know that these values are at least 3 and that is enough for our proofs. Now, in the same way as above, the values in the rows B_3 and B_5 are obtained by successive applying of the transformation Π_1 on the rows B_1 and B_3, respectively. Hence, due to symmetry, in the row B_2 only the values $\mathrm{cr}(B_2, B_4)$ and $\mathrm{cr}(B_2, B_6)$ are unknown. But, since $\overline{B}_2$ cannot be

obtained from $B_4 = (12453)$ or from $B_6 = (14352)$ by one exchange of adjacent elements, it holds that $\mathrm{cr}(B_2, B_4) \geq 2$ and $\mathrm{cr}(B_2, B_6) \geq 2$. This completes the row B_2. Now, the rows B_4 and B_6 can be obtain by successive applying of the transformation Π_1 on the rows B_2 and B_4, respectively.

4. Some Useful Lemmas

In the proof of Theorem 7, the following lemmas related to some restricted drawings of the graph $G^* + D_n$ are needed.

Lemma 1. *Let D be a good drawing of the graph $G^* + D_3$. If $\mathrm{cr}_D\big(G^*, T^1 \cup T^2 \cup T^3\big) = 0$, i.e., $T^1, T^2, T^3 \in R_0$, then $\mathrm{cr}_D\big(T^1 \cup T^2 \cup T^3\big) \geq 6$.*

Proof. Table 2 shows that $\mathrm{cr}(T^i, T^j) \geq 2$ for all $i, j \in \{1, 2, 3\}$, $i \neq j$, except of the three cases when $\{\mathrm{conf}(F^i), \mathrm{conf}(F^j)\} = \{B_r, B_{r+1}\}$ for $r = 1, 3, 5$. This implies that if for each $i, j \in \{1, 2, 3\}$, $i \neq j$, $\{\mathrm{conf}(F^i), \mathrm{conf}(F^j)\} \neq \{B_r, B_{r+1}\}$, $r = 1, 3, 5$, then $\mathrm{cr}(T^1, T^2) \geq 2$, $\mathrm{cr}(T^1, T^3) \geq 2$, and $\mathrm{cr}(T^2, T^3) \geq 2$. So, $\mathrm{cr}_D(T^1 \cup T^2 \cup T^3) \geq 6$ in this case.

Otherwise, assume that some pair F^i, F^j, say F^1 and F^2, have different configurations from the set $\{B_1, B_2\}$. The subgraphs T^1 and T^2 cross at least once. If $\mathrm{conf}(F^3)$ is one of $A_1, \ldots, A_6, B_1, \ldots, B_6$ then, by Table 2, $\mathrm{cr}(T^1 \cup T^2, T^3) \geq 5$. Thus, $\mathrm{cr}_D(T^1 \cup T^2 \cup T^3) \geq 6$. If F^1 and F^2 have different configurations from the set $\{B_3, B_4\}$ or $\{B_5, B_6\}$, the same argument is applied. This completes the proof. ■

Lemma 2. *Let D be a good, antipode-free drawing of $G^* + D_n$, $n \geq 3$, and $X \in \{A, B\}$. Let $M^X_{1_D}$ and $M^X_{2_D}$ be non-empty sets of configurations. If $T^i, T^j \in R_0$ such that $\mathrm{conf}(F^i) \in M^X_{1_D}$ and $\mathrm{conf}(F^j) \in M^X_{2_D}$, then $\mathrm{cr}_D(T^i \cup T^j, T^k) \geq 3$ for any T^k, $k \neq i, j$.*

Proof. By the assumption, $\mathrm{cr}_D(T^i, T^k) \geq 1$ and $\mathrm{cr}_D(T^j, T^k) \geq 1$. Hence, we need to show that there is no T^k with $\mathrm{cr}_D(T^i, T^k) = \mathrm{cr}_D(T^j, T^k) = 1$.

Assume that $\mathrm{cr}_D(T^i, T^k) = \mathrm{cr}_D(T^j, T^k) = 1$. Then, for $\mathrm{conf}(F^i) \in M^A_{1_D}$ and $\mathrm{conf}(F^j) \in M^A_{2_D}$, the rotation of the vertex t_k must be obtained by one interchange of two adjacent elements from some cyclic permutation in $\overline{M}^A_1$ as well as by one interchange of two adjacent elements from some cyclic permutation in $\overline{M}^A_2$. The cyclic permutations in $\overline{M}^A_1$ are of type $(1xyz5)$, where the ordered triple (xyz) is one of (234), (423), and (342). Thus, by one interchange of two adjacent elements only the permutations $(15xyz)$, $(1yz5x)$, $(1yxz5)$, $(1xzy5)$, and $(1xy5z)$ can be obtained. In Table 1 it is easy to verify that these permutations are elements of $M^A_2 \cup \overline{M}^A_2 \cup \overline{M}^B_1$. Similarly, the cyclic

permutations in $\overline{M}_2^A$ are of type $(15xyz)$ and by one interchange of adjacent elements only the permutations $(1xyz5)$, $(1x5yz)$, $(15yxz)$, $(15xzy)$, and $(1z5xy)$ can be obtained. These permutations are elements of $M_1^A \cup \overline{M}_1^A \cup \overline{M}_2^B$. Since $\left(M_2^A \cup \overline{M}_2^A \cup \overline{M}_1^B\right) \cap \left(M_1^A \cup \overline{M}_1^A \cup \overline{M}_2^B\right) = \emptyset$, there is no T^k with $\mathrm{cr}_D(T^i, T^k) = \mathrm{cr}_D(T^j, T^k) = 1$.

If $\mathrm{conf}(F^i) \in M_{1_D}^B$ and $\mathrm{conf}(F^j) \in M_{2_D}^B$, the proof proceeds in the same way. ■

Corollary 3. *Let D be a good, antipode-free drawing of $G^* + D_n$, $n \geq 3$, and $X \in \{A, B\}$. Let $M_{1_D}^X$ and $M_{2_D}^X$ be non-empty sets of configurations. If $T^i, T^j \in R_0$ such that $\mathrm{conf}(F^i) \in M_{1_D}^X$ and $\mathrm{conf}(F^j) \in M_{2_D}^X$, then $\mathrm{cr}_D(G^* \cup T^i \cup T^j, T^k) \geq 4$ for any $T^k \notin R_0$.*

Lemma 4. *Let D be a good and antipode-free drawing of $G^* + D_n$ for $n > 3$. If $T^i, T^j, T^k \in R_0$ such that F^i, F^j, and F^k have different configurations from $\{A_1, B_2, B_6\}$, then $\mathrm{cr}_D(G^* \cup T^i \cup T^j \cup T^k, T^l) \geq 5$ for any $T^l \notin R_0$. The same holds if F^i, F^j, and F^k have different configurations from any of the sets $\{A_3, B_4, B_2\}$, $\{A_5, B_6, B_4\}$, $\{A_2, B_5, B_1\}$, $\{A_4, B_1, B_3\}$, $\{A_6, B_3, B_5\}$, $\{B_1, B_3, B_5\}$, and $\{B_2, B_4, B_6\}$.*

Proof. As the drawing D is antipode-free, $\mathrm{cr}_D(T^i, T^l) \geq 1$, $\mathrm{cr}_D(T^j, T^l) \geq 1$, $\mathrm{cr}_D(T^k, T^l) \geq 1$, and $\mathrm{cr}_D(G^*, T^l) \geq 1$. Thus, if $\mathrm{cr}_D(G^*, T^l) \geq 2$, we are done. Otherwise we need to show that there is no $T^l \notin R_0$ with $\mathrm{cr}_D(T^i, T^l) = \mathrm{cr}_D(T^j, T^l) = \mathrm{cr}_D(T^k, T^l) = 1$.

Let us solve the first case when three different configurations of F^i, F^j, and F^k are elements of the set $\{A_1, B_2, B_6\}$. If, in D, the subgraph T^l crosses each of T^i, T^j, and T^k exactly once, then the permutation which represents the rotation of the vertex t_l must be obtained by one interchange of adjacent elements from each of $\overline{A}_1 = (12345)$, $\overline{B}_2 = (14523)$, and $\overline{B}_6 = (12534)$. It is easy to find out that there is only one such permutation, namely $\overline{A}_2 = (15234)$. The drawing of F^i with $\mathrm{conf}(F^i) = A_1$ in Figure 3(a) shows that the restrictions $\mathrm{cr}_D(G^*, T^l) = 1$ and $\mathrm{cr}_D(T^i, T^l) = 1$ force that the vertex t_l must be placed in one of the regions $\omega_{3,4}$ and ω_4. If $t_l \in \omega_4$, then the edge $t_l v_2$ must cross the edge $t_i v_1$ and the edge $t_l v_3$ crosses the edge $v_1 v_4$. Thus, $\mathrm{rot}_D(t_l) = (13452) = \overline{B}_5$. For the case when $t_l \in \omega_{3,4}$, depending on which of the edges $t_i v_4$ and $v_1 v_4$ is crossed by the edge $t_l v_5$, $\mathrm{rot}_D(t_l) = (12354) = \overline{B}_1$ or $\mathrm{rot}_D(t_l) = (13245) = A_4$. This implies that the edges of T^l cross $G^* \cup T^i$ at least three times if $\mathrm{rot}_D(t_l) = \overline{A}_2 = (15234)$. Hence, the proof is done if F^i, F^j, and F^k have different configurations from $\{A_1, B_2, B_6\}$.

For the case $F^i, F^j, F^k \in \{A_3, B_4, B_2\}$, only the permutation $\overline{A}_4 = (15423)$ can be obtained by one interchange of adjacent elements from each of $\overline{A}_3 = (14235)$, $\overline{B}_4 = (13542)$, and $\overline{B}_2 = (14523)$. As $A_3 = \Pi_1(A_1)$, the corresponding

drawing of F^i with $\mathrm{conf}(F^i) = A_3$ is possible to obtain from the previous drawing of F^i with $\mathrm{conf}(F^i) = A_1$ shown in Figure 3(a) such that the vertices v_2, v_3, and v_4 are replaced by the vertices v_4, v_2, and v_3, respectively. This implies that all rotations for which $\mathrm{cr}_D(G^*, T^l) = \mathrm{cr}_D(T^i, T^l) = 1$ can be obtained from the permutations $\overline{B}_1$, $\overline{B}_5$, and A_4 using the function Π_1. Thus, $\mathrm{cr}_D(G^*, T^l) = \mathrm{cr}_D(T^i, T^l) = 1$ only if $\mathrm{rot}_D(t_l) \in \{\overline{B}_3, \overline{B}_1, A_6\}$. As $\overline{A}_4 = (15423) \notin \{\overline{B}_3, \overline{B}_1, A_6\}$, the proof is done for $F^i, F^j, F^k \in \{A_3, B_4, B_2\}$.

1.	A_1, B_2, B_6	$\overline{A}_2$	$\overline{B}_1, \overline{B}_5, A_4$
2.	A_3, B_4, B_2	$\overline{A}_4$	$\overline{B}_3, \overline{B}_1, A_6$
3.	A_5, B_6, B_4	$\overline{A}_6$	$\overline{B}_5, \overline{B}_3, A_2$
4.	A_2, B_5, B_1	$\overline{A}_1$	$\overline{B}_6, \overline{B}_2, A_5$
5.	A_4, B_1, B_3	$\overline{A}_3$	$\overline{B}_2, \overline{B}_4, A_1$
6.	A_6, B_3, B_5	$\overline{A}_5$	$\overline{B}_4, \overline{B}_6, A_3$

Table 3. The discussed configurations of F^i, F^j, and F^k, respectively, in the first column; the unique $\mathrm{rot}_D(t_l)$ with $Q(\mathrm{rot}_D(t_l), \mathrm{rot}_D(t_s)) = 1$, $s = i, j, k$, in the second column; all $\mathrm{rot}_D(t_l)$ with $\mathrm{cr}_D(T^i, T^l) = \mathrm{cr}_D(G^*, T^l) = 1$ in the third column.

We remark that also $\Pi_1(B_2) = B_4$, $\Pi_1(B_6) = B_2$, and $\Pi_1(\overline{A}_2) = \overline{A}_4$, which confirms that the second row of Table 3 is obtained from the first using transformation Π_1. Now, applying Π_1 on the second row, the permutations in the third row are obtained, which proves the case for $F^i, F^j, F^k \in \{A_5, B_6, B_4\}$.

If $F^i, F^j, F^k \in \{A_2, B_5, B_1\}$, only the permutation $\overline{A}_1 = (12345)$ can be obtained by one interchange of adjacent elements from each of $\overline{A}_2 = (15234)$, $\overline{B}_5 = (13452)$, and $\overline{B}_1 = (12354)$. As $A_2 = \Pi_2(\overline{A}_1)$, in a similar way as in the second paragraph of the proof it can be shown that $\mathrm{cr}(G^*, T^l) = \mathrm{cr}(T^i, T^l) = 1$ only if $\mathrm{rot}_D(t_l) \in \{\Pi_2(B_1), \Pi_2(B_5), \Pi_2(\overline{A}_4)\} = \{\overline{B}_6, \overline{B}_2, A_5\}$. Thus, the fourth row of the Table 3 is obtained by applying the function Π_2 on the inverse configurations from the first row. Of course, the function Π_1 also transforms the fourth row to the fifth and the fifth row to the sixth. Thus, all these cases are equivalent, and the proof is done for the first six cases.

The last two cases are much easier. If $F^i, F^j, F^k \in \{B_1, B_3, B_5\}$, no permutation can be obtained by one interchange of adjacent elements from each of $\overline{B}_1 = (12354)$, $\overline{B}_3 = (14253)$, and $\overline{B}_5 = (13452)$. The same holds when $F^i, F^j, F^k \in \{B_2, B_4, B_6\}$.

So, $\mathrm{cr}_D(T^i \cup T^j \cup T^k, T^l) \geq 4$ in all cases and, as $T^l \notin R_0$, $\mathrm{cr}_D(G^* \cup T^i \cup T^j \cup T^k, T^l) \geq 5$. This completes the proof. ■

Lemma 5. *Let D be a good and antipode-free drawing of $G^* + D_n$, $n > 3$. Let $|R_0| \geq \lceil \frac{n}{2} \rceil + 1$ and let $T^i, T^j, T^k \in R_0$ be three different subgraphs of $G^* + D_n$. If*

$$\text{(4)} \qquad \mathrm{cr}_D\left(G^* \cup T^i \cup T^j \cup T^k, T^l\right) \geq 8 \quad \textit{for any} \;\; T^l \in R_0 \setminus \left\{T^i, T^i, T^k\right\}$$

and

$$\text{(5)} \qquad \mathrm{cr}_D\left(G^* \cup T^i \cup T^j \cup T^k, T^l\right) \geq 5 \quad \textit{for any} \;\; T^l \notin R_0,$$

then there are at least $4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor$ *crossings in* D.

Proof. For easier reading, let $r = |R_0|$. By the assumption of Lemma 5, $r \geq \left\lceil \frac{n}{2} \right\rceil + 1$. As the graph $G^* + D_n$ is the union of two edge disjoint graphs $K_{5,n-3}$ and $G^* \cup T^i \cup T^j \cup T^k$, the number of crossings in D satisfies

$$\begin{aligned} \mathrm{cr}_D(G^* + D_n) \;=\; & \mathrm{cr}_D(K_{5,n-3}) + \mathrm{cr}_D\left(G^* \cup T^i \cup T^j \cup T^k\right) \\ & + \mathrm{cr}_D\left(K_{5,n-3}, G^* \cup T^i \cup T^j \cup T^k\right). \end{aligned}$$

By Lemma 1, $\mathrm{cr}_D(T^i \cup T^j \cup T^k) \geq 6$ and, by the assumption, the subgraph $K_{5,n-3}$ contains $r - 3 \geq 0$ subgraphs T^l which are elements of R_0. By (1), $\mathrm{cr}(K_{5,n-3}) = 4\left\lfloor \frac{n-3}{2} \right\rfloor \left\lfloor \frac{n-4}{2} \right\rfloor$. Thus, using the conditions (4), (5) and the fact $\mathrm{cr}_D(G^* \cup T^i \cup T^j \cup T^k) = \mathrm{cr}_D(T^i \cup T^j \cup T^k) \geq 6$, we have

$$\begin{aligned} & \mathrm{cr}_D(G^* + D_n) \geq 4\left\lfloor \frac{n-3}{2} \right\rfloor \left\lfloor \frac{n-4}{2} \right\rfloor + 6 + 8(r-3) + 5(n-r) \\ & = 4\left\lfloor \frac{n-3}{2} \right\rfloor \left\lfloor \frac{n-4}{2} \right\rfloor + 5n + 3r - 18 \geq 4\left\lfloor \frac{n-3}{2} \right\rfloor \left\lfloor \frac{n-4}{2} \right\rfloor + 5n + 3\left(\left\lceil \frac{n}{2} \right\rceil + 1\right) - 18 \\ & = 4\left\lfloor \frac{n-3}{2} \right\rfloor \left\lfloor \frac{n-4}{2} \right\rfloor + 5n + 3\left\lceil \frac{n}{2} \right\rceil - 15 > 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor. \end{aligned}$$

This completes the proof. ■

Lemma 6. *Let* D *be a good and antipode-free drawing of* $G^* + D_n$, $n > 3$. *Let* $|R_0| \geq \left\lceil \frac{n}{2} \right\rceil + 1$ *and let* $T^i, T^j \in R_0$ *be different subgraphs of* $G^* + D_n$. *If both conditions*

$$\text{(6)} \qquad \mathrm{cr}_D\left(G^* \cup T^i \cup T^j, T^k\right) \geq 5 \quad \textit{for any} \;\; T^k \in R_0 \setminus \left\{T^i, T^j\right\}$$

and

$$\text{(7)} \qquad \mathrm{cr}_D\left(G^* \cup T^i \cup T^j, T^k\right) \geq 4 \quad \textit{for any} \;\; T^k \notin R_0$$

hold, or the condition

$$\text{(8)} \qquad \mathrm{cr}_D\left(G^* \cup T^i \cup T^j, T^k\right) \geq 6 \quad \textit{for any} \;\; T^k \in R_0 \setminus \left\{T^i, T^j\right\}$$

holds, then there are at least $4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor$ *crossings in* D.

Proof. We use the notation $r = |R_0|$ again. By the assumption, $r \geq \lceil \frac{n}{2} \rceil + 1$. The graph $G^* + D_n$ is the union of two edge disjoint subgraphs $K_{5,n-2}$ and $G^* \cup T^i \cup T^j$ such that $K_{5,n-2}$ contains $r - 2 \geq 1$ subgraphs T^k with $\mathrm{cr}_D(G^*, T^k) = 0$. As the graph $G^* + D_2$ contains $K_{3,3}$ as a subgraph, $\mathrm{cr}(G^* + D_2) \geq 1$. Hence, $\mathrm{cr}_D(G^* \cup T^i \cup T^j) \geq \mathrm{cr}(G^* + D_2) \geq 1$. By (1), $\mathrm{cr}(K_{5,n-2}) = 4 \lfloor \frac{n-2}{2} \rfloor \lfloor \frac{n-3}{2} \rfloor$. This fact and the conditions (6) and (7) imply that

$$\begin{aligned}
&\mathrm{cr}_D(G^* + D_n) = \mathrm{cr}_D(K_{5,n-2}) + \mathrm{cr}_D(G^* \cup T^i \cup T^j) + \mathrm{cr}(K_{5,n-2}, G^* \cup T^i \cup T^j) \\
&\geq 4 \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 1 + 5(r-2) + 4(n-r) = 4 \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + r + 4n - 9 \\
&\geq 4 \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + \left(\left\lceil \frac{n}{2} \right\rceil + 1 \right) + 4n - 9 \geq 4 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor .
\end{aligned}$$

Moreover, the condition (8) and the fact that $\mathrm{cr}_D(G^* \cup T^i \cup T^j, T^k) \geq 3$ for any $T^k \notin R_0$ imply that

$$\begin{aligned}
&\mathrm{cr}_D(G^* + D_n) \geq 4 \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 1 + 6(r-2) + 3(n-r) \\
&= 4 \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 3r + 3n - 11 \geq 4 \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 3 \left(\left\lceil \frac{n}{2} \right\rceil + 1 \right) + 3n - 11 \\
&\geq 4 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor .
\end{aligned}$$

This completes the proof. ■

5. The Crossing Number of $G^* + D_n$

Now we are prepared to prove the main result of the paper.

Theorem 7. *For $n \geq 1$, $\mathrm{cr}(G^* + D_n) = 4 \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$.*

Proof. In Figure 1 there are two drawings of the graph $G^* + D_n$ with $4 \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$ crossings. Thus, $\mathrm{cr}(G^* + D_n) \leq 4 \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$. We prove the reverse inequality by induction on n. The graph $G^* + D_1$ is planar. The graph $G^* + D_2$ contains a subgraph isomorphic to $K_{3,3}$. So, Theorem 7 is true for $n = 1$ and $n = 2$. Assume that for $n \geq 3$ there is a good drawing D of the graph $G^* + D_n$ with fewer than $4 \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$ crossings and that $\mathrm{cr}(G^* + D_m) \geq 4 \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor + \lfloor \frac{m}{2} \rfloor$ for every integer $m < n$. Our assumption on D, together with $\mathrm{cr}(K_{5,n}) = 4 \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$, implies that

$$\mathrm{cr}_D(G^*) + \mathrm{cr}_D(G^*, K_{5,n}) < \left\lfloor \frac{n}{2} \right\rfloor .$$

Hence, $|R_0| \geq \lceil \frac{n}{2} \rceil + 1$ because at most $\lfloor \frac{n}{2} \rfloor - 1$ subgraphs T^i can cross G^*. If $n = 3$, this forces that none of T^1, T^2, and T^3 crosses G^* in D. But, in such a case, Lemma 1 implies that D has at least six crossings and Theorem 7 is true. For $n \geq 4$, the drawing D contains at least three subgraphs T^i which are elements of R_0.

Let us show that the considered drawing D must be antipode-free. For a contradiction suppose that, without loss of generality, $\mathrm{cr}_D\big(T^{n-1}, T^n\big) = 0$. As $\mathrm{cr}(G^*+D_2) = 1$ and the edges of G^* cannot cross each other, $\mathrm{cr}_D(G^*, T^{n-1} \cup T^n) \geq 1$. By (1), $\mathrm{cr}(K_{5,3}) = 4$. This implies that any T^i, $i = 1, 2, \ldots, n-2$, crosses $T^{n-1} \cup T^n$ at least four times. So, using the equations $G^* + D_n = (G^* + D_{n-2}) \cup \big(T^{n-1} \cup T^n\big)$ and $G^* + D_{n-2} = G^* \cup \big(\bigcup_{i=1}^{n-2} T^i\big)$, the number of crossings in D satisfies

$$\begin{aligned}
&\mathrm{cr}_D(G^* + D_n) = \mathrm{cr}_D\left(G^* \cup \bigcup_{i=1}^{n-2} T^i\right) + \mathrm{cr}_D\big(T^{n-1} \cup T^n\big) + \mathrm{cr}_D\big(G^*, T^{n-1} \cup T^n\big) \\
&+ \mathrm{cr}_D\left(\textstyle\bigcup_{i=1}^{n-2} T^i, T^{n-1} \cup T^n\right) \\
&\geq 4\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + \left\lfloor \frac{n-2}{2} \right\rfloor + 0 + 1 + 4(n-2) \geq 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor.
\end{aligned}$$

This contradiction confirms that the drawing D is antipode-free.

We know that, in D, at least three subgraphs T^i do not cross G^*. For these $T^i \in R_0$, we will discuss the existence of possible configurations of $F^i = G^* \cup T^i$ in the drawing D. Using the values in Table 2 we show that, in D, both conditions (4) and (5) of Lemma 5 hold, or both conditions (6) and (7) of Lemma 6 hold, or the condition (8) of Lemma 6 holds.

Case 1. $M^A_{1_D} \neq \emptyset$ and $M^A_{2_D} \neq \emptyset$. If we fix any two T^i, $T^j \in R_0$ such that F^i, F^j have configurations from $M^A_{1_D}$, $M^A_{2_D}$, respectively, then the condition (6) holds. This can be easily verified by summing the values in the considered rows for each column of Table 2. Corollary 3 implies that also the condition (7) is fulfilled and therefore, by Lemma 6, in D there are at least $4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor$ crossings. This contradicts the assumption of D.

Case 2. $M^A_{1_D} = \emptyset$ or $M^A_{2_D} = \emptyset$.

(a) $M^B_{1_D} \neq \emptyset$ and $M^B_{2_D} \neq \emptyset$.

We will consider two subcases.

(1) $B_r \in M^B_{1_D} \cup M^B_{2_D}$ and $B_{r+1} \in M^B_{1_D} \cup M^B_{2_D}$ for some $r \in \{1, 3, 5\}$. If we fix any two T^i, $T^j \in R_0$ such that F^i, F^j have the configurations B_r and B_{r+1}, respectively, then Table 2 confirms that the condition (6) holds.

(2) $B_r \notin M^B_{1_D} \cup M^B_{2_D}$ or $B_{r+1} \notin M^B_{1_D} \cup M^B_{2_D}$ for any $r = 1, 3, 5$. Hence, we have $\left|M^B_{1_D}\right| = 1$ or $\left|M^B_{2_D}\right| = 1$. If we fix any two T^i, $T^j \in R_0$ such that F^i, F^j have configurations from $M^B_{1_D}$ and $M^B_{2_D}$, respectively, then the condition (6) holds. This can be verified in the following way. If $M^B_{1_D} = \{B_1\}$, then $M^B_{2_D}$ cannot contain B_2. This forces that, for T^k in the condition (6), the configuration of F^k can be only one of $A_1, A_2, \ldots, A_6, B_1, B_4$, and B_6. For these columns in Table 2, the sum of the values in the rows B_1 and B_4 as well as in the rows B_1 and B_6 is at least five, which implies that the condition (6) holds. The verification for all five other possibilities, i.e., $M^B_{1_D} = \{B_3\}$, $M^B_{1_D} = \{B_5\}$, $M^B_{2_D} = \{B_2\}$, $M^B_{2_D} = \{B_4\}$, and $M^B_{2_D} = \{B_6\}$, proceeds in the same way.

The condition (7) follows from Corollary 3 in both cases. Hence, by Lemma 6, the discussed drawing contradicts the assumption of D again.

(b) $M^B_{1_D} = \emptyset$ and $M^B_{2_D} = \emptyset$. Assume that $M^A_{1_D} \neq \emptyset$. Since $M^A_{2_D} = \emptyset$, we can fix any two different T^i, $T^j \in R_0$ such that F^i, F^j have the configurations from $M^A_{1_D}$. It is easy to verify in Table 2 that the condition (8) of Lemma 6 holds. The same holds also for the case when $M^A_{2_D} \neq \emptyset$. Hence, in D there are at least $4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor$ crossings again.

(c1) $M^B_{1_D} = \emptyset$ and $M^B_{2_D} \neq \emptyset$.

We will discuss three possibilities.

(1) $\left|M^B_{2_D}\right| = 1$. If $M^A_{1_D} \cup M^A_{2_D} = \emptyset$, then we fix any two different T^i, $T^j \in R_0$ such that F^i and F^j have the same configuration from $M^B_{2_D}$. For $T^k \in R_0$, the subgraph F^k must have the same configuration as both F^i and F^j. Thus, the condition (8) holds.

If either $M^A_{1_D} \neq \emptyset$, or $M^A_{2_D} \neq \emptyset$, then we fix any two T^i, $T^j \in R_0$ such that F^i and F^j have configurations from $M^B_{2_D}$ and $M^A_{1_D} \cup M^A_{2_D}$, respectively. For example if $M^A_{1_D} \neq \emptyset$ and $M^B_{2_D} = \{B_2\}$, then the pair of $\mathrm{conf}(F^i)$ and $\mathrm{conf}(F^j)$ can be only one of B_2 and A_r, for $r \in \{1, 3, 5\}$. For $T^k \in R_0$, the configuration of F^k is one of A_1, A_3, A_5, and B_2. In Table 2 it is easy to verify that the condition (8) holds in this case. The condition (8) holds also when $M^B_{2_D} = \{B_4\}$ or $M^B_{2_D} = \{B_6\}$. The verification proceeds in a similar way also for the case when $M^A_{2_D} \neq \emptyset$. Hence, by Lemma 6, a contradiction with the assumption of the drawing D is obtained in all these cases.

(2) $\left|M^B_{2_D}\right| = 2$. Assume first that $M^A_{1_D} = \emptyset$. Let us fix any two T^i, $T^j \in R_0$ such that F^i, F^j have different configurations from $M^B_{2_D}$, for example B_2 and B_4. In this case, for $T^k \in R_0$, also $\mathrm{conf}(F^k)$ can be only element of $\{A_2, A_4, A_6, B_2, B_4\}$ and the condition (8) holds. The same result is obtained when $M^B_{2_D} = \{B_2, B_6\}$ or $M^B_{2_D} = \{B_4, B_6\}$.

If $M^A_{1_D} \neq \emptyset$, then $M^A_{2_D} = \emptyset$. Assume first that $M^B_{2_D} = \{B_2, B_6\}$.

If $A_1 \notin M^A_{1_D}$, let us fix any two T^i, $T^j \in R_0$ such that F^i and F^j have the configurations B_2 and B_6, respectively. Then, for $T^k \in R_0$, $\mathrm{conf}(F^k) \in \{A_3, A_5, B_2, B_6\}$ and the condition (8) holds.

If $A_1 \in M^A_{1_D}$, then we fix any three T^i, T^j, $T^k \in R_0$ such that F^i, F^j, F^k have configurations A_1, B_2, B_6, respectively. Then for any T^l with $\mathrm{conf}(F^l) \in \{A_1, A_3, A_5, B_2, B_6\}$ the condition (4) of Lemma 5 holds. Moreover, Lemma 4 implies that also the condition (5) of Lemma 5 holds for any $T^l \notin R_0$.

For the cases when $M^B_{2_D} = \{B_2, B_4\}$ or $M^B_{2_D} = \{B_4, B_6\}$, the similar discussion for A_3 and A_5, respectively, confirms that holds the condition (8) of Lemma 6, or hold both conditions (4) and (5) of Lemma 5. This also contradicts the assumption of D.

(3) $|M^B_{2_D}| = 3$. If we fix any three T^i, T^j, $T^k \in R_0$ such that F^i, F^j, F^k have the configurations B_2, B_4, B_6, respectively, then the condition (4) of Lemma 5 holds. Lemma 4 confirms that also the condition (5) holds.

Thus, in all three cases, the contradiction with the assumption of the drawing D is obtained.

(c2) $M^B_{2_D} = \emptyset$ and $M^B_{1_D} \neq \emptyset$. Due to symmetry of Table 2, the discussion proceeds in the same way as in the previous case (c1).

Thus, it is shown that there is no good drawing D of the graph of $G^* + D_n$ with fewer than $4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor$ crossings. This completes the proof. ■

6. Some Consequences of the Main Result

Let H be the graph obtained from G^* by adding the edges v_2v_3, v_3v_5 and v_4v_5 and H' be the graph obtained from G^* by adding the edges v_3v_5 and v_4v_5. In [6] it is shown that $\mathrm{cr}(H + D_n) = \mathrm{cr}(H' + D_n) = 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor$ for $n \geq 1$, $\mathrm{cr}(H+P_n) = \mathrm{cr}(H'+P_n) = 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + 1$ for $n \geq 2$ and that $\mathrm{cr}(H+C_n) = \mathrm{cr}(H' + C_n) = 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + 2$ for $n \geq 3$, where P_n and C_n are the path and the cycle on n vertices, respectively.

Let H_1 be the graph obtained from G^* by adding the edge v_3v_5, i.e., $H_1 = G^* \cup \{v_3v_5\}$. Similarly, let $H_2 = G^* \cup \{v_2v_3\}$, $H_3 = G^* \cup \{v_2v_3, v_3v_5\}$, and $H_4 = G^* \cup \{v_2v_3, v_4v_5\}$. Clearly, each of H_i, $i = 1, 2, 3, 4$, is a subgraph of H and therefore each $H_i + D_n$ is a subgraph of $H + D_n$. Thus, $\mathrm{cr}(H_i + D_n) \leq \mathrm{cr}(H + D_n)$ for all $i = 1, 2, 3, 4$. On the other hand, $G^* + D_n$ is a subgraph of each $H_i + D_n$ and therefore, $\mathrm{cr}(H_i + D_n) \geq \mathrm{cr}(G^* + D_n)$ for each $i = 1, 2, 3, 4$. So, we have the next result.

Corollary 8. *For $n \geq 1$, $\mathrm{cr}(H_i + D_n) = 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor$, $i = 1, 2, 3, 4$.*

Into both drawings in Figure 1 we can add the edges $t_1t_2, t_2t_3, \ldots, t_{n-1}t_n$ without additional crossings. Hence, the drawings of the graph $G^* + P_n$ with $4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor$ crossings are obtained. Moreover, by adding the necessary edges without additional crossings, the drawings of $H_2 + P_n$ and $H_4 + P_n$ can be obtained from Figure 1(a), and the drawings of $H_1 + P_n$ and $H_3 + P_n$ can be obtained from Figure 1(b). So, the next result is obvious.

Corollary 9. *For $n \geq 2$, $\mathrm{cr}(G^* + P_n) = \mathrm{cr}(H_i + P_n) = 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor$, $i = 1, 2, 3, 4$.*

Moreover, the edge t_1t_n can be added into both drawings of G^*+P_n in such a way that in the drawing obtained from Figure 1(a) the edge t_1t_n crosses only the edge v_1v_4 and in the drawing obtained from Figure 1(b) this edge crosses only the edge v_1v_3. Thus, the drawings of the graph $G^* + C_n$ with $4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor + 1$ crossings are obtained. It is easy to see that also $\mathrm{cr}(H_i + C_n) \leq 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor + 1$ for $i = 1, 2, 4$ and $n \geq 3$. Only in the drawing of $H_3 + C_n$ obtained from Figure 1(b) the edge t_1t_n crosses the edge v_2v_3 of H_3. This implies that $\mathrm{cr}(H_3 + C_n) \leq 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor + 2$.

Each of the graphs G^*+C_n and H_i+C_n, $i = 1, 2, 3, 4$, contains the subgraph $D_5 + C_n$, where D_5 consists only of the vertices v_1, v_2, v_3, v_4, and v_5. For $x = v_1, v_2, \ldots, v_5$, let T^x denote the subgraph of $G^* + C_n$ ($H_i + C_n$) induced by n edges incident with the vertex x. In the proof of the last theorem of the paper we will need the next results published in [8].

Lemma 10 [8]**.** *Let G be a graph of order m, $m \geq 1$. In an optimal drawing of the join product $G + C_n$, $n \geq 3$, the edges of C_n do not cross each other.*

Lemma 11 [8]**.** *Let D be a good drawing of the join product $D_m + C_n$, $m \geq 2$, $n \geq 3$, in which no edge of C_n is crossed and C_n does not separate the other vertices of the graph. Then, for all $x, y \in \{v_1, v_2, \ldots, v_m\}$, two different subgraphs T^x and T^y cross each other at least $\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$ times in D.*

Theorem 12. *For $n \geq 3$, $\mathrm{cr}(G^* + C_n) = \mathrm{cr}(H_i + C_n) = 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor + 1$ for $i = 1, 2, 4$, and $\mathrm{cr}(H_3 + C_n) = 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor + 2$.*

Proof. It follows from the discussion above that $\mathrm{cr}(G^* + C_n) \leq 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor + 1$. Assume that there is a good drawing D of the graph $G^* + C_n$ with less than $4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor + 1$ crossings. Then none of the edges of C_n is crossed in D, because otherwise removing all the edges of C_n results in a good drawing of the graph $G^* + D_n$ with less than $4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor$ crossings. Hence, the subdrawing of C_n induced by D divides the plane into two regions and, in D, the vertices v_1, v_2, v_3, and v_4 must be placed in one of them. Assume now the subgraph $D_4 + C_n$ of $G^* + C_n$, where D_4 consists of the vertices v_1, v_2, v_3, and v_4. By Lemma 11, the edges of $T^{v_1} \cup T^{v_2} \cup T^{v_3} \cup T^{v_4}$ cross each other at least

$\binom{4}{2}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$ times. But, for $n \geq 3$, $\binom{4}{2}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor > 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor$ which confirms that $\mathrm{cr}(G^* + C_n) = 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor + 1$.

We know that $\mathrm{cr}(H_i + C_n) \leq 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor + 1$ for $i = 1, 2$, and 4. As each $H_i + C_n$ contains $G^* + C_n$ as a subgraph, the opposite inequality applies. We remark that this opposite inequality applies also for the graph $H_3 + C_n$. To prove that $\mathrm{cr}(H_3 + C_n) = 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor + 2$ assume that there is a drawing of the graph $H_3 + C_n$ with $4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor + 1$ crossings. Such a drawing must be optimal and, by Lemma 10, the edges of C_n do not cross each other. Moreover, at most one edge of C_n can be crossed. If no edge of C_n is crossed, then the whole graph H_3 is placed in the same region in the view of the subdrawing of C_n and, by Lemma 11, the edges of $T^{v_1} \cup T^{v_2} \cup T^{v_3} \cup T^{v_4} \cup T^{v_5}$ cross each other at least $\binom{5}{2}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$ times. If C_n is crossed once, regardless of which edge crosses C_n, at least four subgraphs T^x, $x \in \{v_1, v_2, \ldots, v_5\}$, are placed in the same region of C_n and their edges cross each other at least $\binom{4}{2}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$ times and in such a drawing there are at least $\binom{4}{2}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 1$ crossings. Thus, in all considered drawings of $H_3 + C_n$ there are more than $4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor + 1$ crossings and the proof is done. ■

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