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ALTERNATING-PANCYCLISM IN 2-EDGE-COLORED GRAPHS[1]

Narda Cordero-Michel

and

Hortensia Galeana-Sánchez

Instituto de Matemáticas,
Universidad Nacional Autónoma de México
Ciudad Universitaria, México, D.F., C.P. 04510, México

e-mail: narda@matem.unam.mx
hgaleana@matem.unam.mx

Abstract

An *alternating cycle* in a 2-edge-colored graph is a cycle such that any two consecutive edges have different colors. Let $G_1, \dots, G_k$ be a collection of pairwise vertex disjoint 2-edge-colored graphs. The *colored generalized sum* of $G_1, \dots, G_k$, denoted by $\oplus_{i=1}^{k} G_i$, is the set of all 2-edge-colored graphs G such that: (i) $V(G) = \bigcup_{i=1}^{k} V(G_i)$, (ii) $G\langle V(G_i)\rangle \cong G_i$ for $i = 1, \dots, k$ where $G\langle V(G_i)\rangle$ has the same coloring as G_i and (iii) between each pair of vertices in different summands of G there is exactly one edge, with an arbitrary but fixed color. A graph G in $\oplus_{i=1}^{k} G_i$ will be called a *colored generalized sum* (c.g.s.) and we will say that $e \in E(G)$ is an *exterior edge* if and only if $e \in E(G) \setminus \left(\bigcup_{i=1}^{k} E(G_i)\right)$. The set of exterior edges will be denoted by $E_\oplus$. A 2-edge-colored graph G of order $2n$ is said to be an *alternating-pancyclic graph*, whenever for each $l \in \{2, \dots, n\}$, there exists an alternating cycle of length $2l$ in G.

The topics of pancyclism and vertex-pancyclism are deeply and widely studied by several authors. The existence of alternating cycles in 2-edge-colored graphs has been studied because of its many applications. In this paper, we give sufficient conditions for a graph $G \in \oplus_{i=1}^{k} G_i$ to be an alternating-pancyclic graph.

Keywords: 2-edge-colored graph, alternating cycle, alternating-pancyclic graph.

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1. Introduction

Let G be an edge-colored multigraph. An *alternating walk* in G is a walk such that any two consecutive edges have different colors.

Several problems have been modeled by edge-colored multigraphs, the study of applications of alternating walks seems to have started in [15], according to [1], and ever since it has crossed diverse fields, such as genetics [8–10, 16], transportation and connectivity problems [13, 18], social sciences [4] and graph models for conflict resolutions [19–21], as pointed out in [5].

The alternating Hamiltonian path and cycle problems are $\mathcal{NP}$-complete even for $c = 2$, it was proved in [12], and so the problem of deciding if a given graph is alternating pancyclic is as difficult as those two problems.

Let B_r and B'_r be 2-edge-colored complete bipartite graphs with the same partite sets $\{v_1, v_2, \ldots, v_{2r}\}$ and $\{w_1, w_2, \ldots, w_{2r}\}$. The edge set of the red (blue) subrgraph of B_r (B'_r) consists of $\{v_i w_j \mid 1 \leq i, j \leq r\} \cup \{v_i w_j \mid r+1 \leq i, j \leq 2r\}$. In [7], Das proved that a 2-edge-colored complete bipartite multigraphs is vertex alternating-pancyclic if and only if it has an alternating Hamiltonian cycle and is not color-isomorphic to one of the graphs B_r, $B_{r'}$ $(r = 2, 3, \ldots)$.

Figure 1 shows a graph which is isomorphic neither to B_r nor to $B_{r'}$ and has no spanning complete bipartite alternating Hamiltonian graph, so it does not fulfill the hypothesis asked in the theorem by Das. However, by Proposition 22 we can assert that it really is an alternating-pancyclic graph. Clearly, an infinite class of such graphs can be easily constructed.

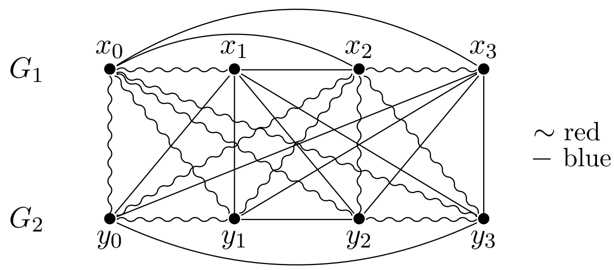
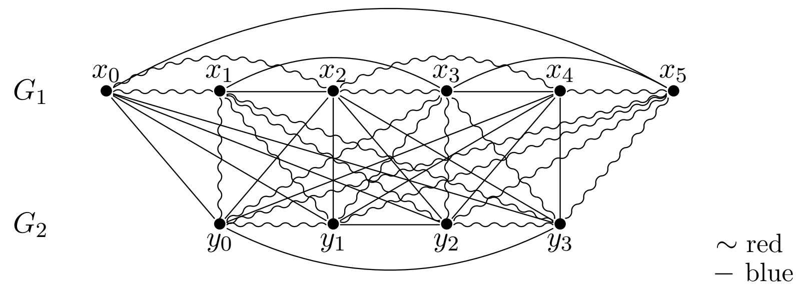


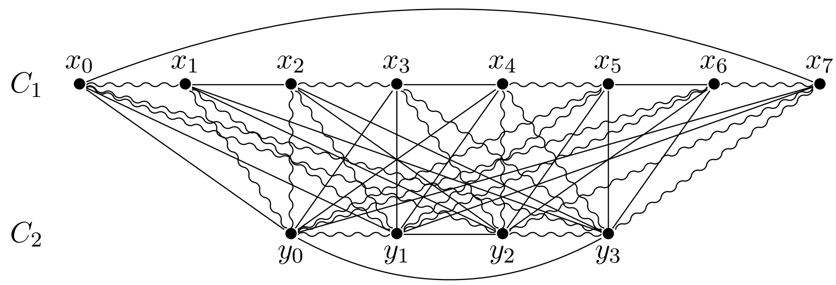
Figure 1. A graph $G \in G_1 \oplus G_2$.

In [1], Bang-Jensen and Gutin characterized 2-edge-colored complete multigraphs which are (vertex) alternating-pancyclic. Clearly, our results do not ask for completeness of the considered graphs.

In [2], Bang-Jensen and Gutin give a polynomial time algorithm to find a longest alternating cycle in a complete 2-edge-colored graph. In our results we do not ask for completeness of the graph and, under certain conditions, not only a longest cycle is found but alternating cycles of each even length.



(i)



(ii)

Figure 2. (i) A graph in $G_1 \oplus G_2$. (ii) A graph in $C_1 \oplus C_2$.

In [3], to prove that a complete bipartite 2-edge-colored graph is (vertex) alternating-pancyclic, Bang-Jensen and Gutin consider the following construction (known as DHM-construction [3]). Given a complete bipartite 2-edge-colored graph G, with partition (X, Y), construct a complete 2-edge-colored graph H from G by adding all edges between vertices in X with red color and all edges between vertices in Y with blue color. That is, the graph induced by X in H is a complete red monochromatic graph and the graph induced by Y in H is a complete blue monochromatic graph. In this way, H is a complete 2-edge-colored graph such that every alternating cycle in H is an alternating cycle in G, as no alternating cycle in H contains edges in $H\langle X\rangle$ or $H\langle Y\rangle$; and thus, H is (vertex) alternating-pancyclic if and only if G is (vertex) alternating-pancyclic.[2] We now give an example which shows that is not possible to give a DHM type construction to prove our results. A DHM type construction would add edges to this graph until a complete graph is obtained, the goal of this construction is that the complete graph and the original graph have the same set of alternating cycles.

[2]This construction is due to Das [7] and later by Häggkvist and Manoussakis [14], it was used to study Hamiltonian alternating cycles in complete bipartite 2-edge-colored graphs.

Notice that in the graph of Figure 1, it does not matter which color is given to an added edge between y_0 and y_2, we obtain alternating cycles which do not exist in the original graph.

However, the graph in Figure 1 satisfies the hypothesis of Proposition 22 and so it is indeed alternating-pancyclic.

A similar analysis can be done for the graphs in Figure 2. They have no spanning complete bipartite alternating Hamiltonian subgraph, so they do not fulfill the hypothesis of Das' theorem and it does not matter which color we give to an added edge between y_0 and y_2, we obtain alternating cycles which do not exist in the original graphs, so we cannot use a DHM type construction to determine if they are alternating pancyclic graphs. However, the graph in Figure 2(i) satisfies the hypothesis of Corollary 20, so it is a vertex alternating-pancyclic graph; and the graph in Figure 2(ii) satisfies the hypothesis of Theorem 1, so it is a vertex alternating-pancyclic graph.

These simple examples show that our results work for different graphs than complete bipartite and complete 2-edge-colored graphs.

In other publications, such as [11] and [17], authors studied the existence of alternating cycles of certain lengths in terms of vertex degrees.

In [6], we proved Theorem 1 and a generalization of it for k summands, Theorem 2.

Theorem 1. *Let G_1 and G_2 be two vertex disjoint graphs with alternating Hamiltonian cycles, $C_1 = x_0x_1 \cdots x_{2n-1}x_0$ and $C_2 = y_0y_1 \cdots y_{2m-1}y_0$, respectively, and $G \in G_1 \oplus G_2$. If there is no good pair in G, and for each $i \in \{1, 2\}$, in C_i there is a non-singular vertex with respect to C_{3-i}, then G is vertex alternating-pancyclic.*

Theorem 2. *Let $G_1, G_2, \ldots, G_k$ be a collection of $k \geq 2$ vertex disjoint graphs with Hamiltonian alternating cycles, $C_1, C_2, \ldots, C_k$, respectively, and $G \in \oplus_{i=1}^{k} G_i$. If there is no good cycle in G and, for each pair of different indices $i, j \in [1, \ldots, k]$, in C_i there is a non-singular vertex with respect to C_j, then G is vertex alternating-pancyclic.*

In this paper we analyze the cases where good pairs, singular vertices or good cycles[3] appear and we give a complete classification of graphs in $G_1 \oplus G_2$ which are alternating-pancyclic graphs, vertex alternating-pancyclic graphs or, simply, Hamiltonian alternating graphs (Figure 3).

Theorem 3. *Let G_1 and G_2 be two vertex disjoint 2-edge-colored graphs with alternating Hamiltonian cycles, $C_1 = x_0x_1 \cdots x_{2n-1}x_0$ and $C_2 = y_0y_1 \cdots y_{2m-1}y_0$, respectively; and $G \in G_1 \oplus G_2$. Then one of the following assertions hold:*

[3]In Section 2 we define *good pair*, in Section 3 we define *singular vertex* and in Section 4 we define *good cycle*.

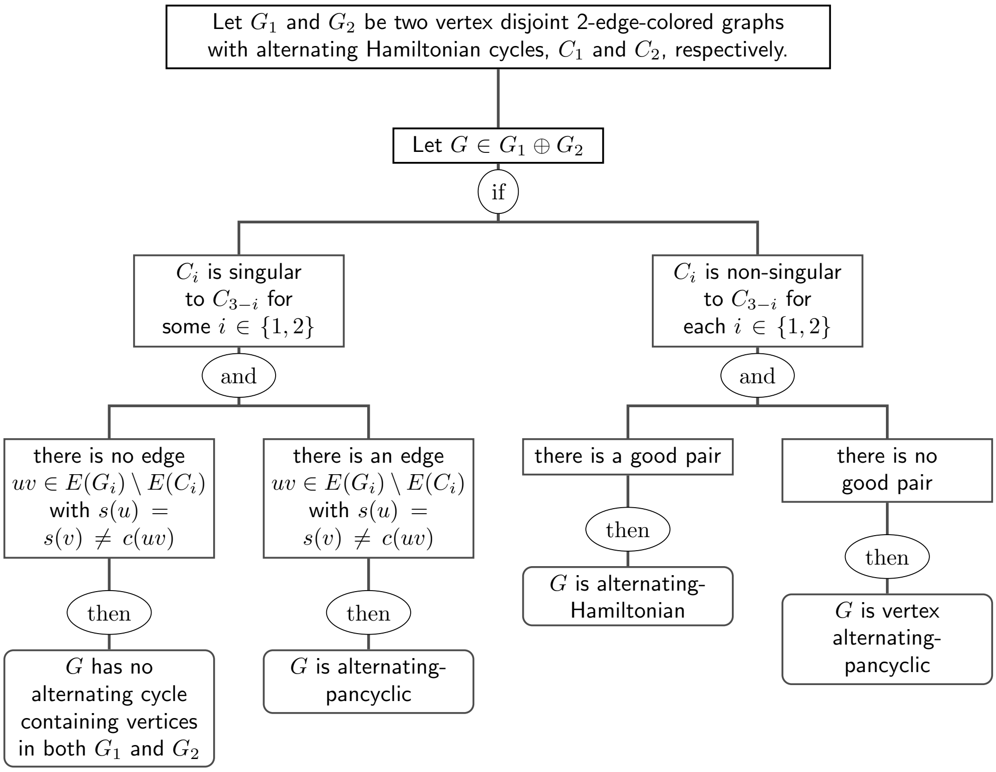


Figure 3. Theorem 3.

(i) *C_i is singular with respect to C_{3-i} for some $i \in \{1,2\}$ and, either,*
   (a) *G has no alternating cycle containing vertices in both G_1 and G_2; or*
   (b) *G is alternating-pancyclic;*

(ii) *C_i is non-singular with respect to C_{3-i} for each $i \in \{1,2\}$ and, either,*
   (a) *there is a good pair and G contains an alternating Hamiltonian cycle; or*
   (b) *there is no good pair and G is vertex alternating-pancyclic.*

We also prove an extension of this result, which provides sufficient conditions for a graph in the c.g.s. of k alternating Hamiltonian graphs to be an alternating Hamiltonian graph or an alternating-pancyclic graph.

Theorem 4. *Let $G_1, G_2, \ldots, G_k$ be a collection of k vertex disjoint 2-edge-colored graphs with alternating Hamiltonian cycles, $C_1, C_2, \ldots, C_k$, respectively, and $G \in \oplus_{i=1}^{k} G_i$.*

(i) *If G contains no good cycle and G contains an alternating cycle γ such that $V(\gamma) \cap V(G_i) \neq \emptyset$ for each $i \in [1, \ldots, k]$, then G is an alternating-pancyclic graph.*

(ii) *G contains an alternating cycle γ such that $V(\gamma) \cap V(G_i) \neq \emptyset$ for each $i \in [1, \ldots, k]$ if and only if G is an alternating Hamiltonian graph.*

It should be noted that the proofs in this paper carry on an implicit algorithm to construct the alternating cycles.

2. Definitions

In this paper $G = (V(G), E(G))$ will denote a simple graph. A *k-edge-coloring* of G is a function c from the edge set, $E(G)$, to a set of k colors, $\{1, 2, \ldots, k\}$. A graph G provided with a k-edge-coloring is a *k-edge-colored-graph.*

A path or a cycle in G will be called an *alternating path* or an *alternating cycle* whenever two consecutive edges have different colors. An alternating cycle containing each vertex of the graph is an *alternating Hamiltonian cycle* and a graph containing an alternating Hamiltonian cycle will be called an *alternating Hamiltonian graph.* A 2-edge-colored graph G of order $2n$ is *alternating-pancyclic* whenever G contains an alternating cycle of length $2k$ for each $k \in \{2, \ldots, n\}$; and G is *vertex alternating-pancyclic* if and only if, for each vertex $v \in V(G)$ and each $k \in \{2, \ldots, n\}$, G contains an alternating cycle of length $2k$ passing through v.

For further details we refer the reader to [3] pages 608–610.

Remark 5. Clearly the c.g.s. of two vertex disjoint graphs is well defined and commutative. Let G_1, G_2, G_3 be three vertex disjoint 2-edge-colored graphs. It is easy to see that the sets $(G_1 \oplus G_2) \oplus G_3$ defined as $\bigcup_{G \in G_1 \oplus G_2} G \oplus G_3$ and $G_1 \oplus (G_2 \oplus G_3)$ defined as $\bigcup_{G' \in G_2 \oplus G_3} G_1 \oplus G'$ are equal, thus $\oplus_{i=1}^{3} G_i = (G_1 \oplus G_2) \oplus G_3 = G_1 \oplus (G_2 \oplus G_3)$ is well defined. By means of an inductive process it is easy to see that the c.g.s. of k vertex disjoint 2-edge-colored graphs is well defined, commutative and associative.

Notation 6. Let k_1 and k_2 be two positive integers, such that $k_1 \leq k_2$. We will denote by $[k_1, k_2]$ the set of integers $\{k_1, k_1 + 1, \ldots, k_2\}$.

Remark 7. Let $G_1, G_2, \ldots, G_k$ be a collection of pairwise vertex disjoint 2-edge-colored graphs; $G \in \oplus_{i=1}^{k} G_i$; and $J \subset [1, k]$. The induced subgraph of G by $\bigcup_{j \in J} V(G_j)$, $H = G\langle \bigcup_{j \in J} V(G_j) \rangle$, belongs to the c.g.s. of $\{G_j\}_{j \in J}$.

Notation 8. Let $C = x_0 x_1 \cdots x_{2n-1} x_0$ be an alternating cycle. For each $v \in V(C)$, we will denote by v^r (respectively, v^b) the vertex in C such that $vv^r \in E(C)$ is red (respectively, $vv^b \in E(C)$ is blue). Notice that if $v = x_i$ then $\{x_{i-1}, x_{i+1}\} = \{v^r, v^b\}$.

If more than one alternating cycle contains v, we will write v_C^r (respectively, v_C^b).

Definition. Let G be a 2-edge-colored graph and let $C_1 = x_0x_1 \cdots x_{2n-1}x_0$ and $C_2 = y_0y_1 \cdots y_{2m-1}y_0$ be two vertex disjoint alternating cycles. Let vw be an edge with $v \in V(C_1)$ and $w \in V(C_2)$. If $c(vw) =$ red (respectively, $c(vw) =$ blue) we will say that vw, v^rw^r (respectively, vw, v^bw^b) is a *good pair of edges* whenever $c(v^rw^r) =$ red (respectively, $c(v^bw^b) =$ blue).

Whenever there is a good pair of edges between two vertex disjoint alternating cycles C_1 and C_2, we simply say that there is a good pair.

Remark 9. Notice that vv^rw^rwv (respectively, vv^bw^bwv) is a monochromatic 4-cycle whenever vw, v^rw^r (respectively, vw, v^bw^b) is a good pair.

Remark 10. Let G be a 2-edge-colored graph and let $C_1 = x_0x_1 \cdots x_{2n-1}x_0$ and $C_2 = y_0y_1 \cdots y_{2m-1}y_0$ be two vertex disjoint alternating cycles. A pair of edges x_sy_t, $x_{s'}y_{t'}$ with $s \in [0, 2n-1]$, $s' \in \{s-1, s+1\}$, $t \in [0, 2m-1]$, $t' \in \{t-1, t+1\}$ where all the subscripts are taken modulo $2n$ and $2m$, respectively, is a good pair whenever $x_sx_{s'}y_{t'}y_tx_s$ is a monochromatic 4-cycle (Figure 4). This is a consequence of the definition of a good pair and Notation 8.

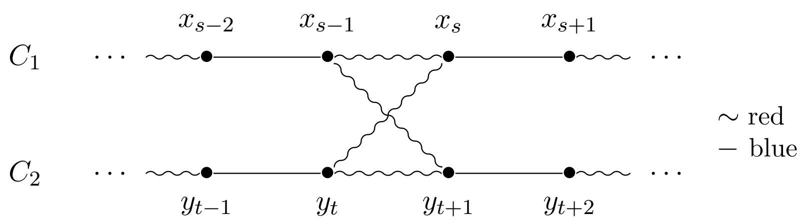


Figure 4. A good pair of edges.

In the study of alternating cycles, the more general case is the one with two colors and so we will work with 2-edge-colored graphs. In what follows, any graph G will denote a 2-edge-colored graph and $c : E(G) \to \{\text{red}, \text{blue}\}$ will denote its edge coloring and we will simply say a graph instead of a 2-edge-colored graph; a 2-edge-colored cycle C which is properly colored will simply be called an alternating cycle. In our figures curly lines will represent red edges while straight lines will represent blue edges, sometimes we will use dotted lines to represent edges that we ignore to construct a cycle, we will use double-dotted lines for red edges and dotted lines for blue edges.

From now on the subscripts for vertices in $C_1 = x_0x_1 \cdots x_{2n-1}x_0$ will be taken modulo $2n$ and for vertices in $C_2 = y_0y_1 \cdots y_{2m-1}y_0$ will be taken modulo $2m$.

3. Preliminary Results

In this section we will, first, state two results from [6], Proposition 11 and Lemma

14, and then we will see a series of results that describe the behavior of exterior edges in a c.g.s. of two 2-edge-colored graphs.

Proposition 11. *Let C_1 and C_2 be two disjoint alternating cycles in a graph G. If there is a good pair of edges between them, then there is an alternating cycle with the vertex set $V(C_1) \cup V(C_2)$ (Figure 5).*

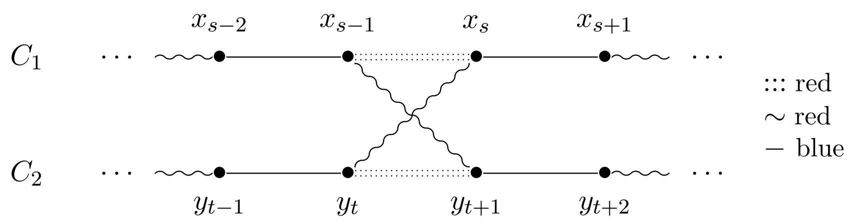


Figure 5. A cycle using a good pair of edges.

Notation 12. Let C be an alternating cycle, given an arbitrary but fixed description of its vertices as $C = x_0x_1 \cdots x_{2n-1}x_0$, we will say that two vertices $x, y \in V(C)$ are *congruent modulo* 2, whenever their subscripts in C are congruent modulo 2, and we will write $x \equiv y \pmod 2$ (or $x \equiv_C y \pmod 2$, when x and y belong to more than one cycle).

Notation 13. Let $G_1, G_2, \ldots, G_k$ be a collection of pairwise vertex disjoint 2-edge-colored graphs and take G in $\oplus_{i=1}^{k} G_i$. For each $v \in V(G)$, we will denote by $d_r(v)$ (respectively, $d_b(v)$) the number of red (respectively, blue) exterior edges of G incident with v.

Lemma 14. *Let $C_1 = x_0x_1 \cdots x_{2n-1}x_0$ and $C_2 = y_0y_1 \cdots y_{2m-1}y_0$ be two vertex disjoint alternating cycles and G be a graph in $C_1 \oplus C_2$ such that G has no good pair. For each vertex $w \in V(C_i)$, if $d_r(w) = t$ and $d_b(w) = |V(C_{3-i})| - t$, then $d_r(x) = |V(C_{3-i})| - t = d_b(w)$ and $d_b(x) = t = d_r(w)$ for each $x \in \{w^r, w^b\}$. Furthermore, if $w, x \in V(C_i)$, then*

$$d_r(x) = \begin{cases} d_r(w) & \textit{if and only if } \; w \equiv x \pmod 2, \\ |V(C_{3-i})| - d_r(w) & \textit{if and only if } \; w \not\equiv x \pmod 2. \end{cases}$$

In what follows we will write w.r.t. instead of "with respect to".

Definition. Let $\mathcal{F}' = \{C, H\}$ be a factor in a graph G, where C is an alternating cycle and H is a subgraph. A vertex $v \in V(C)$ is *red-singular* (*blue-singular*) w.r.t. H, if $\{vu \in E(G) \mid u \in V(H)\}$ is not empty and all the edges in $\{vu \in E(G) \mid u \in V(H)\}$ are red (blue); v is *singular* w.r.t. H if it is either red-singular or blue-singular w.r.t. H.

The color of the exterior edges incident with a singular vertex v w.r.t. H, will be called the *singularity* of v and it will be denoted by $s_H(v)$.

Definition. Let $\mathcal{F}' = \{C, H\}$ be a factor in a graph G, where C is an alternating cycle and H is a subgraph. The cycle C is *singular* w.r.t. H, whenever the vertices in C are alternatively red-singular and blue-singular w.r.t. H.

Remark 15. Let G_1 and G_2 be two vertex disjoint graphs with alternating Hamiltonian cycles, C_1 and C_2, respectively. Let G be a graph in $G_1 \oplus G_2$ such that C_i is singular w.r.t. G_{3-i}. Then for each $x \in V(G_{3-i})$, x is non-singular w.r.t. G_i. In particular C_{3-i} is non-singular w.r.t. G_i.

Lemma 16. *Let $C_1 = x_0x_1 \cdots x_{2n-1}x_0$ and $C_2 = y_0y_1 \cdots y_{2m-1}y_0$ be two vertex disjoint alternating cycles, and $G \in C_1 \oplus C_2$. Then C_i is singular w.r.t. G_{3-i}, for some $i \in \{1, 2\}$, if and only if C_i has at least one singular vertex w.r.t. G_{3-i} and G has no good pair.*

Proof. Assume w.l.o.g. that C_1 is singular w.r.t. G_2. Suppose by contradiction that there is a good pair x_sy_r, $x_{s'}y_{r'}$ with $s \in [0, 2n-1]$, $s' \in \{s-1, s+1\}$, $r \in [0, 2m-1]$, $r' \in \{r-1, r+1\}$, as in Remark 10, and $C = x_sx_{s'}y_{r'}y_rx_s$ is a monochromatic cycle (w.l.o.g., red). So $d_r(x_s) \geq 1$ and $d_r(x_{s'}) \geq 1$, contradicting the singularity of C_1.

The converse follows directly from Lemma 14. ∎

Given two disjoint alternating-pancyclic graphs G_1 and G_2, a graph $G \in G_1 \oplus G_2$ is not necessarily an alternating-pancyclic graph. In fact, we may construct a large family of alternating-pancyclic 2-edge-colored graphs and c.g.s.' members of this family that are not alternating-pancyclic graphs. To prove this assertion, we will use the next proposition.

Proposition 17. *Let G_1 be a graph with an alternating Hamiltonian cycle, $C_1 = x_0x_1 \cdots x_{2n-1}x_0$, G_2 a graph and $G \in G_1 \oplus G_2$ with edge-coloring c. If C_1 is singular w.r.t. G_2 in G and for each edge $x_ix_l \in E(G_1) \setminus E(C_1)$ satisfying $i \equiv l \pmod 2$ we have that $c(x_ix_l) = s_{G_2}(x_i)$, then G has no alternating cycle containing vertices in both G_1 and G_2.*

Proof. Suppose by contradiction that there is an alternating cycle γ in G such that $V(\gamma) \cap V(C_1) \neq \emptyset$ and $V(\gamma) \cap V(G_2) \neq \emptyset$.

Observe that γ contains an alternating subpath of length at least 1 which is contained in G_1. Take $u \in V(\gamma) \cap V(C_1)$. Since u is singular w.r.t. G_2, at least one in $\{u^r_\gamma, u^b_\gamma\}$ belongs to G_1 and thus there is a path of length one contained in G_1.

Let $P = u_0u_1 \cdots u_r$ be a longest alternating subpath of γ contained in G_1. Let v_0 be the predecessor of u_0 in γ and v_r be the successor of u_r in γ belonging to G_2. Observe that $P' = v_0u_0u_1 \cdots u_r$ and $P'' = u_0u_1 \cdots u_rv_r$ are both subpaths of γ and they are alternating.

We may assume w.l.o.g. that u_0 is red-singular and thus v_0u_0 is red and u_0u_1 is blue. Since P is alternating, u_iu_{i+1} is blue if and only if i is even and it is red if and only if i is odd, for each $i \in [0, r-1]$. Moreover, as u_0 is red-singular and C_1 is singular w.r.t. G_2 we have that u_i is red-singular if and only if $i \equiv 0 \pmod 2$ and it is blue singular otherwise. Hence, for each $i \in [0, r-1]$: u_{i+1} is blue-singular and u_iu_{i+1} is blue if and only if i is even; and u_{i+1} is red-singular and u_iu_{i+1} is red if and only if i is odd.

Case 1. r is even. Then $r-1$ is odd and thus u_r is red-singular and $u_{r-1}u_r$ is red. As u_r is red-singular, u_rv_r is red, contradicting that P'' is alternating.

Case 2. r is odd. Then $r-1$ is even and thus u_r is blue-singular and $u_{r-1}u_r$ is blue. As u_r is blue-singular, u_rv_r is blue, contradicting P'' is alternating.

Hence, G has no cycle containing vertices in both G_1 and G_2. ■

In order to construct the example of a c.g.s. of two alternating-pancyclic graphs which is not an alternating-pancyclic graph, we define the following family of complete graphs. First, let $n \geq 2$ and $C_{2n} = x_0x_1 \cdots x_{2n-1}x_0$ be an alternating cycle such that x_ix_{i+1} is red whenever i is even and it is blue whenever i is odd.

Second, let $\mathcal{AP}_{2n}$ be the family of 2-edge-colored complete graphs G, such that $V(G) = V_{2n} = \{x_0, x_1, \ldots, x_{2n-1}\}$ and $E(G)$ satisfies: (i) $E(C_{2n}) \subset E(G)$; (ii) $G\langle\{x_i \mid i \equiv 0 \pmod 2\}\rangle$ is a complete red graph and $G\langle\{x_i \mid i \equiv 1 \pmod 2\}\rangle$ is a complete blue graph, this is, an edge $x_ix_j \in E(G)$ is red whenever $i \equiv j \equiv 0 \pmod 2$ and it is blue whenever $i \equiv j \equiv 1 \pmod 2$; (iii) x_0x_{2i+1} is blue for each $i \in [1, n-1]$; (iv) the remaining edges can be colored in any way.

The set $\mathcal{AP}_{2n}$ is not empty and it consists of alternating-pancyclic graphs, as $x_0x_1 \cdots x_{2i+1}x_0$ is an alternating cycle of length $2i+2$ for each $i \in [1, n-1]$. Set $\mathcal{AP} = \bigcup_{n \geq 2} \mathcal{AP}_{2n}$, $\mathcal{AP}$ is a countable family of alternating-pancyclic graphs with at least one graph of order $2n$ for each $n \geq 2$ (Figure 6).

Remark 18. Let $G_1 \in \mathcal{AP}$ of order $2n$, G_2 be an alternating-pancyclic graph, and $G \in G_1 \oplus G_2$. If C_{2n} is singular w.r.t. G_2 in G with x_0 being a red-singular vertex w.r.t. G_2, then G is not alternating-pancyclic. Moreover, G has no alternating cycle with vertices in both G_1 and G_2.

Proof. Notice that G satisfies the hypothesis of Proposition 17. C_{2n} is an alternating Hamiltonian cycle in G_1 which is singular w.r.t. G_2, and so, C_{2n} is singular w.r.t. an alternating Hamiltonian cycle of G_2; since x_0 is red-singular, x_i is red-singular whenever i is even and it is blue-singular whenever i is odd, $i \in [0, 2n-1]$; each edge $x_ix_l \in E(G_1) \setminus E(C_{2n})$ satisfying $i \equiv l \pmod 2$ is red whenever i is even and it is blue whenever i is odd, by definition of $\mathcal{AP}_{2n}$, this is $c(x_ix_l) = s_{G_2}(x_i)$. Therefore, G has no alternating cycle containing vertices in both G_1 and G_2. ■

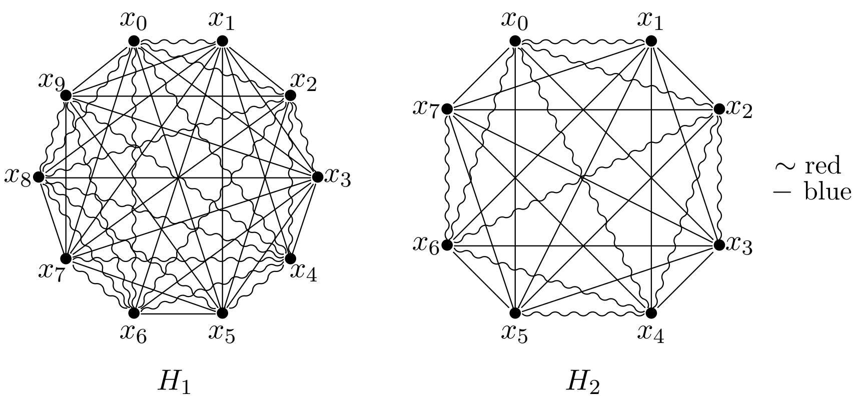


Figure 6. Examples of alternating-pancyclic graphs in $\mathcal{AP}$: $H_1 \in \mathcal{AP}_{10}$ and $H_2 \in \mathcal{AP}_8$.

As a consequence of Proposition 17, we have the following: Let C_1 and C_2 be two vertex disjoint alternating cycles and $G \in C_1 \oplus C_2$. If C_i is singular w.r.t. C_{3-i} in G, then G has no alternating cycle with vertices in both C_1 and C_2, in particular, G has no alternating Hamiltonian cycle. In the next result, we will see that if we add a particular kind of edge between certain vertices in C_i, then G will become an alternating-pancyclic graph.

Proposition 19. *Let G_1 and G_2 be two graphs with alternating Hamiltonian cycles, $C_1 = x_0x_1 \cdots x_{2n-1}x_0$ and $C_2 = y_0y_1 \cdots y_{2m-1}y_0$, respectively, and $G \in G_1 \oplus G_2$ with edge-coloring c. If C_1 is singular w.r.t. G_2 and there exists an edge $x_sx_t \in E(G_1)$ such that $s \equiv t \pmod 2$ and $c(x_sx_t) \neq s_{G_2}(x_s)$, then G is an alternating-pancyclic graph. Moreover, for each even length in $[4, 2n+2m]$ and each $j \in [0, 2m-1]$, there is an alternating cycle passing through x_sx_t and y_j.*

Proof. Suppose w.l.o.g. that x_sx_t is red.

Since C_1 is singular w.r.t. G_2 and $s \equiv t \pmod 2$, we have that x_s and x_t have the same singularity, this is $s_{G_2}(x_s) = s_{G_2}(x_t)$. As red $= c(x_sx_t) \neq s_{G_2}(x_s)$, we obtain that x_s and x_t are both blue-singular vertices w.r.t. G_2.

Assume w.l.o.g. that y_0y_1 is red, then $y_{2j}y_{2j+1}$ is red and $y_{2j+1}y_{2j+2}$ is blue for each $j \in [0, m-1]$, as C_2 is alternating. Therefore, the alternating paths $\rho_h^j = y_{2j}y_{2j+1} \cdots y_{2j+2h+1}$ have odd length $2h+1$ starting and ending at red edges, for each $h \in [0, m-1]$ and each $j \in [0, m-1]$. Then, for each $h \in [0, m-1]$ and each $j \in [0, m-1]$, the cycle $\alpha_h^j = \rho_h^j \cup y_{2j+2h+1}x_sx_ty_{2j}$ is alternating with length $l(\alpha_h^j) = 2h+4$, $x_sx_t \in E(\alpha_h)$ and $y_{2j}, y_{2j+1} \in V(\alpha_h)$ for each $j \in [0, m-1]$. So, for each even length l in $[4, 2m+2]$ and each $j \in [0, 2m-1]$, G contains an alternating cycle of length l passing through x_sx_t and y_j.

Suppose w.l.o.g. that $s < t$. Let P_1 and P_2 be the two alternating subpaths of C_1 determined by x_s and x_t, namely $P_1 = x_s x_{s+1} \cdots x_t$ and $P_2 = x_t x_{t+1} \cdots x_s$. Since $s \equiv t \pmod 2$, we have that P_1 and P_2 have even length and they start at edges of the same color; assume w.l.o.g. that $c(x_s x_{s+1}) = c(x_t x_{t+1}) =$ blue, then we have

- $x_{s+2i}x_{s+2i+1}$ is blue for each $i \in \left[0, \frac{t-s}{2} - 1\right]$,
- $x_{s+2i-1}x_{s+2i}$ is red for each $i \in \left[1, \frac{t-s}{2}\right]$,
- $x_{t+2i}x_{t+2i+1}$ is blue for each $i \in \left[0, n - \frac{t-s}{2} - 1\right]$,
- $x_{t+2i-1}x_{t+2i}$ is red for each $i \in \left[1, n - \frac{t-s}{2}\right]$,
- x_{s+2i-1} is red-singular for each $i \in \left[1, \frac{t-s}{2}\right]$,
- x_{t+2i-1} is red-singular for each $i \in \left[1, n - \frac{t-s}{2}\right]$.

Let $\rho = y_1 y_2 \cdots y_{2m-1} y_0$ be the alternating path of length $2m - 1$ obtained from C_2 by removing the red edge $y_0 y_1$, and consider the alternating paths $\sigma_i = x_{s+2i-1}x_{s+2i-2} \cdots x_{s+1}x_s$, for each $i \in \left[1, \frac{t-s}{2}\right]$, and $\tau_j = x_t x_{t+1} \cdots x_{t+2j-1}$, for each $j \in \left[1, n - \frac{t-s}{2}\right]$. Each of these paths has odd length and its end edges are both blue; the σ_i's start at a red-singular vertex and end at a blue-singular vertex and the τ_j's start at a blue-singular vertex and end at a red-singular vertex (Figure 7).

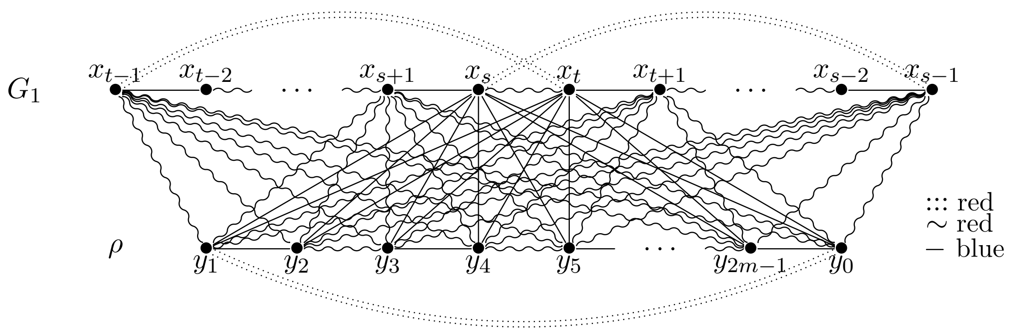


Figure 7. G in the proof of Proposition 19.

Hence, for each $i \in \left[1, \frac{t-s}{2}\right]$ and each $j \in \left[1, n - \frac{t-s}{2}\right]$, the cycle $\beta(\sigma_i, \tau_j) = \rho \cup y_0 x_{s+2i-1} \cup \sigma_i \cup x_s x_t \cup \tau_j \cup x_{t+2j-1} y_1$ is alternating, has length $l\left(\beta(\sigma_i, \tau_j)\right) = (2m-1) + 1 + (2i-1) + 1 + (2j-1) + 1 = 2m + 2i + 2j$ and passes through $x_s x_t$ and y_j for each $j \in [0, 2m-1]$. ■

Corollary 20. *Let G_1 and G_2 be two graphs with alternating Hamiltonian cycles, $C_1 = x_0 x_1 \cdots x_{2n-1} x_0$ and $C_2 = y_0 y_1 \cdots y_{2m-1} y_0$, respectively, and $G \in G_1 \oplus G_2$ with edge-coloring c. If C_1 is singular w.r.t. C_2 and, for each $x_s \in V(C_1)$, there exists an edge $x_s x_t \in E(G_1)$ such that $s \equiv t \pmod 2$ and $c(x_s x_t) \neq s_{C_2}(x_s)$, then G is a vertex alternating-pancyclic graph.*

As a consequence of Propositions 17 and 19 we have the next nice result.

Corollary 21. *Let G_1 and G_2 be two vertex disjoint graphs with alternating Hamiltonian cycles, $C_1 = x_0x_1 \cdots x_{2n-1}x_0$ and $C_2 = y_0y_1 \cdots y_{2m-1}y_0$, respectively, and $G \in G_1 \oplus G_2$ with edge-coloring c. If C_1 is singular w.r.t. G_2, then either, G has no alternating cycle containing vertices in both G_1 and G_2, or G is an alternating-pancyclic graph.*

Proof. If there is an edge $x_sx_t \in E(G_1) \setminus E(C_1)$ such that $s \equiv t \pmod 2$ and $c(x_sx_t) \neq s_{G_2}(x_s)$ then, by Proposition 19, G is alternating-pancyclic.

If there is no such an edge then, by Proposition 17, G has no alternating cycle containing vertices in both C_1 and C_2. ■

If G_1 and G_2 are alternating Hamiltonian graphs and $G \in G_1 \oplus G_2$ has a good pair, then we know G contains an alternating Hamiltonian cycle, by Proposition 11. However, we can say more when $|V(G_1)| = 2^s p$, where s is a positive integer and p is a prime number, and there are two special vertices in G_1.

Proposition 22. *Let G_1 and G_2 be two vertex disjoint graphs with alternating Hamiltonian cycles, $C_1 = x_0x_1 \cdots x_{2n-1}x_0$ and $C_2 = y_0y_1 \cdots y_{2m-1}y_0$, respectively, where $2n = 2^s p$ with s a positive integer and p a prime number, and $G \in G_1 \oplus G_2$ with edge-coloring c. If C_1 is non-singular w.r.t. C_2 and there exist two singular vertices $x, w \in V(C_1)$ such that $xw \in E(G_1) \setminus E(C_1)$, $s_{C_2}(x) = s_{C_2}(w)$ and $c(xw) \neq s_{C_2}(x)$, then G contains alternating cycles of every even length in $[4, 2n+2m] \setminus \{2m + jp + 1 \mid j \in [1, 2^s]$ and $j \equiv 1 \pmod 2\}$, when $p \neq 2$. And G is an alternating-pancyclic graph, whenever $p = 2$.*

Proof. First, notice that there is a good pair. Otherwise, Lemma 16 and the fact that x is a singular vertex w.r.t. C_2 imply C_1 is singular w.r.t. C_2, a contradiction. So, Proposition 11 implies that there is an alternating cycle of length $2n + 2m$ in G.

Now we proceed to prove the existence of the other alternating cycles which are not Hamiltonian.

Assume w.o.l.g. that x and w are both red-singular vertices w.r.t. C_2, and y_0y_1 is blue.

As y_0y_1 is blue, then for each $i \in [1, m]$, the path $P_i = y_0y_1 \cdots y_{2i-1}$, is alternating of odd length $2i - 1$ and so it starts and ends at blue edges. Hence, the cycle $\gamma_i = xy_0 \cup P_i \cup x_{2i-1}wx$ is alternating of length $2i + 2$.

Let $L = 2[2, n-1]$, this is, L is the set of all even numbers between 4 and $2n - 2$ and let $L_p = \{jp + 1 \mid j \in [1, 2^s]$ and $j \equiv 1 \pmod 2\}$. Observe that, whenever $p = 2$ the set L_p is a set of odd integers, so $L \setminus L_p = L$, and whenever p is odd L_p is a set of even integers with 2^{s-1} elements.

We will prove that for each $h \in L \setminus L_p$, G contains an alternating cycle of length $2m + h$.

Proceeding by contradiction, suppose that there is an $h \in L \setminus L_p$ such that G has no alternating cycle of length $2m + h$.

Assume w.l.o.g. that $x = x_0$ and x_0x_1 is blue. Since h is even and C_1 is alternating, then the path $Q_1 = x_0x_1 \cdots x_{h-1}$ is also alternating, has odd length $h - 1$ and so it starts and ends at blue edges. We will prove that x_{h-1} is blue-singular w.r.t. C_2.

If there is a vertex $y \in V(C_2)$ such that $x_{h-1}y$ is red, then taking R_1 to be the yy^r-subpath of C_2 obtained from C_2 by removing the red edge yy^r, we may construct the cycle $\alpha_1 = Q_1 \cup x_{h-1}y \cup R_1 \cup y^r x_0$ which is alternating and has length $l(\alpha_1) = h - 1 + 1 + 2m - 1 + 1 = 2m + h$, a contradiction. Hence, x_{h-1} is a blue-singular vertex w.r.t. C_2.

Now, $x_{h-1}x_h$ is red. Then the alternating path $Q_2 = x_{h-1}x_h \cdots x_{2(h-1)}$ which has odd length $h - 1$ starts and ends at red edges. We will prove that $x_{2(h-1)}$ is red-singular w.r.t. C_2.

If there is a vertex $y \in C_2$ such that $x_{2(h-1)}y$ is blue, then taking R_2 to be the yy^b-subpath of C_2 obtained from C_2 by removing the blue edge yy^b, we may construct the cycle $\alpha_2 = Q_2 \cup x_{2(h-1)}y \cup R_2 \cup y^b x_{h-1}$ which is alternating and has length $l(\alpha_2) = h - 1 + 1 + 2m - 1 + 1 = 2m + h$, a contradiction. Hence, $x_{2(h-1)}$ is a red-singular vertex w.r.t. C_2.

Arguing this way we obtain the sequence, $\{x_{t(h-1)}\}_{t \geq 1}$, of singular vertices in C_1, such that $x_{t(h-1)}$ is red-singular if t is even and blue-singular if t is odd.

Observe that, if $l_h = \frac{\operatorname{lcm}(2^s p, h-1)}{h-1}$, then $x_0, x_{h-1}, \ldots, x_{(l_h-1)(h-1)}$ are all different vertices. Recall that $h \in L \setminus L_p$ and thus h is even and $h \not\equiv 1 \pmod{p}$, which means $h - 1$ is odd and $p \nmid h - 1$. Hence, $\operatorname{lcm}(2^s p, h - 1) = 2^s p(h - 1)$ and thus $l_h = 2^s p$. Therefore, $x_0, x_{h-1}, \ldots, x_{(l_h-1)(h-1)}$ are $l_h = 2^s p$ different singular vertices in C_1, such that $x_{t(h-1)}$ is red-singular if t is even and blue-singular if t is odd. As $h-1$ is odd, $x_{t(h-1)}$ is red-singular if $t(h-1)$ is even and blue-singular if $t(h - 1)$ is odd. This is, C_1 is singular w.r.t. C_2, a contradiction.

Then, G contains alternating cycles of length $2m + h$, for each $h \in L \setminus L_p$, which concludes the proof. ■

Next we will prove a proposition that simplifies the proof of Theorem 4, which is one of our two main theorems.

Proposition 23. *Let $G_1, G_2, \ldots, G_k$ be a collection of pairwise vertex disjoint graphs with Hamiltonian alternating cycles $C_1, C_2, \ldots, C_k$, respectively, and $G \in \oplus_{i=1}^k G_i$. If there is a sequence $\{i_j\}_{j=1}^k$ such that C_{i_j} has a red-singular vertex w.r.t. $C_{i_{j+1}}$ for all $j \in [1, k]$ and where $C_{i_{k+1}} = C_{i_1}$, then G has a Hamiltonian alternating cycle.*

Proof. Suppose w.l.o.g. that, for each $i \in [1, k]$, C_i has a red-singular vertex w.r.t. C_{i+1}, where $C_{k+1} = C_1$.

For each $i \in [1, k]$, let $v_i \in V(C_i)$ be a red-singular vertex w.r.t. C_{i+1}. Recall that we denoted by v_i^r the vertex in C_i such that $v_i v_i^r \in E(C_i)$ is red.

Consider the paths that result by removing the edges $v_i v_i^r$ from the cycles C_i, namely $P_i = v_i^r C v_i$, which are k alternating paths that start and end with blue edges.

Since $v_i \in V(C_i)$ is red-singular w.r.t. C_{i+1}, we have $e_i = v_i v_{i+1}^r$ is red for each $i \in [1, k]$, where $v_{k+1}^r = v_1^r$. Therefore, $\gamma = P_1 \cup e_1 \cup P_2 \cup e_2 \cup \cdots \cup e_{k-1} \cup P_k \cup e_k$ is a Hamiltonian alternating cycle (Figure 8). ■

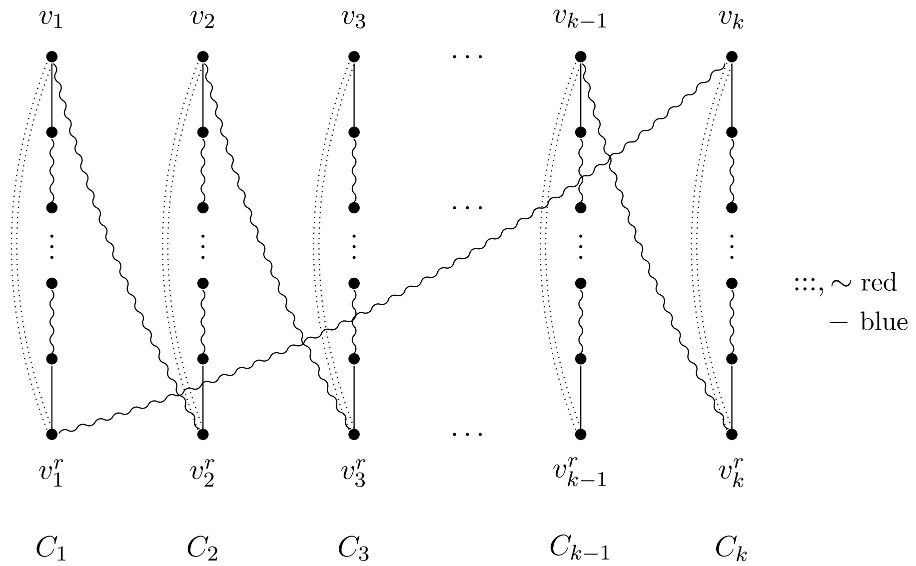


Figure 8. The cycle γ in the proof of Proposition 23.

Notice that the assertion in Proposition 23 holds if we change the hypothesis for red-singular vertices to blue-singular vertices.

4. Main Results-Conclusions

In this section we will see which graphs in the c.g.s. of two alternating Hamiltonian 2-edge-colored graphs are vertex alternating-pancyclic graphs, alternating-pancyclic graphs or, simply, alternating Hamiltonian graphs.

Our main result is Theorem 3, which is consequence of Propositions 17 and 19, and Theorem 1. Next we prove Theorem 3 (Figure 3).

***Proof of Theorem* 3.** (i) Suppose C_i is singular w.r.t. C_{3-i} for some $i \in \{1, 2\}$. Hence, by Corollary 21, G is either alternating-pancyclic or it has no alternating cycle containing vertices in both G_1 and G_2.

(ii) Suppose C_i is non-singular w.r.t. C_{3-i} for each $i \in \{1,2\}$. If there is a good pair of edges then, by Proposition 11, G contains an alternating Hamiltonian cycle. And if there is no good pair then, for each $i \in \{1,2\}$, C_i contains a non-singular vertex w.r.t. C_{3-i}. Otherwise, each vertex $v \in V(C_i)$ is singular w.r.t. C_{3-i}, for some $i \in \{1,2\}$. By Lemma 14, the exterior edges incident with v have different color from the exterior edges incident with v^r and v^b, for each $v \in V(C_i)$. Hence, C_i must be singular w.r.t. C_{3-i}, a contradiction.

Therefore, G satisfies the hypothesis of Theorem 1 and thus G is a vertex alternating-pancyclic graph. ■

Observe that in Theorem 3 three out of four possibilities imply that G is an alternating Hamiltonian graph and two possibilities in the same theorem assert that the graph is alternating-pancyclic, so we have Corollary 24. We will extend the results of Corollary 24 for k summands.

Corollary 24. *Let G_1 and G_2 be two vertex disjoint graphs with alternating Hamiltonian cycles, C_1 and C_2, and $G \in G_1 \oplus G_2$.*

(i) *If G contains no good pair and G contains an alternating cycle γ such that $V(\gamma) \cap V(G_i) \neq \emptyset$ for each $i \in \{1,2\}$, then G is an alternating-pancyclic graph.*

(ii) *G contains an alternating cycle γ such that $V(\gamma) \cap V(G_i) \neq \emptyset$ for each $i \in \{1,2\}$ if and only if G is an alternating Hamiltonian graph.*

Next we prove a result that will be a useful tool in the next part, it is a consequence of Theorem 1, Propositions 11 and 17, and Lemma 16.

Corollary 25. *Let G_1 and G_2 be two vertex disjoint graphs with alternating Hamiltonian cycles, C_1 and C_2, respectively; and $G \in G_1 \oplus G_2$. If G has no alternating Hamiltonian cycle, then C_i is singular w.r.t. C_{3-i} for some $i \in \{1,2\}$.*

Proof. Suppose that C_i is non-singular w.r.t. C_{3-i} for each $i \in \{1,2\}$.

If there is a good pair in G, then Proposition 11 asserts that G is an alternating Hamiltonian graph.

So, we may assume that G has no good pair. Observe that, for each $i \in \{1,2\}$, C_i cannot have singular vertices by Lemma 16. Hence, C_i contains at least one non-singular vertex for each $i \in \{1,2\}$. It follows from Theorem 1 that G is vertex alternating-pancyclic, in particular G contains an alternating cycle of length $2n + 2m = |V(G)|$. ■

From the definition of good pair and Remarks 9 and 10 we obtain the following remark which allow us to define a generalization of a good pair [6].

Remark 26. Let G_1 and G_2 be two vertex disjoint graphs, $\alpha_1 = x_0x_1 \cdots x_{2n-1}x_0$ and $\alpha_2 = y_0y_1 \cdots y_{2m-1}y_0$ be two alternating cycles in G_1 and G_2, respectively, and $G \in G_1 \oplus G_2$. If $x_s y_t$ and $y_{t'} x_{s'}$ is a good pair of edges, then $\mathcal{C} = x_s x_{s'} y_{t'} y_t x_s$ is a monochromatic 4-cycle such that its edges are alternatively in $E_\oplus$ and $E(\alpha_1) \cup E(\alpha_2)$, namely $x_{s'} y_{t'}, y_t x_s \in E_\oplus$, $x_s x_{s'} \in E(\alpha_1)$ and $y_{t'} y_t \in E(\alpha_2)$.

Definition. Let $G_1, G_2, \ldots, G_k$ be a collection of pairwise vertex disjoint 2-edge-colored graphs, $G \in \oplus_{i=1}^k G_i$. A monochromatic 4-cycle $\mathcal{C} = v_0 v_1 v_2 v_3 v_0$ in G will be called a *good cycle* when either $v_0 v_1,\ v_2 v_3 \subset E_\oplus$ or $v_1 v_2,\ v_3 v_0 \subset E_\oplus$, or both. This is, when two opposite edges in $\mathcal{C}$ are exterior.

Now we prove Theorem 4.

***Proof of Theorem* 4.** (i) We will prove the assertion by induction on k. We will first prove the assertion for $k = 2, 3$.

Suppose $k = 2$. If C_i is singular w.r.t. C_{3-i}, for some $i \in \{1, 2\}$. Then, by Theorem 3, we have either G has no alternating cycle containing vertices in both G_1 and G_2, or G is alternating-pancyclic. As G contains an alternating cycle γ such that $V(\gamma) \cap V(G_i) \neq \emptyset$ for each $i \in \{1, 2\}$, it follows that G is an alternating-pancyclic graph.

If C_i is non-singular w.r.t. C_{3-i}, for each $i \in \{1, 2\}$. Then, by Theorem 3, we have either G contains a good pair and it is an alternating Hamiltonian graph, or G has no good pair and it is a vertex alternating-pancyclic graph. As G has no good cycle, G has no good pair w.r.t. C_1 and C_2. Otherwise G would contain a good cycle by Remark 26. Hence, G is a vertex alternating-pancyclic graph.

Suppose $k = 3$. If there exist $i, j \in [1, 3]$ with $i \neq j$ such that the induced graph $H_{ij} = G\langle V(G_i) \cup V(G_j)\rangle$ contains an alternating cycle α such that $V(\alpha) \cap V(G_i) \neq \emptyset$ and $V(\alpha) \cap V(G_j) \neq \emptyset$. Then, by the base case $k = 2$, H_{ij} is an alternating-pancyclic graph, since H_{ij} contains no good cycle (as it is a subgraph of G) and contains α. In particular, H_{ij} contains an alternating Hamiltonian cycle. Notice that $G \in G_h \oplus H_{ij}$, where $h \in [1, 3] \setminus \{i, j\}$, each summand is an alternating Hamiltonian graph and G has no good cycle, then G is an alternating-pancyclic graph, by the base case $k = 2$.

If, for each pair of different indices $i, j \in [1, 3]$, the induced graph $H_{ij} = G\langle V(G_i) \cup V(G_j)\rangle$ contains no alternating cycle containing vertices in both G_i and G_j, and thus it has no alternating Hamiltonian cycle. Then, by the contrapositive of Corollary 25, either C_i is singular w.r.t. C_j or C_j is singular w.r.t. C_i. There are two cases (w.l.o.g.).

Let $C_1 = x_0 x_1 \cdots x_{2n-1} x_0$, $C_2 = y_0 y_1 \cdots y_{2m-1} y_0$ and $C_3 = w_0 w_1 \cdots w_{2l-1} w_0$ be the alternating Hamiltonian cycles of G_1, G_2 and G_3, respectively, and assume w.l.o.g. that $x_0 x_1$, $y_0 y_1$ and $w_0 w_1$ are blue; then $x_i x_{i+1}$, $y_i y_{i+1}$ and $w_i w_{i+1}$ are blue whenever $i \equiv 0 \pmod 2$ and they are red whenever $i \equiv 1 \pmod 2$.

Case 1. C_i is singular w.r.t. C_{i+1} for each $i \in [1, 3]$, where $C_4 = C_1$. Suppose w.l.o.g. that x_0, y_0 and w_0 are red-singular vertices w.r.t. C_2, C_3 and C_1, respectively.

Then x_i, y_i and w_i are red-singular vertices w.r.t. C_2, C_3 and C_1, respectively, whenever $i \equiv 0 \pmod 2$ and blue-singular otherwise.

Case 2. C_i is singular w.r.t. C_j whenever $1 \leq i < j \leq 3$. Take $v_1 \in V(C_1)$. By the definition of singular cycle, we have that v_1 is singular w.r.t. C_i for each $i \in \{2, 3\}$. If $s_{C_2}(v_1) = s_{C_3}(v_1)$, then v_1 is singular w.r.t. $G_0 = G\langle V(G_2) \cup V(G_3)\rangle$, and thus C_1 is singular w.r.t. G_0, as the color of the singularities of vertices in C_1 alternate and C_1 is singular w.r.t. C_i for each $i \in \{2, 3\}$.

As $G \in G_1 \oplus G_0$ and G contains a cycle γ such that $V(\gamma) \cap V(G_i) \neq \emptyset$ for each $i \in [1, 3]$, we have by contrapositive of Proposition 17 that it must exist an edge $xy \in E(G_1) \setminus E(C_1)$ such that $x \equiv y \pmod 2$ in C_1 and $c(xy) \neq s_{G_0}(x)$. Observe that G_1 is singular w.r.t. G_2, $s_{G_0}(v) = s_{G_2}(v)$ for each $v \in V(G_1)$ and thus the edge $xy \in E(G_1) \setminus E(C_1)$ is such that $x \equiv y \pmod 2$ in C_1 and $c(xy) \neq s_{G_2}(x)$. Then if we consider the induced subgraph by $V(G_1) \cup V(G_2)$ in G, namely $H_{12} = G\langle V(G_1) \cup V(G_2)\rangle$, it satisfies the hypothesis of Proposition 19. Hence, H_{12} is an alternating-pancyclic graph and thus it contains an alternating Hamiltonian cycle, contradicting our assumption.

Then, $s_{C_2}(v_1) \neq s_{C_3}(v_1)$. Assume w.l.o.g. that x_0 and y_0 are red-singular vertices w.r.t. C_2 and C_3, respectively. Then x_0 is blue-singular w.r.t. C_3; x_i and y_i are red-singular vertices w.r.t. C_2 and C_3, respectively, whenever $i \equiv 0 \pmod 2$ and blue-singular otherwise; x_i is blue-singular w.r.t. C_3 whenever $i \equiv 0 \pmod 2$ and red-singular otherwise.

Hence, in both cases, $x_{2n-2p}y_{2t}$ is red for each $p \in [1, n]$ and each $t \in [0, m-1]$, y_0w_{2l-1} is red, $w_{2l-2s}x_{2n-1}$ is red for each $s \in [1, l]$; the paths $w_{2l-1}w_{2l-2} \cdots w_{2l-2s}$ and $x_{2n-1}x_{2n-2} \cdots x_{2n-2p}$ are alternating which start and end at blue edges and the paths $y_{2t}y_{2t-1} \cdots y_0$ are alternating which start at a red edge and end at a blue edge.

Therefore, the cycle $\beta_s = x_{2n-1}y_0w_{2l-1}w_{2l-2} \cdots w_{2l-2s}x_{2n-1}$ is alternating of length $1 + 1 + (2s - 1) + 1 = 2 + 2s$ for each $s \in [1, l]$; the cycle $\delta_t = x_{2n-1}y_{2t}y_{2t-1} \cdots y_0w_{2l-1}w_{2l-2} \cdots w_0x_{2n-1}$ is alternating of length $1 + 2t + 1 + (2l - 1) + 1 = 2 + 2l + 2t$ for each $t \in [1, m-1]$; and the cycle $\eta_p = x_{2n-1}x_{2n-2} \cdots x_{2n-2p}y_{2m-1}y_{2m-2} \cdots y_0w_{2l-1}w_{2l-2} \cdots w_0x_{2n-1}$ is alternating of length $(2p-1)+1+(2m-1)+1+(2l-1)+1 = 2l+2m+2p$ for each $p \in [1, n]$.

From the above, G is alternating-pancyclic.

Now, assume that the assertion of this theorem holds for each $k' \leq k - 1$. We will prove it for $k \geq 4$.

Let $G_1, G_2, \ldots, G_k$ be a collection of $k \geq 4$ vertex disjoint graphs with alternating Hamiltonian cycles, $C_1, C_2, \ldots, C_k$, respectively, and take $G \in \oplus_{i=1}^{k} G_i$ as in the hypothesis.

Claim 27. *There exists an alternating cycle β in G such that $V(\beta) \subset \bigcup_{j \in J} V(G_j)$ for some $J \subset [1, k]$, with $2 \leq |J| \leq k - 1$, and $V(\beta) \cap V(G_j) \neq \emptyset$ for each $j \in J$.*

Proof. Suppose by contradiction that there is no such a cycle. Then each alternating cycle β in G satisfies either $V(\beta) \subset V(G_i)$ for some $i \in [1, k]$ or $V(\beta) \cap V(G_i) \neq \emptyset$ for each $i \in [1, k]$.

Then for each pair of different indices $i, j \in [1, k]$ the graph $H_{ij} = G\langle V(G_i) \cup V(G_j)\rangle$ has no alternating Hamiltonian cycle and thus, by the contrapositive of Corollary 25, either C_i is singular w.r.t. C_j or C_j is singular w.r.t. C_i.

Define a digraph T of order k with vertex set $V(T) = \{G_i \mid i \in [1, k]\}$ and (G_i, G_j) is an arc of T if and only if C_i is singular w.r.t. C_j in H_{ij} (and thus, in G).

From the above, between each pair of different summands in G, G_i and G_j, for some $r \in \{i, j\}$ the alternating Hamiltonian cycle C_r of G_r is singular w.r.t. $G_{r'}$ in G, where $r' \in \{i, j\} \setminus \{r\}$. In this way, any two vertices in T are adjacent and, by Remark 15, C_i and C_j cannot be simultaneously singular with respect each other. Hence, there is exactly one arc between G_i and G_j in T and thus T is a tournament.

Claim 28. *T is an acyclic tournament.*

Proof. Suppose by contradiction that T has a cycle, namely $\alpha = (G_{i_1}, G_{i_2}, \ldots, G_{i_s}, G_{i_1})$ where $3 \leq s \leq k-1$. This cycle in T produces a sequence $\{i_j\}_{j=1}^{s}$ such that C_{i_j} is singular w.r.t. $C_{i_{j+1}}$ in G (and so C_{i_j} contains a red-singular vertex w.r.t. $C_{i_{j+1}}$ in G) for each $j \in [1, s]$, where $C_{i_{s+1}} = C_{i_1}$. Therefore, by Proposition 23, $G_0 = G\big\langle \bigcup_{j=1}^{s} V(G_{i_j})\big\rangle$ contains an alternating Hamiltonian cycle C_0. Then $V(C_0) \cap V(G_{i_j}) \neq \emptyset$ for each $j \in [1, s]$, contradicting our assumption.

Now, suppose by contradiction that T has a Hamiltonian cycle, namely $\alpha = (G_{i_1}, G_{i_2}, \ldots, G_{i_k}, G_{i_1})$. Then, as G_{i_1} and G_{i_3} are adjacent, either $(G_{i_1}, G_{i_3}) \in A(T)$ or $(G_{i_3}, G_{i_1}) \in A(T)$. If $(G_{i_1}, G_{i_3}) \in A(T)$, then $(G_{i_1}, G_{i_3}, \ldots, G_{i_k}, G_{i_1})$ is a cycle of length $k-1$ in T, a contradiction. If $(G_{i_3}, G_{i_1}) \in A(T)$, then $(G_{i_1}, G_{i_2}, G_{i_3}, G_{i_1})$ is a cycle of length 3 in T, a contradiction. □

As T is acyclic, it follows that T is transitive and it contains a Hamiltonian path $\tau = (G_{i_1}, G_{i_2}, \ldots, G_{i_k})$ such that $(G_{i_j}, G_{i_{j'}})$ is an arc in T if and only if $1 \leq j < j' \leq k$.

If each vertex $v \in V(G_{i_1})$ is singular w.r.t. $H = G\big\langle \bigcup_{j=2}^{k} V(G_{i_j})\big\rangle$, that is, all exterior arcs incident with v are colored alike. Then, as C_{i_1} is singular w.r.t. C_{i_j} for each $j \in [2, k]$, it follows that C_{i_1} is singular w.r.t. H. Notice that $G \in G_{i_1} \oplus H$; C_{i_1} is an alternating Hamiltonian cycle in G_{i_1} which is singular w.r.t. H and; G contains the alternating cycle γ which contains vertices in both G_{i_1} and H. Therefore, by Proposition 17, it must exist an edge $xy \in E(G_{i_1}) \setminus E(C_{i_1})$ such that $x \equiv y \pmod 2$ in C_{i_1} and $c(xy) \neq s_H(x)$.

Observe that G_{i_1} is singular w.r.t G_{i_2}, $s_H(v) = s_{G_{i_2}}(v)$ for each $v \in V(G_{i_1})$ and thus the edge $xy \in E(G_{i_1}) \setminus E(C_{i_1})$ is such that $x \equiv y \pmod 2$ in C_{i_1}

and $c(xy) \neq s_{G_{i_2}}(x)$. Then if we consider the induced subgraph by $V(G_{i_1}) \cup V(G_{i_2})$ in G, namely $H_{i_1 i_2} = G\langle V(G_{i_1}) \cup V(G_{i_2})\rangle$, it satisfies the hypothesis of Proposition 19. Hence, $H_{i_1 i_2}$ is an alternating-pancyclic graph and thus it contains an alternating Hamiltonian cycle, contradicting our assumption.

Therefore, there is a vertex $v_1 \in V(G_{i_1})$ such that $d_r(v_1) \geq 1$ and $d_b(v_1) \geq 1$, *i.e.*, there are blue exterior and red exterior edges incident with v_1. Since C_{i_1} is singular w.r.t. C_{i_j} for each $j \in [2,k]$, v_1 is singular w.r.t. C_{i_j} for each $j \in [2,k]$ and, as $d_r(v_1) \geq 1$ and $d_b(v_1) \geq 1$, there are $j, j' \in [2,k]$, with $j \neq j'$, such that v_1 is red-singular w.r.t. C_{i_j} and it is blue-singular w.r.t. $C_{i_{j'}}$. We may assume w.l.o.g. that v_1 is red-singular w.r.t. C_{i_2} and let s be the minimum index in $[3,k]$ such that v_1 is blue-singular w.r.t. C_{i_j}. Then v_1 is red-singular w.r.t. $C_{i_{s-1}}$, it is blue-singular w.r.t. C_{i_s}, and v_1^r, the red neighbor of v_1 in C_{i_1}, is red-singular w.r.t. C_{i_s}, by definition of singular cycle.

Consider $v_{s-1} \in V(C_{i_{s-1}})$ such that v_{s-1} is red-singular w.r.t. C_{i_s} and $v_s \in V(C_{i_s})$.

Now, take the red exterior edges $v_1 v_{s-1}^r$, $v_{s-1} v_s^r$ and $v_s v_1^r$ and; P_j, the $v_j^r v_j$-alternating path which is obtained from C_{i_j} by removing the red edge $v_j v_j^r$, for each $j \in \{1, s-1, s\}$. Hence, $C = P_1 \cup v_1 v_{s-1}^r \cup P_{s-1} \cup v_{s-1} v_s^r \cup P_s \cup v_s v_1^r$ is an alternating cycle in $H' = G\langle V(G_{i_1}) \cup V(G_{i_{s-1}}) \cup V(G_{i_s})\rangle$ with $V(C) = V(G_{i_1}) \cup V(G_{i_{s-1}}) \cup V(G_{i_s})$, contradicting our assumption (as $k \geq 4$). □

Let β be an alternating cycle as in the assertion of Claim 27.

Then $G_0 = G\langle \bigcup_{j\in J} V(G_j)\rangle$ is a graph in $\oplus_{j\in J} G_j$ which contains β and has no good cycle (as G_0 is a subgraph of G). Hence, by induction hypothesis G_0 is alternating-pancyclic. In particular, it contains an alternating Hamiltonian cycle C_0.

Notice that $\{G_i\}_{i\in[0,k]\setminus J}$ is a collection of $k+1-|J| \leq k-1$ alternating Hamiltonian graphs; $G \in \oplus_{i\in[0,k]\setminus J} G_i$ as it satisfies the definition of a c.g.s. of $\{G_i\}_{i\in[0,k]\setminus J}$; G contains γ which satisfies $V(\gamma) \cap V(G_i) \neq \emptyset$ for each $i \in [0,k] \setminus J$ and G has no good cycle (the set of exterior edges in G as c.g.s. in $\oplus_{i\in[0,k]\setminus J} G_i$ is contained in the set of exterior edges of G as c.g.s. in $\oplus_{i=1}^k G_i$). Then, by induction hypothesis, G is alternating-pancyclic.

(ii) The proof is similar to that of (i). However, in the induction basis we can only assert that the graph G is alternating Hamiltonian, by Corollary 24, instead of alternating pancyclic, as the hypothesis about good cycles is missing. And thus, the induction process asserts that $G \in \oplus_{i=1}^k G_i$ must be an alternating Hamiltonian graph.

The converse is immediate. ■

Let $G_1, G_2, \ldots, G_k$ be a collection of k 2-edge-colored graphs with alternating Hamiltonian cycles $C_1, C_2, \ldots, C_k$, respectively. In Theorem 4 we characterized graphs in $\oplus_{i=1}^k G_i$ which are alternating Hamiltonian, we gave sufficient conditions for a graph in $\oplus_{i=1}^{k} G_i$ to be alternating-pancyclic; and in Theorem 2 [6], we gave sufficient conditions for a graph in that same set to be vertex alternating-pancyclic. Those conditions are not proved to be necessary, as we used at most one edge in $E(G_i) \setminus E(C_i)$, for each $i \in [1, k]$ and we do not know if there are other edges in those sets, and if there are, we do not know how they behave.

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