

## UNIQUE MINIMUM SEMIPAIRED DOMINATING SETS IN TREES

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### Abstract

Let  $G$  be a graph with vertex set  $V$ . A subset  $S \subseteq V$  is a semipaired dominating set of  $G$  if every vertex in  $V \setminus S$  is adjacent to a vertex in  $S$  and  $S$  can be partitioned into two element subsets such that the vertices in each subset are at most distance two apart. The semipaired domination number is the minimum cardinality of a semipaired dominating set of  $G$ . We characterize the trees having a unique minimum semipaired dominating set. We also determine an upper bound on the semipaired domination number of these trees and characterize the trees attaining this bound.

**Keywords:** paired-domination, semipaired domination number.

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### 1. INTRODUCTION

Paired domination, where each vertex in the paired dominating set must be partnered with an adjacent vertex in the set, was introduced in [14, 15] as a model for security applications. Semipaired domination relaxes the restriction that the partners must be adjacent. Specifically, a set  $S$  of vertices in a graph  $G$  is a *dominating set* of  $G$  if every vertex in  $V(G) \setminus S$  is adjacent to a vertex in  $S$ . Further, a dominating set  $S$  is a *paired dominating set* of  $G$  if the subgraph induced by  $S$ ,

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denoted  $G[S]$ , contains a perfect matching. The *domination number*  $\gamma(G)$  is the minimum cardinality of a dominating set of  $G$  and the *paired domination number*  $\gamma_{\text{pr}}(G)$  is the minimum cardinality of a paired dominating set of  $G$ . For a survey of paired domination, see [5].

The *distance* between two vertices  $u$  and  $v$  in a connected graph  $G$ , denoted by  $d_G(u, v)$ , is the length of a shortest  $(u, v)$ -path in  $G$ . A *semi-matching*  $M$  in a graph  $G$  is a set of pairs of vertices such that every vertex of  $G$  belongs to at most one pair in  $M$  and for every pair  $\{u, v\} \in M$ , either  $u$  and  $v$  are adjacent in  $G$  or  $u$  and  $v$  are at distance 2 apart in  $G$ . Further, if  $\{u, v\} \in M$  and  $d_G(u, v) = 1$ , then  $u$  and  $v$  are said to be *1-paired* in  $M$ , while if  $\{u, v\} \in M$  and  $d_G(u, v) = 2$ , then  $u$  and  $v$  are *2-paired* in  $M$ . A vertex that belongs to the semi-matching  $M$  is called an  *$M$ -matched vertex*, and a vertex not in the semi-matching  $M$  is called an  *$M$ -unmatched vertex*.

A set  $S$  of vertices in a graph  $G$  with no isolated vertices is a *semipaired dominating set*, abbreviated semi-PD-set, of  $G$  if  $S$  is a dominating set of  $G$  and every vertex in  $S$  is paired with exactly one other vertex in  $S$  that is within distance 2 from it. In other words, the vertices in the dominating set  $S$  can be partitioned into 2-sets such that if  $\{u, v\}$  is a 2-set, then  $uv \in E(G)$  or the distance between  $u$  and  $v$  is 2. We say that  $u$  and  $v$  are *paired*, and that  $u$  and  $v$  are *partners* with respect to the resulting semi-matching consisting of the pairings of vertices of  $S$ . The *semipaired domination number*, denoted by  $\gamma_{\text{pr2}}(G)$ , is the minimum cardinality of a semi-PD-set of  $G$ . We call a semi-PD-set of  $G$  of cardinality  $\gamma_{\text{pr2}}(G)$  a  *$\gamma_{\text{pr2}}$ -set of  $G$* . Semipaired domination was introduced in [11] and studied, for example, in [12, 13, 19–21]. From the definitions, we observe the following.

**Observation 1.** *If  $G$  is a graph with no isolated vertices, then  $\gamma(G) \leq \gamma_{\text{pr2}}(G) \leq \gamma_{\text{pr}}(G)$ .*

In this paper, we investigate the trees having unique minimum semi-PD-sets, and call such a tree a *USPD-tree*. Graphs having unique minimum dominating sets have been studied for many graph families (for examples, see [6, 8, 16, 17, 26]). Gunther, Hartnell, Markus and Rall [8] were the first to consider such graphs and they characterized the trees having unique minimum dominating sets. Trees having unique paired dominating sets were characterized in [2]. Graphs having a unique set for other domination parameters have also been much studied, including [1, 3, 4, 7, 9, 10, 18, 24, 25]. It is worth mentioning the related topic of which vertices appear in all or in no minimum dominating sets. Mynhardt [23] introduced an ingenious technique called tree-pruning to characterize these sets in trees.

A *double star*  $S(r, s)$  for  $1 \leq r \leq s$  is the tree having exactly two non-leaf vertices, one of which is adjacent to  $r$  leaves and the other to  $s$  leaves. For

examples of USPD-trees, note that the set containing the two non-leaf vertices of a double star  $T = S(r, s)$ , for  $2 \leq r \leq s$ , is a unique minimum  $\gamma_{\text{pr}2}$ -set of  $T$  as well as a unique minimum dominating and minimum paired dominating set. The double star  $S(1, 1)$ , that is, the path  $P_4: u_1 u_2 u_3 u_4$  has  $\{u_2, u_3\}$  as its unique minimum paired dominating set. But the path  $P_4$  is not a USPD-tree since each of  $\{u_2, u_3\}$ ,  $\{u_2, u_4\}$ , and  $\{u_1, u_3\}$  is a  $\gamma_{\text{pr}2}$ -set of  $P_4$ . On the other hand, for the path  $P_5: u_1 u_2 u_3 u_4 u_5$ , the set  $\{u_2, u_4\}$  is the unique  $\gamma_{\text{pr}2}$ -set of  $P_5$ , but not the unique  $\gamma$ -set of  $P_5$  since each of  $\{u_1, u_4\}$ ,  $\{u_2, u_4\}$ , and  $\{u_2, u_5\}$  is a  $\gamma$ -set of  $P_5$ . We also note that  $\gamma_{\text{pr}}(P_5) = 4$ .

We proceed as follows. In Section 1.1, we describe the graph theory terminology and notation, as well as additional definitions. In Section 2, we present some known results, and give some preliminary results that will be need when proving our main result. Thereafter, in Section 3, we present our main results, including a characterization of the USPD-trees.

### 1.1. Notation and terminology

For notation and graph theory terminology, we in general follow [22]. Specifically, the *order* of a graph  $G$  with vertex set  $V(G)$  and edge set  $E(G)$  is denoted by  $n(G) = |V(G)|$  and its *size* by  $m(G) = |E(G)|$ . If the graph  $G$  is clear from the context, we simply write  $V = V(G)$  and  $E = E(G)$ . The *open neighborhood* of a vertex  $v$  in  $G$  is the set  $N_G(v) = \{u \in V \mid uv \in E\}$ , and its *closed neighborhood* is the set  $N_G[v] = N_G(v) \cup \{v\}$ . For a set  $S \subseteq V$ , its *open neighborhood* is the set  $N_G(S) = \bigcup_{v \in S} N_G(v)$  and its *closed neighborhood* is the set  $N_G[S] = N_G(S) \cup S$ . The *degree* of a vertex  $v$  in  $G$  is  $d_G(v) = |N(v)|$ . If the graph  $G$  is clear from context, we simply write  $n$ ,  $m$ ,  $N(v)$ ,  $N[v]$ , and  $d(v)$  rather than  $n(G)$ ,  $m(G)$ ,  $N_G(v)$ ,  $N_G[v]$ , and  $d_G(v)$ , respectively.

For a subset  $S$  of vertices of  $G$ , the  *$S$ -private neighborhood* of the vertex  $v$  in  $S$  is the set  $\text{pn}(v, S) = \{w \in V(G) \mid N_G[w] \cap S = \{v\}\}$ , while the *external  $S$ -private neighborhood* of  $v$  is  $\text{epn}(v, S) = \text{pn}(v, S) \setminus S$ . An  *$S$ -external private neighbor* of  $v$  is a vertex in  $\text{epn}(v, S)$ . The subgraph induced by the set  $S$  is denoted by  $G[S]$ .

A non-trivial tree is a tree of order at least 2. A *leaf* of a tree is a vertex of degree 1 and its neighbor is called a *support* vertex. A *non-leaf* is a vertex of degree at least 2 in the tree. We denote a path on  $n$  vertices by  $P_n$ . We call  $P_1$  a *trivial star*. A *non-trivial star*  $K_{1,k}$  is a tree having exactly one non-leaf vertex (of degree  $k$ ) with  $k$  leaf neighbors for some  $k \geq 1$ . A trivial star and a non-trivial star, we call a *star*. An *odd star* is a star of odd order; that is, an odd star is either a trivial star (of order 1) or a non-trivial star of odd order (at least 3).

A *rooted tree*  $T$  distinguishes one vertex  $r$  called the *root*. For each vertex  $v \neq r$  of  $T$ , the *parent* of  $v$  is the neighbor of  $v$  on the unique  $(r, v)$ -path, while a

*child* of  $v$  is any other neighbor of  $v$ . The set of children of  $v$  is denoted by  $C(v)$ . A *descendant* of  $v$  is a vertex  $u \neq v$  such that the unique  $(r, u)$ -path contains  $v$ .

## 2. PRELIMINARY RESULTS AND OBSERVATIONS

In this section, we present known results and preliminary results that we will need when proving our main results. We begin with the following observations.

**Observation 2.** *The path  $P_n$  for  $n \geq 2$  is a USPD-tree if and only if  $n = 2$  or  $n \equiv 0 \pmod{5}$ .*

As observed earlier, a star is a USPD-tree if and only if it has order 2, while a double star is a USPD-tree if and only if both its support vertices have at least two leaf neighbors. We state this formally as follows.

**Observation 3.** *A nontrivial tree  $T$  of diameter at most 3 is a USPD-tree if and only if  $T = P_2$  or  $T$  is a double star  $S(r, s)$  where  $r, s \geq 2$ .*

For a vertex  $v$ , let  $L_v$  denote the set of leaves adjacent to  $v$ .

**Proposition 1.** *If  $T$  is a USPD-tree with unique  $\gamma_{\text{pr}2}$ -set  $S$ , then every support vertex of  $T$  is in  $S$ .*

**Proof.** Assume  $S$  is the unique  $\gamma_{\text{pr}2}$ -set of  $T$ , and let  $M$  be a semi-matching of  $S$ . Suppose, to the contrary, that  $v$  is a support vertex and  $v \in V \setminus S$ . The set of leaves  $L_v$  of  $v$  is therefore contained in  $S$ . Let  $u \in L_v$ , and let  $x$  be the partner of  $u$  in  $S$ , and so  $\{u, x\} \in M$  and  $u$  and  $x$  are 2-paired in  $M$ . We note that  $x \in N(v) \setminus \{u\}$ . Hence,  $S' = (S \setminus \{u\}) \cup \{v\}$  with semi-matching  $M' = (M \setminus \{\{u, x\}\}) \cup \{\{v, x\}\}$  is a  $\gamma_{\text{pr}2}$ -set of  $T$ , contradicting the fact that  $S$  is the unique  $\gamma_{\text{pr}2}$ -set of  $T$ . ■

A tight upper bound on the semipaired domination number of connected graphs is established in [13].

**Theorem 1** [13]. *If  $G$  is a connected graph of order  $n \geq 3$ , then  $\gamma_{\text{pr}2}(G) \leq \frac{2}{3}n$ , and this bound is tight.*

To see that the bound of Theorem 1 is sharp, consider the tree  $T_k$  obtained from a star  $K_{1,k+1}$  for  $k \geq 2$  by subdividing  $k$  of the edges exactly twice and the remaining edge exactly once. The resulting tree  $T_k$  has order  $n = 3k + 3$  and  $\gamma_{\text{pr}2}(T_k) = 2k + 2 = \frac{2}{3}n$ .

### 3. MAIN RESULTS

Our immediate aim is to characterize the trees having unique minimum semi-PD-sets; that is, to characterize the USPD-trees. For this purpose, we need some additional notation. For a given  $\gamma_{\text{pr}2}$ -set  $S$  and semi-matching  $M$  of a graph  $G$ , we say that the set  $S$  has properties  $\mathcal{P}_1$  and  $\mathcal{P}_2$  if the following hold.

- (a) Property  $\mathcal{P}_1$  if for every 1-pair  $\{u, v\}$  in  $M$ , we have  $|\text{epn}(u, S)| \geq 2$  and  $|\text{epn}(v, S)| \geq 2$ .
- (b) Property  $\mathcal{P}_2$  if for every 2-pair  $\{u, v\}$  in  $M$ , we have  $|\text{epn}(u, S)| \geq 1$  and  $|\text{epn}(v, S)| \geq 1$ .

Further, we say that a  $\gamma_{\text{pr}2}$ -set  $S$  in the graph  $G$  has Property  $\mathcal{P}$  if every possible semi-matching in  $G[S]$  has both Property  $\mathcal{P}_1$  and Property  $\mathcal{P}_2$ . We call a pair  $\{u, v\}$  in a semi-matching  $M$ , an *M-bad pair* if  $\{u, v\}$  is a 1-pair that does not have Property  $\mathcal{P}_1$  or  $\{u, v\}$  is a 2-pair that does not have Property  $\mathcal{P}_2$ . We observe the following about trees for which every  $\gamma_{\text{pr}2}$ -set has Property  $\mathcal{P}$ .

**Observation 4.** *If  $T$  is a tree such that every  $\gamma_{\text{pr}2}$ -set of  $T$  has Property  $\mathcal{P}$ , then no leaf is in any  $\gamma_{\text{pr}2}$ -set of  $T$ .*

We are now in a position to present our characterization of USPD-trees. A proof of Theorem 2 is presented in Section 4.

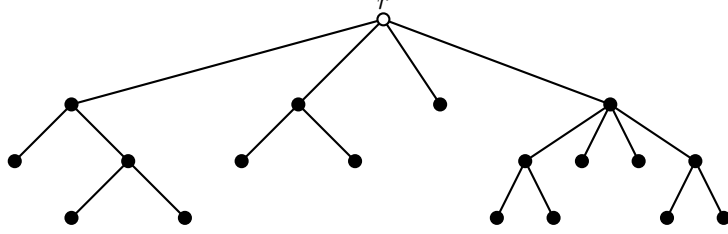
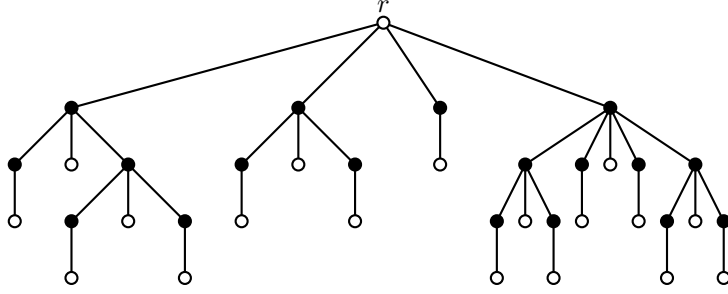
**Theorem 2.** *If  $T$  is a tree of order at least 3, then  $T$  is a USPD-tree if and only if  $T$  has a  $\gamma_{\text{pr}2}$ -set with Property  $\mathcal{P}$ .*

Our second aim is to show that the tight upper bound given in Theorem 1 can be significantly improved for USPD-trees by establishing a tight upper bound on the semipaied domination number of USPD-trees and characterizing the trees attaining this bound. In order to state this characterization, we define an *even rooted tree* as a rooted tree  $T$  of order at least 3 such that every vertex of  $T$  has an even number, including the possibility of zero, of children. By definition, the root of an even rooted tree has even degree at least 2, while every vertex different from the root has odd degree. An example of an even rooted tree with root  $r$  is illustrated in Figure 1.

Let  $\mathcal{T}$  be the family of all trees that can be obtained from an even rooted tree  $T$  by adding a pendant edge to every vertex of  $T$  different from the root. For example, if  $T$  is the even rooted tree shown in Figure 1, then the tree built from  $T$  that belongs to the family  $\mathcal{T}$  is shown in Figure 2.

We are now in a position to present our second main result. A proof of Theorem 3 is given in Section 5.

**Theorem 3.** *If  $T$  is a USPD-tree of order  $n \geq 3$ , then  $\gamma_{\text{pr}2}(T) \leq \frac{1}{2}(n - 1)$ , with equality if and only if  $T \in \mathcal{T}$ .*

Figure 1. An even rooted tree  $T$ .Figure 2. A tree in the family  $\mathcal{T}$ .

#### 4. PROOF OF THEOREM 2

In this section, we present a proof of Theorem 2. For this purpose, we first prove two lemmas.

**Lemma 1.** *If  $T$  is a USPD-tree of order at least 3 with unique  $\gamma_{\text{pr}2}$ -set  $S$ , then the set  $S$  has Property  $\mathcal{P}$ .*

**Proof.** Let  $T$  be a USPD-tree of order  $n \geq 3$  and let  $S$  be the unique  $\gamma_{\text{pr}2}$ -set of  $T$ . Suppose, to the contrary, that  $S$  does not have Property  $\mathcal{P}$ . Then there exists a semi-matching  $M$  of  $S$  with an  $M$ -bad pair  $\{u, v\}$ . We consider two cases.

*Case 1.*  $\{u, v\}$  is a 1-pair in  $M$  that does not have Property  $\mathcal{P}_1$ . Renaming the vertices  $u$  and  $v$  if necessary, we may assume that  $|\text{epn}(u, S)| \leq 1$ . If  $|\text{epn}(u, S)| = 1$  where  $\text{epn}(u, S) = \{u'\}$ , then  $(S \setminus \{u\}) \cup \{u'\}$  with semi-matching  $(M \setminus \{\{u, v\}\}) \cup \{\{v, u'\}\}$  is a  $\gamma_{\text{pr}2}$ -set of  $T$ , contradicting the uniqueness of the  $\gamma_{\text{pr}2}$ -set  $S$  of  $T$ . Thus,  $\text{epn}(u, S) = \emptyset$ . A similar argument shows that  $|\text{epn}(v, S)| \neq 1$ , and so,  $|\text{epn}(v, S)| \geq 2$  or  $\text{epn}(v, S) = \emptyset$ . If  $\text{epn}(v, S) \geq 2$  and  $v' \in \text{epn}(v, S)$ , then  $(S \setminus \{u\}) \cup \{v'\}$  with semi-matching  $(M \setminus \{\{u, v\}\}) \cup \{\{v, v'\}\}$  is a  $\gamma_{\text{pr}2}$ -set of  $T$ , contradicting the uniqueness of the  $\gamma_{\text{pr}2}$ -set  $S$  of  $T$ . Hence,  $\text{epn}(v, S) = \emptyset$ .

We note that  $|S| > 2$ , for otherwise since  $T$  is a tree and  $n \geq 3$ , at least one of  $u$  and  $v$  has an  $S$ -external private neighbor. If either  $u$  or  $v$ , say  $u$ , has a neighbor

$u'$  in  $V \setminus S$ , then  $(S \setminus \{u\}) \cup \{u'\}$  with semi-matching  $(M \setminus \{\{u, v\}\}) \cup \{\{v, u'\}\}$  is a  $\gamma_{\text{pr2}}$ -set of  $T$  different from  $S$ , contradicting the uniqueness of the  $\gamma_{\text{pr2}}$ -set  $S$  of  $T$ . Hence, neither  $u$  nor  $v$  has a neighbor in  $V \setminus S$ .

If both  $u$  and  $v$  have a neighbor in  $S \setminus \{u, v\}$ , then  $S \setminus \{u, v\}$  with semi-matching  $M \setminus \{\{u, v\}\}$  is a semi-PD-set of  $T$  with cardinality less than  $\gamma_{\text{pr2}}(T)$ , a contradiction, implying by our earlier observations that exactly one of  $u$  and  $v$  has a neighbor in  $S \setminus \{u, v\}$ . Renaming the vertices  $u$  and  $v$ , if necessary, we may assume that  $v$  has a neighbor in  $S \setminus \{u, v\}$ . Therefore, the vertex  $u$  is a leaf of  $T$ ,  $N[v] \subseteq S$ , and the vertex  $v$  has at least one neighbor, say  $w$ , in  $S \setminus \{u\}$ . Let  $x$  be the partner of  $w$  in  $M$ , and so  $\{w, x\} \in M$  and  $x \in S \setminus \{u, v, w\}$ .

If  $w$  has a neighbor  $w'$  in  $V \setminus S$ , then  $(S \setminus \{u\}) \cup \{w'\}$  with semi-matching  $(M \setminus \{\{u, v\}\}) \cup \{\{v, w'\}\}$  is a semi-PD-set of  $T$  different from  $S$ , again a contradiction. Hence,  $N(w) \subset S$ , and so  $\text{epn}(w, S) = \emptyset$ . If  $\{w, x\}$  is a 1-pair in  $M$ , then  $S \setminus \{u, w\}$  with semi-matching  $(M \setminus \{\{u, v\}, \{w, x\}\}) \cup \{v, x\}$  is a semi-PD-set of  $T$  with cardinality less than  $\gamma_{\text{pr2}}(T)$ , a contradiction. Hence,  $\{w, x\}$  is a 2-pair in  $M$ . Let  $y$  be the common neighbor of  $w$  and  $x$ . Since  $w$  has no neighbor in  $V \setminus S$ , we note that  $y \in S$ . If  $\text{epn}(x, S) = \emptyset$ , then  $S \setminus \{w, x\}$  with semi-matching  $M \setminus \{\{w, x\}\}$  is a semi-PD-set of  $T$  with cardinality less than  $\gamma_{\text{pr2}}(T)$ , a contradiction. Thus,  $x'$  is an  $S$ -external private neighbor of  $x$ ; that is,  $x' \in \text{epn}(x, S)$ . But then  $(S \setminus \{w\}) \cup \{x'\}$  with semi-matching  $(M \setminus \{\{w, x\}\}) \cup \{\{x, x'\}\}$  is a semi-PD-set of  $T$  different from  $S$ , a contradiction.

*Case 2.*  $\{u, v\}$  is a 2-pair in  $M$  that does not have Property  $\mathcal{P}_2$ . Renaming the vertices  $u$  and  $v$  if necessary, we may assume that  $\text{epn}(u, S) = \emptyset$ . Let  $x$  be the common neighbor of  $u$  and  $v$ . If  $x \notin S$ , then  $(S \setminus \{u\}) \cup \{x\}$  with semi-matching  $(M \setminus \{\{u, v\}\}) \cup \{\{v, x\}\}$  is a  $\gamma_{\text{pr2}}$ -set of  $T$ , contradicting the uniqueness of the  $\gamma_{\text{pr2}}$ -set  $S$  of  $T$ . Hence,  $x \in S$  and  $\{x, y\} \in M$  for some vertex  $y \in S \setminus \{u, v, x\}$ . If  $\text{epn}(v, S) = \emptyset$ , then  $S \setminus \{u, v\}$  with semi-matching  $M \setminus \{\{u, v\}\}$  is a semi-PD-set of  $T$  having cardinality less than  $\gamma_{\text{pr2}}(T)$ , a contradiction. Hence,  $|\text{epn}(v, S)| \geq 1$ . Let  $v' \in \text{epn}(v, S)$ . Then  $(S \setminus \{u\}) \cup \{v'\}$  with semi-matching  $(M \setminus \{\{u, v\}\}) \cup \{\{v, v'\}\}$  is a  $\gamma_{\text{pr2}}$ -set of  $T$ , contradicting the uniqueness of the  $\gamma_{\text{pr2}}$ -set  $S$  of  $T$ .  $\blacksquare$

**Lemma 2.** *If  $T$  is a tree of order at least 3 that contains a  $\gamma_{\text{pr2}}$ -set with Property  $\mathcal{P}$ , then  $T$  is a USPD-tree.*

**Proof.** Let  $T$  be a tree of order  $n \geq 3$  and let  $S$  be a  $\gamma_{\text{pr2}}$ -set of  $T$  with Property  $\mathcal{P}$ . Let  $M$  be a semi-matching associated with  $S$ . We wish to prove that  $S$  is the unique  $\gamma_{\text{pr2}}$ -set of  $T$ . We proceed by induction on  $n$ . Since  $\gamma_{\text{pr2}}(T) \geq 2$  and every vertex  $S$  has at least one  $S$ -external private neighbor, it follows that  $n \geq 4$ . If  $S$  contains a 1-pair, then since  $S$  has Property  $\mathcal{P}_1$ , it follows that  $n \geq 6$ . If  $S$  contains a 2-pair, then  $n \geq 5$ . The only tree with  $n = 5$  for which  $S$  has

Property  $\mathcal{P}_2$  is the path  $P_5$ , which is a USPD-tree by Observation 2, establishing the base case. Henceforth, we may assume that  $T$  is a tree of order  $n \geq 6$ .

Assume that if  $T'$  is a tree of order  $n'$ , where  $3 \leq n' < n$ , with a  $\gamma_{\text{pr2}}$ -set having Property  $\mathcal{P}$ , then  $T'$  is a USPD-tree. Suppose, to the contrary, that  $T$  is not a USPD-tree. Let  $D$  be a  $\gamma_{\text{pr2}}$ -set of  $T$  different from  $S$ . We proceed further with the following claim.

**Claim 1.** *The set  $V \setminus S$  is an independent set.*

**Proof.** Suppose, to the contrary, that  $V \setminus S$  contains two adjacent vertices, say  $u_1, u_2 \in V \setminus S$  such that  $u_1 u_2 \in E(T)$ . Let  $T_i$  be the component of  $T - u_1 u_2$  that contains the vertex  $u_i$  for  $i \in [2]$  and let  $S_i$  be the restriction of  $S$  to  $T_i$ , and so  $S_i = S \cap V(T_i)$ . Let  $M_i$  be the semi-matching subset of  $M$  consisting of pairs from  $S_i$  for  $i \in [2]$ .

**Claim 1.1** *The set  $S_i$  is the unique  $\gamma_{\text{pr2}}$ -set of  $T_i$  and for  $i \in [2]$ . Further, the  $\gamma_{\text{pr2}}$ -set  $S_i$  has Property  $\mathcal{P}$  in  $T_i$ .*

**Proof.** Since neither  $u_1$  nor  $u_2$  belongs to the set  $S$ , we note that the distance between a vertex of  $S_1$  and a vertex of  $S_2$  is at least 3 in  $T$ . Thus since  $S$  is a semi-PD-set of  $T$ , the set  $S_i$  with semi-matching  $M_i$  is a semi-PD-set of  $T_i$ , and so  $\gamma_{\text{pr2}}(T_i) \leq |S_i|$  for  $i \in [2]$ . The union of a  $\gamma_{\text{pr2}}$ -set of  $T_1$  and a  $\gamma_{\text{pr2}}$ -set of  $T_2$  is a semi-PD-set of  $T$ , implying that  $\gamma_{\text{pr2}}(T) \leq \gamma_{\text{pr2}}(T_1) + \gamma_{\text{pr2}}(T_2) \leq |S_1| + |S_2| = |S| = \gamma_{\text{pr2}}(T)$ . Hence we must have equality throughout this inequality chain, implying that  $\gamma_{\text{pr2}}(T_i) = |S_i|$  and therefore that the set  $S_i$  is a  $\gamma_{\text{pr2}}$ -set of  $T_i$  for  $i \in [2]$ . If  $v \in S_i$ , then we note that  $\text{epn}_{T_i}(v, S_i) \subset V(T_i)$ , and so  $\text{epn}_{T_i}(v, S_i) = \text{epn}_T(v, S)$  for  $i \in [2]$ . Thus, since the  $\gamma_{\text{pr2}}$ -set  $S$  of  $T$  has Property  $\mathcal{P}$  in  $T$ , the  $\gamma_{\text{pr2}}$ -set  $S_i$  has Property  $\mathcal{P}$  in  $T_i$ . Further, the tree  $T_i$  has order at least 5. Applying the inductive hypothesis to  $T_i$ , the tree  $T_i$  is a USPD-tree and the set  $S_i$  is the unique  $\gamma_{\text{pr2}}$ -set of  $T_i$  for  $i \in [2]$ .  $\square$

We now consider the  $\gamma_{\text{pr2}}$ -set  $D$  of  $T$  and let  $D_i$  be the restriction of  $D$  to  $T_i$ , and so  $D_i = D \cap V(T_i)$  for  $i \in [2]$ . For a semi-matching  $X$  associated with  $D$ , let  $X_i$  be the pairs of the vertices of  $D_i$  in  $X$  for  $i \in [2]$ . Note that  $X$  may contain pairs that are not in  $X_1 \cup X_2$ , that is, pairs that contain one vertex from  $D_1$  and one vertex from  $D_2$ . We call such a pair a *cross pair*. Among all semi-matchings of  $D$ , let  $X$  be one with the fewest cross pairs.

Analogously as with the set  $S$ , if neither  $u_1$  nor  $u_2$  is in  $D$ , then the set  $D_i$  is a semi-PD-set of  $D_i$  for  $i \in [2]$ . Thus,  $\gamma_{\text{pr2}}(T_i) = |S_i| \leq |D_i|$  for  $i \in [2]$ , and so  $\gamma_{\text{pr2}}(T) = \gamma_{\text{pr2}}(T_1) + \gamma_{\text{pr2}}(T_2) \leq |D_1| + |D_2| = \gamma_{\text{pr2}}(T)$ , implying that  $D_i$  is a  $\gamma_{\text{pr2}}$ -set of  $T_i$ . Since  $S_i$  is the unique  $\gamma_{\text{pr2}}$ -set of  $T_i$  for  $i \in [2]$ , this implies that  $D_i = S_i$ , and so  $D = S$ , a contradiction.



Hence, at least one of  $u_1$  and  $u_2$  is in  $D$ . Renaming vertices if necessary, we may assume that  $u_1 \in D$ . Let  $x$  be the vertex paired with  $u_1$  in  $X$ . If  $u_2 \in D$ , then let  $y$  be the vertex paired with  $u_2$  in  $X$ . We note that  $y$  could be  $u_1$ .

**Claim 1.2.** *The semi-matching  $X$  of the  $\gamma_{\text{pr}2}$ -set  $D$  of  $T$  has exactly one cross pair.*

**Proof.** We show firstly that the semi-matching  $X$  has at least one cross pair. Suppose, to the contrary, that  $X$  has no cross pairs. Thus the vertex  $x \in D_1$ , and if  $u_2 \in D_2$ , then the vertex  $y \in D_2$ . That is, every vertex in  $D_1$  is paired with a vertex of  $D_1$  and every vertex of  $D_2$  is paired with a vertex of  $D_2$ . Thus,  $D_1$  with semi-matching  $X_1$  is a semi-PD-set of  $T_1$ . Since  $S_1$  is the unique  $\gamma_{\text{pr}2}$ -set of  $T_1$  and  $u_1 \notin S_1$ , it follows that  $|S_1| = \gamma_{\text{pr}2}(T_1) < |D_1|$ . If  $D_2$  dominates  $T_2$ , then  $D_2$  with semi-matching  $X_2$  is a semi-PD-set of  $T_2$ , and so  $|S_2| = \gamma_{\text{pr}2}(T_2) \leq |D_2|$ . If  $D_2$  does not dominate  $T_2$ , then no vertex in  $N[u_2]$  is in  $D_2$ . In this case, the set  $D'_2 = D_2 \cup \{u_2, z\}$ , where  $z \in N(u_2) \setminus \{u_1\}$ , with semi-matching  $X_2 \cup \{\{u_2, z\}\}$  is a semi-PD-set of  $T_2$ . Since  $S_2$  is the unique  $\gamma_{\text{pr}2}$ -set of  $T_2$  and  $u_2 \notin S_2$ , it follows that  $|S_2| = \gamma_{\text{pr}2}(T_2) < |D'_2|$ . Furthermore, since both  $|S_2|$  and  $|D'_2|$  are even,  $|S_2| = \gamma_{\text{pr}2}(T_2) \leq |D'_2| - 2 = |D_2|$ . Hence in both cases, we have  $|S_2| \leq |D_2|$ . As observed earlier,  $|S_1| < |D_1|$ . Thus,  $\gamma_{\text{pr}2}(T) = |S_1| + |S_2| < |D_1| + |D_2| = \gamma_{\text{pr}2}(T)$ , a contradiction. Hence, the semi-matching  $X$  has at least one cross pair.

We show next that the semi-matching  $X$  has exactly one cross pair. If this is not the case, then  $X$  has exactly two cross pairs, namely  $\{u_1, x\}$  and  $\{u_2, y\}$ , where  $x \neq u_2$  and  $y \neq u_1$ . In this case, we note that  $x \in D_2$  and  $x$  is a neighbor of  $u_2$  in  $T_2$ , while  $y \in D_1$  and  $y$  is a neighbor of  $u_1$  in  $T_1$ . Hence, the set  $D$  with semi-matching  $(X \setminus \{\{u_1, x\}, \{u_2, y\}\}) \cup \{\{u_1, y\}, \{u_2, x\}\}$  is a  $\gamma_{\text{pr}2}$ -set of  $T$  having no cross pairs, contradicting our choice of  $X$ .  $\square$

By Claim 1.2, the semi-matching  $X$  of the  $\gamma_{\text{pr}2}$ -set  $D$  of  $T$  has exactly one cross pair. Renaming the vertices  $u_1$  and  $u_2$  if necessary, we may assume that  $\{u_1, x\}$  is the cross pair of  $X$ . Thus,  $x \in D_2$ , and so  $x = u_2$  or  $x$  is a neighbor of  $u_2$  in  $T_2$ . We note that if  $u_2 \in D_2$ , then either  $y = u_1$  or  $y \in D_2$ .

**Claim 1.3.**  $\gamma_{\text{pr}2}(T_1) \leq |D_1| - 1$ .

**Proof.** If  $D_1 \setminus \{u_1\}$  with semi-matching  $X_1$  is a semi-PD-set of  $T_1$ , then  $\gamma_{\text{pr}2}(T_1) \leq |D_1| - 1$ . Hence we may assume that  $D_1 \setminus \{u_1\}$  with semi-matching  $X_1$  is not a semi-PD-set of  $T_1$ , for otherwise the desired result of the claim follows. This implies that some vertex in the  $N[u_1]$  in  $T_1$  is not dominated by  $D_1 \setminus \{u_1\}$ . In this case, there is a vertex  $z_1 \in N(u_1) \cap (V(T_1) \setminus D_1)$ . Hence,  $D'_1 = D_1 \cup \{z_1\}$  with semi-matching  $X'_1 = X_1 \cup \{\{u_1, z_1\}\}$  is a semi-PD-set of  $T_1$ , and so  $|S_1| = \gamma_{\text{pr}2}(T_1) \leq |D'_1| = |D_1| + 1$ . Since  $S_1$  is the unique  $\gamma_{\text{pr}2}$ -set of  $T_1$  and  $u_1 \notin S_1$ ,

it follows that  $|S_1| = \gamma_{\text{pr2}}(T_1) < |D'_1|$ . Since both  $|D'_1|$  and  $|S_1|$  are even, this implies that  $|S_1| \leq |D'_1| - 2 = |D_1| - 1$ . Thus,  $\gamma_{\text{pr2}}(T_1) = |S_1| \leq |D_1| - 1$ .  $\square$

**Claim 1.4.**  $\gamma_{\text{pr2}}(T_2) \leq |D_2| - 1$ .

**Proof.** If  $D_2 \setminus \{x\}$  with semi-matching  $X_2$  is a semi-PD-set of  $T_2$ , then  $|S_2| = \gamma_{\text{pr2}}(T_2) \leq |D_2| - 1$ . Hence we may assume that  $D_2 \setminus \{x\}$  with semi-matching  $X_2$  is not a semi-PD-set of  $T_2$ , for otherwise the desired result of the claim follows. Since  $\{u_1, x\}$  is the unique cross pair of  $X$ , we note that every vertex in  $D_2 \setminus \{x\}$  is paired in  $X$  with a vertex of  $D_2 \setminus \{x\}$ . Thus since  $D_2 \setminus \{x\}$  is not a semi-PD-set of  $T_2$ , this implies that at least one neighbor of  $x$  in  $T_2$  does not belong to the set  $D_2$ .

We now define the vertex  $z_2$  as follows. If  $u_2 = x$  or if  $u_2 \neq x$  and  $u_2 \in D_2$ , let  $z_2$  be an arbitrary neighbor of  $x$  in  $T_2$  that does not belong to the set  $D_2$ . If  $u_2 \notin D_2$ , let  $z_2 = u_2$ . In both cases, the vertex  $z_2 \notin D_2$ . We now let  $D'_2 = D_2 \cup \{z_2\}$  and  $X'_2 = X_2 \cup \{\{x, z_2\}\}$ . The resulting set  $D'_2$  with semi-matching  $X'_2$  is a semi-PD-set of  $T_2$  satisfying  $|D'_2| = |D_2| + 1$ . Further by our choice of the vertex  $z_2$ , we note that  $u_2 \in D'_2$ . Since  $S_2$  is the unique  $\gamma_{\text{pr2}}$ -set of  $T_2$  and  $u_2 \notin S_2$ , it follows that  $|S_2| = \gamma_{\text{pr2}}(T_2) < |D'_2|$ . Since both  $|D'_2|$  and  $|S_2|$  are even, this implies that  $|S_2| \leq |D'_2| - 2 = |D_2| - 1$ . Hence,  $\gamma_{\text{pr2}}(T_2) = |S_2| \leq |D_2| - 1$ .  $\square$

We now return to the proof of Claim 1. By Claim 1.3,  $\gamma_{\text{pr2}}(T_1) \leq |D_1| - 1$ . By Claim 1.4,  $\gamma_{\text{pr2}}(T_2) \leq |D_2| - 1$ . Thus,  $\gamma_{\text{pr2}}(T) = |S_1| + |S_2| = \gamma_{\text{pr2}}(T_1) + \gamma_{\text{pr2}}(T_2) \leq |D_1| - 1 + |D_2| - 1 < |D_1| + |D_2| = |D| = \gamma_{\text{pr2}}(T)$ , a contradiction. This completes the proof of Claim 1.  $\square$

By Claim 1, the set  $V \setminus S$  is an independent set. Thus every vertex outside  $S$  has all its neighbors in the set  $S$ ; that is, for every vertex  $w \in V \setminus S$ , we have  $N(w) \subseteq S$ . Since the  $\gamma_{\text{pr2}}$ -set  $S$  has Property  $\mathcal{P}$ , by Observation 4 all the leaves of  $T$  are in  $V \setminus S$  and therefore all the support vertices of  $T$  are in  $S$ . Moreover, since no edge is in  $T[V \setminus S]$ , all the  $S$ -external private neighbors of vertices of  $S$  are leaves of  $T$ . Since every vertex in  $S$  has an  $S$ -external private neighbor, every vertex in  $S$  is a support vertex of  $T$ . But since every  $\gamma_{\text{pr2}}$ -set of  $T$  contains the support vertices of  $T$ , it follows that  $S$  is the unique  $\gamma_{\text{pr2}}$ -set of  $T$ , and so  $T$  is a USPD-tree. This completes the proof of Lemma 2.  $\blacksquare$

Theorem 2 is an immediate consequence of Lemmas 1 and 2.

## 5. PROOF OF THEOREM 3

In this section we prove Theorem 3 which gives an upper bound on the semipaired domination number of USPD-trees in terms of their order and characterizes the

trees achieving equality in this upper bound. Before we present a proof of Theorem 3, we first prove five lemmas.

**Lemma 3.** *If  $T$  is a USPD-tree of order  $n \geq 3$ , then  $\gamma_{\text{pr2}}(T) \leq \frac{1}{2}(n-1)$ .*

**Proof.** Let  $T$  be a USPD-tree of order  $n \geq 3$  with unique  $\gamma_{\text{pr2}}$ -set  $S$  and an associated semi-matching  $M$ . Further, let  $a_1$  be the number of 1-pairs and  $a_2$  be the number of 2-pairs in  $M$ . Thus,

$$(1) \quad |S| = 2a_1 + 2a_2.$$

Let  $\bar{S} = V \setminus S$ , and so  $|\bar{S}| = n - |S|$ . By Theorem 2, the  $\gamma_{\text{pr2}}$ -set  $S$  has Property  $\mathcal{P}$  in the tree  $T$ . Hence, for every 1-pair  $\{u, v\}$  in  $M$ , we have  $|\text{epn}(u, S)| \geq 2$  and  $|\text{epn}(v, S)| \geq 2$ , and for every 2-pair  $\{u, v\}$  in  $M$ , we have  $|\text{epn}(u, S)| \geq 1$  and  $|\text{epn}(v, S)| \geq 1$ . Thus,

$$(2) \quad |\bar{S}| \geq 4a_1 + 2a_2.$$

If there is at least one 1-pair in  $M$ , then by inequalities (1) and (2), we have

$$\begin{aligned} n &= |S| + |\bar{S}| \geq (2a_1 + 2a_2) + (4a_1 + 2a_2) \\ &= (4a_1 + 4a_2) + 2a_1 = 2|S| + 2a_1 \geq 2\gamma_{\text{pr2}}(T) + 2, \end{aligned}$$

implying that  $\gamma_{\text{pr2}}(T) \leq \frac{1}{2}(n-2) < \frac{1}{2}(n-1)$ . Hence, we may assume that every pair in  $M$  is a 2-pair, for otherwise the desired result follows. With this assumption,  $a_1 = 0$ . Thus, by inequalities (1) and (2), we have  $|S| = 2a_2$  and  $|\bar{S}| \geq 2a_2 = |S|$ . Thus,  $|\bar{S}| = 2a_2 + \ell$  for some integer  $\ell \geq 0$ . If  $\ell \geq 1$ , then

$$n = |S| + |\bar{S}| = 2a_2 + (2a_2 + \ell) = 2|S| + \ell = 2\gamma_{\text{pr2}}(T) + \ell,$$

or, equivalently,  $\gamma_{\text{pr2}}(T) \leq \frac{1}{2}(n - \ell) \leq \frac{1}{2}(n - 1)$ . Further if  $\ell > 1$ , then  $\gamma_{\text{pr2}}(T) < \frac{1}{2}(n - 1)$ . Hence, we may assume that  $\ell = 0$ , for otherwise the desired upper bound holds. With this assumption and our earlier observations,  $|\text{epn}(v, S)| = 1$  for every vertex  $v \in S$  and

$$(3) \quad \bar{S} = \bigcup_{v \in S} \text{epn}(v, S) \quad \text{and} \quad |\bar{S}| = \sum_{v \in S} |\text{epn}(v, S)| = |S|.$$

Let  $\{u_1, v_1\}$  be an arbitrary 2-pair in  $M$ , and let  $u_2$  be the common neighbor of  $u_1$  and  $v_1$ . We note that  $u_2 \in S$ . Let  $v_2$  be the partner of  $u_2$  in  $M$ , and so  $\{u_2, v_2\}$  is a 2-pair in  $M$ . Let  $x$  be the common neighbor of  $u_2$  and  $v_2$ . Suppose that  $x \in \{u_1, v_1\}$ . Renaming  $u_1$  and  $v_1$  if necessary, we may assume in this case that  $x = v_1$ . We now replace the 2-pairs  $\{u_1, v_1\}$  and  $\{u_2, v_2\}$  in  $M$  with the 1-pairs  $\{u_1, u_2\}$  and  $\{v_1, v_2\}$ . Proceeding as before when there is at least one 1-pair in  $M$ , we have  $\gamma_{\text{pr2}}(T) < \frac{1}{2}(n-1)$ . Hence, we may assume that  $x \notin \{u_1, v_1\}$ . Let

$x = u_3$  and let  $v_3$  be the partner of  $u_3$  in  $M$ , and so  $\{u_3, v_3\}$  is a 2-pair in  $M$ . Continuing in this manner, since  $S$  is finite, there must exist two 2-pairs in  $M$ , say  $\{u_i, v_i\}$  and  $\{u_j, v_j\}$  for some  $i$  and  $j$  where  $1 \leq i < j$  such that  $u_i u_j v_i v_j$  is a path in  $T$ . We now replace the 2-pairs  $\{u_i, v_i\}$  and  $\{u_j, v_j\}$  in  $M$  with the 1-pairs  $\{u_i, u_j\}$  and  $\{v_i, v_j\}$ . Proceeding as before when there is at least one 1-pair in  $M$ , we have  $\gamma_{\text{pr2}}(T) < \frac{1}{2}(n-1)$ . ■

**Lemma 4.** *The following properties hold in an even rooted tree  $T$ .*

- (a) *The tree  $T$  contains a semi-matching  $M$  such that every vertex different from the root is an  $M$ -matched vertex.*
- (b) *Every semi-matching of  $T$  satisfying part (a) contains only 2-pairs.*

**Proof.** (a) Let  $T$  be an even rooted tree with root  $r$ . Let  $X$  be the set of vertices of  $T$  of degree at least 2. Thus,  $X$  consists of all vertices of  $T$  that are not leaves. In particular, the root  $r$  belongs to the set  $X$ . Since  $T$  is an even rooted tree, we note that every vertex in  $X$  is the parent of an even number (at least two) of children in  $T$ . We can therefore partition  $V(T) \setminus \{r\}$  into  $|X|$  sets, namely the sets  $C(v)$  for each vertex  $v \in X$  where recall that  $C(v)$  denotes the set of children of  $v$ . Since  $|C(v)| \geq 2$  is even, we can partition the set  $C(v)$  into  $\frac{1}{2}|C(v)|$  pairs of vertices. We note that if  $\{v_1, v_2\}$  is such a pair of children of  $v$ , then  $d(v_1, v_2) = 2$ . Let  $M$  be the resulting set of all such pairs of vertices over all vertices  $v$  that belong to the set  $X$ . The resulting set  $M$  is a semi-matching of  $T$  in which every vertex of  $T$  different from the root is an  $M$ -matched vertex. Further, every pair in  $M$  is a 2-pair. This completes the proof of part (a).

To prove part (b), let  $T$  be a counterexample of smallest order  $n \geq 3$ . Thus, the tree  $T$  contains a semi-matching  $M$  such that every vertex of  $T$  different from the root is an  $M$ -matched vertex and at least one pair in  $M$  is a 1-pair.

Suppose that  $T$  is a star. In this case,  $T$  is an odd star of order at least 3. The set  $V(T) \setminus \{r\}$  of vertices different from the root is an independent set of even order and consists of all leaves in  $T$ . Thus, every semi-matching of  $T$  satisfying part (a) contains only 2-pairs, a contradiction to tree  $T$  being a counterexample. Hence,  $T$  is not a star, implying that at least one child of the root is not a leaf. Let  $v$  be a descendant of the root  $r$  that is not a leaf and is at maximum distance from  $r$ . Let  $w$  denote the parent of  $v$  in  $T$ . (Possibly,  $w = r$ .) Since  $T$  is an even rooted tree, the vertex  $v$  has an even number of children. By our choice of  $v$ , we note that every child of  $v$  is a leaf.

Let  $u_1$  and  $u_2$  be two arbitrary children of  $v$ , and let  $T' = T - \{u_1, u_2\}$ . We note that  $T'$  is an even rooted tree with root  $r$ . Let  $T'$  have order  $n'$ , and so  $3 \leq n' < n$ . Since  $T'$  is not a counterexample to our theorem, every semi-matching of  $T'$  satisfying part (a) contains only 2-pairs. Suppose that some pair in  $M$  contains two children of  $v$ . Renaming vertices if necessary, we may assume that  $\{u_1, u_2\}$  is such a pair in the semi-matching  $M$ . Thus,  $M' = M \setminus \{\{u_1, u_2\}\}$

is a semi-matching in  $T'$  satisfying part (a) that contains at least one 1-pair, a contradiction. Hence, no pair in  $M$  contains two children of  $v$ . This implies that  $v$  has exactly two children, namely  $u_1$  and  $u_2$ . Further, renaming  $u_1$  and  $u_2$  if necessary, we may assume that  $\{u_1, v\}$  and  $\{u_2, w\}$  are pairs in  $M$ . Since the root  $r$  is  $M$ -unmatched, we note that in this case  $w \neq r$ . Since  $u_1$  and  $v$  are adjacent, we note that  $\{u_1, v\}$  is a 1-pair in  $M$ . But then  $M' = (M \setminus \{\{u_1, v\}, \{u_2, w\}\}) \cup \{\{v, w\}\}$  is a semi-matching in  $T'$  satisfying part (a) that contains at least one 1-pair, a contradiction. This proves part (b) and completes the proof of Lemma 4. ■

We call a semi-matching of a graph that matches every vertex a *perfect semi-matching* of the graph.

**Lemma 5.** *If  $T$  is a tree of even order  $n \geq 2$ , then  $T$  contains a perfect semi-matching with at least one 1-pair.*

**Proof.** We proceed by induction on the order  $n \geq 2$  of a tree  $T$  of even order. If  $n = 2$ , then there is a unique perfect semi-matching in  $T$ , and such a semi-matching is a matching (in the ordinary sense). This establishes the base case. Let  $n \geq 4$  and assume that every tree  $T'$  of even order  $n'$  where  $2 \leq n' < n$  contains a perfect semi-matching with at least one 1-pair. Let  $T$  be a tree of even order  $n \geq 4$ , and let  $P: v_1 v_2 \cdots v_k$  be a longest path in  $T$ , where we note that  $k \geq 3$ .

Suppose that  $v_2$  has degree at least 3 in  $T$ . Let  $u_1$  be a neighbor of  $v_2$  different from  $v_1$  and  $v_3$ . Since  $P$  is a longest path in  $T$ , we note that  $u_1$  is a leaf. We now let  $T' = T - \{u_1, v_1\}$ . Applying the inductive hypothesis to the tree  $T'$  of even order strictly less than  $n$ , the tree  $T'$  contains a perfect semi-matching  $M'$  with at least one 1-pair. We can now extend the semi-matching  $M'$  to a perfect semi-matching  $M = M' \cup \{\{u_1, v_1\}\}$  of  $T$  with at least one 1-pair (namely, a 1-pair that belongs to  $M'$ ), as desired. Hence, we may assume that the vertex  $v_2$  has degree 2 in  $T$ . In this case, we let  $T' = T - \{v_1, v_2\}$ . Applying the inductive hypothesis to  $T'$ , the tree  $T'$  contains a perfect semi-matching  $M'$  with at least one 1-pair. We can now extend the semi-matching  $M'$  to a perfect semi-matching  $M = M' \cup \{\{v_1, v_2\}\}$  of  $T$  with at least one 1-pair, as desired. ■

We next establish properties of a tree that belongs to the family  $\mathcal{T}$ .

**Lemma 6.** *If  $T$  is an arbitrary tree of order  $n$  that belongs to the family  $\mathcal{T}$ , then the following hold.*

- (a)  $\gamma_{\text{pr2}}(T) = \frac{1}{2}(n - 1)$ .
- (b) *The tree  $T$  is a USPD-tree whose unique  $\gamma_{\text{pr2}}$ -set is the set of support vertices in  $T$ .*

**Proof.** Let  $T$  be a tree of order  $n$  in the family  $\mathcal{T}$ . Thus,  $T$  is built from an even rooted tree  $H$  with root  $r$  by adding a pendant edge to every vertex of  $H$  different from the root. If  $S = V(H) \setminus \{r\}$ , then by Lemma 4, the tree  $H$  contains a semi-matching  $M$  such that every vertex of  $S$  is an  $M$ -matched vertex. Since the set  $S$  dominates the tree  $T$ , we note that  $S$  with semi-matching  $M$  is a semi-PD-set of  $T$ , implying that  $\gamma_{\text{pr2}}(T) \leq |S|$ . Since every vertex in  $S$  is a support vertex in  $T$ , we note that  $|S| \leq \gamma(T) \leq \gamma_{\text{pr2}}(T) \leq |S|$ . Thus we must have equality throughout this inequality chain. In particular,  $\gamma_{\text{pr2}}(T) = |S| = \frac{1}{2}(n-1)$  and  $S$  is a  $\gamma_{\text{pr2}}$ -set of  $T$ . We note that  $S$  is precisely the set of support vertices of  $T$ . This completes the proof of part (a).

To prove part (b), suppose, to the contrary, that  $S$  is not the unique  $\gamma_{\text{pr2}}$ -set of  $T$ . Let  $S'$  be a  $\gamma_{\text{pr2}}$ -set of  $T$  different from  $S$ , and let  $M'$  be the associated semi-matching of  $T$ . Since every  $\gamma_{\text{pr2}}$ -set of  $T$  contains either a support vertex or its leaf neighbor, this implies that the set  $S'$  contains at least one leaf, say  $v'$ , of  $T$ . Let  $v$  be the support vertex with  $v'$  as its leaf neighbor. By our earlier observations, we note that  $v \notin S'$  and  $r \notin S'$ . Let  $w$  be the partner of  $v'$  in  $S'$ , and so  $\{v', w\}$  is a 2-pair in  $M'$  and the vertex  $w$  is a support vertex that belongs to the set  $S$ . We now replace every such 2-pair  $\{v', w\}$  in  $M'$  with the 1-pair  $\{v, w\}$  to produce a new semi-matching  $M^*$  in  $T$ . By construction, the semi-matching  $M^*$  consisting entirely of support vertices and contains at least one 1-pair. Hence,  $M^*$  is a semi-matching of the even rooted tree  $H$  such that every vertex different from the root  $r$  of  $H$  is an  $M^*$ -matched vertex. However,  $M^*$  contains at least one 1-pair, namely the pair  $\{v, w\}$ , contradicting Lemma 4. This proves part (b) and completes the proof of Lemma 6. ■

**Lemma 7.** *Let  $T$  be a tree of odd order  $n \geq 3$  and let  $r$  be a specified vertex of  $T$  of degree at least 2. If the tree satisfies both properties (a) and (b) below, then  $T$  is an even rooted tree with root  $r$ .*

- (a) *The tree  $T$  contains a semi-matching  $M$  such that every vertex different from  $r$  is an  $M$ -matched vertex.*
- (b) *Every semi-matching of  $T$  satisfying part (a) contains only 2-pairs.*

**Proof.** Suppose, to the contrary, that the lemma is false, and let  $T$  be a counterexample of smallest order  $n \geq 3$ . Thus, the tree  $T$  satisfies both properties (a) and (b), but  $T$  is not an even rooted tree with root  $r$ . Suppose that  $T$  is a star. In this case,  $T$  is an odd star of order at least 3. By assumption, the vertex  $r$  has degree at least 2 in  $T$ , and hence the vertex  $r$  is the center of the star, implying that  $T$  is an even rooted tree with root  $r$ , a contradiction to  $T$  being a counterexample. Hence,  $T$  is not a star.

We now root the tree  $T$  at the vertex  $r$ . Since  $T$  is not a star, at least one child of the vertex  $r$  is not a leaf. Let  $v$  be a descendant of the vertex  $r$  at maximum distance from  $r$ , and let  $w$  denote the parent of  $v$  and let  $x$  denote the

parent of  $w$ . We note that  $w \neq r$ . Further by our choice of the vertex  $v$ , every child of  $w$  is a leaf. In particular, the vertex  $v$  is a leaf. Let  $M$  be an arbitrary semi-matching of  $T$  satisfies both properties (a) and (b). Let  $\{v, v^*\}$  and  $\{w, w^*\}$  be the 2-pairs in  $M$  containing  $v$  and  $w$ , respectively. Since  $M$  contains no 1-pair, we note that  $v^* \neq w$ .

Suppose that  $v^*$  is a leaf neighbor of  $w$  in the rooted tree  $T$  with root  $r$ . In this case, we let  $T' = T - \{v, v^*\}$  and we let  $M' = M \setminus \{\{v, v^*\}\}$ . We note that  $T'$  is a tree of odd order  $n'$  where  $3 \leq n' = n - 2$ . Further,  $M'$  is a semi-matching in  $T'$  such that every vertex different from  $r$  is an  $M'$ -matched vertex. If  $T'$  has a semi-matching that matches every vertex different from  $r$  and contains a 1-pair, then such a semi-matching can be extended to a semi-matching of  $T$  that satisfies property (a) but not property (b) by adding to it the pair  $\{v, v^*\}$ , a contradiction. Hence, the tree  $T'$  satisfies both properties (a) and (b) (with  $T$  replaced by  $T'$ , and with  $M$  replaced by  $M'$ ). Since  $T'$  is not a counterexample to our lemma, the tree  $T'$  is an even rooted tree with root  $r$ . By restoring the tree  $T$  by adding back the two deleted children  $v$  and  $v^*$  of  $w$  (both of which are leaf neighbors of  $w$ ), the tree  $T$  is therefore is an even rooted tree with root  $r$ , a contradiction.

Hence, we may assume that  $v^*$  is not a leaf neighbor of  $w$  in the rooted tree  $T$  with root  $r$ . Thus,  $v^*$  is necessarily the parent,  $x$ , of the vertex  $w$  in  $T$ . In particular, this implies that  $x \neq r$  since  $r$  is an  $M$ -unmatched vertex. Since  $M$  contains no 1-pair, and since every child of  $w$  is a leaf, we note that the vertex  $w^*$  paired with  $w$  in  $M$  is a neighbor of  $x$  (different from  $w$ ). But then  $M^* = (M \setminus \{\{v, v^*\}, \{w, w^*\}\}) \cup \{\{v, w\}, \{v^*, w^*\}\}$  is a semi-matching in  $T$  such that every vertex different from  $r$  is an  $M^*$ -matched vertex but such that  $M^*$  contains at least two 1-pairs, contradicting the fact that  $T$  satisfies property (b). This completes the proof of Lemma 7.  $\blacksquare$

We are now in a position to prove Theorem 3. Recall its statement.

**Theorem 3.** *If  $T$  is a USPD-tree of order  $n \geq 3$ , then  $\gamma_{\text{pr2}}(T) \leq \frac{1}{2}(n - 1)$ , with equality if and only if  $T \in \mathcal{T}$ .*

**Proof.** Let  $T$  be a USPD-tree of order  $n \geq 3$ . By Lemma 3,  $\gamma_{\text{pr2}}(T) \leq \frac{1}{2}(n - 1)$ . By Lemma 6, if  $T \in \mathcal{T}$  has order  $n$ , then  $\gamma_{\text{pr2}}(T) = \frac{1}{2}(n - 1)$ . Hence it suffices for us to prove that if  $T$  is a USPD-tree of order  $n \geq 3$  satisfying  $\gamma_{\text{pr2}}(T) = \frac{1}{2}(n - 1)$ , then  $T \in \mathcal{T}$ . Suppose therefore that  $T$  is a USPD-tree of order  $n \geq 3$  such that  $\gamma_{\text{pr2}}(T) = \frac{1}{2}(n - 1)$ . In this case, we must have equality throughout the inequalities in the proof of Lemma 3. Adopting the notation in the proof of Lemma 3, this implies that  $a_1 = 0$ , and so every pair in  $M$  is a 2-pair. We state this formally as follows.

**Claim 2.** *Every pair in  $M$  is a 2-pair.*

Adopting our earlier notation, this also implies that  $|S| = 2a_2$  and that  $\ell = 1$ ; that is,  $|\bar{S}| = 2a_2 + 1$  and  $n = |S| + |\bar{S}| = 4a_2 + 1$ . Further,  $|\text{epn}(v, S)| = 1$  for every vertex  $v \in S$ , except for possibly one vertex  $v \in S$  for which  $|\text{epn}(v, S)| = 2$ . Recall that  $\bar{S} = V \setminus S$ .

**Claim 3.** *The set  $\bar{S}$  is an independent set.*

**Proof.** Suppose, to the contrary, that  $\bar{S}$  contains two adjacent vertices  $u_1$  and  $u_2$ . Let  $T_i$  be the component of  $T - u_1u_2$  that contains the vertex  $u_i$  for  $i \in [2]$  and let  $S_i$  be the restriction of  $S$  to  $T_i$ , and so  $S_i = S \cap V(T_i)$ . Also, let  $M_i$  be pairs of  $M$  that contain two vertices from  $T_i$  for  $i \in [2]$ . Since neither  $u_1$  nor  $u_2$  belongs to the set  $S$ , we note that the distance between a vertex of  $S_1$  and a vertex of  $S_2$  is at least 3 in  $T$ . Thus, since  $S$  is a semi-PD-set of  $T$ , the set  $S_i$  with semi-matching  $M_i$  is a semi-PD-set of  $T_i$ , and so  $\gamma_{\text{pr2}}(T_i) \leq |S_i|$  for  $i \in [2]$ . The union of a  $\gamma_{\text{pr2}}$ -set of  $T_1$  and a  $\gamma_{\text{pr2}}$ -set of  $T_2$  is a semi-PD-set of  $T$ , implying that  $\gamma_{\text{pr2}}(T) \leq \gamma_{\text{pr2}}(T_1) + \gamma_{\text{pr2}}(T_2) \leq |S_1| + |S_2| = |S| = \gamma_{\text{pr2}}(T)$ . Hence we must have equality throughout this inequality chain, implying that  $\gamma_{\text{pr2}}(T_i) = |S_i|$  and therefore that the set  $S_i$  is a  $\gamma_{\text{pr2}}$ -set of  $T_i$  for  $i \in [2]$ .

We note that if  $T_1$  has a  $\gamma_{\text{pr2}}$ -set  $D_1$  different from  $S_1$ , then  $D_1 \cup S_2$  is a  $\gamma_{\text{pr2}}$ -set of  $T$  different from  $S$ , contradicting the fact that  $T$  is a USPD-tree. Hence,  $S_1$  is the unique  $\gamma_{\text{pr2}}$ -set of  $T_1$ , and a similar argument shows that  $S_2$  is the unique  $\gamma_{\text{pr2}}$ -set of  $T_2$ . Let  $n_1$  and  $n_2$  be the order of  $T_1$  and  $T_2$ , respectively. Thus each tree  $T_i$  is a USPD-tree of order  $n_i \geq 3$ , implying by our earlier result that  $\gamma_{\text{pr2}}(T_i) \leq \frac{1}{2}(n_i - 1)$  for  $i \in [2]$ . This implies that  $\gamma_{\text{pr2}}(T) = \gamma_{\text{pr2}}(T_1) + \gamma_{\text{pr2}}(T_2) \leq \frac{1}{2}(n_1 - 1) + \frac{1}{2}(n_2 - 1) = \frac{1}{2}(n - 2)$ , a contradiction to our supposition that  $\gamma_{\text{pr2}}(T) = \frac{1}{2}(n - 1)$ .  $\square$

We present properties of the sets  $S$  and  $\bar{S}$ .

**Claim 4.** *The following holds.*

- (a) *Every vertex  $v \in S$  is a support vertex with exactly one leaf neighbor, say  $v'$ . Further,  $\text{epn}(v, S) = \{v'\}$ .*
- (b) *The set  $\bar{S}$  consists of  $|S|$  leaves and exactly one non-leaf vertex.*

**Proof.** By Claim 3, the set  $\bar{S}$  is an independent set. Thus every vertex outside  $S$  has all its neighbors in the set  $S$ ; that is, for every vertex  $w \in \bar{S}$ , we have  $N(w) \subseteq S$ . By Theorem 2, the  $\gamma_{\text{pr2}}$ -set  $S$  has Property  $\mathcal{P}$ . Thus, Observation 4 implies that all the leaves of  $T$  are in  $\bar{S}$  and therefore all the support vertices of  $T$  are in  $S$ . Moreover, since no edge is in  $T[\bar{S}]$ , all the  $S$ -external private neighbors of vertices of  $S$  are leaves of  $T$ . Since every vertex in  $S$  has an  $S$ -external private neighbor, every vertex in  $S$  is a support vertex of  $T$ . By our earlier observations, recall that  $|\bar{S}| = 2a_2 + 1$  and  $n = |S| + |\bar{S}| = 4a_2 + 1$ . Further,  $|\text{epn}(v, S)| = 1$  for every vertex  $v \in S$ , except for possibly one vertex  $v \in S$  for which  $|\text{epn}(v, S)| = 2$ .



Suppose that  $|\text{epn}(v, S)| = 2$  for some vertex  $v \in S$ . Thus, every vertex in  $\bar{S}$  is a leaf of  $T$ . In particular, this implies that  $T[S]$  is a tree of even order  $|S| = 2a_2$ . By Lemma 5, there is a perfect semi-matching of  $T$  with at least one 1-pair. Thus, we can choose a semi-matching of the  $\gamma_{\text{pr}2}$ -set  $S$  to contain at least one 1-pair. But then by our earlier observations this would imply that  $\gamma_{\text{pr}2}(T) < \frac{1}{2}(n - 1)$ , a contradiction. Hence,  $|\text{epn}(v, S)| = 1$  for every vertex  $v \in S$ . Thus, every vertex  $v \in S$  is a support vertex with exactly one leaf neighbor, say  $v'$ . Further,  $\text{epn}(v, S) = \{v'\}$ . There is therefore exactly one vertex in  $\bar{S}$  that is not a leaf and has at least two neighbors in  $S$ .  $\square$

By Claim 4, every vertex  $v \in S$  is a support vertex with exactly one leaf neighbor, say  $v'$ . Further,  $\text{epn}(v, S) = \{v'\}$ . Further, the set  $\bar{S}$  consists of  $|S|$  leaves and exactly one non-leaf vertex. Let  $r$  be the non-leaf that belongs to  $\bar{S}$ . By Claim 3,  $N_T(r) \subseteq S$ . Let  $H$  be the subtree of  $T$  induced by the set  $S \cup \{r\}$ . Thus,  $H$  is the tree obtained from  $T$  by deleting all leaves of  $T$ . By Claim 2 and Claim 4,  $H$  is a tree of odd order at least 3 and with a specified vertex  $r$  of degree at least 2 that satisfies both properties (a) and (b) in the statement of Lemma 7 (with  $T$  replaced by  $H$ ). Thus by Lemma 7, the tree  $H$  is an even rooted tree with root  $r$ . Hence by our earlier observation that every vertex of  $S$  is a support vertex of  $T$ , we have that  $T \in \mathcal{T}$ . This completes the proof of Theorem 3.  $\blacksquare$

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