Discussiones Mathematicae
Graph Theory 43 (2023) 17–34
https://doi.org/10.7151/dmgt.2348

ON CONDITIONAL CONNECTIVITY OF THE CARTESIAN PRODUCT OF CYCLES

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Abstract

The conditional h-vertex (h-edge) connectivity of a connected graph H of minimum degree $k > h$ is the size of a smallest vertex (edge) set F of H such that $H - F$ is a disconnected graph of minimum degree at least h. Let G be the Cartesian product of $r \geq 1$ cycles, each of length at least four and let h be an integer such that $0 \leq h \leq 2r-2$. In this paper, we determine the conditional h-vertex-connectivity and the conditional h-edge-connectivity of the graph G. We prove that both these connectivities are equal to $(2r-h)a_h^r$, where a_h^r is the number of vertices of a smallest h-regular subgraph of G.

Keywords: fault tolerance, hypercube, conditional connectivity, cut, Cartesian product.

2010 Mathematics Subject Classification: 05C40, 68R10.

1. Introduction

One of the feature of a good interconnection network is its high fault tolerance capacity. Interconnection network can be modelled into a graph with the help of which we can study many properties of the network. Connectivity of a modelled graph measures the fault tolerance capacity of the interconnection network. High fault tolerance capacity of the network plays an important role in practice. Traditional connectivities have some limitations to measure the fault tolerance capacity of a network accurately. In order to compute traditional edge connectivity, one allows failure of all the links incident with the same processor, practically which is rare. One can overcome this limitation effectively by considering the conditional connectivity of graphs introduced by Harary [6].

Let G be a connected graph with minimum degree at least $k \geq 1$ and let h be an integer such that $0 \leq h < k$. A set F of vertices (edges) of G such that $G - F$ is disconnected and each component of it has minimum degree at least h is an *h-vertex* (*edge*) *cut* of G. The *conditional h-vertex* (*edge*) connectivity of G, denoted by $\kappa^h(G)$ ($\lambda^h(G)$), is the minimum cardinality $|F|$ of an h-vertex(edge) cut F of G. Clearly, $h = 0$ gives the traditional vertex (edge) connectivity.

Many researchers have worked on the problem of determining the conditional connectivities for various classes of graphs and determined these parameters for smaller values of h [4, 5, 7, 9]. Exact values of one or both conditional connectivities are known for some classes of graphs. For the n-dimensional hypercube Q_n, the conditional connectivities λ^h and κ^h are same and their common value is $2^h(n - h)$; see [3, 7]. Li and Xu [10] proved that λ^h of any n-dimensional hypercube-like network G_n is also $2^h(n - h)$. Ye and Liang [16] established that κ^h is also $2^h(n-h)$ for some members of hypercube-like networks such as Crossed cubes, Locally twisted cubes, Möbius cubes. Independently, Wei and Hsieh [14] determined κ^h for the Locally twisted cubes. Ning [13] obtained κ^h for the exchanged crossed cubes. Both λ^h and κ^h are determined for the class of (n, k)-star graphs by Li *et al.* [11].

An r-dimensional torus is the Cartesian product of r cycles. The *k-ary r-cube*, denoted by Q^k_r, is the Cartesian product of r cycles each of length k. In particular, the hypercube Q_{2r} is Q^4_r. Hypercubes, k-ary r-cubes and multidimensional tori are widely used interconnection networks; see [2, 8, 12, 15].

It is easy to see that an r-dimensional torus is a $2r$-regular graph with traditional vertex connectivity and edge connectivity $2r$; see [15]. In this paper, we determine the conditional h-edge-connectivity as well as the conditional h-vertex-connectivity of the given multidimensional torus.

By C_k we mean a cycle of length k. For integers $h, r, k_1, k_2, \ldots, k_r$ with $0 \leq h \leq 2r$ and $4 \leq k_1 \leq k_2 \leq \cdots \leq k_r$, we define a quantity a^r_h as follows.

Definition 1.1.

$$a_h^r = \begin{cases} 2^h & \text{if } 0 \le h \le r, \\ 2^{r-i}\ k_1 k_2 \cdots k_i & \text{if } h = r+i,\ 1 \le i \le r. \end{cases}$$

We prove that both the conditional connectivities λ^h and k^h are equal to $a_h^r(2r-h)$ for the Cartesian product of cycles $C_{k_1}, C_{k_2}, \ldots, C_{k_r}$.

The following is the main theorem of the paper.

Theorem 1.2. *Let $h, r, k_1, k_2, \ldots, k_r$ be integers such that $0 \le h \le 2r-2$ and $4 \le k_1 \le k_2 \le \cdots \le k_r$ and let G be the Cartesian product of the cycles $C_{k_1}, C_{k_2}, \ldots, C_{k_r}$. Then $\lambda^h(G) = \kappa^h(G) = a_h^r(2r-h)$.*

Corollary 1.3. *Let h, r, k be integers such that $0 \le h \le 2r-2$, $4 \le k$ and let Q_r^k be the k-ary r-cube. Then $\lambda^h(Q_r^k) = k^h(Q_r^k) = a_h^r(2r-h)$, where $a_h^r = 2^h$ if $0 \le h \le r$ and $a_h^r = 2^{r-i}k^i$ if $h = r+i$ and $1 \le i \le r$.*

Corollary 1.4 [3, 7]**.** *For integers h and r with $0 \le h \le 2r-2$, $\lambda^h(Q_{2r}) = k^h(Q_{2r}) = 2^h(2r-h)$.*

The proof of our main result, Theorem 1.2 is divided into three sections. In Section 2, we characterize the h-regular subgraph of the graph G with minimum number of vertices and explore some of its properties. Using these properties we determine the conditional h-vertex connectivity and the conditional h-edge connectivity of G in Sections 3 and 4, respectively.

2. Smallest h-Regular Subgraph

In this section, we define a smallest h-regular subgraph of the Cartesian product of r cycles and obtain some properties of it. We first introduce some notations.

For a graph K, let $V(K)$ denote the set of all vertices of K. If H is a subgraph K, then $\delta(K)$ is the minimum degree of K while $\delta_K(H)$ is the minimum degree of H in K. The *Cartesian product* of two graphs H and K is a graph $H \Box K$ with vertex set $V(H) \times V(K)$. Two vertices (x, y) and (u, v) are adjacent in $H \Box K$ if and only if either $x = u$ and y is adjacent to v in K, or $y = v$ and x is adjacent to u in H. The hypercube Q_n is the Cartesian product of n copies of the complete graph K_2.

We use the following notations about the structure of the multidimensional torus.

Notation.

Consider the graph G of Theorem 1.2. We have $G = C_{k_1} \Box C_{k_2} \Box \cdots \Box C_{k_r}$, where C_{k_i} is a cycle of length k_i for $i = 1, 2, \ldots, k_r$ and $4 \le k_1 \le k_2 \le \cdots \le k_r$. We can write G as $G = H\Box C_{k_r}$, where $H = C_{k_1}\Box C_{k_2}\Box\cdots\Box C_{k_{r-1}}$. Label by $1, 2, \ldots, k_r$ the vertices of the cycle C_{k_r} so that i is adjacent to $(i+1) \pmod{k_r}$. Hence G can be obtained by replacing i^{th} vertex of C_{k_r} by a copy H^i of H and replacing edge joining i and $i+1$ of C_{k_r} by the perfect matching M_i between the corresponding vertices of H^i and H^{i+1}. Thus $G = H^1 \cup H^2 \cup \cdots \cup H^{k_r} \cup (M_1 \cup M_2 \cup \cdots \cup M_{k_r})$; see Figure 1.

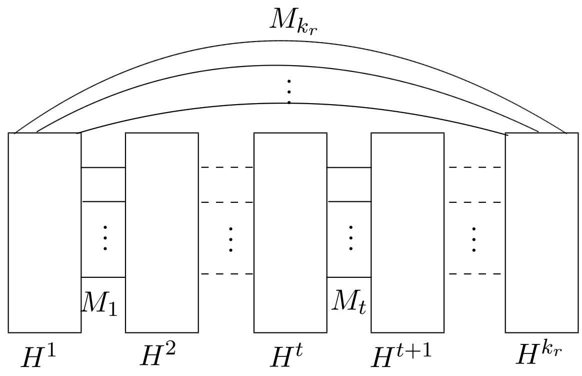


Figure 1. $G = H\Box C_{k_r}$.

Henceforth, by G we mean the graph $C_{k_1}\Box C_{k_2}\Box\cdots\Box C_{k_r}$ with $4 \le k_1 \le k_2 \le \cdots \le k_r$, that is, the graph of Theorem 1.2.

From the following lemma, it is clear that G is a $2r$-regular and $2r$-connected graph on $k_1k_2\cdots k_r$ vertices.

Lemma 2.1. *If G_i is an m_i-regular and m_i-connected graph on n_i vertices for $i = 1, 2$, then $G_1\Box G_2$ is an (m_1+m_2)-regular and (m_1+m_2)-connected graph on n_1n_2 vertices.*

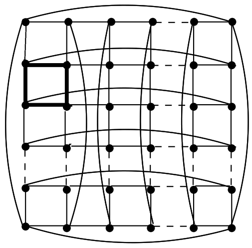
We now define an h-regular subgraph, denoted by W_h^r, of the graph G.

Definition 2.2. For $4 \le k_1 \le k_2 \le \cdots \le k_r$ and $0 \le h \le 2r$, let

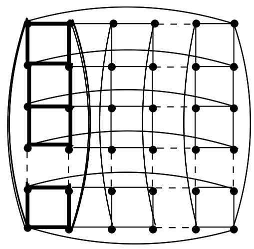
$$W_h^r = \begin{cases} Q_h & \text{if } 0 \le h \le r, \\ Q_{r-i}\Box C_{k_1}\Box C_{k_2}\Box\cdots\Box C_{k_i} & \text{if } h = r+i \text{ and } 1 \le i \le r. \end{cases}$$

In the following figure, a 2-regular subgraph W_2^2 and a 3-regular subgraph W_3^2 of the graph $C_5\Box C_5$ are shown by bold lines.

It is known that a smallest h-regular subgraph of the hypercube Q_n is isomorphic to Q_h (see [1]). We prove the analogous result for the Cartesian product of cycles. In fact, we establish that W_h^r is a smallest h-regular subgraph of the above graph G.



(a) W_2^2



(b) W_3^2

Figure 2. The subgraph W_2^2 and W_3^2 of $C_5 \square C_5$.

The following lemma follows from Lemma 2.1, Definition 1.1 of the number a_h^r and the fact that the hypercube Q_n is an n-regular, n-connected graph on 2^n vertices for any integer $n \geq 0$.

Lemma 2.3. *The graph W_h^r is h-regular and h-connected with a_h^r vertices.*

We need the following lemma that gives relations between different values of a_h^r.

Lemma 2.4. *Let $r \geq 2$ and let a_h^r be the quantity given in Definition* 1.1. *Then the following statements hold.*

1. $a_h^r = 2a_{h-1}^{r-1}$ *if* $1 \leq h \leq 2r-1$;
2. $k_r a_{h-2}^{r-1} \geq a_h^r$ *if* $2 \leq h \leq 2r$;
3. $a_h^{r-1} \geq a_h^r$ *if* $0 \leq h \leq 2r-2$.

Proof. Recall that $a_h^r = 2^h$ if $0 \leq h \leq r$ and $a_h^r = 2^{r-i}k_1k_2\cdots k_i$ if $h = r+i$ with $1 \leq i \leq r$, where $4 \leq k_1 \leq k_2 \leq \cdots \leq k_r$.

(1) If $1 \leq h \leq r$, then $a_h^r = 2^h = 2(2^{h-1}) = 2a_{h-1}^{r-1}$. For $r+1 \leq h \leq 2r-1$, we have $h = r+i$ for some $1 \leq i \leq r-1$. Hence $h-1 = (r-1)+i$ gives $a_{h-1}^{r-1} = 2^{(r-1)-i}k_1k_2\cdots k_i$. Therefore $2a_{h-1}^{r-1} = a_h^r$.

(2) Suppose $2 \leq h \leq r+1$. Then $a_{h-2}^{r-1} = 2^{h-2}$, and $a_h^r = 2^h$ if $h < r+1$ and $a_h^r = 2^{r-1}k_1$ if $h = r+1$. For $r+2 \leq h \leq 2r$, we have $h-2 = (r-1)+(i-1)$ for some $2 \leq i \leq r$ and so, $a_{h-2}^{r-1} = 2^{r-i}k_1k_2\cdots k_{i-1}$. Therefore, $k_r a_{h-2}^{r-1} \geq a_h^r$ in each case as $k_r \geq k_i \geq k_1 \geq 4$.

(3) Note that $a_h^{r-1} = 2^h$ for $1 \leq h \leq r-1$, and $a_h^{r-1} = 2^{(r-2)}k_1$ for $h = r = (r-1)+1$, and finally, $a_h^{r-1} = 2^{r-i-2}k_1k_2\cdots k_ik_{i+1}$ for $h = (r-1)+(i+1)$ for $1 \leq i \leq r$. Since $k_{i+1} \geq k_1 \geq 4$, we have $a_h^{r-1} \geq a_h^r$ in all the three cases. ∎

Lemma 2.5. *Every subgraph of the graph G of minimum degree at least h has at least a_h^r vertices.*

Proof. The graph G is the product of r cycles. We prove the result by induction on r. The result holds obviously for $h = 0$ and $h = 1$ and so it holds for $r = 1$. Suppose $r \geq 2$ and $h \geq 2$. Assume that the result holds for the product of $r - 1$ cycles. We have $G = C_{k_1} \Box C_{k_2} \Box \cdots \Box C_{k_r}$, where $4 \leq k_1 \leq k_2 \leq \cdots \leq k_r$. Write G as $G = H \Box C_{k_r}$, where $H = C_{k_1} \Box C_{k_2} \Box \cdots \Box C_{k_{r-1}}$. Then $G = H^1 \cup H^2 \cup \cdots \cup H^{k_r} \cup (M_1 \cup M_2 \cup \cdots \cup M_{k_r})$, where H^i is the copy of H corresponding to vertex i of C_{k_r} and M_i is the perfect matching between the corresponding vertices of H^i and H^{i+1}.

Let K be a subgraph of G with $\delta(K) \geq h$. We prove that $|V(K)| \geq a_h^r$. Clearly, K intersects at least one H^i. Let $K^i = K \cap H^i$ for $i = 1, 2, \ldots, k_r$. We have the following three cases.

(1) Suppose only one K^i is non-empty. Due to symmetry in G, we may assume K^1 is non-empty and K^j is empty for every $j \neq 1$. Therefore K is a subgraph of H^1 and it has minimum degree at least h in H^1. Since H^1 is $2(r-1)$-regular, $h \leq 2r - 2$. Suppose $h = 2r - 2$. Then $K = H^1$ and so, $|V(K)| = k_1 k_2 \cdots k_{r-1}$. If $r = 2$, then $|V(K)| = k_1 \geq 4 = a_2^2 = a_h^r$. If $r \geq 3$, then $|V(K)| \geq 4k_1 k_2 \cdots k_{r-2} = a_h^r$ as $k_{r-1} \geq 4$. If $h < 2r - 2$, then, by induction and Lemma 2.4(3), we have $|V(K)| \geq a_h^{r-1} \geq a_h^r$.

(2) Suppose K^i is non-empty for all i. Note that in the graph G, every vertex of H^i has exactly one neighbour in H^{i-1} and one in H^{i+1}. Hence the minimum degree of K^i is at least $h-2$. By induction, $|V(K^i)| \geq a_{h-2}^{r-1}$. Therefore, by Lemma 2.4(2),

$$|V(K)| = \left|V\left(K^1\right)\right| + \left|V\left(K^2\right)\right| + \cdots + \left|V\left(K^{k_r}\right)\right| \geq k_r a_{h-2}^{r-1} \geq a_h^r.$$

(3) Suppose at least two K^i are non-empty and at least one K^i is empty. Hence, we may assume that $K^1 \neq \emptyset$ but $K^{k_r} = \emptyset$. Further, we get an integer $1 < t < k_r$ such that $K^t \neq \emptyset$ but $K^{t+1} = \emptyset$. Then $\delta(K^j) \geq h - 1$ and so, by induction, $|V(K^j)| \geq a_{h-1}^{r-1}$ for $j = 1, t$. Now, by Lemma 2.4(1),

$$|V(K)| \geq \left|V\left(K^1\right)\right| + \left|V\left(K^t\right)\right| \geq 2a_{h-1}^{r-1} = a_h^r.$$

This completes the proof. ■

The following result is an immediate consequence of Lemmas 2.3 and 2.5.

Corollary 2.6. W_h^r *is a smallest subgraph of the graph* G *of minimum degree at least* h.

We obtain some more properties of the subgraph W_h^r of G to obtain an upper bound on the conditional connectivity of the graph G.

First, we introduce some notations. Let K be a graph and let Y be a subgraph of K. A *neighbour* of Y in K is a vertex in $V(K) \setminus V(Y)$ that is adjacent to a vertex of Y. Let $N(Y)$ denote the set of all neighbours of Y in K and let $N[Y] = N(Y) \cup V(Y)$. Also, for a subgraph H of K, let $N_H(Y)$ be the set of all neighbours of Y that are present in H and let $N_H[Y] = N_H(Y) \cup V(Y)$.

The following result is analogous to the result of hypercubes which states that if K is a subgraph of the hypercube Q_n isomorphic to Q_h, then every vertex of Q_n which is not in K has at most one neighbour in K; see [1].

Lemma 2.7. *If $0 \leq h < 2r-1$ and K is a subgraph of G isomorphic to the graph W_h^r, then every vertex of G belonging to $V(G) \setminus V(K)$ has at most one neighbour in the subgraph K.*

Proof. We argue by induction on r. If $r = 1$, then G is just a cycle and so the result holds obviously. Suppose $r \geq 2$. Assume that the result holds for the product of any $r-1$ cycles. We have $G = H \Box C_{k_r}$. Then $G = H^1 \cup H^2 \cup \cdots \cup H^{k_r} \cup (M_1 \cup M_2 \cup \cdots \cup M_{k_r})$, where H^i is the copy of H corresponding to vertex i of C_{k_r} and M_i is the perfect matching between the corresponding vertices of H^i and H^{i+1}. Since the graph W_h^r is isomorphic to $W_{h-1}^{r-1} \Box K_2$, we may assume that W_h^r is a subgraph of $H \Box K_2$ by considering W_{h-1}^{r-1} as a subgraph of H. Hence, we may assume that K is a subgraph of $H^2 \cup H^3 \cup M_2$, where M_2 is the perfect matching between H^2 and H^3.

Let $K^i = K \cap H^i$ for $i = 2, 3$. Then K^i is isomprphic to W_{h-1}^{r-1}. Let x be any vertex of $V(G) \setminus V(K)$. If x is in $V(H^2)$, then, by induction, x has at most one neighbour in K^2. Then x has no neighbour in K^3 and so, it has at most one neighbour in K. Similarly, x has at most one neighbour in K if it belongs to $V(H^3)$. Suppose x is in H^j for some $j \notin \{2, 3\}$. Then x has exactly one neighbour in H^{j+1} and one in H^{j-1} each and no neighbour in H^i for any $i \notin \{j-1, j+1\}$. This shows that x has at most one neighbour in $H^2 \cup H^3$ and hence in K as $k_r \geq 4$. This completes the proof. ■

Lemma 2.8. *If $0 \leq h \leq 2r-1$ and $Y = W_h^r$, then any vertex of G which is not in $N[Y]$ has at most two neighbours in $N[Y]$.*

Proof. We proceed by induction on r. The result holds trivially for $r = 1$ as G is just a cycle in this case. Suppose $r \geq 2$. Assume that the result holds for the product of any $r-1$ cycles. Write G as $H \Box C_{k_r}$, where $H = C_{k_1} \Box C_{k_2} \Box \cdots \Box C_{k_{r-1}}$. Since the graph W_h^r is isomorphic to $W_{h-1}^{r-1} \Box K_2$, we may assume that $Y = W_h^r$ is a subgraph of $H^2 \cup H^3 \cup M_2$. Then Y has neighbours in H^1 and H^4. Let $Y_i = W_h^r \cap H^i$ for $i = 2, 3$. Let $S_1 = N_{H^1}(Y_2)$, $S_2 = N_{H^2}[Y_2]$, $S_3 = N_{H^3}[Y_3]$ and $S_4 = N_{H^4}(Y_3)$. Then $N[Y] = S_1 \cup S_2 \cup S_3 \cup S_4$.

Let $x \in V(G) \setminus N[Y]$. Then x is a vertex of H^j for some j. If $j > 4$, then x has at most two neighbours in the set $V(H^1) \cup V(H^2) \cup V(H^3) \cup V(H^4)$ and so in its subset $N[Y]$. Suppose $j \in \{1, 2, 3, 4\}$. Then $h \leq 2r-2$ as for $h = 2r-1$, we have $Y = H^2 \cup H^3 \cup M_2$ and so, $N[Y] = V(H^1) \cup V(H^2) \cup V(H^3) \cup V(H^4)$.

The subgraph of G induced by the set S_i is isomorphic to the graph W_{h-1}^{r-1} for $i = 1, 4$. If $j \in \{1, 4\}$, then x has at most one neighbour in $S_1 \cup S_4$ and at most one in $V(H^2) \cup V(H^3)$ by Lemma 2.7. If $j = 2$, then, by induction, x has at most two neighbours in S_2 and no neighbour in $S_1 \cup S_3 \cup S_4$. Similarly, if $j = 3$, then x has at most two neighbours in S_3 and no neighbour in $S_1 \cup S_2 \cup S_4$. Thus, in any case, x has at most two neighbours in $N[Y]$. ■

Lemma 2.9. *For* $0 \le h \le 2r - 1$, *the inequality* $(2r - h + 1)a_h^r \le k_1 k_2 \cdots k_r$ *holds. Moreover, the inequality is strict if* $h < 2r - 1$.

Proof. Recall that $4 \le k_1 \le k_2 \cdots \le k_r$, and $a_h^r = 2^h$ if $h \le r$ and $a_h^r = 2^{(r-i)} k_1 k_2 \cdots k_i$ if $h = r + i$. For convenience, let $L = (2r - h + 1)a_h^r$ and $R = k_1 k_2 \cdots k_r$. Then $L = 2a_h^r = 4k_1 k_2 \cdots k_{r-1} \le R$ for $h = 2r-1$. Suppose $h \le 2r-2$. If $h = 0$ or $h = 1$, then $L < 4^r \le R$. Similarly, if $2 \le h \le r$, then $L < 2ra_h^r = 2r2^h \le 2r2^r \le 4^r \le R$ as $2r \le 2^r$. Suppose $h = r + i$ with $1 \le i \le r - 2$. Then $L = (r - i + 1)2^{r-i} k_1 k_2 \cdots k_i < 2^{2(r-i)} k_1 k_2 \cdots k_i$, as $2l \le 2^l$ if $l \ge 1$. This shows that $L \le 4^{r-i} k_1 k_2 \cdots k_i \le k_1 k_2 \cdots k_r = R$. ■

3. Conditional Vertex Connectivity

Recall from Section 2 the graph $G = C_{k_1} \Box C_{k_2} \Box \cdots \Box C_{k_r}$ and its h-regular subgraph W_h^r with a_h^r vertices, where $4 \le k_1 \le k_2 \le \cdots \le k_r$. In this section, we prove that the conditional h-vertex connectivity $\kappa^h(G)$ the graph G is $(2r-h)a_h^r$. Using Lemmas 2.7, 2.8 and 2.9, it easily follows that $\kappa^h(G) \le (2r - h)a_h^r$.

Lemma 3.1. *If* $0 \le h \le 2r - 2$, *then* $\kappa^h(G) \le (2r - h)a_h^r$.

Proof. We have $G = C_{k_1} \Box C_{k_2} \Box \cdots \Box C_{k_r}$. We simply denote the subgraph W_h^r of G by Y. Then $|V(Y)| = a_h^r$. Since G is $2r$-regular and Y is h-regular, every vertex of Y has $2r - h$ neighbours in the $G - V(Y)$. By Lemma 2.7, $|N(Y)| = (2r - h)|V(Y)| = (2r - h)a_h^r$. This gives $|N[Y]| = |V(Y) \cup N(Y)| = |V(Y)| + |N(Y)| = (2r - h + 1)a_h^r$. Therefore, by Lemma 2.9, $|N[Y]| < k_1 k_2 \cdots k_r = |V(G)|$. Hence $V(G) \setminus N[Y]$ is a non-empty set and by Lemma 2.8, every member of this set has at most two neighbours in $N[Y]$. Consequently, the minimum degree of the subgraph of G induced by this set is at least $2r - 2 \ge h$. Already, the minimum degree of the graph Y is h. Hence the graph $G - N(Y)$ is disconnected and every component of it has minimum degree at least h. Thus $N(Y)$ is an h-vertex cut of G. Therefore $\kappa^h(G) \le |N(Y)| = (2r - h)a_h^r$. ■

To prove the reverse inequality for $\kappa^h(G)$, we obtain the following lemma.

Lemma 3.2. *If* $0 \le h \le 2r-1$ *and* Y *is a subgraph of the graph* G *with minimum degree at least* h, *then* $|N[Y]| \ge a_h^r(2r - h + 1)$.

Proof. If $N[Y] = V(G)$, then the result follows obviously from Lemma 2.9. Suppose $N[Y] \neq V(G)$. We prove the result by induction on r. Since G is $2r$-regular, all $2r$ neighbours of any vertex of Y belong to the set $N[Y]$. Hence $|N[Y]| \geq 2r+1$. Therefore the result holds for $h=0$. Also, the result trivially follows for $r=1$ and $h=1$ as in this case G is a cycle of length $k_1 \geq 4$, Y is a path on at least two vertices and $a_1^1 = 2$.

Suppose $r \geq 2$ and $h \geq 1$. Assume that the result holds for a graph that is the product of $r-1$ cycles. Let $G = C_{k_1}\Box C_{k_2}\Box \cdots \Box C_{k_r}$. Then $G = H\Box C_{k_r}$, where $H = C_{k_1}\Box C_{k_2}\Box \cdots \Box C_{k_{r-1}}$. Then G contains k_r vertex-disjoint copies $H^1, H^2, \ldots, H^{k_r}$ of H. Then every vertex of H^i has one neighbour in H^{i-1} and H^{i+1}, where the addition and subtraction in the superscript is carried out modulo k_r. Let Y be a subgraph of G with $\delta(Y) \geq h$ and $N[Y] \neq V(G)$. Then Y intersects at least one copy of H^i. Let $Y_i = Y \cap H^i$ for $i = 1, 2, \ldots, k_r$.

Case 1. $Y_i \neq \emptyset$ for only one value of i. Without loss of generality we may assume that only Y_1 is non-empty. Then $Y = Y_1$ is contained in the graph H^1. Since H^1 is $(2r-2)$-regular, $h \leq 2r-2$. Also, the minimum degree of Y in H^1 is at least h. Hence, by Lemma 2.5, Y has at least a_h^{r-1} vertices. We have $N[Y] = N_{H^1}[Y] \cup N_{H^{k_r}}(Y) \cup N_{H^2}(Y)$. If $h = 2r-2$, then $Y = H^1$ and so, $N[Y] = V(H^1) \cup V(H^2) \cup V(H^{k_r})$. Therefore

$$|N[Y]| \geq 3\big|V\big(H^1\big)\big| = 3k_1k_2\cdots k_{r-1} \geq 12k_1k_2\cdots k_{r-2} = (2r-h+1)a_h^r.$$

Suppose $0 \leq h \leq 2r-3 = 2(r-1)-1$. Then, by induction, $|N_{H^1}[Y]| \geq a_h^{r-1}(2r-h-1)$. As $|N_{H^{k_r}}(Y)| = |N_{H^2}(Y)| = |V(Y)| \geq a_h^{r-1}$, by Lemma 2.4(3) we have

$$|N[Y]| \geq a_h^{r-1}(2r-h-1) + 2a_h^{r-1} = (2r-h+1)a_h^{r-1} \geq (2r-h+1)a_h^r.$$

Case 2. $Y_i \neq \emptyset$ for all $i = 1, 2, \ldots, k_r$. In this case, $N[Y] \supseteq N_{H^1}[Y_1] \cup N_{H^2}[Y_2] \cup \cdots \cup N_{H^{k_r}}[Y_{k_r}]$. If $h = 1$, then $\delta_{H^i}(Y_i) \geq 0$ and so, by induction, $|N_{H^i}[Y_i]| \geq a_0^{r-1}(2(r-1)-0+1) = 2r-1$ implying

$$\begin{aligned} |N[Y]| &\geq \big|N_{H^1}[Y_1]\big| + \big|N_{H^2}[Y_2]\big| + \cdots + \big|N_{H^{k_r}}[Y_{k_r}]\big| \\ &\geq k_r(2r-1) \geq 8r-4 \geq 4r \geq a_1^r(2r) = a_h^r(2r-h+1). \end{aligned}$$

Suppose $h \geq 2$. Then $\delta_{H^i}(Y_i) \geq h-2 \geq 0$ and so, by induction, $|N_{H^i}[Y_i]| \geq a_{h-2}^{r-1}(2r-h+1)$ for all i. Therefore, by Lemma 2.4(2),

$$\begin{aligned} |N[Y]| &\geq \big|N_{H^1}[Y_1]\big| + \big|N_{H^2}[Y_2]\big| + \cdots + \big|N_{H^{k_r}}[Y_{k_r}]\big| \\ &\geq k_r a_{h-2}^{r-1}(2r-h+1) \geq a_h^r(2r-h+1). \end{aligned}$$

Case 3. $Y_i \neq \emptyset$ for more than one but not all values of i. Without loss of generality, we may assume that Y_1 is non-empty but Y_{k_r} is empty. Let t be the largest integer such that Y_t is non-empty. Then $1 < t < k_r$; see Figure 3. Suppose that $h = 2r - 1$. Then $Y_1 = H^1$ and $Y_t = H^t$. Hence $N[Y] \supseteq V(H^1) \cup V(H^2) \cup V(H^t) \cup V(H^{t+1}) \cup V(H^{k_r})$. Since $k_r \geq 4$, $t \neq 2$ or $t+1 \neq k_r$ and $|V(H^1)| = |V(H^i)|$ for all $i > 1$. By Lemma 2.9,

$$|N[Y]| \geq 4\big|V\big(H^1\big)\big| = 4|V(H)| \geq 4k_1k_2\cdots k_{r-1} = (2r-h+1)a_h^r.$$

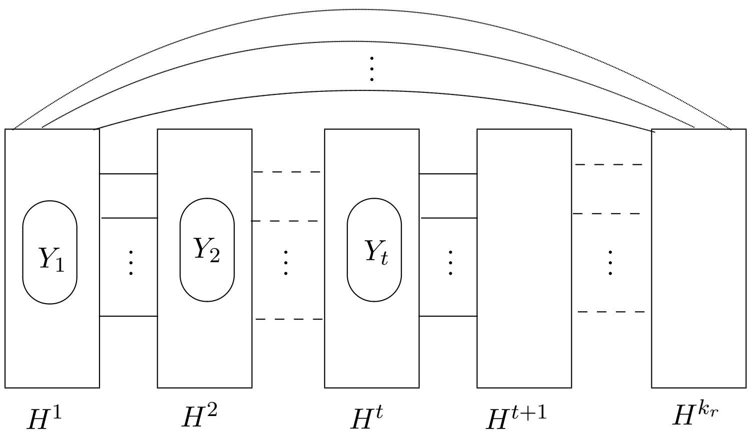


Figure 3. The graph G with $Y_j = \emptyset$ for $t < j \leq k_r$.

Suppose that $0 \leq h \leq 2r-2$. The graph Y_i has $|V(Y_i)|$ neighbours in H^{i-1} and H^{i+1} for $i = 1, t$. Therefore $|N[Y]| \geq |N_{H^1}[Y_1]| + |N_{H^t}[Y_t]| + |V(Y_1)| + |V(Y_t)|$. If $i \in \{1, t\}$, then $\delta_{H^i}(Y_i) \geq h-1$ and so, by induction, $|N_{H^i}[Y_i]| \geq a_{h-1}^{r-1}(2r-h)$. Also, by Lemma 2.5, $|V(Y_i)| \geq a_{h-1}^{r-1}$. Hence, by Lemma 2.4(1), we have

$$|N[Y]| \geq 2a_{h-1}^{r-1}(2r-h) + 2a_{h-1}^{r-1} = a_h^r(2r-h) + a_h^r = a_h^r(2r-h+1).$$

Thus $|N[Y]| \geq a_h^r(2r-h+1)$ in each case. This completes the proof. ■

Proposition 3.3. *If $0 \leq h \leq 2r-2$ and S is an h-vertex cut of the graph G, then $|S| \geq a_h^r(2r-h)$.*

Proof. We argue by induction on r. Suppose $h = 0$. Then S is a traditional vertex cut of G. Therefore $|S| \geq 2r = a_0^r(2r-0)$ as G is $2r$-connected by Lemma 2.1. Hence the result holds for $h = 0$ and so for $r = 1$. Suppose $r \geq 2$ and $h \geq 1$. Assume that the result is true for the Cartesian product of $r-1$ cycles, each of length at least 4. Let $G = C_{k_1}\square C_{k_2}\square\cdots\square C_{k_r}$. Then $G = H\square C_{k_r}$, where $H = C_{k_1}\square C_{k_2}\square\cdots\square C_{k_{r-1}}$. Then G is obtained by replacing i^{th} vertex of C_{k_r} by the copy H^i of H and replacing each edge C_{k_r} by the matching between the two copies of H^i corresponding to the end vertices of that edge.

As S is an h-vertex cut of G, the graph $G-S$ is disconnected and each component of it has minimum degree h. Let Y be a subgraph of $G-S$ consisting of at least one but not all components of $G-S$ and let Z be the subgraph consisting of the remaining components. Thus $G-S=Y\cup Z$ and further, $\delta(Y)\geq h$ and $\delta(Z)\geq h$. As S is a cut, $N(Y)\subseteq S$ and $N(Z)\subseteq S$ and so, $|S|\geq|N(Y)|$ and $|S|\geq|N(Z)|$. Note that Y and Z each intersects H^i for at least one i. Let $S_i=S\cap V(H^i)$, $Y_i=Y\cap V(H^i)$ and $Z_i=Z\cap V(H^i)$. Depending upon the nature of Y and Z, the proof is divided into several cases.

Case 1. Suppose $Y_i\neq\emptyset$ for only one i. Without loss of generality, we may assume that only Y_1 is non-empty. Then $Y=Y_1$ is contained in H^1. Therefore $\delta_{H^1}(Y)\geq h-1$. As H^1 is $(2r-2)$-regular, $0\leq h\leq 2r-2$. If $h=2r-2$, then $Y=H^1$, $N(Y)=V(H^{k_r})\cup V(H^2)$ and therefore,

$$\begin{aligned}|S|&\geq|N(Y)|=\left|V\left(H^{k_r}\right)\right|+\left|V\left(H^2\right)\right|=2k_1k_2\cdots k_{r-2}k_{r-1}\\&\geq 8k_1k_2\cdots k_{r-2}=a_h^r(2r-h).\end{aligned}$$

Suppose $0\leq h\leq 2r-3=2(r-1)-1$. The graph Y has $|V(Y)|$ neighbours in each of H^{k_r} and H^2. Therefore $|N(Y)|=|N_{H^1}(Y)|+|V(Y)|+|V(Y)|=|N_{H^1}[Y]|+|V(Y)|$. By Lemmas 2.4(3), 2.5 and 3.2,

$$|S|\geq|N(Y)|\geq a_h^{r-1}(2r-h-1)+a_h^{r-1}=a_h^{r-1}(2r-h)\geq a_h^r(2r-h).$$

Case 2. Suppose $Y_i\neq\emptyset$ for more than one but not all values of i. Without loss of generality, we may assume that Y_1 is non-empty but Y_{k_r} is empty. Suppose there is an integer t with $1<t<k_r$ such that Y_t is non-empty. Note that $\delta_{H^1}(Y_1)\geq h-1$. Further, the set S_{k_r} contains all $|V(Y_1)|$ neighbours of Y_1 present in H^{k_r} and S_1 contains the set $N_{H^1}(Y_1)$ of neighbours of Y_1 in H^1. Therefore, by Lemma 3.2,

$$|S_1\cup S_{k_r}|\geq\left|N_{H^1}(Y_1)\right|+|V(Y_1)|=\left|N_{H^1}[Y_1]\right|\geq(2r-h)a_{h-1}^{r-1}.$$

Suppose Y_i is empty for more than one values of i. Suppose Y_{k_r-1} is empty. Then we can choose t so that Y_{t+1} is empty. Then $\delta_{H^t}(Y_t)\geq h-1$. The set S contains $|N_{H^1}(Y_1)|$ neighbours of Y_t present in H^t and the $|V(Y_t)|$ neighbours of Y_t that are present in H^{t+1}. Thus, by Lemmas 2.4(1) and 3.2,

$$\begin{aligned}|S|&\geq\left|S_1\cup S_{k_r}\right|+\left|N_{H^t}(Y_t)\right|+\left|V(Y_t)\right|\geq(2r-h)a_{h-1}^{r-1}+\left|N_{H^t}[Y_t]\right|\\&\geq 2(2r-h)a_{h-1}^{r-1}\geq a_h^r(2r-h).\end{aligned}$$

Similarly, if Y_{k_r-1} is non-empty, then we can choose t so that Y_{t-1} is empty and so, in this case S contains $N_{H^t}(Y_t)$ and $N_{H^{t-1}}(Y_t)$ implying $|S|\geq a_h^r(2r-h)$.

Suppose Y_i is non-empty for all $1\leq i\leq k_r-1$. Here we calculate $|S_i|$ by using Lemma 3.2 or induction. To use induction, we need to consider the nature of the graph Z also. If $Z_i \neq \emptyset$ for only one value of i, then result follows from Case 1. Suppose $Z_i \neq \emptyset$ for more than one values of i. If $Z_i = \emptyset$ for at least two values of i, then the result follows from the above paragraph by replacing Y with Z. It remains to consider the two subcases depending on whether Z_i is empty for exactly one value of i or no value of i.

Subcase 1. $Z_i = \emptyset$ for exactly one value of i. We have two subcases depending on $i = k_r$ or $i < k_r$.

(i) Suppose Z_{k_r} is empty. Then Z_j is non-empty like Y_j for $1 \leq j < k_r$; Figure 4(a). Suppose $h = 1$. Then $\delta_{H^i}(Y_i) \geq 0$ and $\delta_{H^i}(Z_i) \geq 0$ for all i. Hence $|S_1 \cup S_{k_r}| \geq (2r-1)a_0^{r-1} = (2r-1)$ and by induction, $|S_i| \geq (2(r-1)-0)a_0^{r-1} = 2r-2$ for $i \in \{2, 3, \ldots, k_r - 1\}$. Therefore, as $S = S_1 \cup S_2 \cup \cdots \cup S_{k_r}$, we have

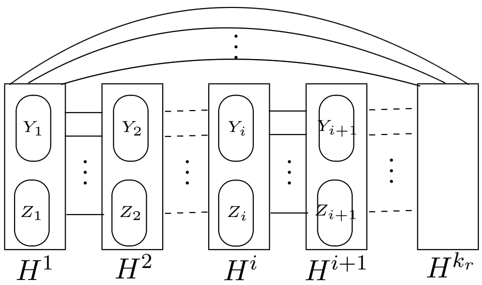
$$\begin{aligned}
|S| &= (|S_1 \cup S_{k_r}|) + \sum_{i=2}^{k_r-1} |S_i| \geq (2r-1) + \sum_{i=2}^{k_r-1} (2r-2) \\
&= (2r-1) + (2r-2)(k_r-2) \geq 2(2r-1) = (2r-h)a_h^r.
\end{aligned}$$

Suppose $h \geq 2$. Since Y_{k_r} and Z_{k_r} are empty, $\delta_{H^{k_r-1}}(Y_{k_r-1}) \geq h-1 > h-2$ and $\delta_{H^{k_r-1}}(Z_{k_r-1}) \geq h-1 > h-2$. Thus S_{k_r-1} is an $(h-2)$-cut in H^{k_r-1}. For $i \in \{2, 3, \ldots, k_r-2\}$, as both Y_i and Z_i are non-empty subgraphs of H^i of minimum degree at least $h-2$, S_i is an $(h-2)$-cut in H^i. Hence, by induction, $|S_i| \geq (2r-h)a_{h-2}^{r-1}$ for $i \in \{2, 3, \ldots, k_r - 1\}$. Therefore

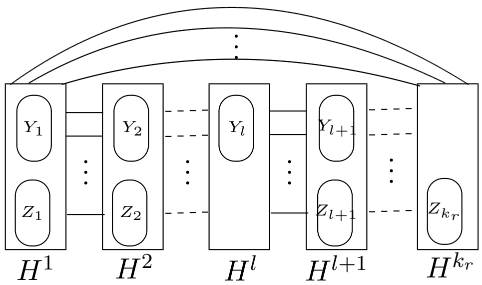
$$\begin{aligned}
|S| &= (|S_1 \cup S_{k_r}|) + \sum_{i=2}^{k_r-1} |S_i| \geq (2r-h)a_{h-1}^{r-1} + \sum_{i=2}^{k_r-1} (2r-h)a_{h-2}^{r-1} \\
&= (2r-h)a_{h-1}^{r-1} + (k_r-2)(2r-h)a_{h-2}^{r-1} \\
&\geq (2r-h)a_{h-1}^{r-1} + \frac{k_r}{2}a_{h-2}^{r-1}(2r-h) && \text{(since } k_r \geq 4) \\
&\geq (2r-h)a_{h-1}^{r-1} + \frac{1}{2}a_h^r(2r-h) && \text{(by Lemma 2.4(2))} \\
&= (2r-h)a_{h-1}^{r-1} + a_{h-1}^{r-1}(2r-h) && \text{(by Lemma 2.4(1))} \\
&= 2a_{h-1}^{r-1}(2r-h) \\
&= a_h^r(2r-h) && \text{(by Lemma 2.4(1)).}
\end{aligned}$$

(ii) Suppose Z_{k_r} is non-empty. Then Z_l is empty for some l with $1 \leq l < k_r$ and Z_j is non-empty for every $j \neq l$; see Figure 4(b). Then the minimum degree of Z_{l+1} is at least $h-1$ in H^{l+1}. Also, the neighbours of Z_{l+1} present in H^i are contained in S_i for $i = l, l+1$. Hence $|S_l \cup S_{l+1}| \geq |N_{H^{l+1}}[Z_{l+1}]| \geq a_{h-1}^{r-1}(2r-h)$ by Lemma 3.2. Thus, if $l \notin \{1, k_r - 1\}$, then

$$|S| \geq |S_1 \cup S_{k_r}| + |S_l \cup S_{l+1}| \geq 2a_{h-1}^{r-1}(2r-h) = a_h^r(2r-h).$$



(a) $Z_{k_r} = \emptyset$.



(b) $Z_l = \emptyset$ for some $l < k_r$.

Figure 4. The graph G with $Y_{k_r} = \emptyset$.

Suppose $l = 1$. Then by using similar arguments, we see that $S_1 \cup S_2 \supseteq N_{H^2}(Z_2) \cup N_{H^1}(Z_2)$ and $S_{k_r} \cup S_{k_r-1} \supseteq N_{H^{k_r-1}}(Y_{k_r-1}) \cup N_{H^{k_r}}(Y_{k_r-1})$. Hence

$$|S| \geq |S_1 \cup S_2| + \left|S_{k_r-1} \cup S_{k_r}\right| \geq 2a_{h-1}^{r-1}(2r-h) = a_h^r(2r-h).$$

Similarly, for $l = k_r - 1$,

$$|S| \geq |S_1 \cup S_{k_r}| + \left|S_{k_r-2} \cup S_{k_r-1}\right| \geq 2a_{h-1}^{r-1}(2r-h) = a_h^r(2r-h).$$

Subcase 2. Suppose that $Z_i \neq \emptyset$ for $i = 1, 2, \ldots, k_r$. Then $|S_1 \cup S_{k_r}| \geq a_{h-1}^{r-1}(2r-h)$ and $|S_i| \geq (2r-h)a_{h-2}^{r-1}$ for $i \in \{2, 3, \ldots, k_{r-1}\}$. As in Subcase 1(i), we have $|S| \geq a_h^r(2r-h)$.

Case 3. Suppose $Y_i \neq \emptyset$ for $i = 1, 2, \ldots, k_r$. If Z does not intersect H^i for some i, then the result follows by replacing Y by Z in Case 1 and Case 2. Suppose that Z intersects H^i for all $i = 1, 2, \ldots, k_r$. If $h = 1$, then the minimum degree of Y_i and Z_i is at least 0 and so, by induction, $|S_i| \geq a_0^{r-1}(2(r-1)-0) = 2r-2$, also as $r \geq 2$ implies

$$\begin{aligned} |S| &= \sum_{i=1}^{k_r} |S_i| \geq \sum_{i=1}^{k_r}(2r-2) = k_r(2r-2) \geq 4(2r-2) \\ &= 8(r-1) > 2(2r-1) = a_1^r(2r-1). \end{aligned}$$

Suppose $h \geq 2$. The minimum degree of Y_i and Z_i is at least $h-2 \geq 0$. This shows that S_i is an $(h-2)$-vertex cut of the graph H^i for $i = 1, 2, \ldots, k_r$. Therefore, by induction and by Lemma 2.4(2), we have

$$|S| = \sum_{i=1}^{k_r} |S_i| \geq \sum_{i=1}^{k_r} a_{h-2}^{r-1}(2r-h) = k_r a_{h-2}^{r-1}(2r-h) \geq a_h^r(2r-h).$$

Thus $|S| \geq a_h^r(2r-h)$ in all the above cases. This completes the proof. ■

Corollary 3.4. *For the graph G of Theorem* 1.2, $\kappa^h(G) = a^r_h(2r - h)$.

Proof. By Lemma 3.1, $\kappa^h(G) \leq a^r_h(2r - h)$. Since $\kappa^h(G)$ is the cardinality of a smallest h-vertex cut of G, by Proposition 3.3, $\kappa^h(G) \geq a^r_h(2r - h)$. Hence $\kappa^h(G) = a^r_h(2r - h)$. ■

4. Conditional Edge Connectivity

In this section, we prove that the conditional edge connectivity $\lambda^h(G)$ of the graph G of Theorem 1.2 is same as its conditional vertex connectivity $\kappa^h(G)$.

Recall that $G = C_{k_1} \square C_{k_2} \square \cdots \square C_{k_r}$ with $4 \leq k_1 \leq k_2 \leq \cdots \leq k_r$ and W^r_h is an h-regular subgraph of G with a^r_h vertices. We get an upper bound for $\lambda^h(G)$ from the set of edges of G each of which has exactly one end vertex in W^r_h. For such edge sets we introduce the following notation. For a subgraph K of a graph H, let

$$E_H(K) = \{xy \colon x \in V(K) \text{ and } y \in V(H) \setminus V(K)\}.$$

Lemma 4.1. *For* $0 \leq h \leq 2r - 1$, $\lambda^h(G) \leq (2r - h)a^r_h$.

Proof. Let $K = W^r_h$. Then K is h-regular and G is $2r$-regular. Hence $|E_G(K)| = (2r - h)|V(K)|$ and $G - E_G(K)$ is disconnected with K as one of its components. By Lemma 2.7, the minimum degree of every component of $G - E_G(K)$ other than K is at least $2r - 1 \geq h$. Therefore $E_G(K)$ contains an h-edge cut of G. This shows that $\lambda^h(G) \leq |E_G(K)| = (2r - h)a^r_h$. ■

Lemma 4.2. *For a subgraph Y of G of minimum degree at least h,* $|V(Y)| + |E_G(Y)| \geq a^r_h(2r - h + 1)$.

Proof. If Y spans G, then $|V(Y)| = k_1k_2 \cdots k_r \geq a^r_h(2r - h + 1)$ by Lemma 2.9. Suppose Y is not a spanning subgraph of G. Since for every x in $N(Y)$ there is a vertex y of Y adjacent to x so that the edge xy belongs to the edge set $E_G(Y)$. This implies that $|E_G(Y)| \geq |N(Y)|$. Hence, by Lemma 3.2, $|V(Y)| + |E_G(Y)| \geq |N[Y]| \geq a^r_h(2r - h + 1)$. ■

Using this lemma we now obtain the reverse inequality for $\lambda^h(G)$.

Proposition 4.3. *Let F be an h-edge cut of the graph G. Then* $|F| \geq a^r_h(2r - h)$.

Proof. Since the graph G is $2r$-regular, $0 \leq h \leq 2r$. The result holds obviously for $h = 2r$. Suppose $h = 0$. Then F is a set of edges G such that $G - F$ is a disconnected graph. It follows from Lemma 2.1 that G is $2r$-edge connected and so, $|F| \geq 2r = a^r_0(2r - 0)$. Thus the result holds for $h = 0$ also. Suppose $1 \leq h \leq 2r - 1$. We prove the result by induction on r. The result follows trivially for $r = 1$. Suppose $r \geq 2$. Assume that the result holds for the product of $r-1$ cycles. Let F be an h-edge cut of G. Then $G - F$ is disconnected and every component of it has minimum degree at least h.

Let Y be a subgraph of $G-F$ consisting of at least one but not all components of $G-F$ and let Z be the subgraph consisting of the remaining components. Then Y and Z are vertex disjoint subgraphs of $G-F$ of minimum degree at least h and their union is $G-F$. Note that F contains both edge sets $E_G(Y)$ and $E_G(Z)$. Hence $|F| \geq |E_G(Y)|$ and $|F| \geq |E_G(Z)|$.

Write G as $H \square C_{k_r}$, where $H = C_{k_1} \square C_{k_2} \square \cdots \square C_{k_{r-1}}$. Then G is obtained by replacing vertex i of the cycle C_{k_r} by a copy H^i of H and replacing the edge joining i and $i+1 \pmod{k_r}$ by the perfect matching M_i between the corresponding vertices of H^i and $H^{i+1 \pmod{k_r}}$. Then Y intersects at least one H^i. Similarly, Z intersects at least one H^i. Let $Y_i = Y \cap H^i$ and $Z_i = Z \cap H^i$ for $i = 1, 2, \ldots, k_r$.

For a subgraph K of G, let $M_i(K)$ be the set of all edges in the matching M_i each having exactly one end vertex in K.

Case 1. Suppose $Y_i \neq \emptyset$ for only one value of i. Without loss of generality, we may assume that Y_i is non-empty for only $i = 1$. Then Y is contained in the graph H^1 and $\delta_{H^1}(Y) \geq h$. Since H^1 is $(2r-2)$-regular, $h \leq 2r-2$. If $h = 2r-2$, then $Y = H^1$ and so, $|E_G(Y)| = |M_1| + |M_{k_r}| = 2|V(H^1)| = 2k_1k_2 \cdots k_{r-1}$. As $4 \leq k_{r-1}$, we have

$$\begin{aligned} a_h^r(2r-h) &= 2a_h^r 2(2^{r-(r-2)}k_1k_2\cdots k_{r-2}) = 2(4k_1k_2\cdots k_{r-2}) \\ &\leq 2k_1k_2\cdots k_{r-2}k_{r-1} = |E_G(Y)| \leq |F|. \end{aligned}$$

Suppose $h < 2r-2$. Then $E_G(Y) \supseteq E_{H^1}(Y) \cup M_1(Y) \cup M_{k_r}(Y)$. As $|M_1(Y)| = |M_{k_r}(Y)| = |V(Y)|$, by Lemmas 2.4(3), 2.5 and 4.2, we have

$$\begin{aligned} |E_G(Y)| &\geq \big(|E_{H^1}(Y)| + |V(Y)|\big) + |V(Y)| \geq a_h^{r-1}(2r-h-1) + a_h^{r-1} \\ &= a_h^{r-1}(2r-h) \geq a_h^r(2r-h). \end{aligned}$$

Case 2. Suppose $Y_i \neq \emptyset$ for more than one but not all values of i. Without loss of generality, we may assume that Y_1 is non-empty but Y_{k_r} is empty. Let t be the largest integer such that Y_t is non-empty. Then $1 < t < k_r$. The minimum degree of Y_i in H^i is at least $h-1$ for $i = 1, t$. The graph Y_1 has $|V(Y_1)|$ neighbours in H^{k_r} and Y_t has $|V(Y_t)|$ neighbours in H^{t+1}. Hence $E_G(Y) \supseteq E_{H^1}(Y_1) \cup E_{H^t}(Y_t) \cup M_{k_r}(Y_1) \cup M_t(Y_t)$.

Suppose $h = 2r-1$. Then $Y_j = H^j$ for $j = 1, t$ giving $M_{k_r}(Y_1) = M_{k_r}(H^1) = M_{k_r}$ and $M_t(Y_t) = M_t(H^t) = M_t$. Hence

$$\begin{aligned} a_h^r(2r-h) &= a_h^r = 2k_1k_2\cdots k_{r-1} = |V(H^1)| + |V(H^t)| \\ &= |M_{k_r}| + |M_t| \leq |E_G(Y)| \leq |F|. \end{aligned}$$

Suppose $h \leq 2r-2$. Then $h-1 \leq 2r-3$ and so, by Lemmas 4.2 and 2.4(1),

$$\begin{aligned}|F| &\geq |E_G(Y)| \geq \left(\left|E_{H^1}(Y_1)\right| + |V(Y_1)|\right) + \left(\left|E_{H^t}(Y_t)\right| + |V(Y_t)|\right)\\ &\geq 2a_{h-1}^{r-1}(2r-h) = (2r-h)a_h^r.\end{aligned}$$

Case 3. Suppose $Y_i \neq \emptyset$ for all $i = 1, 2, \ldots, k_r$. If the graph Z does not intersect H^i for some i, then the result follows easily by replacing Y by Z in Case 1 and Case 2. Suppose Z intersects H^i for all $i = 1, 2, \ldots, k_r$. Suppose $h = 1$. As $r \geq 2$, $\delta(Y_i) \geq 0$ and $\delta(Z_i) \geq 0$, by induction, we have

$$\begin{aligned}|E_G(Y)| &= \sum_{i=1}^{k_r} |E_{H^i}(Y_i)| \geq \sum_{i=1}^{k_r}(2r-2) \geq k_r(2r-2)\\ &\geq 4(2r-2) = 8(r-1) > 2(2r-1) = a_1^r(2r-1).\end{aligned}$$

Suppose $h \geq 2$. The minimum degree of Y_i and Z_i is at least $h-2 \geq 0$. Therefore the edge set $E_{H^i}(Y_i)$ is an $(h-2)$-edge cut of H^i. By induction, $|E_{H^i}(Y_i)| \geq a_{h-2}^{r-1}(2r-h)$ for $i = 1, 2, \ldots, k_r$. By Lemma 2.4(2),

$$\begin{aligned}|F| &\geq |E_G(Y)| = \sum_{i=1}^{k_r} |E_{H^i}(Y_i)| \geq \sum_{i=1}^{k_r} a_{h-2}^{r-1}(2r-h)\\ &\geq k_r a_{h-2}^{r-1}(2r-h) \geq a_h^r(2r-h).\end{aligned}$$

This completes the proof. ■

Corollary 4.4. *For the graph G of Theorem* 1.2, $\lambda^h(G) = a_h^r(2r-h) = \kappa^h(G)$.

Proof. By Proposition 4.3, $\lambda^h(G) \geq a_h^r(2r-h)$ and by Lemma 4.1, $\lambda^h(G) \leq a_h^r(2r-h)$. Hence $\lambda^h(G) = a_h^r(2r-h) = \kappa^h(G)$ by Corollary 3.4. ■

This completes the proof of Theorem 1.2.

It is worth mentioning that the edge connectivity part of Theorem 1.2 proves that the following conjecture of Xu [7] holds for the classes of multidimensional tori and k-ary r-cubes.

Conjecture 4.5. *Let k, h be two non-negative integers and G be a connected graph with minimum degree at least k and $a_h(G)$ be the minimum cardinality of a vertex set of an h-regular subgraph of G. If $\lambda^h(G)$ exists, then $\lambda^h(G) \leq a_h(G)(k-h)$.*

Concluding Remarks.

We determine the conditional h-vertex connectivity and the conditional h-edge connectivity of a multidimensional torus G which is the Cartesian product of r

cycles each of length at least four, for all possible values of h. We first characterize the h-regular subgraph of G with minimum number of vertices and then establish that both these conditional connectivities of G are equal to $(2r - h)$ times the number of vertices of this subgraph.

Acknowledgement

The second author is financially supported by DST-SERB, Government of India through the project MTR/2018/000447.

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Received 5 February 2020
Revised 1 July 2020
Accepted 6 July 2020