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GRAPH OPERATIONS AND NEIGHBORHOOD POLYNOMIALS

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Abstract

The neighborhood polynomial of graph G is the generating function for the number of vertex subsets of G of which the vertices have a common neighbor in G. In this paper, we investigate the behavior of this polynomial under several graph operations. Specifically, we provide an explicit formula for the neighborhood polynomial of the graph obtained from a given graph G by vertex attachment. We use this result to propose a recursive algorithm for the calculation of the neighborhood polynomial. Finally, we prove that the neighborhood polynomial can be found in polynomial-time in the class of k-degenerate graphs.

Keywords: neighborhood complex, neighborhood polynomial, domination polynomial, graph operations, graph degeneracy.

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1. Introduction

All graphs considered in this paper are simple, finite, and undirected. Let $G = (V, E)$ be a graph where V is its vertex set and E its edge set, suppose $v \in V$ is a vertex of G. The *open neighborhood* of v, denoted by $N(v)$, is the set of all vertices that are adjacent to v,

$$N(v) = \{u \mid \{u, v\} \in E\},$$

and the *open neighborhood* of U, for a vertex subset $U \subseteq V$ is

$$N(U) = \bigcup_{u \in U} N(u) \setminus U.$$

The *neighborhood complex* of a graph G, denoted by $\mathcal{N}(G)$, has been introduced in [15]. It is the family of all subsets of open neighborhoods of vertices of the graph G,

$$\mathcal{N}(G) = \{A \mid A \subseteq V, \exists\, v \in V : A \subseteq N(v)\}.$$

The interested reader is referred to [15] and [5] for more properties of the neighborhood complex $\mathcal{N}(G)$.

The *neighborhood polynomial* of a graph G, denoted by $N(G, x)$, is the ordinary generating function for the neighborhood complex of G. It has been introduced in [5] and is defined as follows

$$N(G, x) = \sum_{U \in \mathcal{N}(G)} x^{|U|}. \tag{1}$$

Suppose $|V| = n$, and let $n_k(G) = |\{A \mid A \in \mathcal{N}(G), |A| = k\}|$. Then we can rephrase the Equation (1) as follows

$$N(G, x) = \sum_{k=0}^{n-1} n_k(G) x^k.$$

The neighborhood polynomial of a graph is of special interest as it has a close relation to the domination polynomial of a graph. A *dominating set* of a graph $G = (V, E)$ is a vertex set $W \subseteq V$ such that the closed neighborhood of W is equal to V, where the *closed neighborhood* is defined by

$$N[W] = N(W) \cup W.$$

We denote by $\mathcal{D}(G)$ the family of all dominating sets of a graph G. The *domination polynomial* of a graph, introduced in [3], is

$$D(G, x) = \sum_{W \in \mathcal{D}(G)} x^{|W|}.$$

For further properties of the domination polynomial, see [1, 6, 7, 12, 13]. It has been observed in [4] that a vertex set W of G belongs to the neighborhood complex of G if and only if W is non-dominating in the complement ($\bar{G}$) of G, which implies

$$D(G, x) + N(\bar{G}, x) = (1 + x)^{|V|}. \tag{2}$$

A proof of this relation is given in [10].

In Section 2 of this paper, we investigate the effect of several graph operations on the neighborhood polynomial of a graph. The vertex (or edge) addition plays an important role in this context.

In Section 3, we present a recursion formula for the neighborhood polynomial of a graph based on deleting vertices. We apply this recursion to several graph classes and prove that the neighborhood polynomial of *planar* graphs can be computed efficiently.

In Section 4, we investigate the complexity of calculation of the neighborhood polynomial of a graph and finally in Section 5, a conclusion as well as some interesting open problems are discussed.

2. Graph Operations

Having defined a graph polynomial, one of the first natural problems is its calculation. Often, *local* graph operations, such as edge or vertex deletions, prove useful. In addition, *global* operations, like complementation or forming the line graph, might be beneficial. Finally, graph products, for instance the disjoint union, the join, or the Cartesian product, can be employed to simplify graph polynomial calculations.

The *disjoint union* of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with disjoint vertex sets V_1 and V_2, denoted by $G_1 \cup G_2$ is a graph with the vertex set $V_1 \cup V_2$, and the edge set $E_1 \cup E_2$.

Theorem 1 [5]**.** *Let G_1 and G_2 be simple undirected graphs. The disjoint union $G_1 \cup G_2$ satisfies*

$$N(G_1 \cup G_2, x) = N(G_1, x) + N(G_2, x) - 1.$$

The *join* of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with disjoint vertex sets V_1 and V_2, denoted by $G_1 + G_2$, is the disjoint union of G_1 and G_2, together with all those edges that join vertices in V_1 to vertices in V_2.

Theorem 2 [5]**.** *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be simple undirected graphs. The neighborhood polynomial of the join $G_1 + G_2$ of these two graphs satisfies*

$$\begin{aligned} N(G_1 + G_2, x) = {} & (1 + x)^{|V_1|} N(G_2, x) + (1 + x)^{|V_2|} N(G_1, x) \\ & - N(G_1, x) N(G_2, x). \end{aligned}$$

The *Cartesian product* of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with disjoint vertex sets V_1 and V_2, denoted by $G_1 \Box G_2$, is a graph with vertex set $V_1 \times V_2 = \{(u, v) \mid u \in V_1, v \in V_2\}$, where the vertices $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are adjacent in $G_1 \Box G_2$ if and only if $[x_1 = y_1$ and $\{x_2, y_2\} \in E_2]$ or $[x_2 = y_2$ and $\{x_1, y_1\} \in E_1]$.

Theorem 3 [5]**.** *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be simple undirected graphs. Then*

$$\begin{aligned} N(G_1 \Box G_2, x) &= 1 + |V_1|(N(G_2, x) - 1) + |V_2|(N(G_1, x) - 1) \\ &\quad + \sum_{(u,v) \in V(G_1 \Box G_2)} \left((1+x)^{|N_{G_1}(u)|} - 1\right)\left((1+x)^{|N_{G_2}(v)|} - 1\right) \\ &\quad - |V_1||V_2|x - 2|E_1||E_2|x^2. \end{aligned}$$

2.1. Cut vertices

In this section, we consider connected graphs with cut vertices and prove that there exists an interesting relation between the neighborhood polynomial of a graph G with the neighborhood polynomials of its split components.

Let $G = (V, E)$ be a simple undirected graph and let $v \in V$. By $G - v$ we denote the subgraph of G induced by $V \setminus \{v\}$. A vertex v in a connected graph G is called a *cut vertex* (or an *articulation*) of G if $G - v$ is disconnected. More generally, a vertex v is a cut vertex of a graph G if v is a cut vertex of a component of G.

Let v be a cut vertex of the graph $G = (V, E)$ and suppose $G - v$ has two components. Then we can find two subgraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ of G such that

$$V_1 \cup V_2 = V, \qquad V_1 \cap V_2 = \{v\}, \qquad E_1 \cup E_2 = E, \qquad E_1 \cap E_2 = \emptyset.$$

We call the graphs G_1 and G_2 the *split components* of G.

Theorem 4. *Let $G = (V, E)$ be a simple connected graph where $v \in V$ is a cut vertex of G such that $G - v$ has two components. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be the split components of G. Then the neighborhood polynomial of G is*

$$\begin{aligned} N(G, x) &= N(G_1, x) + N(G_2, x) - (1 + x) \\ &\quad + \left((1+x)^{|N_{G_1}(v)|} - 1\right)\left((1+x)^{|N_{G_2}(v)|} - 1\right). \end{aligned}$$

Proof. Let X be a vertex subset of G with $X \in \mathcal{N}(G)$, which implies that the vertices of X have a common neighbor in G. Then we can distinguish the following cases.

(a) Assume $X \subseteq V_1$ or $X \subseteq V_2$. Then all possibilities for the selection of X are generated by the polynomial $N(G_1, x) + N(G_2, x) - (1 + x)$, in which we prevent the double-counting of the empty set and the vertex v by subtracting $1 + x$.

(b) Now assume that X contains at least one vertex from each of $V_1 \setminus \{v\}$ and $V_2 \setminus \{v\}$. In this case, v is the common neighbor, so we need to count all subsets of the open neighborhood of v in G which include at least one vertex from each of the open neighborhoods of v in G_1 and G_2, which is performed by the generating function $((1+x)^{|N_{G_1}(v_1)|}-1)((1+x)^{|N_{G_2}(v_2)|}-1)$. ■

Let $G=(V,E)$ be a simple undirected connected graph. A *vertex cut* (or a *separator*) is a set of vertices of G which, if removed together with any incident edges, makes that the remaining graph is disconnected. More generally, a vertex subset W is a vertex cut if W is a vertex cut of a component of G.

Let $W \subseteq V$ be a vertex cut of G which is an independent set in G such that $G-W$ has two components, where $G-W$ denotes the subgraph of G induced by $V \setminus W$. We can find two subgraphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ of G such that

$$V_1 \cup V_2 = V, \qquad V_1 \cap V_2 = W, \qquad E_1 \cup E_2 = E, \qquad E_1 \cap E_2 = \emptyset.$$

We call the graphs G_1 and G_2 the *split components* of G. In the following theorem we prove that similar to Theorem 4, there is a relation between the neighborhood polynomial of a graph containing a vertex cut and the neighborhood polynomials of its split components.

Theorem 5. *Let $G=(V,E)$ be a simple connected graph, $W \subseteq V$ a vertex cut with $|W|=k$ that is also an independent set in G such that $G-W$ has two components. Let $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ be the split components of G. Then*

$$\begin{aligned} N(G,x) &= N(G_1,x)+N(G_2,x)-\sum_{U\subseteq W} A_U \\ &\quad + \sum_{\substack{U\subseteq W\\ U\neq\emptyset}} (-1)^{|U|+1}\left((1+x)^{|\bigcap_{u\in U} N_{G_1}(u)|}-1\right)\left((1+x)^{|\bigcap_{u\in U} N_{G_2}(u)|}-1\right)\end{aligned}$$

where

$$A_U = \begin{cases} x^{|U|} & \text{if } \; [U=\emptyset] \text{ or } \left[\bigcap_{u\in U} N_{G_1}(u)\neq\emptyset \;\text{ and }\; \bigcap_{u\in U} N_{G_2}(u)\neq\emptyset\right], \\ 0 & \text{otherwise.}\end{cases}$$

Proof. Let X be a vertex subset of G with $X \in \mathcal{N}(G)$, which implies that the vertices of X have a common neighbor in G. Then we can distinguish the following cases.

(a) If $X \subseteq V_1$ or $X \subseteq V_2$, all possibilities to select X are generated by the polynomial

$$N(G_1,x)+N(G_2,x)-\sum_{U\subseteq W} A_U,$$

in which we avoid any subset of the set W having common neighbors in both split components G_1 and G_2, to be counted more than once by subtracting $\sum_{U\subseteq W} A_U$.

(b) Now assume that X contains at least one vertex from each of $V_1 \setminus W$ and $V_2 \setminus W$. In this case the common neighbor is in W. We generate all subsets of the open neighborhoods of vertices of W by the generating function

$$\sum_{\substack{U\subseteq W\\ U\neq\emptyset}} (-1)^{|U|+1}\left((1+x)^{|\bigcap_{u\in U} N_{G_1}(u)|}-1\right)\left((1+x)^{|\bigcap_{u\in U} N_{G_2}(u)|}-1\right),$$

where by applying the principle of inclusion-exclusion we prevent any double counting. ■

2.2. Matching edge cuts

Theorem 6. *Let $F = \{\{a,b\} \mid a \in A, b \in B\}$ be a minimal edge cut of a connected graph $G = (V,E)$ such that F is a matching of G. The components of $G - F$ are denoted by H and K. Assume that $A \subseteq V(H)$ and $B \subseteq V(K)$. Let $H' = H \cup (A \cup B, F)$ and $K' = K \cup (A \cup B, F)$. Then*

$$N(G,x) = N(H',x) + N(K',x) - 2|F|x - 1.$$

Figure 1 illustrates the situation of Theorem 6.

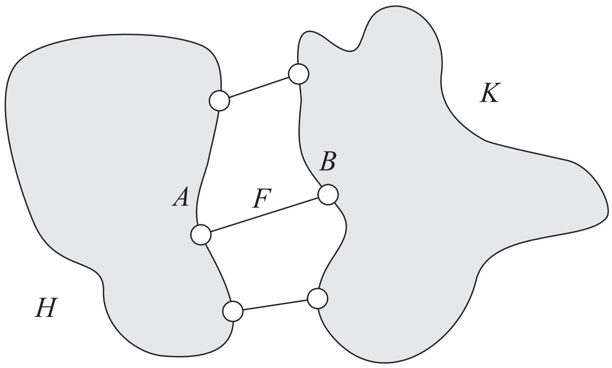

Figure 1. A graph with a matching cut.

Proof. The open neighborhoods of all vertices of $V(H)$ are the same in G and in H'. We also have $N_G(v) = N_{K'}(v)$ for any $v \in V(K)$. This implies

$$\mathcal{N}(G) = \mathcal{N}(H') \cup \mathcal{N}(K').$$

Each singleton $\{v\}$ for $v \in A \cup B$ is contained in $\mathcal{N}(H')$ and in $\mathcal{N}(K')$, which yields $\mathcal{N}(H') \cap \mathcal{N}(K') = \{\{v\} \mid v \in A \cup B\} \cup \{\emptyset\}$. ■

2.3. Edge addition

Let $G = (V, E)$ be a graph, and $u, v \in V$ such that $\{u, v\} \notin E$. Let $G + uv = (V, E \cup \{\{u, v\}\})$ be the graph obtained from G by insertion of the edge $\{u, v\}$. In this case, the neighborhoods of the two vertices u, v have been changed. If there is a path of length 3 between the vertices u and v in G, after the edge insertion, a 4-cycle is being formed which causes some difficulties in counting neighborhood sets.

In Theorem 7, we suppose that there is no path of length 3 between the vertices u and v to prevent any over-counting and relate the neighborhood polynomial of the new graph after addition the edge $\{u, v\}$ to the neighborhood polynomial of the original graph. In Theorem 8, we investigate the general case.

Theorem 7. *Let $G = (V, E)$ be a graph, $u, v \in V$, and $\{u, v\} \notin E$. Suppose there is no path of length 3 between u and v in G. Let $G' = (V, E \cup \{\{u, v\}\})$. Then*

$$N(G', x) = N(G, x) + x\left((1 + x)^{|N_G(u)|} - 1\right) + x\left((1 + x)^{|N_G(v)|} - 1\right).$$

Proof. Observe that any vertex subset with a common neighbor in G forms a vertex subset with a common neighbor in G'. Such vertex subsets are generated by $N(G, x)$. The addition of the edge $\{u, v\}$ to G results in the fact that any non-empty subset of the open neighborhood of u together with v forms a vertex subset with u as the common neighbor in G' and the same arguments applies to the open neighborhood of v. Such subsets are generated by $x((1 + x)^{|N_G(u)|} - 1) + x((1 + x)^{|N_G(v)|} - 1)$.

Now, assume there is a vertex subset W with a common neighbor in G' and suppose it is counted once in $x((1 + x)^{|N_G(u)|} - 1) + x((1 + x)^{|N_G(v)|} - 1)$ and once in $N(G, x)$, and let $y \in W$. This implies that W, which is a subset of the open neighborhood of one of the vertices u (or v) together with v (or u) have a common neighbor in G. Let x be that common neighbor. The existence of such set results in the existence of a path of length 3 between u_1 and u_2 through vertices x and y which contradicts the assumption that there is no path of length 3 between those two vertices in G. So there is no vertex subset with a common neighbor in G' that is counted more than once and this completes the proof. ∎

Theorem 8. *Let $G = (V, E)$ be a graph, $u_1, u_2 \in V$, and $\{u_1, u_2\} \notin E$. Suppose u_1 and u_2 are not isolated vertices and let $G' = (V, E \cup \{\{u_1, u_2\}\})$. Then*

$$N(G', x) = N(G, x) + x \sum_{\substack{\emptyset \neq U_1 \subseteq N_G(u_1) \\ N_G(U_1) \cap N_G(u_2) = \emptyset}} x^{|U_1|} + x \sum_{\substack{\emptyset \neq U_2 \subseteq N_G(u_2) \\ N_G(U_2) \cap N_G(u_1) = \emptyset}} x^{|U_2|}.$$

Proof. If $X \in \mathcal{N}(G)$, then $X \in \mathcal{N}(G')$ and all such vertex subsets are generated by $N(G, x)$. After insertion of the edge $\{u_1, u_2\}$, the vertices in any subset of the open neighborhood of u_1 together with u_2 have a common neighbor in G' (which is the vertex u_1), and analogously the vertices in any subset of the open neighborhood of u_2 together with u_1 share a neighbor in G' (which is the vertex u_2). Any of such vertex sets adds a term like $xx^{|U|}$ to the neighborhood polynomial of G', where x represents u_1 or u_2, and $x^{|U|}$ stands for U as a subset of $N_G(u_2)$ or $N_G(u_1)$, respectively.

In order to prevent counting a vertex set of this form that already has a common neighbor in G (and therefore being counted in $N(G,x)$), we consider only those subsets of $N_G(u_2)$ or $N_G(u_1)$ which do not have a common neighbor with u_1 or u_2 in G, respectively. All such vertex subsets are generated by the term

$$x \sum_{\substack{\emptyset \neq U_1 \subseteq N_G(u_1) \\ N_G(U_1) \cap N_G(u_2) = \emptyset}} x^{|U_1|} + x \sum_{\substack{\emptyset \neq U_2 \subseteq N_G(u_2) \\ N_G(U_2) \cap N_G(u_1) = \emptyset}} x^{|U_2|},$$

which completes the proof. ■

2.4. Vertex attachment

Suppose $G = (V, E)$ is a simple graph and let $U \subseteq V$. Let

$$G_{U \rhd v} = (V \cup \{v\}, E \cup \{\{u, v\} \mid u \in U\}),$$

be the graph obtained from G by adding a new vertex v to V and attaching v to all vertices of U, so the degree of v in $G_{U \rhd v}$ is $|U|$; in this case the set U is called the *vertex attachment set*. In case of $U = \emptyset$, the graph $G_{U \rhd v}$ is the disjoint union of G and the single vertex v, which implies $\mathcal{N}(G_{U \rhd v}) = \mathcal{N}(G)$ and therefore $N(G_{U \rhd v}, x) = N(G, x)$. In the following theorem, we investigate the neighborhood polynomial of $G_{U \rhd v}$ for any $U \subseteq V$.

Theorem 9. *Let $G = (V, E)$ be a graph and $U \subseteq V$. For each vertex subset $W \subseteq U$, we define a monomial A_W by*

$$A_W = \begin{cases} x^{|W|} & \text{if } \bigcap_{w \in W} N_G(w) = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$N(G_{U \rhd v}, x) = N(G, x) + \sum_{\substack{W \subseteq U \\ W \neq \emptyset}} (-1)^{|W|+1} x (1+x)^{|\bigcap_{w \in W} N_G(w)|} + \sum_{\substack{W \subseteq U \\ W \neq \emptyset}} A_W.$$

Proof. Assume $X \in \mathcal{N}(G_{U \rhd v})$, which implies that the vertices of X have a common neighbor in $G_{U \rhd v}$. Then we can distinguish the following cases.

(a) Assume $v \notin X$ and v is not a common neighbor of the vertices in X. In this case $X \in \mathcal{N}(G)$ and all possibilities for the choice of X are counted in $N(G, x)$.

(b) Assume $v \notin X$ but v is a common neighbor of the vertices in X. Since the only neighbors of v are the vertices in U, then we only need to count those subsets of U which do not have a common neighbor in G. We do so by

$$\sum_{\substack{W \subseteq U \\ W \neq \emptyset}} A_W,$$

where

$$A_W = \begin{cases} x^{|W|} & \text{if } \bigcap_{w \in W} N_G(w) = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

(c) Finally assume $v \in X$. In this case, we need to count all vertex subsets of the open neighborhoods of vertices $u \in U$ where each one of those subsets together with v forms a vertex subset, with u as their common neighbor in $G_{U \triangleright v}$. To exclude double counting subsets of intersections of the neighborhoods of u's, we use the principle of inclusion-exclusion. Such subsets are generated by

$$\sum_{\substack{W \subseteq U \\ W \neq \emptyset}} (-1)^{|W|+1} x (1+x)^{|\bigcap_{w \in W} N_G(w)|},$$

where the factor x accounts for the vertex v itself.

Finally, the arguments in (a), (b), and (c) together prove the theorem. ∎

In the following corollary we rephrase Theorem 9 in a way that the neighborhood polynomial of a graph resulting from vertex removal (instead of vertex attachment) is investigated.

Corollary 10. *Let $G = (V, E)$ be a graph and $v \in V$. Then we have*

$$N(G, x) = N(G - v, x) + \sum_{\substack{U \subseteq N_G(v) \\ U \neq \emptyset}} (-1)^{|U|+1} x (1+x)^{|\bigcap_{u \in U} N_{G-v}(u)|} + \sum_{\substack{U \subseteq N_G(v) \\ U \neq \emptyset}} A_U,$$

where for $U \subseteq N_G(v)$

$$A_U = \begin{cases} x^{|U|} & \text{if } \bigcap_{u \in U} N_{G-v}(u) = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

3. Graph Degeneracy

Corollary 10 suggests a recursion for the neighborhood polynomial of a graph based on removing the vertices of the graph one by one. In this section, we introduce k-degenerate graphs in which the calculation of the neighborhood polynomial is efficiently possible using the mentioned recursion.

A simple undirected graph $G = (V, E)$ of order n is called *k-degenerate* if every non-empty subgraph of G, including G itself, has at least one vertex of degree at most k for $0 < k < n$. Suppose $G = (V, E)$ is a k-degenerate graph of order n. To apply Corollary 10 we need to select a vertex in G to remove.

Since G is k-degenerate, the existence of a vertex of degree at most k is guaranteed. Suppose v_1 is a vertex of degree k in G, then we remove it and by using Corollary 10, we have

$$\begin{aligned}N(G,x) = N(G-v_1,x) &+ \sum_{\substack{U\subseteq N_G(v_1)\\ U\neq\emptyset}} (-1)^{|U|+1}x(1+x)^{|\bigcap_{u\in U} N_{G-v_1}(u)|}\\ &+ \sum_{\substack{U\subseteq N_G(v_1)\\ U\neq\emptyset}} A_U.\end{aligned}$$

We define $G_n := G$ where n is the number of vertices of G. For each i, $0 < i < n$, let G_{n-i} be the graph obtained from $G_{n-(i-1)}$ after removing a vertex v_i of degree at most k. The existence of such vertex is guaranteed by k-degeneracy of the graph G. We define for every non-empty $U \subseteq N_{G_{n-(i-1)}}(v_i)$

$$A_U^i = \begin{cases} x^{|U|} & \text{if } \bigcap_{u\in U} N_{G_{n-i}}(u) = \emptyset,\\ 0 & \text{otherwise.}\end{cases}$$

and for each i, $0 < i < n$

$$X_i = \sum_{U\subseteq N_{G_{n-(i-1)}}(v_i),U\neq\emptyset} \left[(-1)^{|U|+1}x(1+x)^{|\bigcap_{u\in U} N_{G_{n-i}}(u)|} + A_U^i\right].$$

Then we have

$$\begin{aligned}N(G_n,x) &= N(G_{n-1},x) + X_1\\ &= (N(G_{n-2},x) + X_2) + X_1\\ &\;\vdots\\ &= (\cdots((N(G_1,x) + X_{n-1}) + X_{n-2}) + \cdots) + X_1)\\ &= 1 + \sum_{i=1}^{n-1} X_i,\end{aligned}$$

where the last equation is a result of the fact that G_1 is equal to a single vertex which has neighborhood polynomial 1.

Clearly, in each step (in other words for each i, $0 < i < n$), we remove a vertex v_i of degree at most k and calculate the term X_i that is a sum over all non-empty subsets of $N_{G_{n-(i-1)}}(v_i)$, but since this set has at most k elements due to the degree of v_i in $G_{n-(i-1)}$, the calculation of $N(G,x)$ can be done in polynomial time for any fixed k which yields the following theorem.

Theorem 11. *Let k be a fixed positive integer. The calculation of the neighborhood polynomial can be performed in polynomial-time in the class of k-degenerate graphs.*

The recursive procedure for the calculation of the neighborhood polynomial becomes for graphs with regular structure especially simple. We give here an example of a $2 \times n$-grid graph that can easily be generalized to similar graphs with regular structure.

Example 12. A *ladder graph* of order $2n$, denoted by L_n, is the Cartesian product of two path graphs P_2 and P_n, see Figure 2. The neighborhood polynomial of L_n is

$$N(L_n, x) = 1 + 4x + 2x^2 + (2n-4)x(1+x)^2 \quad \text{for } n \geq 2.$$

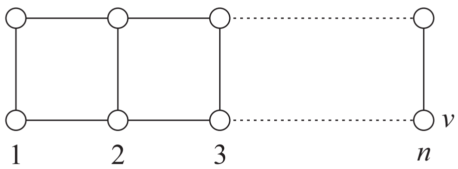

Figure 2. A ladder graph L_n.

Proof. To prove this equation we can calculate the neighborhood polynomial of L_n by applying Theorem 3, but since L_n is 2-degenerate, we can also apply Corollary 10. To do so, we need to specify a vertex to remove. In the following, we remove a vertex of degree 2, denoted by v in Figure 2. After removing the vertex v, we obtain a modified ladder graph that we denote by M_n, see Figure 3. The graph M_n has exactly one vertex of degree 1. This is the vertex to be removed in the next step; it is denoted by v in Figure 3. By Corollary 10, we obtain

$$N(L_n, x) = N(M_n, x) + x(1+x)^2 \quad \text{for } n \geq 3 \tag{3}$$

and

$$N(M_n, x) = N(L_{n-1}, x) + x(1+x)^2 \quad \text{for } n \geq 3. \tag{4}$$

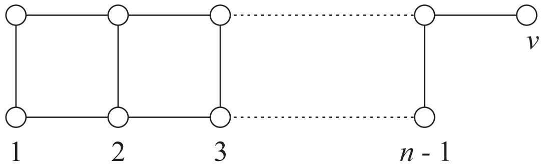

Figure 3. A modified ladder graph M_n.

Substituting $N(M_n, x)$ in Equation (3) according to Equation (4) yields

$$N(L_n, x) = N(L_{n-1}, x) + 2x(1 + x)^2 \quad \text{for } n \geq 3,$$

which provides together with the initial value

$$N(L_2, x) = N(C_4, x) = 1 + 4x + 2x^2$$

the above given result. ■

As any k-regular graph is k-degenerate, we obtain the following result.

Corollary 13. *Let k be a fixed positive integer. The calculation of the neighborhood polynomial can be performed in polynomial-time in the class of k-regular graphs.*

As a consequence of Euler's polyhedron formula, each simple planar graph contains a vertex of degree at most 5. This implies that any simple planar graph is 5-degenerate, which provides the next statement.

Corollary 14. *The neighborhood polynomial of a simple planar graph can be found in polynomial-time.*

There is an interesting generalization of planar graphs which was introduced in [9] that belongs to the class of k-degenerate graphs, too. An *almost planar graph* is a non-planar graph $G = (V, E)$ in which for every edge $e \in E$, at least one of the graphs $G - e$ (obtained from G after removing e) and G/e (obtained from G by contraction of e) is planar. It can be easily shown that every finite almost-planar graph is 6-degenerate, which implies that its neighborhood polynomial can be efficiently calculated.

4. Complexity

In [11], it is shown that counting dominating sets in several graph classes is #P-complete, which implies that calculating the domination polynomial and hence, by Equation 2, the neighborhood polynomial of a graph is #P-hard [14]. For the problem of counting dominating sets, even parameterized counting is hard. It has been shown in [8] that counting dominating sets of size k, for a given positive integer k, is #W[2]-hard.

Consequently, any result that shows that the neighborhood polynomial can be calculated in polynomial time in some graph class is highly welcome. Brown and Nowakowski showed in their seminal paper [5] that the neighborhood polynomial can be calculated in polynomial time in any C_4-free graph. The results given in [10] show that we can generalize this result to graphs that exclude small complete bipartite subgraphs. (A cycle C_4 can also be considered as a complete bipartite subgraph $K_{2,2}$.) Equation 2 shows that whenever the neighborhood polynomial can be found in polynomial time for some graph class $\mathcal{H}$, then the domination polynomial can be calculated efficiently in the class of all complements of graphs in $\mathcal{H}$.

All positive complexity results found so far concern sparse graphs, as we either restrict degrees of the graph or exclude some small subgraphs, which corresponds to the known fact that the domination polynomial can be calculated more efficiently in graphs of high edge density. Hence an interesting open problem is to find some edge-dense graph classes that allow a polynomial-time computation of the neighborhood polynomial.

5. Conclusions and Open Problems

The presented decomposition and reduction methods for the calculation of the neighborhood polynomial work well for graphs of bounded degree or, more generally, for k-degenerate graphs. They can easily be extended to derive the neighborhood polynomials of further graphs with regular structure such as grid graphs with additional diagonal edges. The splitting formula for vertex separators also suggests that the neighborhood polynomial should be polynomial-time computable in the class of graphs of bounded treewidth.

A main open problem remains the calculation of the neighborhood polynomial of graphs for which the number of edges is not linearly bounded by its order. Can we find an efficient way to calculate the neighborhood polynomial of graphs of bounded clique-width?

For some graph polynomials, like the matching polynomial, we can find nice relations between the corresponding polynomial of a graph G and the polynomial of its complement $\bar{G}$. Does there exist a similar relation for the neighborhood polynomial?

For a given graph $G = (V, E)$, a nice approach to calculate a graph polynomial is splitting along a separating vertex set, say $U \subseteq V$. This method has been successfully applied to the *Tutte polynomial*, see [2], from which we easily obtain similar splitting formulae for chromatic, flow, and reliability polynomials as well. In this paper, we proved a splitting formula for the special case that one of the split components with respect to the separator U is a star (the vertex addition formula). However, the main obstacle to generalize this idea is that we cannot restrict the consideration to subsets of the separator, but also all neighborhoods of those sets. Is there a general splitting formula for the neighborhood polynomial that avoids this difficulty?

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