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GRAPH CLASSES GENERATED BY MYCIELSKIANS

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Abstract

In this paper we use the classical notion of weak Mycielskian $M'(G)$ of a graph G and the following sequence: $M'_0(G) = G$, $M'_1(G) = M'(G)$, and $M'_n(G) = M'\big(M'_{n-1}(G)\big)$, to show that if G is a complete graph of order p, then the above sequence is a generator of the class of p-colorable graphs. Similarly, using Mycielskian $M(G)$ we show that analogously defined sequence is a generator of the class consisting of graphs for which the chromatic number of the subgraph induced by all vertices that belong to at least one triangle is at most p. We also address the problem of characterizing the latter class in terms of forbidden graphs.

Keywords: Mycielski graphs, graph coloring, chromatic number.

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1. Introduction

The notion of the Mycielski graph (also called the Mycielskian of a graph) was introduced in 1955 by Mycielski [17], which led to the famous construction of

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triangle-free graphs with arbitrarily large chromatic number. Since then Mycielski's graphs have been the subject of diverse studies. In particular, their properties have been investigated in relation to various coloring problems—vertex as well as edge related invariants have been considered (see, e.g., [4, 5, 7, 8, 11, 13, 14, 16]). Other invariants like packing, domination and biclique partition numbers, not to mention some transformations and generalizations of Mycielskians (see, e.g., [12, 15, 18, 19]), have also attracted much attention. Because of their inherent lack of susceptibility to most coloring algorithms the Mycielski graphs are well-known examples of the so-called benchmark graphs [6].

The *Mycielskian* of a graph G with the vertex set $V = \{v_1, \ldots, v_n\}$ is a graph, denoted by $M(G)$, obtained from G by adding an independent set of vertices $V' = \{v'_1, \ldots, v'_n\}$ and joining each $v'_i \in V'$ with all neighbors of v_i in G, and adding a *special* vertex x with the edges making it adjacent to all vertices in V'. In what follows for vertices v_i, v'_i we say that v_i is the *parent* of v'_i, which in turn is the *child* of v_i. In a natural way Mycielskians can be used to define a sequence $\{M_n(G)\}_{n\geq 0}$ of graphs with $M_0(G) = G$, $M_1(G) = M(G)$, and $M_n(G) = M\big(M_{n-1}(G)\big)$ for $n \geq 2$. Obviously, by definition for every $n \geq 1$, it holds that $M_{n-1}(G)$ is an induced subgraph of $M_n(G)$; in symbols $M_{n-1}(G) \leq M_n(G)$. For every Mycielskian in the sequence we say that the graph G is its *initial* graph.

One of the main topics of this paper is the embeddability of graphs in Mycielskians. We say that a graph H can be *embedded* in G if there exists an induced subgraph H' of G such that H' is isomorphic to H. Along their investigations on the Hall ratio of graphs Cropper *et al.* [8] gave a positive answer to the question of whether for every triangle-free graph there exists $n \geq 0$ such that G can be embedded in $M_n(K_2)$.

Theorem 1 (Cropper *et al.* [8]). *Every connected triangle-free graph with n vertices is an induced subgraph of $M_n(K_2)$.*

An analogous result for subgraphs of $M_n(K_2)$ was proved by Berger *et al.* [1]. In fact, since every $M_n(K_2)$ is triangle-free, Theorem 1 leads to a characterization of the class of graphs that can be embedded in $M_n(K_2)$ for some $n \geq 0$. In what follows, the class of such graphs and the class of triangle-free graphs is denoted by $\mathcal{M}(K_2)$ and $\mathcal{I}_1$, respectively, so the above characterization can be also written as follows:

$$\mathcal{M}(K_2) = \mathcal{I}_1.$$

In this paper we ask about characterizations of classes of graphs that for some $n \geq 0$ can be embedded in $M_n(K_p)$ and those embeddable in the so-called weak Mycielskians $M'_n(K_p)$, where $p \geq 2$ (for the definition of weak Mycielskian see Section 2). The classes of such graphs we denote by $\mathcal{M}(K_p)$ and $\mathcal{M}'(K_p)$, respectively.

We prove that the class of graphs embeddable in $M'_n(K_p)$ is exactly the class $\mathcal{O}^p$ of p-colorable graphs—the sequence $\{M'_n(K_p)\}_{n\geq 0}$ is a generator of $\mathcal{O}^p$. Moreover, we prove that the graphs embeddable in $M_n(K_p)$ constitute the class $\mathcal{B}_p$, i.e., the class of graphs for which the chromatic number of the subgraph induced by all vertices that belong to at least one triangle is at most p—the sequence $\{M_n(K_p)\}_{n\geq 0}$ is a generator of $\mathcal{B}_p$. Our main contribution can be roughly summarized as follows:

$$\mathcal{M}'(K_p) = \mathcal{O}^p \text{ and } \mathcal{M}(K_p) = \mathcal{B}_p.$$

In the last section we address the problem of characterizing the class $\mathcal{B}_p$ in terms of forbidden graphs.

Our notation on graph classes follows Borowiecki *et al.* [2]. Any other unexplained notation is as found in Diestel [9] or West [20].

2. Preliminaries

For every Mycielskian $M_n(G)$ in the sequence $\{M_n(G)\}_{n\geq 0}$ by X_n we denote a subset of $V\big(M_n(G)\big)$ consisting of all special vertices and their ancestors resulting from the recursive definition of Mycielskians. More formally, X_n can be defined as follows: $X_0 = \emptyset$, $X_1 = \{x_1\}$, and $X_n = X_{n-1} \cup X'_{n-1} \cup \{x_n\}$, where x_n is the special vertex of $M_n(G)$ and X'_{n-1} is the set of the children of vertices in X_{n-1}, $n \geq 2$. In what follows X_n is called the *special set* of $M_n(G)$.

Property 2. *If G is a graph and X_n is the special set of $M_n(G)$, then the graph induced by X_n is triangle-free and no vertex in X_n belongs to a triangle in $M_n(G)$.*

Besides special sets, we will also need the notions of level and generation. Namely, referring to the definition of Mycielskian $M_n(G)$ with the children set V', for $n \geq 1$ we say that the vertices in $V' \setminus X_n$ form a set called the *n-th level* in $M_n(G)$, denoted by $L_n(G)$ (we write L_n for short, and for convenience we assume that $L_0(G) = V(G)$). Naturally, for all $k < n$ by the k-th level of $M_n(G)$ we mean the k-th level of $M_k(G)$. Note that each level is an independent set. The *generation* of a vertex v of the initial graph G of $M_n(G)$, denoted by $[v]$, is a set containing v and every vertex with the parent in $[v]$. As we will see later on it is crucial to observe that generation is an independent set.

The graph $M_n(G) - X_n$, denoted by $M'_n(G)$, is called the *weak Mycielskian*. Naturally, the sequence $\{M'_n(G)\}_{n\geq 0}$ based on weak Mycielskians is defined analogously to $\{M_n(G)\}_{n\geq 0}$. Similarly, it holds $M'_{n-1}(G) \leq M'_n(G)$ and it is not hard to see that $M'_n(G) \leq M_n(G)$.

Since in this work we consider Mycielskians with initial graphs being complete, with K_p as the initial graph we gain the ability of stating the following property.

Property 3. *Each vertex v of the initial graph K_p is adjacent to all vertices of $M'_n(K_p)$ except the vertices in $[v]$.*

We conclude this section with another basic property of weak Mycielskians.

Property 4. *For every $p \geq 2$ and $n \geq 0$ it holds $\chi\big(M'_n(K_p)\big) = p$.*

Indeed, it is not hard to argue that if G' is a graph obtained from G by adding a vertex v' such that in G' the neighborhood of v' and some vertex $v \neq v'$ are the same, then $\chi(G') = \chi(G)$. Therefore, following the definition of weak Mycielskian for every $n \geq 0$ it holds $\chi\big(M'_{n+1}(K_p)\big) = \chi\big(M'_n(K_p)\big)$, and since $M'_0(K_p) = K_p$, it follows that weak Mycielskians $M'_n(K_p)$ are p-chromatic.

3. Generators

Theorem 5. *Let $p \geq 2$. A graph G can be embedded in $M'_n(K_p)$ for some $n \geq 1$ if and only if $\chi(G) \leq p$.*

Proof. Since by Property 4, $\mathcal{M}'(K_p) \subseteq \mathcal{O}^p$, it remains to prove that every p-colorable graph G of order n is an induced subgraph of $M'_m(K_p)$ for some $m \geq n$.

Let $V_1, \ldots, V_p$ be the color classes of G with cardinalities $n_1, \ldots, n_p$, respectively. First, we will show that the complete p-partite graph $K{=}K_{n_1,\ldots,n_p}$ of order n is an induced subgraph of $M'_n(K_p)$. For this purpose let $V(K_p) = \{v_1, \ldots, v_p\}$. Now, for the class V_1 choose n_1 children of the vertex v_1 from levels $L_1, \ldots, L_{n_1}$ (note that such a choice is unique). Next, for each class V_i with $i \in \{2, \ldots, p\}$ choose n_i children of the vertex v_i from the levels $L_{n_1+\cdots+n_{i-1}+1}, \ldots, L_{n_1+\cdots+n_i}$. Observe that each V_i obtained in this way is a subset of the generation $[v_i]$ and hence an independent set. Moreover, each vertex in the set V_i is adjacent to all vertices in each V_j with $j \neq i$. To see this we argue that if vertex v'_i (vertex v'_j) is the child of the vertex v_i (vertex v_j) and $i \neq j$, then $v'_iv'_j$ is an edge in $M'_n(K_p)$ if and only if v'_i and v'_j belong to different levels. Let $v'_i \in L_k$ and let $v'_j \in L_t$ with $t > k$. From Property 3 it follows that $v_jv'_i$ is an edge in $M'_n(K_p)$ while by the definition of weak Mycielskian the neighborhoods of v_j and v'_j in $M'_{t-1}(K_p)$ are the same. Therefore, since v'_i is a vertex in $M'_{t-1}(K_p)$, it follows that $v'_iv'_j$ is an edge in $M'_n(K_p)$. Clearly, the graph G is a spanning subgraph of K.

Now, we claim that for each edge ab of a subgraph H of $M'_n(K_p)$ there exists an *induced subgraph* H' of $M'_{n+1}(K_p)$ such that $H' = H - ab$. Namely, let vertex a' (vertex b') be the child of the vertex a (vertex b) such that a' and b' belong to the level L_{n+1} in $M'_{n+1}(K_p)$. Observe that by the definition of Mycielskian the neighborhoods of a and a', as well as b and b', in $M'_{n+1}(K_p)$ are the same. Therefore a subgraph induced by $\big(V(H)\setminus\{a,b\}\big)\cup\{a',b'\}$ is isomorphic to $H-ab$.

Let $X = E(K) \setminus E(G)$. Clearly, if X is empty, then G is isomorphic to K. Otherwise, the existence of an embedding of G in $M'_{n+|X|}(K_p)$ follows by the above argument applied to subsequent edges in X. ■

Corollary 6. *For every $p \geq 2$ it holds*

$$\mathcal{M}'(K_p) = \mathcal{O}^p.$$

Let $\{G_n\}_{n\geq 0}$ denote a sequence of graphs such that $G_n \leq G_{n+1}$. The class consisting of all induced subgraphs of graphs in the above sequence is uniquely determined and it is known to be closed with respect to taking induced subgraphs—the sequence $\{G_n\}_{n\geq 0}$ is called the *generator* of such a class.

Theorem 7. *For every $p \geq 2$ the sequence $\{M'_n(K_p)\}_{n\geq 0}$ is a generator of $\mathcal{O}^p$.*

Having characterized the graphs embeddable in weak Mycielskians, in the remaining part of this section we focus on embeddability in Mycielskians. As we will see, in this context, the role of a certain subgraph that we call the base of a graph cannot be overestimated. Namely, the *base* of a graph G is a subgraph induced by the set consisting of all vertices of G such that each of them belongs to at least one triangle in G. By $\beta(G)$ we denote the chromatic number of the base of a graph G. In this context, it is worth mentioning that 3-colorability is NP-complete even for certain subclasses of planar graphs with vertex degree at most 4 and each vertex in at least one triangle (see Borowiecki [3]). Interestingly, our new invariant β turns out to be a decisive factor for embeddability of graphs in Mycielskians $M_n(K_p)$.

Theorem 8. *Let $p \geq 3$. A graph G can be embedded in $M_n(K_p)$ for some $n \geq 1$ if and only if $\beta(G) \leq p$.*

Proof. First, we prove that if $\beta(G) \geq p+1$, then there does not exist $n \geq 1$ for which G could be embedded in $M_n(K_p)$. Suppose that, to the contrary, for some $n \geq 1$ the graph G has been embedded in $M_n(K_p)$. Since by definition each vertex of the base B of G belongs to a triangle and by Property 2 there is no triangle containing a vertex in X_n, the base B is embedded in $M_n(K_p) - X_n$. However, by Property 4 all subgraphs of $M_n(K_p) - X_n$, in particular those isomorphic to B, are p-colorable, which contradicts $\beta(G) \geq p+1$. Thus we have proved that if G can be embedded in $M_n(K_p)$ for some $n \geq 1$, then $\beta(G) \leq p$.

Next, we show that if $\beta(G) \leq p$, then G can be embedded in $M_n(K_p)$ for some $n \geq 1$. Since $\chi(B) \leq p$, from Theorem 5 it follows that the base B of G can be embedded in $M'_m(K_p)$ for some $m \geq 1$. Let $S = \{z_1, \ldots, z_s\}$ be defined as $V(G) \setminus V(B)$ and let B' be an induced subgraph of $M'_m(K_p)$ that is isomorphic to B. Assuming $M'_m(K_p) \leq M_n(K_p)$, where $n > m$, we define $S' = \{x_1, \ldots, x_s\}$ as a subset of $V\big(M_n(K_p)\big) \setminus V(B')$. In what follows we are going to argue that S' can be chosen such that the graph G' induced by $V(B') \cup S'$ is isomorphic to G, or in other words, we are interested in the existence of an isomorphism $f : V(G) \to V(G')$.

Let $G_0 = B$ and let G_i be a graph induced by $V(G_{i-1}) \cup \{z_i\}$, where $i > 0$. The case $s = 0$ is obvious, so let us assume that $s \geq 1$ and $z_1, \ldots, z_s$ is an ordering of S under which $N_{G_{i-1}}(z_i) \neq \emptyset$ for each $i \in \{1, \ldots, s\}$. Note that since G is connected, such an ordering always exists. Let G'_{i-1} with $i \leq s$ be an embedding of G_{i-1} in $M_{m+i-1}(K_p)$ that is given by $f|_{G_{i-1}}$ (we write f_{i-1} for short). For an inductive step we need to define f_i (in what follows if $f_i(v)$ is not specified explicitly, we assume $f_i(v) = f_{i-1}(v)$). Let $Z_i = N_{G_{i-1}}(z_i)$. Naturally, Z_i is independent, for otherwise z_i would belong to a triangle and hence to the base B of G, which is not the case. Consequently, the set $Z'_i = \{u \,|\, u = f_{i-1}(v)$, where $v \in Z_i\}$ is independent. By the definition of Mycielskian, the following set $Z''_i = \{v \in L_{m+i} \,|\, v$ is the child of $u \in Z'_i\}$ exists and is independent. Moreover, for each $u \in Z'_i$ and its child $v \in Z''_i$ it holds $N_{G'_{i-1}}(u) = N_{G'_{i-1}}(v)$. Hence the graph induced by $V(G'_{i-1}) \cup Z''_i \setminus Z'_i$ is isomorphic to G_{i-1}. Thus for all $w \in Z_i$ we set $f_i(w) = v$, where $v \in Z''_i$ is a child of $f_{i-1}(w)$. Let $f_i(z_i) = x_{m+i}$. Since by the definition of Mycielskian the neighborhood of x_{m+i} in $M_{m+i}(K_p)$ is L_{m+i}, the neighborhood of $f_i(z_i)$ in G'_i is Z''_i. Thus the graph induced by $V(G'_{i-1}) \cup Z''_i \cup \{x_{m+i}\} \setminus Z'_i$ is an embedding of G_i. It remains to set $f = f_s$. ■

Corollary 9. *For every $p \geq 3$ it holds*

$$\mathcal{M}(K_p) = \mathcal{B}_p.$$

For the statement of the next theorem, we assume that for triangle-free graphs $\beta \equiv 2$. This slightly alters the original definition of β thus allowing the following extension of the result of Cropper *et al.* [8].

Theorem 10. *For every $p \geq 2$ the sequence $\big\{M_n(K_p)\big\}_{n\geq 0}$ is a generator of $\mathcal{B}_p$.*

4. Forbidden Graphs

After determining the generator of the class $\mathcal{B}_p$ a natural direction is to consider a characterization of this class in terms of *minimal forbidden graphs*, i.e., the graphs G that do not belong to $\mathcal{B}_p$ but, independently of the choice of a vertex v, the deletion of v from G results in a graph $G - v$ in $\mathcal{B}_p$.

In our investigation of minimal forbidden graphs we rely on the closely related notion of critical graphs. Namely, for a positive integer k we say that a graph G is (k, β)-*critical* if $\beta(G) = k$ but for each vertex v the graph $G - v$ belongs to

$\mathcal{B}_{k-1}$. This resembles the definition of (k,χ)-critical graph requiring $\chi(G) = k$ and $(k-1)$-colorability of $G - v$ for each v. Recall that for every (k,χ)-critical graph G a graph $G - v$ is $(k-1)$-chromatic. Hence, for every p the class of graphs for which $\chi(G) \leq p$ is simply defined by forbidding all (k,χ)-critical graphs with $k = p + 1$. Moreover, for distinct p_1, p_2 the sets of minimal forbidden graphs characterizing the classes of p_1- and p_2-colorable graphs are disjoint. In this context we ask if analogous properties hold for (k,β)-critical graphs.

Problem 1. Given $k \geq 3$, what is the minimum integer p such that there is a (k,β)-critical graph for which $G - v \in \mathcal{B}_p$ for every vertex v?

As we will see, though β strongly depends on the chromatic number of certain subgraphs, its properties seem different from that of the chromatic number. In this section we present some approach to the solution of Problem 1. Observe that there is only one $(3,\beta)$-critical graph, namely K_3. Hence the solution of Problem 1 for $k = 3$ is $p = 0$. We show that for every $k \geq 4$ it holds $p \geq k - 2$.

Let us start with two simple properties leaving their routine proofs to the reader. First, we observe that every (k,β)-critical graph is the base of itself.

Proposition 11. *If $k \geq 4$ and G is a (k,β)-critical graph, then each vertex of G belongs to some triangle.*

We also note that in some cases χ-criticality of a graph yields its β-criticality.

Proposition 12. *Let $k \geq 4$. If G is a (k,χ)-critical graph and every vertex of G belongs to some triangle, then G is (k,β)-critical.*

A well known fact is that every vertex of a (k,χ)-critical graph has degree at least $k-1$. In contrast, we shall show that (k,β)-critical graphs can have vertices of degrees smaller than $k - 1$. For this purpose, let the *t-core* of a graph G be defined as a graph resulting from G by successive deletion of vertices of degrees less than t. Also observe that every (k,χ)-critical graph is the $(k-1)$-core of itself.

In our next theorem we use (k,χ)-critical graphs to gain a slightly deeper insight into the structure of (k,β)-critical graphs.

Theorem 13. *Let $k \geq 4$ and let the $(k-1)$-core H of a graph G be (k,χ)-critical. Moreover, let $A(H) = V(H) \setminus V(B)$, where B is the base of H. A graph G is (k,β)-critical if and only if every vertex of G belongs to some triangle and for each $v \in V(G) \setminus V(H)$ there is a vertex $w \in A(H)$ such that all triangles of G containing w also contain v.*

Proof. First, suppose that G is (k,β)-critical. By Proposition 11, every vertex of G belongs to some triangle. Suppose that there exists a vertex $v \in V(G) \setminus V(H)$

such that each vertex $w \in A(H)$ belongs to some triangle in $G - v$. Since each vertex in $V(H) \setminus A(H)$ belongs to a triangle in H, we conclude that H is a subgraph of the base of $G - v$, contradicting the (k, β)-criticality of G.

Now, assume that every vertex of G belongs to a triangle and for each $v \in V(G) \setminus V(H)$ there is a vertex $w \in A(H)$ such that all triangles of G containing w also contain v. Observe that $A(H)$ is empty if and only if $V(G) \setminus V(H)$ is empty, i.e., in this case $G = H$ and hence the assertion holds by Proposition 12. Suppose that $A(H) \neq \emptyset$. We would like to prove (k, β)-criticality of G. Since G is the base of itself and H is its induced subgraph with $\chi(H) = k$ and because each k-coloring of H can be extended to G (by the definition of $(k-1)$-core), we have $\beta(G) = k$. Thus, we must show that $\beta(G - v) \leq k - 1$ for every vertex of G. If $v \in V(H)$, then $H - v$ is $(k-1)$-colorable and consequently $G - v$ is $(k-1)$-colorable. Thus $\beta(G - v) \leq k - 1$ in this case. If $v \in V(G) \setminus V(H)$, then there is a vertex $w \in A(H)$ such that w does not belong to the base of $G - v$. Clearly, $H - w$ has a $(k-1)$-coloring, by (k, χ)-criticality of H. We construct a $(k-1)$-coloring of the base of $G - v$ extending a $(k-1)$-coloring of the graph induced by those vertices of $H - w$ that belong to the base of $G - v$. ■

In the next part of this section we prove that for $k \geq 4$ every (k, β)-critical graph G has a vertex v for which $\beta(G - v) \geq k - 2$. This shows that the integer p in Problem 1 is at least $k - 2$. To prove this fact we need the following property.

Property 14. *Let $\chi(G) = k$ and let K be an r-clique of G such that $\chi(G - K) = k - r$. For each color $i \in \{1, \ldots, k - r\}$, in an arbitrary $(k - r)$-coloring of $G - K$, there exists a vertex w_i adjacent to all vertices of K, i.e., $V(K) \subseteq N_G(w_i)$.*

The above property can be simply proved by observing that if there existed in $G - K$ a color class C_i with $i \in \{1, \ldots, k - r\}$ such that each vertex colored i is non-adjacent to some vertex of K, then it would be possible to recolor each vertex v in C_i with a color of a vertex in K to which v is not adjacent. Note that such recoloring is always possible since C_i is an independent set, all colors in K are distinct and different from that in $G - K$. Since in the new coloring there are no vertices colored i, we get a $(k-1)$-coloring of G, which contradicts $\chi(G) = k$.

Theorem 15. *For $k \geq 4$ there does not exist a (k, β)-critical graph G such that $G - v$ belongs to $\mathcal{B}_{k-3}$ for every $v \in V(G)$.*

Proof. Suppose that there exists a (k, β)-critical graph G such that $G - v$ belongs to $\mathcal{B}_{k-3}$ for every $v \in V(G)$ (i.e., $\beta(G - v) \leq k - 3$ for every $v \in V(G)$). By Proposition 11, every vertex v of G belongs to some triangle. Thus, using the fact that $\beta(G - v) \leq 3$, we get that $G - v$ has at least two vertices which are not in the base of $G - v$ (they do not belong to any triangle in $G - v$).

First, we claim that in G there exists a vertex that is in exactly one triangle. Suppose this is not true. By our assumption in $G - v$ there is a vertex that does not belong to a triangle. Let u be such a vertex. Thus all the triangles in G, containing u, must contain the edge vu. Let $w_1, \ldots, w_t$ be vertices such that v, u, w_i form a triangle for $i \in \{1, \ldots, t\}$. Observe that for every w_i the triangle formed by v, u, w_i is the only one that contains uw_i. Again by our assumption in $G - u$ there is a vertex that does not belong to a triangle. If w_i is such a vertex, then w_i is in exactly one triangle, a contradiction. Otherwise there is no triangle in $G-u$ containing v, because also in $G-u$ must be a vertex that is in no triangle. Thus, for every w_i the triangle v, u, w_i is the only triangle that contains vw_i. Since every w_i is in some triangle in $G - u$, every w_i is in the triangle in $G - \{v, u\}$ and consequently $G - \{v, u\}$ is the base of itself. Thus $G - v$ and $G - u$ have only one vertex which is not in the base, a contradiction.

Let $v \in V(G)$ be a vertex belonging to exactly one triangle T in G, and let u, w be its neighbors such that $uw \in E(G)$. We see that also u and w are in exactly one triangle. Namely, they belong to the triangle T, because, as we have already observed $G - v$ has at least two vertices which do not belong to a triangle in $G - v$. Thus $G - T$ is the base of itself. By our assumption $G - T \in \mathcal{B}_{k-3}$, so $\chi(G-T) \leq k-3$. On the other hand, since $\chi(G) = k$, we have $\chi(G-T) \geq k-3$. Thus $\chi(G-T) = k-3$ and hence by Property 14 in each color class of each $(k-3)$-coloring of $G - T$ there is a vertex adjacent to v, u and w, which contradicts the assumption that there is only one triangle containing v. ∎

As we mentioned earlier, the class $\mathcal{B}_p$ can be characterized by the family of forbidden graphs. Let $\mathcal{C}(\mathcal{B}_p)$ denote the family of forbidden graphs for $\mathcal{B}_p$. Comparing the notions of forbidden graphs and β-critical graphs and due to Theorem 15 we see that $\mathcal{C}(\mathcal{B}_p)$ may only consist of $(p+1, \beta)$-critical or $(p+2, \beta)$-critical graphs. Observe that if a graph is $(p+2, \beta)$-critical and belongs to $\mathcal{C}(\mathcal{B}_p)$, then it also belongs to $\mathcal{C}(\mathcal{B}_{p+1})$. Theorem 15 shows that $\bigcap_{i \in J} \mathcal{C}(\mathcal{B}_i)$ can be nonempty only if $J = \{p, p+1\}$. However, we do not know any graph that is $(p+2, \beta)$-critical and belongs to $\mathcal{C}(\mathcal{B}_p)$.

Problem 2. Is there an integer p such that $\mathcal{C}(\mathcal{B}_p) \cap \mathcal{C}(\mathcal{B}_{p+1}) \neq \emptyset$?

References

[1] A.J. Berger, I. Broere, S.J.T. Moagi and P. Mihók, *Meet- and join-irreducibility of additive hereditary properties of graphs*, Discrete Math. **251** (2002) 11–18. doi:10.1016/S0012-365X(01)00323-5

[2] M. Borowiecki, I. Broere, M. Frick, P. Mihók and G. Semanišin, *A survey of hereditary properties of graphs*, Discuss. Math. Graph Theory **17** (1997) 5–50. doi:10.7151/dmgt.1037

[3] P. Borowiecki, *Computational aspects of greedy partitioning of graphs*, J. Comb. Optim. **35** (2018) 641–665. doi:10.1007/s10878-017-0185-2

[4] B. Brešar, J. Ferme, S. Klavžar and D.F. Rall, *A survey on packing colorings*, Discuss. Math. Graph Theory **40** (2020) 923–970.
doi:10.7151/dmgt.2320

[5] B. Brešar, S. Klavžar, D.F. Rall and K. Wash, *Packing chromatic number versus chromatic and clique number*, Aequationes Math. **92** (2018) 497–513.
doi:10.1007/s00010-017-0520-9

[6] M. Caramia and P. Dell'Olmo, *A lower bound on the chromatic number of Mycielski graphs*, Discrete Math. **235** (2001) 79–86.
doi:10.1016/S0012-365X(00)00261-2

[7] G.J. Chang, L. Huang and X. Zhu, *Circular chromatic number of Mycielski's graphs*, Discrete Math. **205** (1999) 23–37.
doi:10.1016/S0012-365X(99)00033-3

[8] M. Cropper, A. Gyárfás and J. Lehel, *Hall ratio of the Mycielski graphs*, Discrete Math. **306** (2006) 1988–1990.
doi:10.1016/j.disc.2005.09.020

[9] R. Diestel, Graph Theory (Springer, Berlin, 1997).

[10] T. Došlić, *Mycielskians and matchings*, Discuss. Math. Graph Theory **25** (2005) 261–266.
doi:10.7151/dmgt.1279

[11] D.C. Fisher, *Fractional colorings with large denominators*, J. Graph Theory **20** (1995) 403–409.
doi:10.1002/jgt.3190200403

[12] D.C. Fisher, P.A. McKenna and E.D. Boyer, *Hamiltonicity, diameter, domination, packing, and biclique partitions of Mycielski's graphs*, Discrete Appl. Math. **84** (1998) 93–105.
doi:10.1016/S0166-218X(97)00126-1

[13] Y.S. Kwon, J. Lee and Z. Zhang, *Edge-chromatic numbers of Mycielski graphs*, Discrete Math. **312** (2012) 1222–1225.
doi:10.1016/j.disc.2011.12.011

[14] M. Larsen, J. Propp and D. Ullman, *The fractional chromatic number of Mycielski's graphs*, J. Graph Theory **19** (1995) 411–416.
doi:10.1002/jgt.3190190313

[15] W. Li, J. Wu, P.C.B. Lam and G. Gu, *Several parameters of generalized Mycielskians*, Discrete Appl. Math. **154** (2006) 1173–1182.
doi:10.1016/j.dam.2005.11.001

[16] D.D.-F. Liu, *Circular chromatic number for iterated Mycielski graphs*, Discrete Math. **285** (2004) 335–340.
doi:10.1016/j.disc.2004.01.020

[17] J. Mycielski, *Sur le coloriage des graphes*, Colloq. Math. **3** (1955) 161–162.
doi:10.4064/cm-3-2-161-162

[18] H.P. Patil and R. Pandiya Raj, *On the total graph of Mycielski graphs, central graphs and their covering numbers*, Discuss. Math. Graph Theory **33** (2013) 361–371. doi:10.7151/dmgt.1670

[19] G. Simons, C. Tardif and D. Wehlau, *Generalised Mycielski graphs, signature systems, and bounds on chromatic number*, J. Combin. Theory Ser. B **122** (2017) 776–793. doi:10.1016/j.jctb.2016.09.007

[20] D.B. West, Introduction to Graph Theory (Prentice Hall, Upper Saddle River, 2001).

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