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ON SMALL BALANCEABLE, STRONGLY-BALANCEABLE AND OMNITONAL GRAPHS

Yair Caro

Department of Mathematics
University of Haifa-Oranim, Israel

Josef Lauri and Christina Zarb

Department of Mathematics
University of Malta, Malta

e-mail: christina.zarb@um.edu.mt

Abstract

In Ramsey Theory for graphs we are given a graph G and we are required to find the least n_0 such that, for any $n \geq n_0$, any red/blue colouring of the edges of K_n gives a subgraph G all of whose edges are blue or all are red. Here we shall be requiring that, for any red/blue colouring of the edges of K_n, there must be a copy of G such that its edges are partitioned equally as red or blue (or the sizes of the colour classes differs by one in the case when G has an odd number of edges). This introduces the notion of balanceable graphs and the balance number of G which, if it exists, is the minimum integer $\mathrm{bal}(n, G)$ such that, for any red/blue colouring of $E(K_n)$ with more than $\mathrm{bal}(n, G)$ edges of either colour, K_n will contain a balanced coloured copy of G as described above. The strong balance number $\mathrm{sbal}(n, G)$ is analogously defined when G has an odd number of edges, but in this case we require that there are copies of G with both one more red edge and one more blue edge.

These parameters were introduced by Caro, Hansberg and Montejano. These authors also introduce the more general omnitonal number $\mathrm{ot}(n, G)$ which requires copies of G containing a complete distribution of the number of red and blue edges over $E(G)$.

In this paper we shall catalogue $\mathrm{bal}(n, G)$, $\mathrm{sbal}(n, G)$ and $\mathrm{ot}(n, G)$ for all graphs G on at most four edges. We shall be using some of the key results of Caro *et al.* which we here reproduce in full, as well as some new results which we prove here. For example, we shall prove that the union of two bipartite graphs with the same number of edges is always balanceable.

Keywords: edge-colouring, zero-sum Ramsey, balanceable graphs, omnitonal graphs.

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1. Introduction

The problem we consider here, introduced in [6], lies in the intersection of several graph theory problems such as Ramsey Theory, extremal Graph Theory (Turan numbers) and Zero-sum Ramsey Theory. It can be described as follows: we first suppose that there is a 2-edge-colouring $f : E(K_n) \to \{red, blue\}$, and we denote by $R = R_f$ and $B = B_f$ the set of edges of K_n coloured red and blue, respectively. For short we shall also denote by R and B the graphs induced by the edge sets R and B, respectively. A subgraph G of such a coloured complete graph is said to be (r, b)*-coloured* if r edges of G are coloured red and b edges are coloured blue with $r + b = e(G)$, where $e(G)$ denotes the number of edges of G. We denote by $\deg_{red}(v)$ the number of vertices adjacent to v by a red edge, $N_{red}(v) = \{u : uv$ is a red edge$\}$ so that $\deg_{red}(v) = |N_{red}(v)|$, while $N_{red}[v] = N_{red}(v) \cup \{v\}$. For other standard graph theoretic notation we refer to [19].

In Ramsey Theory for graphs, we require that one of R or B contains a particular graph, say a complete graph of certain order, and we ask what is the smallest value of n such that any 2-edge-colouring f gives us the graph we want either as a subgraph of R or as a subgraph of B. In this paper, inspired by [6], the graphs we are searching for will be required to have some particular mix of colours on its edge-set. There are three main problems we consider.

Balanceable graphs

For a given graph G we say that the colouring contains a *balanced* copy of G if f induces a coloured copy of G in which the number of edges in each colour is equal (if G has an even number of edges) or differs by one in the other case.

We therefore let, for $n \geq |V(G)|$, $\mathrm{bal}(n, G)$ be the minimum integer, if it exists, such that any 2-edge-colouring of $E(K_n)$ with $\min\{|R|, |B|\} > \mathrm{bal}(n, G)$ contains a balanced copy of G. If $\mathrm{bal}(n, G)$ exists for every sufficiently large n, we say that G is balanceable.

If G has an odd number of edges we then introduce the notion of *strong balance*. That is, for $n \geq |V(G)|$, we let $\mathrm{sbal}(n, G)$ be the minimum integer, if it exists, such that any 2-edge-colouring of $E(K_n)$ with $\min\{|R|, |B|\} > \mathrm{sbal}(n, G)$ contains both a $\left(\left\lfloor \frac{e(G)}{2} \right\rfloor, \left\lceil \frac{e(G)}{2} \right\rceil\right)$-coloured and $\left(\left\lceil \frac{e(G)}{2} \right\rceil, \left\lfloor \frac{e(G)}{2} \right\rfloor\right)$-coloured copy of G. If $\mathrm{sbal}(n, G)$ exists for every sufficiently large n, we say that G is strongly-balanceable.

Omnitonal graphs

Omnitonal graphs are those graphs G for which different copies of G appear in a 2-edge-coloured complete graph such that all the copies carry between them all possible distributions of the two colours on the edges of G. More formally, we define, for a given graph G, and for $n \geq |V(G)|$, $\mathrm{ot}(n, G)$ to be the minimum integer, if it exists, such that any 2-edge-colouring of $E(K_n)$ with $\min\{|R|, |B|\} > \mathrm{ot}(n, G)$ contains an (r, b)-coloured copy of G for any $r \geq 0$ and $b \geq 0$ such that $r + b = e(G)$. If $\mathrm{ot}(n, G)$ exists for every sufficiently large n, we say that G is omnitonal.

A source of motivation for [6], that belongs to the recent developments in zero-sum extremal problems, is the close connection between the concepts of balanceable and omnitonal graphs and zero-sum problems with weights over $\{-p, q\}$ and in particular over $\{-1, 1\}$ (see Remark 1.3 in [6]) [1, 2, 4–9, 14, 16–18].

There are two main goals as background to the present paper: the first one is to compute the functions $\mathrm{sbal}(n, G)$, $\mathrm{bal}(n, G)$ and $\mathrm{ot}(n, G)$ for all the graphs with up to 4 edges, checking which of the results already obtained in [6] can be applied in this task. When no such result from [6] does the job we give our complementary ad-hoc theorems that allow us to complete the various tables.

The second goal of this paper is to find out, while working on completing the tables, whether we can gain some further insight not covered in [6]. In fact we have found at least one such instance. It is mentioned in [6] that not all bipartite graphs are balanceable (or strongly balanceable). Nevertheless we prove here that if G and H are bipartite graphs with $e(G) = e(H)$, then $G \cup H$ is balanceable. We remark that several theorems about the union of balanceable or omnitonal graphs are known (and will probably appear in [3]), but all of them assume that at least one of the graphs is balanceable or omnitonal.

Our paper is organized as follows.

In Section 2 we collect the theorems proved in [6] which we need here and also give our complementary results that allow us to complete the first task — namely to compute $\mathrm{sbal}(n, G)$, $\mathrm{bal}(n, G)$ and $\mathrm{ot}(n, G)$ for all the graphs on up to four edges.

In Section 3 we prove the union theorem mentioned above, and give an example to illustrate its use. We then introduce the triple property, and use this, Theorem 3.1 and the fact that the graphs tK_2 are amoebas (to be defined later), to show that for $n \geq 7t-1$, $\mathrm{sbal}(n, (2t-1)K_2) = \mathrm{bal}(n, 2tK_2) = \mathrm{bal}(n, (2t+1)K_2) = (t-1)(n-t+1) + \binom{t-1}{2}$.

Section 4 contains the tables with the results of our computations.

2. Theorems and Further Definitions

In this section we state those theorems from [6] which we shall use in the presentation of our results, and some of the definitions required to state these theorems. We shall also present a few results which do not appear in [6].

For notation not defined here we refer the reader to [19]. We here note that we shall oftern denote the edge $\{a, b\}$ by ab.

2.1. Known results

The results presented here are taken from [5] and [6].

Theorem 2.1 (Theorem 2.8 in [6]). *Omnitonal graphs are bipartite.*

Theorem 2.2 (Theorem 2.10 in [6]). *Every tree is omnitonal.*

An edge replacement is defined as follows: given a graph G embedded in a complete graph K_n where $n \geq |V(G)|$, we say that $H \simeq G$ (also embedded in K_n) is obtained from G by an *edge-replacement* if there is some $e_1 \in E(G)$ and $e_2 \in E(K_n) \backslash E(G)$, such that $H = (G - e_1) + e_2$. A graph G is called an *amoeba* if there exists $n_0 = n_0(G) \geq |V(G)|$, such that for all $n \geq n_0$ and any two copies F and H of G in K_n, there is a chain $F = G_0, G_1, \ldots, G_t = H$ such that for every $i \in \{1, 2, \ldots, t\}$, $G_i \simeq G$ and G_i is obtained from G_{i-1} by an edge-replacement. For example, it is easy to see that tK_2 is an amoeba for $t \geq 1$, and a little more effort reveals that P_k, the path on k vertices, is also an amoeba. A paper developing in depth structures and properties of amoebas is under preparation [5].

For a given graph G, we denote by $R(G, G)$ the 2-colour Ramsey number, that is, the minimum integer $R(G, G)$ such that, whenever $n \geq R(G, G)$, any 2-edge-colouring of $E(K_n) = E(R) \cup E(B)$ contains either a red or a blue copy of G. For a given graph G, we denote by $\mathrm{ex}(n, G)$ the Turan number for G, that is, the maximum number of edges in a graph with n vertices containing no copy of G [3, 13].

Theorem 2.3 (Theorem 2.14 in [6]). *There is some $n_0 = n_0(G)$ such that every bipartite amoeba G on $n \geq n_0(G)$ is omnitonal with* $\mathrm{ot}(n, G) = \mathrm{ex}(n, G)$.

Theorem 2.4 (Theorem 2.15 in [6]). *Every amoeba G is balanceable/strongly balanceable.*

The final two results that we state consider stars.

Theorem 2.5 (Theorem 3.2 in [6]). *Let $k \geq 2$ and n be integers with k even and such that $n \geq \max\left\{3, \frac{k^2}{4} + 1\right\}$. Then*

$$\mathrm{bal}(n, K_{1,k}) = n\left(\frac{k}{2} - 1\right) - \frac{k^2}{8} + \frac{k}{4}.$$

Theorem 2.6 (Theorem 4.1 in [6])**.** *Let n and k be positive integers such that $n \geq 4k$. Then*

$$\mathrm{ot}(n, K_{1,k}) = \begin{cases} \left\lfloor \frac{(k-1)}{2} n \right\rfloor, & \text{for } k \leq 3, \\ (k-2)n - \frac{k^2}{2} + \frac{3}{2}k - 1, & \text{for } k \geq 4. \end{cases}$$

The following is a simple lemma about amoebas that was observed in [5]. We give the proof for completeness.

Lemma 2.7 (Lemma in [5])**.** *Let G be an amoeba without isolated vertices. Then $\delta(G) = 1$ and, for every k, $1 \leq k \leq \Delta(G)$, there is a vertex v in G with* $\deg(v) = k$.

Proof. Let G be an amoeba. Consider K_n where n sufficiently large and let H be a copy of G such that H and G are vertex disjoint in K_n. Let v be a vertex in G such that $\deg(v) = \Delta(G)$. Suppose u is the vertex in H to which v is to arrive via edge-replacement.

So initially $\deg(u) = 0$ and clearly the first time an edge is replaced to be incident with u, $\deg(u) = 1$ — but that means that in G there is a vertex of degree equal to 1. Now as the process of edge-replacements that carry v to u continues, then every time the degree of u can only increase by 1. So all the numbers between 1 and $\Delta(G)$ are present as degrees of u along the process, and for each k, $1 \leq k \leq \Delta$, there must therefore be a vertex in G of degree k. ■

Finally, we require the following definitions as described in [6]. Let t and n be integers with $1 \leq t < n$. A 2-edge-coloured complete graph K_n is said to be of *type-A*(t) if the edges of one of the colours induce a complete graph K_t, and it is of *type-B*(t) if the edges of one of the colours induce a complete bipartite graph $K_{t,n-t}$. A *type-A*(t) *colouring* (respectively, *type-B*(t) *colouring*) of K_n is called *balanced* if the number of red edges equals the number of blue edges. The following lemma is used here when we consider whether the graph C_4 is omnitonal and K_3 strongly balanceable.

Lemma 2.8 (Lemma 3.1 and Lemma 3.2 in [7], Lemma 2.3 in [6])**.**

1. *For infinitely many positive integers n, we can choose $t = t(n)$ in a way that the type-A(t) colouring of K_n is balanced.*
2. *For infinitely many positive integers n, we can choose $t = t(n)$ in a way that the type-B(t) colouring of K_n is balanced.*

We end this section with the following observations.

Observation 2.9. *A type-B(t) balanced colouring prevents an (r, b)-colouring of $C_4 = K_{2,2}$ where $r + b = 4$ and where both r and b are odd, showing that C_4 is not omnitonal.*

A type-A(t) balanced colouring with red edges forming the induced K_t prevents a $(2, 1)$-coloured K_3, showing that K_3 is not strongly balanced.

Observation 2.10. *It is a well-known simple fact that the only graphs on at least three edges containing no two independent edges are K_3 and $K_{1,k}$ for $k \geq 3$. Therefore if a graph on $n \geq 4$ vertices has at least n edges then it must have at least one pair of independent edges.*

2.2. New results

We now give some direct proofs of results which were not covered by the above theorems.

Theorem 2.11. *For $n \geq 10$, $\mathrm{ot}(n, K_{1,3} \cup K_2) = n$.*

Proof. *Lower bound.* For $n \geq 10$ we have to show a colouring with $\min\{|R|, |B|\} = n$ and some $r, b \geq 0$ such that $r + b = 4$ but no copy of $K_{1,3} \cup K_2$ has r red edges and b blue edges.

So, for $n \geq 10$, any colouring in which the red colour forms a 2-factor in K_n avoids a red $K_{1,3}$, hence avoids a red $K_{1,3} \cup K_2$. Observe that, for $n = 9$, a colouring in which the red edges form a clique K_5 and the rest are blue avoids a red $K_{1,3} \cup K_2$ but has 10 red edges hence the restriction to $n \geq 10$ is necessary.

Upper bound. We have to show that, for $n \geq 10$, any colouring of $E(K_n)$ in which $\min\{|R|, |B|\} \geq n+1$ contains an (r, b)-coloured copy of $K_{1,3} \cup K_2$ for any choice of $r, b \geq 0$ such that $r + b = 4$.

Case 1. The existence of a $(4, 0)$-coloured copy (the proof for a $(0, 4)$-coloured copy follows by symmetry). Clearly as $|R| \geq n + 1$ there is a vertex v with $\deg_{red}(v) \geq 3$. Let X be the subgraph of K_n induced by the vertex-set $V(K_n) \backslash N_{red}[v]$. We consider three possible cases:

(i) $\deg_{red}(v) \geq 5$. If there is a red edge in $V(K_n) \backslash \{v\}$ then clearly we have red $K_{1,3} \cup K_2$, because we always have at least three vertices in $N_{red}(v)$ disjoint from the vertices of this red edge. Hence the only red edges in $E(K_n)$ are those incident with v and it follows that $|R| \leq n - 1$, a contradiction.

(ii) $\deg_{red}(v) = 4$. Clearly $|X| \geq 5$. If there is a red edge from $N_{red}(v)$ to X we are done. So all red edges are contained in $N_{red}[v]$. But then, $|N_{red}[v]| = 5$ hence $|R| \leq 10 < n + 1$ since $n \geq 10$.

(iii) $\deg_{red}(v) = 3$. Clearly we may assume that all edges in X are blue otherwise we are done. Also $|X| = n - 4 \geq 6$. Since no vertex has red degree at least four (otherwise we are back to Case (ii), it follows that the four vertices in $N_{red}[v]$ are incident with at most 9 red edges altogether, a contradiction since $n \geq 10$.

Case 2. The existence of a $(3, 1)$-coloured copy (the proof for a $(1, 3)$-coloured copy follows by symmetry). In fact we will show the existence of $(3, 1)$-coloured copy with a red $K_{1,3}$ and a blue K_2. As before, let v be a vertex with $\deg_{red}(v) \geq 3$.

(i) $\deg_{red}(v) \geq 5$. If there is a blue edge in the subgraph of K_n induced by $V(K_n)\backslash\{v\}$, then clearly we have a red $K_{1,3}$ union a blue K_2, because we always have at least three vertices in Nred(v) disjoint from the vertices of this blue edge.

Hence the only blue edges in $E(Kn)$ are those incident with v and it follows $|B| \leq n-6$, a contradiction.

(ii) $\deg_{red}(v) = 4$. Clearly, we can assume that all edges in X are red otherwise we are done by taking a blue edge in X with a red $K_{1,3}$ in $N_{red}[v]$. Moreover $|X| = n-5 \geq 5$, implying X is a complete red graph on at least five vertices.

If there is a blue edge from $N_{red}[v]$ to X we are done as we can take this blue edge and a vertex-disjoint red $K_{1,3}$ in X.

If there is a blue edge in $N_{red}(v)$ we are done as we can take this blue edge and a red $K_{1,3}$ in X. So no blue edges are possible, a contradiction to $|B| \geq n+1$.

(iii) $\deg_{red}(v) = 3$. Clearly we may assume that all edges in X are red otherwise we are done. Also $|X| = n-4 \geq 6$.

If there is a blue edge in $N_{red}(v)$ we are done by taking it with the red $K_{1,3}$ from X. If there is a blue edge e from $N_{red}[v]$ to X we can still take a red $K_{1,3}$ in X which is vertex-disjoint from the blue edge e and we are done. Hence no blue edges are possible, a contradiction to $|B| \geq n+1$.

Case 3. The existence of a $(2,2)$-coloured copy. We will show the existence of $K_{1,3}$ with two red edges and one blue edge and a vertex-disjoint blue K_2. Suppose first that there is no vertex v incident with two red edges and a blue edge. Then in each vertex, either all edges are blue, or all edges except one are blue, or all edges are red.

Suppose there is a vertex v with $\deg_{blue}(v) = n-1$. Since there is a vertex u with $\deg_{red}(v) \geq 3$, it follows that in u there is the required coloured $K_{1,3}$.

Suppose there is a vertex v with $\deg_{red}(v) = n-1$. Then there must be a red edge in the subgraph of K_n induced by $V(K_n)\backslash v$, say $e = yz$. Then $\deg_{red}(y) = n-1$, otherwise we are done. But that forces all vertices in $V(K_n)$ to have red-degree at least 2. Hence all edges are red, a contradiction.

The only case that remains is that all red edges are vertex disjoint but then $|R| \leq n/2$ a contradiction.

So assume that there is a coloured copy of $K_{1,3}$ with two red edges va and vb and a blue edge vc. Clearly all edges in Y, the subgraph of K_n induced by $V(K_n)\backslash\{v,a,b,c\}$, must be red otherwise we are done. Also Y has $n-4 \geq 6$ vertices.

If there is a blue edge e_1 from a or from b to $z \in V(Y)$, we are done by taking $K_{1,2}$ centered on z with e_1 and vc. So all blue edges are incident with c with at most one more possible blue edge ab, making the total number of blue edges at most n a contradiction. ■

Theorem 2.12. *For* $n \geq 10$, $\mathrm{bal}(n, 4K_2) = n - 1$.

Proof. *Lower bound.* Pick a vertex v in K_n and colour all the edges incident to it red and the rest of the edges blue. Clearly (as there are no two independent edges of red colour) there is no balanced $4K_2$ hence $\mathrm{bal}(n, 4K_2) \geq n - 1$ always holds.

Upper bound. We have to show that for $n \geq 10$ and any 2-edge colouring of K_n with $\min\{R, B\} \geq n$, there exists a balanced $4K_2$.

We may assume without loss of generality that we have K_n with its edges coloured red and blue such that there are at least n edges of each colour. Hence we may assume without loss of generality that there are two independent red edges ab and cd. Let us remove the vertices a, b, c, d and all edges incident to them to leave a graph X isomorphic to K_{n-4} with its edges coloured red and blue. If there are two independent blue edges in this remaining graph then we are done. So we may assume that no such two edges exist in X, which implies that there are at most $n - 5$ blue edges in X, and hence at least five blue edges among the deleted edges. Therefore, there exists at least one blue edge joining a vertex u in X to one of the vertices a, b, c or d, say a, without loss of generality. We then have two cases.

Case 1. There are no blue edges in X. This implies that besides au there are at least four other blue edges among the deleted edges. This gives us two subcases. Either (i) there is another blue edge joining vertex u in X to one of $\{b, c, d\}$; or (ii) there is a blue edge in the complete graph induced by $\{a, b, c, d\}$ and another independent edge joining u to one of these vertices. We consider these two subcases separately.

Subcase (i) We may assume without loss of generality that v in X is joined to b with a blue edge. If we remove u, v and all the edges incident to them in X we are left with K_{n-6} in which all the edges are red. This graph has two independent red edges since $\binom{n-6}{2} \geq n - 6$ because $(n - 7)/2 > 1$ since we are assuming that $n \geq 10$. These two red edges, together with the two blue edges au and bv form the required balanced $4K_2$.

Subcase (ii) If we remove u and its edges from X we have K_{n-5} whose edges are all red, and hence there are two independent red edges in this graph for $n \geq 8$, and again we can form the balanced $4K_2$ using these two edges and the two independent blue edges.

Case 2. There is at least one blue edges, say ux, in X. Therefore there must be a blue edge e among the deleted blue edges which is not incident to u, since there are four such deleted edges incident to u, and at least another four blue edges. If e is incident to x then there must be another edge among the deleted ones not incident to x because again there are at most three remaining deleted edges incident to x. So there are two independent blue edges, one of which may be ux. Again, if we remove u, x and all edges incident to these vertices from X, then we have two independent red edges in this remaining graph if $\binom{n-6}{2} - (n-5) \geq n-6$, that is, $n^2 - 15n + 52 \geq 0$, which is true when $n \geq 10$. These two red edges, together with the two independent blue edges, form the required balanced $4K_2$. ■

Theorem 2.13. *For $n \geq 8$, $\mathrm{bal}(n, 2K_2 \cup K_{1,2}) = 1$.*

Proof. *Lower bound.* Colour one edge of K_n red and the rest blue. Clearly no copy of $2K_2 \cup K_{1,2}$ can have two red edges.

Upper bound. Clearly there must be $K_2 \cup K_{1,2}$ with the two edges of $K_{1,2}$ of distinct colour, say without loss of generality, there are 5 vertices a, b, c, d, e such that ab, is red cd is red and de is blue (where the vertex d is the centre of $K_{1,2}$). Let $X = V \backslash \{a, b, c, d, e\}$, where, since $n \geq 8$, $|X| \geq 3$. Let $u_1, \ldots, u_{n-5}$ be the vertices of X.

If there is a blue edge in X then we are done. So let us assume all edges of X are red.

If there is a blue edge from either a or b to X, say to u_1, then we are done by taking the edge au_1 or bu_1, the edge u_2u_3 and $K_{1,2}$ on the vertices $\{c; d, e\}$ (where the notation $\{p; q, r, s, \ldots\}$ denotes a star with centre p). So let us assume all edges from a and b to x are red.

If there is a blue edge from d to X, say to u_1 we are done by the edges ab, u_2u_3 and $K_{1,2}$ on the vertices $\{e; d, u_1\}$. So let us assume all edges from d to X are red.

If there is a blue edge from c to X, say to u_1 we are done by the edges ab, ed and $K_{1,2}$ on the vertices $\{c; u_1, u_2\}$. So let us assume all edges from c to X are red.

If there is a blue edges from e to X say to u_1 we are done by the edges ab, u_2u_3 and $K_{1,2}$ on the vertices $\{d; e, u_1\}$. So let us assume all edges from e to X are red.

If edge ce is blue we are done by the edges ab, u_1u_2 and $K_{1,2}$ on the vertices $\{d; e, c\}$. So assume edge ce is red.

If either edge ae or be is blue we are done by bu_1, cu_2 and $K_{1,2}$ on the vertices $\{a; e, d\}$ in the case ae is blue, or au_1, cu_2 and $K_{1,2}$ on the vertices $\{b; e, d\}$ in the case be is blue. So let us assume these two edges are red.

If there is a blue edge from c to either a or b, we are done by the edges ac, de and $K_{1,2}$ on the vertices $\{u_1; u_2, u_3\}$ or by bc, de and $K_{1,2}$ on the vertices $\{u_1; u_2, u_3\}$. So let us assume both these edges are red.

Then this implies that either da or db is blue, as there must be at least two blue edges. If da is blue we are done by bu_1, cu_2 and $K_{1,2}$ on the vertices $\{e; d, a\}$, while if db is blue we are done by au_1, cu_2 and $K_{1,2}$ on the vertices $\{e; d, b\}$. ■

Theorem 2.14. *For $n \geq 7$,* $\mathrm{bal}(n, K_2 \cup K_3) = 3$.

Proof. *Lower bound.* Consider a coloring of K_n in which there is a red coloured K_3 and the rest of the edges are blue. Clearly there is no $K_2 \cup K_3$ with exactly two red edges hence $\mathrm{bal}(n, K_2 \cup K_3) \geq 3$.

Upper bound. Suppose now $n \geq 7$ and we have an (r, b)-colouring of K_n with $\min\{R, B\} \geq 4$. We will show the existence of a balanced $K_2 \cup K_3$. Without loss of generality, there must be a K_3 on vertices $\{a, b, c\}$ with two red edges ab and bc and one blue edge ac.

Let $X = V \backslash \{a, b, c\}$. Then $|X| \geq 4$ and let $u_1, \ldots, u_{n-3}$ be the vertices of X.

Now, if there is a blue edge in X we are done. So we may assume that all edges in X are red.

Suppose there are at least two blue edges from b to X say to u_1 and u_2. Then we are done by K_3 induced on $\{b, u_1, u_2\}$ and the edge u_3u_4.

If there is one blue edge from b to X say to u_1 then we are done by K_3 induced on $\{b, u_1, u_2\}$ and the edge ac.

So we may assume b is connected to X by red edges only. Suppose there are at least two blue edges from a to X, say to u_1 and u_2. Then we are done by K_3 induced on $\{a, u_1, u_2\}$ and the edge bc.

Finally suppose there are at least two blue edges from c to X, say to u_1 and u_2. We are done by K_3 induced on $\{c, u_1, u_2\}$ and the edge ab.

So there is at most one blue edge from a to X and one from c to X. But then the total number of blue edges is at most 3, a contradiction.

The colouring with $\{a, b, c\}$ a blue triangle and all other edges red shows that $\mathrm{bal}(n, K_2 \cup K_3) \geq 3$ ■

Theorem 2.15. *For $n \geq 9$,* $\mathrm{bal}(n, K_{1,3} \cup K_2) = n - 1$.

Proof. *Lower bound.* Clearly $\mathrm{bal}(n, K_{1,3} \cup K_2) \geq n - 1$: consider K_n and choose a vertex v. Colour all edges incident with v red and all other edges blue. No balanced copy of $K_{1,3} \cup K_2$ exists.

Upper bound. Suppose now that $\min\{|R|, |B|\} \geq n$. We have to show the existence of a balanced copy of $K_{1,3} \cup K_2$ (two red edges and two blue edges). Since $n \geq 9$ and $\min\{|R|, |B|\} \geq n$, it follows that there are at least two independent red, respectively blue, edges.

Clearly there is a vertex v incident to both red and blue edges. We assume, without loss of generality, that $\deg_{red}(v) \geq \deg_{blue}(v) \geq 1$. Clearly $\deg_{red}(v) \geq (n-1)/2 \geq 4$. We consider the following cases:

Case 1. $\deg_{blue}(v) = 1$ with the blue edge $e = vw$. If there is a blue edge $e^* = xy$ where both x and y are in $N_{red}(v)$ then since $\deg_{red}(v) \geq 4$, we can take $e = vw$ and two red edges incident with v but vertex-disjoint from x and y, and add e^* to get a balanced $K_{1,3} \cup K_2$. Otherwise, if there is no such edge as e^* then all blue edges are incident with w, which would imply that $|B| \leq n - 1$, a contradiction.

Case 2. $\deg_{blue}(v) \geq 2$ — this forces all edges induced in $N_{red}(v)$ to be blue, for otherwise there is a red edge say $e = xy$ in $N_{red}(v)$ and a red edge e^* incident with v but not with x nor y, and we take two blue edges incident to v together with e and e^* to get a balanced $K_{1,3} \cup K_2$. So, all edges induced in $N_{red}(v)$ are blue. We take $e = xy$ to be such a blue edge and since $\deg_{red}(v) \geq 4$ we have two red edges va and vb which are vertex disjoint from x and y. We now take one blue edge incident with v and the edges va and vb together with the edge $e = xy$ to get a balanced $K_{1,3} \cup K_2$. ■

Theorem 2.16. *For* $n \geq 5$, $\mathrm{sbal}(n, K_{1,3}) = n - 1$.

Proof. *Lower bound.* We consider K_n and fix a vertex v in it, colouring all edges incident with v blue and all other edges red. Clearly no $K_{1,3}$ with two blue edges and one red edge exists, and hence $\mathrm{sbal}(n, K_{1,3}) \geq n - 1$.

Upper bound. We now prove that if $\min\{|R|, |B|\} \geq n$ (which is possible as $n \geq 5$), there must be a $K_{1,3}$ with two blue edges and one red edge, as well as one with two red edges and one blue edge.

So let $n \geq 4$ and consider the blue edges. Since there are at least n blue edges there must be a vertex v incident with at least two blue edges. Now if v is also incident to a red edge we have a copy of $K_{1,3}$ with two blue edges and one red edge.

Hence all edges incident with v are blue and hence all vertices of K_n are incident to at least one blue edge. But as there are at least n blue edges there must be another vertex u already incident to v such that uv is blue, but also incident with another vertex w with uw coloured blue. Now if u is also incident to a red edge we are done with $K_{1,3}$ centered at u having two blue edges and one red edge. Otherwise all edges incident to u are blue and it follows that all vertices of K_n have degree at least 2 in the blue graph. But there must be a red edge incident with some vertex z, and we have $K_{1,3}$ centred at z with two blue edges and one red edge. The case for $K_{1,3}$ with two red and one blue edge follows by symmetry atrting with a $K_{1,3}$ with at least two red edges. ■

Finally we shall find this simple observation about graphs on three edges useful for results given in the tables.

Observation 2.17. *If G has three edges, then G is balanceable.*

Proof. Suppose $\min\{|R|, |B|\} \geq 1$ then clearly there is a vertex v in K_n incident with a red edge say vx and blue edge say vy. So no matter how we complete this red and blue coloured $K_{1,2}$ to any of the graphs on 3 edges except for $3K_2$, we have a balanced copy of G. If $G = 3K_2$ and $n \geq 6$ then as before there is red and blue coloured $K_{1,2}$ with vertices $\{v; x, y\}$. Since $n \geq 6$ there is an edge ab disjoint from $\{v, x, y\}$ hence without loss of generality we may assume that edge vx is red and edge ab is blue. Since $n \geq 6$, every edge disjoint from $\{a, b, v, x\}$ gives a balanced $3K_2$. Hence G is balanceable with $\mathrm{bal}(n, G) = 0$ for $n \geq 6$. ■

3. The Triple Property and Union of Bipartite Graphs

It is well known [6] that there are bipartite graphs which are not balanceable however the following theorem shows an interesting property: if G and H are any bipartite graphs with $e(G) = e(H)$ then $G \cup H$ is balanceable. The example following this theorem shows a direct use of it in the case where $G = 2C_{4t+2}$, while it is known that C_{4t+2} is not a balanceable graph [6].

We next develop the triple property which together with Theorem 3.1 allows us to compute $\mathrm{sbal}(n, (2t-1)K_2)$, $\mathrm{bal}(n, 2tK_2)$ and $\mathrm{bal}(n, (2t+1)K_2)$ in one stroke.

Theorem 3.1. *Suppose G and H are bipartite graphs such that $e(G) = e(H)$ and that $n \geq |G| + |H| + R(G, H)$ where $R(G, H)$ is the Ramsey number for a red copy of G or a blue copy of H. Then there exists n_0 such that for $n \geq n_0$, $\mathrm{bal}(n, G \cup H) \leq \max\{\mathrm{ex}(n, G), \mathrm{ex}(n, H)\}$.*

Proof. Since G and H are bipartite graphs and it is well-known that $\mathrm{ex}(n, G)$ and $\mathrm{ex}(n, H)$ are sub-quadratic [13], it follows that, for n large enough, if $\min\{|R|, |B|\} > \max\{\mathrm{ex}(n, G), \mathrm{ex}(n, H)\}$ then a 2-edge-coloured copy of K_n contains both a red copy of G and a blue copy of H, [4, 6, 14].

Let G_1 be the red copy of G and H_1 be the blue copy of H. If G_1 and H_1 are vertex-disjoint we are done, having a balanced $G \cup H$ since $e(G) = e(H)$.

Otherwise let $S = V(G_1) \cup V(H_1)$. Clearly $|S| < |V(G)| + |V(H)|$. Let $X = V(K_n) \backslash S$. Then $|X| \geq R(G, H)$ and hence in the induced colouring on X there is either a red copy of G or a blue copy of H (or both).

If there is a blue copy of H take it with G_1, and if there is a red copy of G take it with H_1 and in both cases we get a balanced $G \cup H$. ■

We now give an example of the applicability of this theorem.

Example 3.2. An illustration of Theorem 3.1.

It is known that C_{4n+2} is not balanceable by Remark 2.9 in [6]. Also $R(C_{4t+2}, C_{4t+2}) = 6t+2$ by a result in [12]. Also Turan numbers for even cycles are bounded above by $\mathrm{ex}(n, C_{2k}) \leq (k-1)n^{1+1/k} + 16(k-1)n$, a result proved in [15].

So applying Theorem 3.1, together with these facts, we get the following.

Let $G = 2C_{4t+2}$. Then G is balanceable and for $n \geq 14t + 6$,

$$\mathrm{bal}(n, G) \leq \mathrm{ex}(n, C_{4t+2}) \leq (4t + 1)n^{1+1/(4t+2)} + 16(4t + 1)n.$$

We define the following property. Let G, H and F be three graphs such that $e(G)$ is odd, H is obtained from G by adding a new edge, and F is obtained from H by adding another new edge. We say that (G, H, F) has *the triple property* if there is some n_0 such that for $n \geq n_0$, $\mathrm{sbal}(n, G) = \mathrm{bal}(n, H) = \mathrm{bal}(F, n)$.

Theorem 3.3. *Let $G = (2t-1)K_2$, $H = 2tK_2$ and $F = (2t+1)K_2$. Then (G, H, F) has the triple property.*

Proof. Clearly $\mathrm{bal}(n, F) \geq \mathrm{bal}(n, H)$, for suppose we have a colouring of $E(K_n)$ with $\min\{|R|, |B|\} \geq \mathrm{bal}(n, F) + 1$. Then by definition there is a balanced (r, b)-coloured copy of F with either $r = t+1$ and $b = t$ or $r = t$ and $b = t+1$. Ignoring an edge with the most frequent colour gives a balanced colouring of H.

Conversely, suppose we have a colouring of $E(K_n)$ with $\min\{|R|, |B|\} \geq \mathrm{bal}(n, H) + 1$. Then by definition there is a balance (r, b)-coloured copy of H with $r = b = t$. If we take $n > n_0 = 4t + 2$, the edges of the balanced copy of H cover $4t$ vertices, but there is at least one further edge independent from all these $2t$ edges, and no matter what the colour of this edge is, we can add it to H to get a balanced F. Therefore $\mathrm{bal}(n, F) \leq \mathrm{bal}(n, H)$.

We now need to show that $\mathrm{bal}(n, H) \geq \mathrm{sbal}(n, G)$. Suppose we have a colouring of $E(K_n)$ with $\min\{|R|, |B|\} \geq \mathrm{bal}(n, H) + 1$. Then by definition there is a balanced copy of H, and we can drop either a red or a blue edge to get a balanced copy of G.

For the converse consider $n_0 = 5t + 2 = n(G) + t + 4$. Suppose we have a colouring of $E(K_n)$ with $\min\{|R|, |B|\} \geq \mathrm{sbal}(n, G) + 1$. Then by definition there is either a $(t, t-1)$-coloured copy of G or a $(t-1, t)$-coloured copy of G.

Consider a $(t, t-1)$-coloured copy of G. Let the t red edges be $e_1, \ldots, e_t$ and the $t-1$ blue edges be $f_1, \ldots, f_{t-1}$. Let S be the complete graph induced by $V(K_n) \backslash V(G)$; clearly $|V(S)| \geq t + 4$ since $n_0 \geq 5t + 2$. Clearly all edges in S must be red for otherwise we can add a blue edge to get a balanced copy of H.

If there is a blue edge not incident with any of $f_1, \ldots, f_{t-1}$, then either it is adjacent to two red edges of $e_1, \ldots, e_t$, or there is a blue edge adjacent with an edge from $e_1, \ldots, e_t$ and an edge from S. In the first case we drop these two red edges and add the blue edge and two independent red edges from S since $|S)| \geq t + 4$. In the second case we drop the red edge and replace it by a red edge from S not incident with the blue edge and add also the blue edge to get balanced H.

So we know that S contains only red edges, therefore together with $e_1, \ldots, e_t$, we have a graph L on at least $3t+4$ vertices, all of whose edges are red.

So we may conclude that there are remaining blue edges and that each is adjacent to at least one blue edge from $f_1, \ldots, f_{t-1}$.

Now take an $(t-1, t)$-coloured copy of G. The t blue edges are incident with at most t vertices from L, leaving in L at least $2t+4$ vertices from which we can choose t red edges not adjacent with the t blue edges of G and we get a balanced H. ∎

Theorem 3.4. *For* $n \geq 7t-1$, $\mathrm{sbal}(n, (2t-1)K_2) = \mathrm{bal}(n, 2tK_2) = \mathrm{bal}(n, (2t+1)K_2) = \mathrm{ex}(n, tK_2) = \binom{t-1}{2} + (t-1)(n-t+1)$.

Proof. It suffices to prove that $\mathrm{bal}(n, 2tK_2) = \mathrm{ex}(n, tK_2)$ for $n = 7t-1$ (by the former triple property all other equality signs were proved).

From the theorem above we know that $\mathrm{bal}(n, 2tK_2) \leq \mathrm{ex}(n, tK_2)$. However in this case the reverse inequality holds as well because we take K_n and colour its edges with $|R| = \mathrm{ex}(n, tK_2)$ forming the extremal graph for tK_2 (not having tK_2).

If there is a balanced $2tK_2$ it must contains a red tK_2 which is a contradiction.

We observe an old result of Erdős and Gallai [11]

$$\begin{aligned}\mathrm{ex}(n, tK_2) &= \max\left\{\binom{2t-1}{2}, \binom{t-1}{2} + (t-1)(n-t+1)\right\}\\ &= \binom{t-1}{2} + (t-1)(n-t+1)\end{aligned}$$

for $n \geq \frac{7t-6}{2}$.

Also, we observe that $R(tK_2, tK_2) = 3t-1$ by the classical result in [10].

Hence putting all these facts together we got the last required equality for $n \geq 2|V(G)| + R(G, G) = 4t + 3t - 1 = 7t - 1$. ∎

One question which this proof raises is whether these equalities hold for n much less that $7t-1$.

4. Tables

We can now give, in this section, the values of $\mathrm{ot}(n, G)$, $\mathrm{bal}(n, G)$ and $\mathrm{sbal}(n, G)$, when they exist, for all graphs G on at most four edges.

Graphs	Amoeba	Omnitonal	$\mathrm{ot}(n,G)$	Valid n	Comments
$4K_2$	Y	Y	$\mathrm{ex}(n,G)$	$n \geq n_0$	Theorem 2.3
$2K_2 \cup K_{1,2}$	Y	Y	$\mathrm{ex}(n,G)$	$n \geq n_0$	Theorem 2.3
$2K_{1,2}$	Y	Y	$\mathrm{ex}(n,G)$	$n \geq n_0$	Theorem 2.3
$K_2 \cup P_4$	Y	Y	$\mathrm{ex}(n,G)$	$n \geq n_0$	Theorem 2.3
P_5	Y	Y	$\mathrm{ex}(n,G)$	$n \geq n_0$	Theorem 2.3
$K_{1,3}$ with extended leaf	Y	Y	$\mathrm{ex}(n,G)$	$n \geq n_0$	Theorem 2.3
$K_2 \cup K_3$	N	N		—	Theorem 2.1
C_4	N [1]	N		—	Lemma 2.8
$K_{1,3} \cup K_2$	N [1]	Y	n	$n \geq 10$	Theorem 2.11
$K_{1,4}$	N [1]	Y	$2n-3$	$n \geq 16$	Theorem 2.6
$K_3 + e$	Y	N		—	Theorem 2.1
$K_{1,3}$	N [1]	Y	n	$n \geq 12$	Theorem 2.6
P_4	Y	Y	$\mathrm{ex}(n,G)$	$n \geq n_0$	Theorem 2.3
K_3	N [1]	N		—	Theorem 2.1
$3K_2$	Y	Y	$\mathrm{ex}(n,G)$	$n \geq n_0$	Theorem 2.3
$P_3 \cup K_2$	Y	Y	$\mathrm{ex}(n,G)$	$n \geq n_0$	Theorem 2.3
P_3	Y	Y	$\mathrm{ex}(n,G)$	$n \geq 3$	Theorem 2.3
$2K_2$	Y	Y	$\mathrm{ex}(n,G)$	$n \geq n_0$	Theorem 2.3

[1] By Lemma 2.7.

Table 1. Amoebas and Omnitonal graphs on at most four edges.

Graphs	$\mathrm{bal}(n,G)$	Valid n	Comments
$4K_2$	$n-1$	$n \geq 10$	Theorem 2.12
$2K_2 \cup K_{1,2}$	1	$n \geq 8$	Theorem 2.13
$2K_{1,2}$	1	$n \geq 7$	[1]
$K_2 \cup P_4$	1	$n \geq 7$	[1]
P_5	1	$n \geq 6$	[1]
$K_{1,3}$ with extended leaf	1	$n \geq 7$	[1]
$K_2 \cup K_3$	3	$n \geq 7$	Theorem 2.14
C_4	1	$n \geq 4$	[1]
$K_{1,3} \cup K_2$	$n-1$	$n \geq 9$	Theorem 2.15
$K_{1,4}$	$n-1$	$n \geq 5$	Theorem E
$K_3 + e$	1	$n \geq 5$	[1]
$K_{1,3}$	0		Observation 2.17
P_4	0		Observation 2.17
K_3	0		Observation 2.17
$3K_2$	0		Observation 2.17
$P_3 \cup K_2$	0		Observation 2.17

[1] The proofs are in nature very similar to the proof of Theorem 2.13 and are left to the interested reader to verify.

Table 2. Balanced graphs on at most four edges.

Graphs	Strong balanced	sbal(n, G)	Valid n	Comments
$K_{1,3}$	Y	$n-1$	$n \geq 4$	Theorem 2.16
P_4	Y	1	$n \geq 7$	Theorem 2.4
K_3	N			Lemma 2.8
$3K_2$	Y	$n-1$	$n \geq 7$	Theorem 3.4
$P_3 \cup K_2$	Y	1	$n \geq 7$	Theorem 2.4

Table 3. Strongly balanced graphs on at most four edges.

5. Conclusion

In this paper, by computing the values of bal(n, G), sbal(n, G) and ot(n, G) for all graphs G on at most four edges we have tried to convey the flavour of the results in [6] and the techniques used to obtain them. We have also tried to obtain some new techniques which could shed more insight on these problems. We hope that this paper will be an invitation to the interested reader to delve into [6] for a more comprehensive treatment of balanceable and omnitonal graphs.

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