Discussiones Mathematicae
Graph Theory 42 (2022) 727–746
https://doi.org/10.7151/dmgt.2341

NOWHERE-ZERO UNORIENTED 6-FLOWS ON CERTAIN TRIANGULAR GRAPHS

Fan Yang

School of Physical and Mathematical Sciences
Nanjing University of Technolgy
Nanjing, Jiangsu 211816, China

e-mail: fanyang_just@163.com

Liangchen Li

Department of Mathematics
Luoyang Normal University
Luoyang 471022, China

and

Sizhong Zhou

Department of Science
Jiangsu University of Science and Technology
Zhenjiang, Jiangsu 212003, China

e-mail: zsz_cumt@163.com

Abstract

A nowhere-zero unoriented flow of graph G is an assignment of non-zero real numbers to the edges of G such that the sum of the values of all edges incident with each vertex is zero. Let k be a natural number. A nowhere-zero unoriented k-flow is a flow with values from the set $\{\pm 1, \ldots, \pm(k-1)\}$, for short we call it NZ-unoriented k-flow. Let H_1 and H_2 be two graphs, $H_1 \oplus H_2$ denote the 2-sum of H_1 and H_2, if $E(H_1 \oplus H_2) = E(H_1) \cup E(H_2)$, $|V(H_1) \cap V(H_2)| = 2$, and $|E(H_1) \cap E(H_2)| = 1$. A triangle-path in a graph G is a sequence of distinct triangles $T_1, T_2, \ldots, T_m$ in G such that for $1 \leq i \leq m$, $|E(T_i) \cap E(T_{i+1})| = 1$ and $E(T_i) \cap E(T_j) = \emptyset$ if $j > i+1$. A triangle-star is a graph with triangles such that each triangle having one common edges with other triangles. Let G be a graph which can be partitioned into some triangle-paths or wheels $H_1, H_2, \ldots, H_t$ such that $G = H_1 \oplus H_2 \oplus \cdots \oplus H_t$. In this paper, we prove that G except a triangle-star admits an NZ-unoriented 6-flow. Moreover, if each H_i is a triangle-path, then G except a triangle-star admits an NZ-unoriented 5-flow.

Keywords: nowhere-zero k-flow, triangle-tree, triangle-star, bidirected graph.
2010 Mathematics Subject Classification: 05C21.

1. Introduction

All graphs in this paper are finite and undirected without loops, possibly with parallel edges. A *nowhere-zero k-flow* in a graph with orientation is an assignment of an integer from $\{\pm 1, \ldots, \pm(k-1)\}$ to each of its edges such that Kirchhoff's law is respected, that is, the total incoming flow is equal to the total outgoing flow at each vertex. As noted in [8], the existence of a nowhere-zero flow of a graph G is independent of the choice of the orientation. Nowhere-zero flows in graphs were introduced by Tutte [12] in 1949. A great deal of research in the area has been motivated by Tutte's 5-Flow Conjecture which asserts that every 2-edge-connected graph admits a nowhere-zero 5-flow. In 1983, Bouchet [5] generalized this concept to bidirected graphs. A *bidirected graph G* is a graph with vertex set $V(G)$ and edge set $E(G)$ such that each edge is oriented as one of the four possibilities in Figure 1.

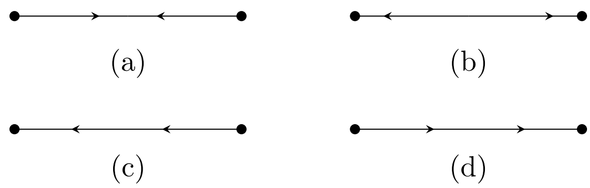


Figure 1. Orientations of edges in bidirected graph

An edge with orientation as (a) (respectively, (b)) is called an *in-edge* (respectively, *out-edge*). An edge that is neither an in-edge nor an out-edge is called an *ordinary edge* as in (c) or (d). An integer-valued function f on $E(G)$ is a nowhere-zero bidirected k-flow if for every $e \in E(G)$, $0 < |f(e)| < k$, and at every vertex v, the sum of values of f on all coming-in edges incident with v is equal to the sum of values of f on all going-out edges incident with v. The following conjecture, posed by Bouchet [5] and known as Bouchet's 6-flow conjecture, is one of the most important problems on nowhere-zero integer flows in bidirected graphs.

Conjecture 1 [5]**.** *If a bidirected graph admits a nowhere-zero k-flow for some positive integer k, then it admits a nowhere-zero 6-flow.*

In [5], Bouchet showed that the value 6 in this conjecture is best possible. Raspaud and Zhu [11] proved that, if a 4-edge-connected bidirected graph admits a nowhere-zero integer flow, then it admits a nowhere-zero 4-flow, which is best possible. In [13], Zyka proved the result in Bouchet's Conjecture is true if 6 is replaced by 30.

A *nowhere-zero unoriented flow* (also called *zero-sum flow*) in a graph is an assignment of non-zero integers to the edges of G such that the total sum of the assignments of all edges incident with any vertex of G is zero. A nowhere-zero unoriented k-flow in a graph G is an unoriented flow with flow values from the set $\{\pm 1, \ldots, \pm(k-1)\}$, for short we call it NZ-unoriented k-flow. The following conjecture is known as Zero-Sum Conjecture.

Conjecture 2 [1]**.** *If a graph G admits an NZ-unoriented flow, then it admits an NZ-unoriented 6-flow.*

Indeed an NZ-unoriented k-flow in G is exactly a nowhere-zero k-flow in the bidirected graph with underlying graph G such that each edge is an in-edge or out-edge. The following theorem shows the equivalence between Bouchet's Conjecture and Zero-Sum Conjecture.

Theorem 3 [2]**.** *Bouchet's Conjecture and Zero-Sum Conjecture are equivalent.*

There are many results about Zero-Sum Conjecture recently. Akbari, Daemi *et al.* [3] proved that Zero-Sum Conjecture is true for hamiltonian graphs, with 6 replaced by 12. Akbari *et al.* [2] proved that every r-regular graph ($r \geq 3$) admits an NZ-unoriented 7-flow. Moreover, Akbari *et al.* [4] showed that every r-regular graph, where $r \geq 3$ and $r \neq 5$, admits an NZ-unoriented 5-flow. Recently, Yang and Li [14] proved that every 5-regular graph admits an NZ-unoriented 6-flow.

In this paper we prove the existence of NZ-unoriented k-flow in certain triangular graphs.

Theorem 4. *Let $G = H_1 \oplus H_2 \oplus \cdots \oplus H_t$, where H_i is a triangle-path or a wheel. If G is not a triangle-star, then G admits an NZ-unoriented 6-flow. Moreover, if each H_i is a triangle-path, then G admits a NZ-unoriented 5-flow.*

We will prove this result in Section 3. In Section 2, we establish some lemmas which are crucial to our result.

2. Preliminaries

Given a subgraph H of G, we use G/H to denote the graph obtained from G by contracting all edges in H and deleting the resulting loops. Let n-path be a path with n edges. For a graph G and $X \subseteq E(G)$, we use $G - X$ denote a graph obtained from G by deleting edge set X. When $X = \{e\}$, we use $G - e$

for conveniences. For $u_i \in V(G)$ and each $i = 1, 2$, $G + u_1u_2$ means a new graph obtained from G by adding edge u_1u_2.

Akbari, Ghareghani [1] *et al.* deduce the following two results which present a geometric interpretation for graphs having an NZ-unoriented flow.

Theorem 5. *Suppose G is not a bipartite graph. Then G has an NZ-unoriented flow if and only if for any edge e of G, $G - e$ has no bipartite component.*

Lemma 6. *Let G be a 2-edge-connected bipartite graph. Then G has an NZ-unoriented 6-flow.*

A graph G is called *even* (respectively, *odd*), if its number of edges is even (respectively, odd). A *circuit* in G is a closed walk with no repeated edge. A *cycle* in this paper is a connected 2-regular graph. Two parallel edges form a cycle of length two. An n-cycle denotes a cycle with n edges. The following easy lemma will be useful.

Lemma 7 [2]**.** *Every even circuit admits an NZ-unoriented 2-flow.*

Let H_1 and H_2 be two graphs, $H_1 \oplus H_2$ denote the 2-sum of H_1 and H_2, if $E(H_1) \cup E(H_2) = E(H_1 \oplus H_2)$, $|V(H_1) \cap V(H_2)| = 2$, and $|E(H_1) \cap E(H_2)| = 1$. We define that $H_1 \oplus H_2 \oplus \cdots \oplus H_t \cong (H_1 \oplus H_2 \oplus \cdots \oplus H_{t-1}) \oplus H_t$. A triangle-path in a graph G is a sequence of distinct triangles $T_1, T_2, \ldots, T_m$ such that for $1 \le i \le m$, $|E(T_i) \cap E(T_{i+1})| = 1$ and $E(T_i) \cap E(T_j) = \emptyset$ if $j > i + 1$. For simplify, in the rest of this paper we use T^m to denote a triangle-path with m triangles. An n-triangle-star H is a graph with n triangles such that each triangle has one common edge with other triangles (see Figure 2). By definition, a triangle is a 1-triangle-star and K_4^- is a 2-triangle-star. A triangle-tree G is a graph which can be partitioned into some triangle-paths $H_1, H_2, \ldots, H_t$ such that $G = H_1 \oplus H_2 \oplus \cdots \oplus H_t$. By the definition of triangle-tree, we know that a triangle-star is also a special triangle-tree with each H_i as a triangle.

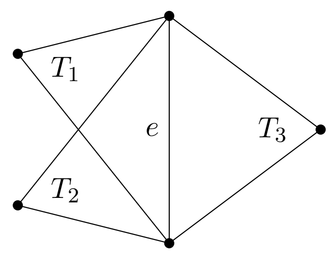


Figure 2. A 3-triangle-star with common edge e.

By Theorem 5, the following lemma is obvious.

Lemma 8. *A triangle-star does not admit an NZ-unoriented flow.*

Proof. Let G be a n-triangle-star. If $n = 1$, then G is a triangle which is an odd cycle. Clearly, G does not admit an NZ-unoriented flow. Assume that $n \geq 2$ and e_0 is the common edge of G. Then $G - e_0$ is a graph containing no odd cycles, that is a bipartite graph. By Theorem 5, G does not admit an NZ-unoriented flow. ■

Definition [11]**.** A *barbell* is a graph if it consists of two odd cycles that have exactly one common vertex, or two vertex-disjoint odd cycles together with a path with exactly one end-vertex on each of them and all internal vertices outside of them.

By applying Lemma 2.6 in [11], the following result is obvious.

Lemma 9. *Any barbell admits an NZ-unoriented* 3*-flow that assigns* 1 *or* -1 *to the edges on the odd cycles and* 2 *or* -2 *to all other edges.*

The following results are technical key to prove Theorem 4.

Lemma 10. *Let G admit an NZ-unoriented k-flow ($k \geq 4$) and $u_1, u_2 \in V(G)$. If there is a* 3*-path between u_1 and u_2 in G, then $G + u_1u_2$ admits a NZ-unoriented $(k+1)$-flow.*

Proof. Let f_1 be an NZ-unoriented k-flow of G. Since there is a 3-path between u_1 and u_2, there exists two vertices, say u_3, u_4, such that $u_2u_3, u_3u_4, u_4u_1 \in E(G)$. In this case, this 3-path plus the new added edge u_1u_2 is a 4-cycle $u_1u_2u_3u_4u_1$ of $G + u_1u_2$. In the rest of the proof, u_1u_2 means the new added edge. By Lemma 7, $u_1u_2u_3u_4u_1$ admits an NZ-unoriented flow f_2 such that $f_2(u_1u_4) = f_2(u_3u_2) = a$, $f_2(u_1u_2) = f_2(u_4u_3) = -a$ and $a \in \{\pm 1, \pm 2\}$. Let f be a function on $G + u_1u_2$ as follows: $f(e) = f_1(e)$ if $e \in E(G) \setminus \{u_1u_4, u_4u_3, u_2u_3\}$; $f(e) = f_2(e)$ if $e \in \{u_1u_2\}$; $f(e) = f_1(e) + f_2(e)$ if $e \in \{u_1u_4, u_4u_3, u_2u_3\}$. Clearly, if $f(e) \in \{\pm 1, \pm 2, \ldots, \pm k\}$ for each $e \in \{u_1u_4, u_4u_3, u_2u_3\}$, then f is an NZ-unoriented $(k+1)$-flow of $G + u_1u_2$. In order to satisfy that $f(e) \in \{\pm 1, \pm 2, \ldots, \pm k\}$, for $e \in \{u_1u_4, u_4u_3, u_2u_3\}$, we have the next observation (see the following table).

$f_1(e)$	$f_2(e)$	$f(e) = f_1(e) + f_2(e)$
1	$S_1 = S \setminus \{-1\}$	
-1	$S_1^- = S \setminus \{1\}$	
$k-1, -2$	$S_2 = S \setminus \{2\}$	$\pm 1, \ldots, \pm k$
$-(k-1), 2$	$S_2^- = S \setminus \{-2\}$	
$\pm 3, \ldots, \pm(k-2)$	$S = \{\pm 1, \pm 2\}$	

Observation. (a) *If $f_1(e) = 1$ (or -1), then $f_2(e) \in S_1 = \{1, \pm 2\}$ (or $f_2(e) \in S_1^- = \{-1, \pm 2\}$);*

(b) *If* $f_1(e) \in \{k-1, -2\}$ (*or* $\{2, -(k-1)\}$), *then* $f_2(e) \in S_2 = \{\pm 1, -2\}$ (*or* $f_2(e) \in S_2^- = \{\pm 1, 2\}$);

(c) *If* $f_1(e) \in \{\pm 3, \dots, \pm(k-2)\}$, *then* $f_2(e) \in S = \{\pm 1, \pm 2\}$.

By the above observation, we have that $\{\pm 1\} \subseteq S_2, S_2^-, S$ and $1 \in S_1, -1 \in S_1^-$. If there is one edge in $\{u_1u_4, u_2u_3, u_3u_4\}$, say u_1u_4, such that the value of f_1 on this edges is in the set $\{\pm 2, \pm 3, \dots, \pm(k-1)\}$, then we only need to prove the case that the values of other two edges are not equal and are in the set $\{\pm 1\}$. Since $\pm 2 \subseteq S_1, S_1^-, S$, we only need to prove the case that $f_1(u_1u_4) \in \{\pm 2, \pm(k-1)\}$. If $f_1(u_1u_4) \in \{2, -(k-1)\}$ (or $\{-2, k-1\}$), then $a = 2$ (or $a = -2$), we have done, f is an NZ-unoriented $(k+1)$-flow.

Thus suppose $f_1(u_1u_4), f_1(u_2u_3), f_1(u_3u_4) \in \{\pm 1\}$. Then in this case, $\{\pm 2\} \subseteq S_1, S_1^-$. Thus $a \in \{\pm 2\}$, we have done, f is an NZ-unoriented $(k+1)$-flow. ■

Let G and H be two different graphs with $E(G) \cap E(H) = \emptyset$. In this paper, $G \cup H$ denote a graph with vertex-set $V(G) \cup V(H)$ and edge-set $E(G) \cup E(H)$.

Lemma 11. *Let G admit an NZ-unoriented k-flow ($k \geq 4$) and let T be a triangle such that $E(G) \cap E(T) = \emptyset$ and $u_1 \in V(G) \cap V(T)$. If there is a triangle in G containing u_1 as a vertex, then $G \cup T$ admits a NZ-unoriented $(k+1)$-flow.*

Proof. Let f_1 be an NZ-unoriented k-flow of G. Without loss of generality, we assume that $T_1 = u_1e_1u_2e_2u_3e_3u_1$ is a triangle in G which contains u_1. In this case, $T_1 \cup T$ is an even circuit.

By Lemma 7, $T_1 \cup T$ admits an NZ-unoriented flow f_2 such that $f_2(e_1) = f_2(e_3) = a$, $f_2(e_2) = -a$ and $a \in \{\pm 1, \pm 2\}$. Let f be a function on $G \cup T$ as follows: $f(e) = f_1(e)$ if $e \in E(G) \setminus \{e_1, e_2, e_3\}$; $f(e) = f_2(e)$ if $e \in E(T)$; $f(e) = f_1(e) + f_2(e)$ if $e \in \{e_1, e_2, e_3\}$. Clearly, if $f(e) \in \{\pm 1, \pm 2, \dots, \pm k\}$ for each $e \in \{e_1, e_2, e_3\}$, then f is an NZ-unoriented $(k+1)$-flow of $G \cup T$. By the similar discussion in Lemma 10, we can find a proper a such that f is a NZ-unoriented $(k+1)$-flow of $G \cup T$. ■

Lemma 12. *Let G admit an NZ-unoriented k-flow and let H be a t-triangle-star. Assume that $E(G) \cap E(H) = \{e_0\}$. If e_0 is an edge of a triangle of G, then $G \oplus H$ admits an NZ-unoriented $(k+1)$-flow such that there is at most one edge with value k or $-k$ for each $k \geq 4$. Moreover, if t is odd, lemma holds for $k \geq 3$.*

Proof. Let f_1 be an NZ-unoriented k-flow of G and without loss of generality, we assume that $T = u_1e_1u_2e_2u_3e_0u_1$ is a triangle of G containing e_0. Clearly, $t \geq 1$.

Case 1. $t = 1$. In this case, H is a triangle. Without loss of generality, we assume that H is $u_1e_0u_3e_3ve_4u_1$. Then $e_1e_2e_3e_4$ is a 4-cycle of $G \oplus H$. By Lemma 7, $e_1e_2e_3e_4$ admits a NZ-unoriented k-flow f_2 such that $f_2(e_1) = f_2(e_3) =$

a, $f_2(e_2) = f_2(e_4) = -a$ and $a \in \{\pm 1, \pm 2\}$. Let f be a function on $G \oplus H$ as follows: $f(e) = f_1(e)$ if $e \in E(G) \setminus \{e_1, e_2\}$; $f(e) = f_2(e)$ if $e \in \{e_3, e_4\}$; $f(e) = f_1(e) + f_2(e)$ if $e \in \{e_1, e_2\}$. Clearly, if $f(e_1), f(e_2) \in \{\pm 1, \pm 2, \ldots, \pm k\}$, then f is an NZ-unoriented $(k+1)$-flow of $G \oplus H$. Next we prove that there exists f_2 on $e_1e_2e_3e_4$ such that $f(e_1), f(e_2) \in \{\pm 1, \pm 2, \ldots, \pm k\}$.

By reversing the value of f_1 on each edge of G, we can assume that $f_1(e_1) \in \{1, 2, \ldots, k-1\}$.

When $f_1(e_1) = 1$. If $f_1(e_2) \in \{-(k-1), 1\}$, then let $a = -2$ and f is a NZ-unoriented k-flow of $G \oplus H$ if $k \geq 4$. If $k = 3$ and $f_1(e_2) = -(k-1) = -2$, then let $a = 1$ and f is an NZ-unoriented $(k+1)$-flow of $G \oplus H$ just with $f(e_2) = -3 = -k$. If $k = 3$ and $f_1(e_2) = 1$, then let $a = -2$ and f is an NZ-unoriented $(k+1)$-flow of $G \oplus H$ just with $f(e_2) = 3 = k$. If $f_1(e_2) \notin \{-(k-1), 1\}$, then let $a = 1$ and f is an NZ-unoriented k-flow of $G \oplus H$ for $k \geq 3$.

When $f_1(e_1) \in \{2, 3, \ldots, k-2\}$. If $f_1(e_2) \in \{-(k-1), 1\}$, then $a = -1$ and f is an NZ-unoriented k-flow of $G \oplus H$ for $k \geq 3$. If $f_1(e_2) \notin \{-(k-1), 1\}$, then $a = 1$ and f is an NZ-unoriented k-flow of $G \oplus H$ for $k \geq 3$.

When $f_1(e_1) = k-1$. If $f_1(e_2) = -1$, then $a = -2$ and f is an NZ-unoriented k-flow of $G \oplus H$ for $k \geq 4$. If $k = 3$, then let $a = 1$ and f is a NZ-unoriented $(k+1)$-flow of $G \oplus H$ just with $f(e_1) = 3 = k$. If $f_1(e_2) = k-1$, then $a = 1$ and f is an NZ-unoriented $(k+1)$-flow of $G \oplus H$ just with $f(e_1) = k$. If $f_1(e_2) \notin \{-1, k-1\}$, then $a = -1$ and f is an NZ-unoriented k-flow of $G \oplus H$.

Case 2. $t \geq 3$ and t is odd. In this case, $G \oplus H = (G \oplus T) \cup C$, where T is a triangle of H containing the edge e_0 and $C = H - E(T)$ is an even circuit. By Case 1, $G \oplus T$ admits an NZ-unoriented $(k+1)$-flow for $k \geq 3$ with at most one edge having value k or $-k$. By Lemma 7, $G \oplus H = (G \oplus T) \cup C$ admits an NZ-unoriented $(k+1)$-flow with at most one edge having value k or $-k$ for each $k \geq 3$.

By Case 1 and 2, if t is odd, then $G \oplus H$ admits an NZ-unoriented $(k+1)$-flow such that there is at most one edge with value k or $-k$ for each $k \geq 3$.

Case 3. $t \geq 2$ and t is even. Without loss of generality, we assume that e_1 is the common edge of H. If $e_1 = e_0$, then $G \oplus H \cong G \cup C$, where $C = H - e_1$ is an even circuit. Since G admits an NZ-unoriented k-flow, by Lemma 7, $G \oplus H$ admits an NZ-unoriented k-flow for each $k \geq 3$. Thus we assume that $e_1 \neq e_0$. In this case, $G \oplus H = (G \oplus C) + e_1$, where $C = H - e_1$ is an even circuit. By Lemma 7, C admits an NZ-unoriented 2-flow, say f_2. Without loss of generality, we assume that $f_2(e_0) = 1$. In this case, if $f_1(e_0) \in \{1, -2, -3, \ldots, -(k-1)\}$, then the combination of f_1 and f_2 is also an NZ-unoriented k-flow of $G \oplus C$. Hence we have done, because $f_1(e_0)$ can be an element in the set $\{1, -2, -3, \ldots, -(k-1)\}$ by reversing the value of f_1 on each edge of G. By Lemma 10, $G \oplus H = (G \oplus C) + e_1$ admits an NZ-unoriented $(k+1)$-flow for each $k \geq 4$. ■

Lemma 13. *Let $G = G_1 \oplus G_2$, where G_i admits an NZ-unoriented k-flow for each $i \in \{1, 2\}$.*

(i) *If $k \geq 5$ and there exists $i \in \{1, 2\}$ such that G_i has a NZ-unoriented k-flow with value on $E(G_1) \cap E(G_2)$ in the set $\{\pm 1, \pm 2\}$, then G admits an NZ-unoriented k-flow.*

(ii) *If $k \geq 3$ and there exists $i \in \{1, 2\}$ such that G_i has a NZ-unoriented k-flow with value on $E(G_1) \cap E(G_2)$ in the set $\{\pm 1\}$, then G admits an NZ-unoriented k-flow.*

Proof. We only prove (i), since (ii) can be proved similarly.

Since $G = G_1 \oplus G_2$, $|E(G_1) \cap E(G_2)| = 1$. Without loss of generality, we assume that $E(G_1) \cap E(G_2) = \{e_0\}$. By the assumption of lemma, we can assume that f_i is an NZ-unoriented k-flow of G_i for each $i \in \{1, 2\}$ satisfying $f_1(e_0) \in \{\pm 1, \pm 2\}$. Define a flow f on G as follows.

$$f(e) = \begin{cases} f_1(e) + f_2(e), & \text{if } e = e_0, \\ f_i(e), & \text{if } e \in E(G_i) \setminus \{e_0\}, \text{ for } i \in \{1, 2\}. \end{cases} \tag{1}$$

We only need to discuss cases that $f_1(e_0) \in \{1, 2\}$ by reversing values of f_1. When $f_1(e_0) = 1$. In this case, if $f_2(e_0) \in \{1, -2, -3, \ldots, -(k-1)\}$, then G is an NZ-unoriented k-flow. By reversing value of flow on G_2, we can get a desired flow of G_2 such that $f_2(e_0) \in \{1, -2, -3, \ldots, -(k-1)\}$, hence f is an NZ-unoriented k-flow of G.

When $f_1(e_0) = 2$. In this case, if $f_2(e_0) \in \{1, 2, -3, -4, \ldots, -(k-1)\}$, then G is an NZ-unoriented k-flow. By reversing value of flow on G_2, we can get a desired flow of G_2 such that $f_2(e_0) \in \{1, 2, -3, -4, \ldots, -(k-1)\}$, hence f is an NZ-unoriented k-flow of G. ■

Next we discuss the existence of NZ-unoriented flow on triangle-paths and wheels.

Lemma 14. *Triangle-path T^m ($m \geq 3$) admits an NZ-unoriented k-flow, where*

$$k = \begin{cases} 3, & \text{if } m \equiv 0 \pmod 3, \\ 4, & \text{otherwise.} \end{cases} \tag{2}$$

Moreover, for $m \equiv 1 \pmod 3$, T^m admits an unoriented 3-flow such that there is at most one edge with value 0, where this edge is an arbitrary edge of the second triangle and not the edge in the first triangle.

Proof. Without loss of generality, assume that T^m is a triangle-path $T_1 \oplus T_2 \oplus \cdots \oplus T_m$. Suppose $m \equiv 0 \pmod 3$. If $m = 3$, then T^3 admits an NZ-unoriented 3-flow shown in Figure 3. By induction hypothesis, we suppose that if $m = 3(t-1)$

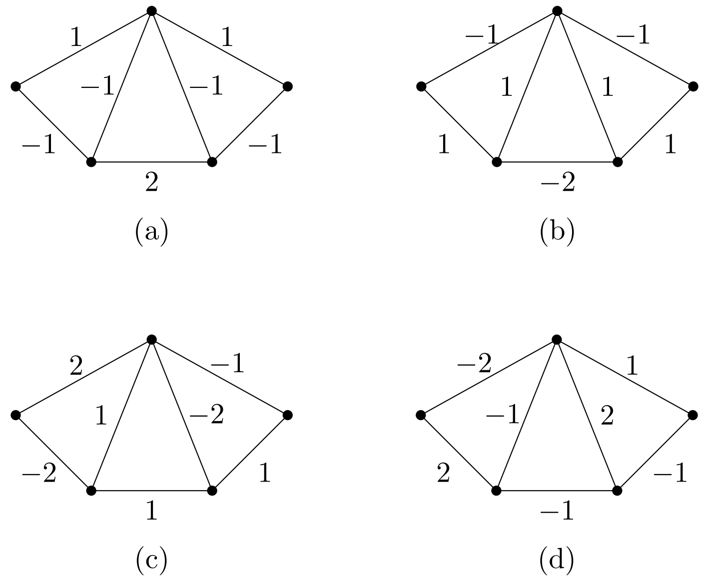


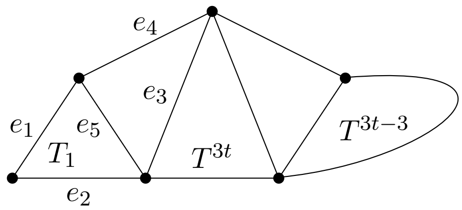
Figure 3. NZ-unoriented 3-flow of graph T^3

($t > 1$), T^m admits an NZ-unoriented 3-flow. Next we assume that $m = 3t$. By the definition of triangle-path, $T^m = T^{m-3} \oplus T^3$. By Figure 3, there is an NZ-unoriented 3-flow of T^3 such that the value on edge $E(T^{m-3}) \cap E(T^3)$ is in the set $\{\pm 1\}$. By Lemma 13(ii), T^m admits an NZ-unoriented 3-flow.

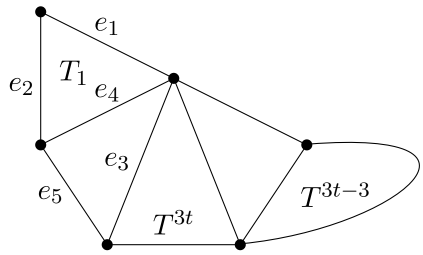
Suppose $m = 3t + 1 \equiv 1 \pmod 3$. In this case, $T^m = T_1 \oplus T^{3t}$. By the above discussion, T^{3t} admits an NZ-unoriented 3-flow. By Lemma 12, T^m admits NZ-unoriented 4-flow.

Next we prove that $T^m = T_1 \oplus T^{3t}$ admits an unoriented 3-flow such that at most one edge with value 0, where this edge is an arbitrary edge of the second triangle and not the edge in the first triangle. Without loss of generality, we assume that $e_1, e_2 \in E(T_1)$, $e_3, e_4, e_5 \in T^{3t}$, see Figure 4. Clearly, we can get an NZ-unoriented flow f_1 of T^{3t} such that $f_1(e_4) = 1$, $f_1(e_3) = f_1(e_5) = -1$. If $e_5 \in E(T_1)$ (see Figure 4(a)), then assign $-1, 1, -1, 1$ on edges e_1, e_2, e_3, e_4, respectively. Then combining with f_1, we get an NZ-unoriented 3-flow of T^m. If $e_4 \in E(T_1)$ (see Figure 4(b)), then assign $-1, 1, -1, 1$ (or $1, -1, 1, -1$) on edges e_1, e_2, e_5, e_3, respectively. Then combining with f_1, we get a unoriented 3-flow of T^m such that there is just edge e_3 (or e_5) with value 0. We know that e_3 and e_5 are edges in T_2 but not edges in T_1.

Suppose $m = 3t+2 \equiv 2 \pmod 3$. In this case, $T^m = T_1 \oplus T_2 \oplus T^{3t} = K_4^- \oplus T^{3t}$. By the above discussion, T^{3t} admits an NZ-unoriented 3-flow with value on edge of T_3 in the set $\{\pm 1\}$ as shown in Figure 3(a) (b). By the similar discussion in



(a) $e_5 \in E(T_1)$



(b) $e_4 \in E(T_1)$

Figure 4. Two types of graph T^m, where $m \equiv 1 \pmod 3$.

Case 3 of Lemma 12, we can deduce that T^m admits an NZ-unoriented 4-flow. ■

Suppose that a wheel denotes a graph consisting of a vertex v and a cycle C such that v is adjacent to all vertices of C, where v is called the center of this wheel. A m-wheel is a wheel such that the cycle is m-cycle. In this paper, a wheel means a m-wheel with $m \geq 3$.

Lemma 15. *Let W_m be a m-wheel $(m \geq 3)$. Then W_m admits an NZ-unoriented k-flow,*

$$k = \begin{cases} 3, & \text{if } m \equiv 0 \pmod 3, \\ 4, & \text{if } m \equiv 1 \pmod 3, \\ 5, & \text{otherwise.} \end{cases} \tag{3}$$

Proof. Clearly, $W_3 \cong K_4$. For any given edge $e_0 \in E(K_4)$, we can find a perfect matching of K_4 containing e_0. Let $\{e_0, e_1\}$ be a perfect matching of K_4. Thus define $f \to E(K_4)$ as follows: $f(e_0) = f(e_1) = 2$ (or -2), and the other edges e of K_4, $f(e) = -1$ (or 1). Clearly, f is an NZ-unoriented 3-flow of K_4. Suppose $m \geq 4$. W_m is a graph obtained from m triangles. We assume that W_m contains triangles $T_1, T_2, \ldots, T_m$, where $V(T_i) = \{u_i, u_{i+1}, v\}$ for each $i \in \{1, 2, \ldots, m\}$ $\pmod m$.

If $m = 3t$ $(t \geq 2)$, then define a function f_1 on $E(T_1 \oplus T_2 \oplus T_3)$ as follows: $f_1(vu_1) = -2$, $f_1(vu_2) = -1$, $f_1(vu_3) = 2$, $f_1(vu_4) = 1$. Then $f_1(u_1u_2) = 2$, $f_1(u_2u_3) = -1$, $f_1(u_3u_4) = -1$. Then f_1 is a NZ-unoriented 3-flow on $E(T_1 \oplus T_2 \oplus T_3)$. Then similarly define an NZ-unoriented 3-flow f_j on $E(T_{3j-2} \oplus T_{3j-1} \oplus T_{3j})$, where $j \in \{1, 2, \ldots, t\}$ such that $f_j(vu_{3j-2}) = -2$, $f_j(vu_{3j-1}) = -1$, $f_j(vu_{3j}) = 2$, $f_j(vu_{3j+1}) = 1$. Then $f_j(u_{3j-2}u_{3j-1}) = 2$, $f_j(u_{3j-1}u_{3j}) = -1$, $f_j(u_{3j}u_{3j+1}) = -1$. Let f be a function of W_m as follows: if $e \in E(T_{3j-2} \oplus T_{3j-1} \oplus T_{3j})$ and $e \notin \{vu_{3j-2}, vu_{3j+1}\}$, then $f(e) = f_j(e)$ for each $j \in \{1, 2, \ldots, t\}$; if $e \in E(T_{3j-2} \oplus T_{3j-1} \oplus T_{3j}) \cap E(T_{3j+1} \oplus T_{3j+2} \oplus T_{3j+3})$ for each $j \in \{1, 2, \ldots, t-1, t\}$ (mod $3t$), then $f(e) = f_j(e) + f_{j+1}(e) = 1 - 2 = -1$. In this case, f is an NZ-unoriented 3-flow of W_m.

If $m = 3t + 1$ $(t \geq 1)$, then W_m is a graph obtained from T^{3t} by adding one edge. Let $T^{3t} = T_1 \oplus T_2 \oplus \cdots \oplus T_{3t}$. Then $W_m = T^{3t} + u_1u_{3t+1}$. By Lemma 14, T^{3t} admits an NZ-unoriented 3-flow f_1 with $f_1(u_1u_2) = 2, f_1(u_2v) = -1$ and $f_1(vu_{3t+1}) = 1$. We know that $u_1u_2vu_{3t+1}u_1$ is a 4-cycle. Assign $1, -1, 1, -1$ on edges of this cycle starting from u_1u_2. By this way, the combination of these two flows is an NZ-unoriented 4-flow of W_m.

If $m = 3t+2$ $(t \geq 1)$, then W_m is a graph obtained from T^{3t+1} by adding one edge. Then $W_m = T^{3t+1} + u_1u_{3t+2}$. By Lemma 14 and Lemma 10, W_m admits an NZ-unoriented 5-flow. ■

3. Proof of Theorem 4

Lemma 16. *A triangle-tree except triangle-star admits a nowhere-zero unoriented* 5*-flow.*

Proof. We prove this lemma by induction on $|V(G)|$. Since G is not a triangle-star, $n \geq 5$. If $n = 5$, then G is a T^3, by Lemma 14, G admits a NZ-unoriented 3-flow, so NZ-unoriented 5-flow.

By induction hypothesis, we assume that G admits an NZ-unoriented 5-flow if G is a triangle-tree except triangle-star with less than n vertices $(n \geq 6)$.

Suppose $n \geq 6$ and our theorem holds for G with the number of vertices less than n vertices. Now we prove our theorem holds for $|V(G)| = n$.

Since G is a triangle-tree with $n \geq 6$, G contains at least three triangles. Since G is not a triangle-star, there exist three triangles constituting a T^3 which is an induced subgraph of G with one 2-vertex. Without loss of generality, we assume that $T^3 = T_1 \oplus T_2 \oplus T_3$ and T_1 has one vertex with degree two in G. Without loss of generality, we assume that u is the 2-vertex of T_1, the other two vertices are u_1, u_2 (see Figure 5). Without loss of generality, we assume that $V(T_2) = \{u_1, u_2, u_3\}, V(T_3) = \{u_2, u_3, u_4\}$. In this case, $G = T^3 \oplus G_1 \oplus G_2 \oplus G_3 \oplus G_4 \oplus G_5$, where G_i is a triangle-tree with less then n vertices, and may be an empty graph. Note that $E(G_i) \cap E(T^3) = e_i$ and $E(G_i) \cap E(G_j) = \emptyset$ for $i \neq j$, where $i, j \in \{1, 2, 3, 4, 5\}$.

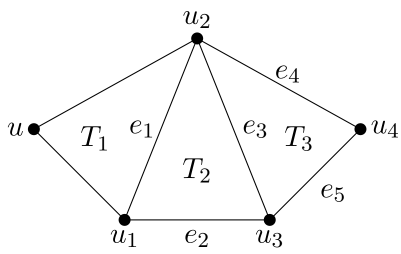


Figure 5. G contains an induced subgraph T^3 with a 2-vertex u.

Claim 1. *If G_i is not a triangle-star for each $i \in \{1, 2, 3, 4, 5\}$, then G admits an NZ-unoriented 5-flow.*

Proof. In this case, by induction hypothesis, each G_i admits a NZ-unoriented 5-flow, say f_i, respectively. (If G_i is an empty graph, then G_i is not needed to discuss in this case.) By Lemma 14, T^3 admits an NZ-unoriented 3-flow, which means the value of each edge is in $\{\pm 1, \pm 2\}$. Since $E(G_i) \cap E(G_j) = \emptyset$, $E(G_i) \cap E(T^3) = e_i$ for each $i \neq j \in \{1, 2, 3, 4, 5\}$, by Lemma 13(i), G admits an NZ-unoriented 5-flow. □

By Claim 1, we assume that there is at least one G_i, such that G_i is a triangle-star.

Claim 2. *If G_i is a 2-triangle-star with the common edge e_i or a k-triangle-star with $k \geq 3$ for some $i \in \{1, 2, 3, 4, 5\}$, then G admits an NZ-unoriented 5-flow.*

Proof. In this case, G_i contains a 4-cycle, say C. Hence $G = C \cup H$, where H is a triangle-tree containing T^3. By induction hypothesis, H admits an NZ-unoriented 5-flow. By Lemma 7, C admits an NZ-unoriented 2-flow. Thus G admits an NZ-unoriented 5-flow. □

Claim 3. *If G_1 is a k-triangle-star, then G admits a NZ-unoriented 5-flow.*

Proof. By Claim 2, we only need to prove cases that $k = 1$ or $k = 2$ and the common edge of G_1 is not e_1. If $k = 1$, then $G_1 \oplus T_1 - e_1$ is a 4-cycle, say C. Thus $G = C \cup H$, and H is a triangle-tree containing $T_2 \oplus T_3$. If H is not a triangle-star, then by induction hypothesis, H admits an NZ-unoriented 5-flow. By Lemma 7, C admits a NZ-unoriented 2-flow. Thus G admits an NZ-unoriented 5-flow. If H is a triangle-star, then H is a 2-triangle-star or a 3-triangle-star by Claim 2. This means $G = T_1 \oplus T_2 \oplus T_3 \oplus G_1$ or $G = T_1 \oplus T_2 \oplus T_3 \oplus G_1 \oplus G_3$, where G_1, G_3 are triangles. By Lemma 14, $T_1 \oplus T_2 \oplus T_3$ admits an NZ-unoriented 3-flow. By Lemma 12, $G = T_1 \oplus T_2 \oplus T_3 \oplus G_1$ admits an NZ-unoriented 4-flow. By Lemma

12, $G = T_1 \oplus T_2 \oplus T_3 \oplus G_1 \oplus G_3$ admits a NZ-unoriented 5-flow. Hence, in either case, G admits an NZ-unoriented 5-flow.

If $k = 2$ and the common edge of G_1 is not e_1, then $G_1 \oplus T_1$ is a T^3. Thus $G = T^3 \oplus H$, where H is a triangle-tree containing $T_2 \oplus T_3$. If H is not a triangle-star, then by induction hypothesis, H admits an NZ-unoriented 5-flow. By Lemma 14 and Lemma 13(i), G admits an NZ-unoriented 5-flow. If H is a triangle-star, then H is a 2-triangle-star or a 3-triangle-star by Claim 2. By Lemma 14, T^3 admits an NZ-unoriented 3-flow. By Lemma 12, $T^3 \oplus H$ admits an NZ-unoriented 5-flow. □

Claim 4. *If G_2 is a k-triangle-star, then G admits a NZ-unoriented 5-flow.*

Proof. By Claim 2, $k = 1$ or $k = 2$ and the common edge of G_2 is not e_2. If $k = 1$, then $T_1 \oplus T_2 \oplus G_2$ is a T^3. Thus $G = T_1 \oplus T_2 \oplus G_2 \oplus H \oplus G_1 = T^3 \oplus H \oplus G_1$, where H is a triangle-tree with less than n vertices. By Claim 3, G_1 is either an empty graph or a triangle-tree except a triangle-star. If G_1 is a triangle-tree except a triangle-star, then by induction hypothesis, G_1 admits an NZ-unoriented 5-flow.

If H is not a triangle-star, then by induction hypothesis, H admits a NZ-unoriented 5-flow. By Lemma 14 and 13(i), $G = T^3 \oplus H \oplus G_1$ admits an NZ-unoriented 5-flow whatever G_1 is a triangle-tree except triangle-star or an empty graph. If H is a triangle-star, by Claim 2, H is a 1-triangle-star or 2-triangle-star with common edge e_4 or e_5. Then $G_2 \oplus T_2 \oplus H$ is a T^3 or a T^4. Thus $G = T^3 \oplus T_1 \oplus G_1$ or $G = T^4 \oplus T_1 \oplus G_1$. If G_1 is an empty graph, then either $G = T^3 \oplus T_1$ or $G = T^4 \oplus T_1$, both of which admit an NZ-unoriented 5-flow by Lemma 14 and Lemma 12. Thus we assume that G_1 is not an empty graph. Clearly, $T_1 \oplus G_1$ is a triangle-tree except a triangle-star. By induction hypothesis, $T_1 \oplus G_1$ admits an NZ-unoriented 5-flow. By Lemma 14 and Lemma 13(i), $G = T^3 \oplus T_1 \oplus G_1$ admits an NZ-unoriented 5-flow. By Lemma 14, either T^4 has an NZ-unoriented 3-flow or T^4 admits an unoriented 3-flow just with edge e_1 having value zero. If T^4 has an NZ-unoriented 3-flow, by Lemma 13(i), $G = T^4 \oplus T_1 \oplus G_1$ admits an NZ-unoriented 5-flow. If T^4 admits a unoriented 3-flow with edge e_1 having value zero, then combining this flow with NZ-unoriented 5-flow of $T_1 \oplus G_1$, $G = T^4 \oplus T_1 \oplus G_1$ admits an NZ-unoriented 5-flow.

Thus we assume that $k = 2$ and the common edge of G_2 is not e_2. In this case, $T_2 \oplus G_2$ is a T^3. Thus $G = T_1 \oplus G_1 \oplus T^3 \oplus H$, where H is a triangle-tree with less than n vertices.

If H is not a triangle-star, then by induction hypothesis, H admits a NZ-unoriented 5-flow. When G_1 is an empty graph. By Lemma 14, $T_1 \oplus G_1 \oplus T^3 = T^4$ admits an NZ-unoriented 3-flow or an unoriented 3-flow such that the value just on edge e_3 is zero. In either case, by Lemma 13(i), G admits an NZ-unoriented 5-flow. When G_1 is a triangle-tree except a triangle-star, $T_1 \oplus G_1$ is a triangle-tree except triangle-star, $T_1 \oplus G_1$ admits an NZ-unoriented 5-flow. By Lemma 14 and Lemma 13(i), $G = T_1 \oplus G_1 \oplus T^3 \oplus H$ admits a NZ-unoriented 5-flow.

If H is a triangle-star, then by Lemma 14 and Lemma 12, $G = T_1 \oplus G_1 \oplus T^3 \oplus H$ admits an NZ-unoriented 5-flow when G_1 is an empty graph. Next we assume that G_1 is not an empty graph. In this case, by Claim 2, H is a 1-triangle-star or a 2-triangle-star. Then $T^3 \oplus H = G_2 \oplus T_2 \oplus H$ is T^4 or T^5. If $T^3 \oplus H = T^4$, then by Lemma 14, $T^3 \oplus H$ admits an NZ-unoriented 3-flow or a unoriented 3-flow just with edge e_1 having value zero. By Lemma 13, $G = T_1 \oplus G_1 \oplus T^3 \oplus H$ admits an NZ-unoriented 5-flow. If $T^3 \oplus H = T^5$, then in this case $T^3 \oplus H = G_2 \oplus T_2 \oplus T_3 \oplus G_i$, where G_i is a triangle and $i = 4$ or $i = 5$. By Lemma 14, $G_2 \oplus T_2 \oplus T_3 = T^4$ admits an NZ-unoriented 3-flow or an unoriented 3-flow just with edge e_1 having value zero. Then Lemma 12, $T^5 = G_2 \oplus T_2 \oplus T_3 \oplus G_i$ admits an NZ-unoriented 4-flow f_1 with $f_1(e_1) \in \{\pm 1, \pm 2\}$ or an unoriented 4-flow just with edge e_1 having value zero. By Lemma 13(i), $G = T_1 \oplus G_1 \oplus T^5$ admits an NZ-unoriented 5-flow. □

Claim 5. *If G_3 is a k-triangle-star, then G admits a NZ-unoriented 5-flow.*

Proof. By Claim 2, $k = 1$ or $k = 2$ and the common edge of G_3 is not e_3. If both G_4 and G_5 are empty graphs, then we can discuss this case similarly as Claim 3. Hence we can assume that at least one of G_4 and G_5 is not an empty graph. By Claims 3 and 4, G_i is either a triangle-tree except triangle-star or an empty graph for each $i \in \{1, 2\}$. If $k = 1$, then $T_1 \oplus T_2 \oplus G_3$ is a T^3. Then $G = T^3 \oplus G_1 \oplus G_2 \oplus H$, where H is a triangle-tree containing T_3, G_4 and G_5. If H is not a triangle-star, then by induction hypothesis, H admits an NZ-unoriented 5-flow. By Lemma 14 and Lemma 13(i), G admits an NZ-unoriented 5-flow. Thus we assume H is a triangle-star. By Claim 2, H is a 2-triangle-star. In this case, $G_3 \oplus H$ is a T^3 and $G = G_3 \oplus H \oplus L = T^3 \oplus L$, where L contains G_1, G_2, T_1, T_2. If L is a triangle-tree except a triangle-star, then by induction hypothesis and Lemma 14, Lemma 13(i), G admits an NZ-unoriented 5-flow. If L is a triangle-star, then L is a 2-triangle-star. By Lemma 14 and Lemma 12, G admits an NZ-unoriented 5-flow.

If $k = 2$, then $T_1 \oplus T_2 \oplus G_3$ is a T^4. Then $G = T^4 \oplus G_1 \oplus G_2 \oplus H$, where H is a triangle-tree containing T_3, G_4 and G_5. If H is not a triangle-star, then by induction hypothesis, H admits a NZ-unoriented 5-flow. By Lemma 14, $T_1 \oplus T_2 \oplus G_3$ admits a NZ-unoriented 3-flow or an unoriented 3-flow just with edge e_3 having value zero. By Lemma 13(i), G admits an NZ-unoriented 5-flow. Thus we assume H is a triangle-star. By Claim 2, H is a 2-triangle-star. In this case, $G_3 \oplus H$ is a T^4 and $G = G_3 \oplus H \oplus L = T^4 \oplus L$, where L contains G_1, G_2, T_1, T_2. If L is a triangle-tree except a triangle-star, then by induction hypothesis, L admits an NZ-unoriented 5-flow. By Lemma 14, T^4 admits an NZ-unoriented 3-flow or an unoriented 3-flow with edge e_3 having value zero. By Lemma 13(i), G admits a NZ-unoriented 5-flow. If L is a triangle-star, by Lemma 14 and Lemma 12, G

admits an NZ-unoriented 5-flow. □

Claim 6. *If G_4 is a k-triangle-star, then G admits a NZ-unoriented 5-flow.*

Proof. By Claim 2, $k = 1$ or $k = 2$ and the common edge of G_4 is not e_4. By Claims 3, 4, 5, G_i is an empty graph or triangle-tree except a triangle-star for $i \in \{1, 2, 3\}$. If $k = 1$, then $T_2 \oplus T_3 \oplus G_4$ is a T^3. Then $G = T^3 \oplus (G_1 \oplus T_1) \oplus G_2 \oplus G_3 \oplus G_5$.

Suppose first G_5 is either an empty graph or a triangle-tree except a triangle-star. If G_1 is not an empty graph, then G admits an NZ-unoriented 5-flow by induction hypothesis, Lemma 14 and Lemma 13(i). If G_1 is an empty graph, then $T_1 \oplus T_2 \oplus T_3 \oplus G_4$ is a T^4. If G_2, G_3, G_5 are empty graphs, then $G = T^4$, which admits an NZ-unoriented 4-flow by Lemma 14. If there exists one G_i such that G_i is not empty for some $i \in \{2, 3, 5\}$, then by Lemma 14, T^4 admits an NZ-unoriented 3-flow or an unoriented 3-flow just with edge e_i having value zero. By Lemma 13(i), G admits an NZ-unoriented 5-flow.

Suppose G_5 is a triangle-star. By Claim 2, G_5 is a 1-triangle-star or a 2-triangle-star. In this case, $G = G_4 \oplus T_3 \oplus G_5 \oplus H$, where H is a triangle-tree containing T_1, T_2, G_1, G_2, G_3. Clearly, $G_4 \oplus T_3 \oplus G_5$ is a T^3 or a T^4. If H is not triangle-star, then by induction hypothesis, H admits an NZ-unoriented 5-flow. By Lemma 14, $G_4 \oplus T_3 \oplus G_5$ admits an NZ-unoriented 3-flow or an unoriented 3-flow just with edge e_3 having value zero. Then G admits an NZ-unoriented 5-flow by Lemma 13(i). Thus we can assume that H is a triangle-star. By Lemma 14 and Lemma 12, G admits an NZ-unoriented 5-flow.

If $k = 2$ and the common edge of G_4 is not e_4, then $T_3 \oplus G_4$ is a T^3. Then $G = T^3 \oplus H \oplus G_5$, where H contains T_1, T_2, G_1, G_2, G_3. If H, G_5 are triangle-trees except triangle-star, then by induction hypothesis, H and G_5 admit NZ-unoriented 5-flow, respectively. By Lemma 14 and Lemma 13(i), G admits an NZ-unoriented 5-flow. If G_5 is a triangle-star, then by Claim 2, G_5 is a 1-triangle-star or a 2-triangle-star with common edge which is not e_5. In either case, by Lemmas 14 and 12, $T^3 \oplus G_5$ admits an NZ-unoriented 4-flow with the value on e_3 in $\{\pm 1, \pm 2\}$ or an unoriented 4-flow just with edge e_3 having value 0. (If G_5 is a 1-triangle-star, then $T^3 \oplus G_5$ is a T^4, by Lemma 14, $T^3 \oplus G_5$ admits an NZ-unoriented 3-flow or an unoriented 3-flow just with edge e_3 having value 0, we have done. If G_5 is a 2-triangle-star, then $T^3 \oplus G_5$ is a T^5, by Lemma 14, T^4 admits an NZ-unoriented 3-flow or an unoriented 3-flow just with edge e_3 having value 0. By Lemma 12, T^5 admits an NZ-unoriented 4-flow with the value on e_3 in $\{\pm 1, \pm 2\}$ or an unoriented 4-flow just with edge e_3 having value 0.) When H is not a triangle-star. By induction hypothesis and Lemma 13(i), G admits an NZ-unoriented 5-flow. When H is a triangle-star, by Lemma 14, $T^3 \oplus G_5$ admits an NZ-unoriented 4-flow. By Lemma 12, G admits an NZ-unoriented 5-flow. Next we only need to prove the case that G_5 is not a triangle-star and H is a triangle-star. By Claim 2, H is a 2-triangle-star. This means $H \oplus T_3 \oplus G_4$ is a T^5. By Lemma 14, $H \oplus T_3 \oplus G_4$ admits an NZ-unoriented 4-flow with value on edge e_5 in $\{\pm 1, \pm 2\}$ or an unoriented 4-flow just with edge e_5 having value 0. By induction hypothesis and Lemma 13(i), G admits an NZ-unoriented 5-flow. □

Claim 7. *If G_5 is a k-triangle-star, then G admits a NZ-unoriented 5-flow.*

Proof. By Claim 2, $k = 1$ or $k = 2$ and the common edge of G_5 is not e_5. If $k = 1$, then $T_2 \oplus T_3 \oplus G_5$ is a T^3. Then $G = T^3 \oplus (G_1 \oplus T_1) \oplus G_2 \oplus G_3 \oplus G_4$. By Claims 3, 4, 5, 6, G_i is not a triangle-star, for each $i \in \{1, 2, 3, 4\}$.

If G_1 is not empty graph, then G admits an NZ-unoriented 5-flow by induction hypothesis, Lemma 14 and Lemma 13(i). If G_1 is an empty graph, then $T_1 \oplus T_2 \oplus T_3 \oplus G_5$ is a T^4. If G_2, G_3, G_4 are empty, then $G = T^4$, which admits an NZ-unoriented 4-flow by Lemma 14. If there exists one G_i such that G_i is not empty for some $i \in \{2, 3, 4\}$, then by Lemma 14, T^4 admits an NZ-unoriented 3-flow or an unoriented 3-flow just with edge e_i having value zero. By Lemma 13(i), G admits a NZ-unoriented 5-flow.

If $k = 2$ and the common edge of G_5 is not e_5, then $T_3 \oplus G_5$ is a T^3. Then $G = T^3 \oplus H \oplus G_4$, where H contains T_1, T_2, G_1, G_2, G_3. If H is a triangle-tree except triangle-star, then by induction hypothesis, H admits an NZ-unoriented 5-flow. By Lemma 14 and Lemma 13(i), G admits an NZ-unoriented 5-flow. If H is a triangle-star, then H is a 2-triangle-star by Claim 2. In this case, $T_3 \oplus G_5 \oplus H$ is a T^5. If G_4 is empty, then G is a T^5. Hence G admits an NZ-unoriented 4-flow by Lemma 14. Thus G_4 is a triangle-tree except triangle-star. By induction hypothesis, G_4 admits an NZ-unoriented 5-flow. By Lemma 14, $T_3 \oplus G_5 \oplus H$ admits an unoriented 4-flow just with edge e_4 having value zero or an NZ-unoriented 4-flow with value on edge e_4 in the set $\{\pm 1, \pm 2\}$. Thus $T_3 \oplus G_5 \oplus H \oplus G_4$ admits a NZ-unoriented 5-flow. □

By Claims 3, 4, 5, 6, 7, we can assume that G_i is an empty graph or a triangle-tree except a triangle-star for $i \in \{1, 2, 3, 4, 5\}$. By induction hypothesis, Lemma 14, Lemma 13(i), G admits an NZ-unoriented 5-flow. ■

In the rest of this paper, we assume that G is not a triangle-star and can be partitioned into some triangle-paths or wheels $H_1, H_2, \ldots, H_t$ such that $G = H_1 \oplus H_2 \oplus \cdots \oplus H_t$, where H_i is a triangle-path or a wheel, where $i \in \{1, 2, \ldots, t\}$ and $t \geq 1$.

***Proof of Theorem* 4.** Let m be the number of wheels in $H_1, H_2, \ldots, H_t$. If $m = 0$, then G is a triangle-tree except a triangle-star. By Lemma 16, G admits a NZ-unoriented 5-flow, hence an NZ-unoriented 6-flow. If $m = 1$, then without loss of generality, we assume that H_j is a wheel for some $1 \leq j \leq t$. By the definition of G, H_j and $H_1 \oplus H_2 \oplus \cdots \oplus H_{j-1}$ just have one common edge. Since H_j is a wheel, H_j has at least three edges which is not adjacent to its center. Then we can choose an edge, say e_0, of H_j such that e_0 is not adjacent to the center of H_j and is not an edge of $H_1 \oplus H_2 \oplus \cdots \oplus H_{j-1}$. Then $H_j - e_0$ is a triangle-path with at least two triangles. If e_0 is not an edge of $H_{j+1} \oplus H_{j+2} \oplus \cdots \oplus H_t$, then $G = T + e_0$, where $T = H_1 \oplus H_2 \oplus \cdots \oplus H_{j-1} \oplus (H_j - e_0) \oplus H_{j+1} \oplus \cdots \oplus H_t$. If T is not a triangle-star, then T is a triangle-tree. By Lemma 16, T admits an NZ-unoriented 5-flow. By Lemma 10, $G = T + e$ admits an NZ-unoriented 6-flow. If T is a triangle-star, then H_j is 3-wheel and other H_is are triangle-star with a common edge. In this case, $G = W_3 \oplus S$, where S is a k-triangle-star and W_3 is a 3-wheel, $E(W_3) \cap E(S)$ is the common edge of S. If $k = 0$, then $G = W_3$. By Lemma 15, G admits an NZ-unoriented 3-flow, hence an NZ-unoriented 6-flow. Thus $k \geq 1$. By Lemma 15 and Lemma 12, G admits an NZ-unoriented 5-flow, hence NZ-unoriented 6-flow.

If e_0 is also an edge of $H_{j+1} \oplus H_{j+2} \oplus \cdots \oplus H_t$, then set $e_0 = uv$ and $G - \{u, v\}$ contains at least two connected components such that one component, say H, contains the center of H_j. Let G_1 be the subgraph comprised of H, $\{u, v\}$ and $E(H, \{u, v\})$, $G_2 = G - E(G_1)$. Clearly, $G = G_1 \cup G_2$ and $e_0 \in E(G_2)$, $E(G_1) \cap E(G_2) = \emptyset$. By the definition of G_1 and G_2, G_i is a triangle-tree for each $i \in \{1, 2\}$.

If G_i is not a triangle-star for each $i \in \{1, 2\}$, then by Lemma 16, G_i admits an NZ-unoriented 5-flow, say f_i, for each $i \in \{1, 2\}$. Since $E(G_1) \cap E(G_2) = \emptyset$, we can define f on $E(G)$ as follows. Let $f(e) = f_1(e)$ when $e \in E(G_1)$, $f(e) = f_2(e)$ when $e \in E(G_2)$. Clearly, f is an NZ-unoriented 5-flow of G, hence an NZ-unoriented 6-flow.

Next we assume that there exists one G_i such that G_i is a triangle-star. Without loss of generality, we assume that G_1 is not a triangle-star and G_2 is a k-triangle-star. In this case, G_1 is a triangle-tree except a triangle-star and without loss of generality, we assume that the common edge of G_2 is e_1. (The proofs are the same whatever e_1 is e_0 or not.) By Lemma 16, G_1 admits an NZ-unoriented 5-flow. If $k = 1$, then by Lemma 11, G admits an NZ-unoriented 6-flow. If $k \geq 2$ and k is even, then $G_2 - e_1$ is an even circuit. By Lemma 7, $G_2 - e_1$ admits a NZ-unoriented 2-flow, hence the combination of flows on G_1 and $G_2 - e_1$ is an NZ-unoriented 5-flow of $G - e_1 = G_1 \cup (G_2 - e_1)$. Thus G admits a NZ-unoriented 6-flow by Lemma 10. If $k \geq 2$ and k is odd, then $G_2 - E(T)$ is an even circuit, where T is an arbitrary triangle of G_2. By Lemma 7, $G_2 - E(T)$ admits an NZ-unoriented 2-flow, hence the combination of flows on G_1 and $G_2 - E(T)$ is an NZ-unoriented 5-flow of $G - E(T)$. Thus G admits an NZ-unoriented 6-flow by Lemma 11.

If G_i is a triangle-star for each $i \in \{1, 2\}$, then without loss of generality, we assume that G_i is a k_i-triangle-star. Since $H_j - e_0$ is contained in G_1, $k_1 \geq 2$ and H_j is a 3-wheel. In this case, $G = S_1 \oplus W_3 \oplus G_2$, where S_1 is a $(k_1 - 2)$-triangle-star and $E(S_1) \cap E(G_2) = \emptyset$. If $k_1 = 2$, then $G = W_3 \oplus G_2$. By Lemma 15 and Lemma 12, G admits an NZ-unoriented 5-flow. If $k_1 \geq 3$, by Lemma 15 and Lemma 12, $S_1 \oplus W_3$ admits an NZ-unoriented 5-flow. By Lemma 12, $G = S_1 \oplus W_3 \oplus G_2$ admits an NZ-unoriented 6-flow.

Thus by induction hypothesis, theorem holds for less then m wheels in $\{H_1, H_2, \ldots, H_t\}$. Next we prove the case that there are m wheels in $\{H_1, H_2, \ldots, H_t\}$.

If there exists an edge on cycle of a wheel such that this edge is also an edge of another connected subgraph except a triangle-star, that is, $G = G_1 \oplus G_2$ and $E(G_1) \cap E(G_2) = \{e_0\}$, where G_1 contains a wheel which e_0 is an edge of the cycle of this wheel, G_2 is not a triangle-star. In this case, we can choose G_1 such that e_0 is not a common edge in G_1. Then $G = (G_1 - e_0) \cup G_2$. If $G_1 - e_0$ is not a triangle-star, then by induction hypothesis, $G_1 - e_0$ and G_2 admit an NZ-unoriented 6-flow. Then G admits an NZ-unoriented 6-flow. If $G_1 - e_0$ is a triangle-star, then $G_1 = W_3 \oplus S$, where S is a triangle-star and $E(S) \cap E(G_2) = \emptyset$. By Lemma 15 and Lemma 12, G_1 admits an NZ-unoriented 4-flow with e_0 having value in the set $\{\pm 1, \pm 2\}$. By Lemma 13(i), G admits an NZ-unoriented 6-flow. Then we can assume that each edge of a cycle of wheel is not contained in other subgraph of G or is an edge of another connected subgraph which is a triangle-star. This means edges of cycle of wheel can be divided into two types.

Type 1. edge is contained in just one H_i, where H_i is a wheel;

Type 2. edge is contained in one H_i and other H_js, where H_i is a wheel, other H_js constitute a triangle-star.

For each wheel of G, we can choose one edge from cycles of wheel, say $e_1, e_2, \ldots, \ldots, e_m$, such that the number of edges of type 1 is maximal and e_i, e_j are different edges from different wheels. If all e_is are edges of type 1, then $G - \{e_1, \ldots, e_m\}$ is a triangle-tree. If $G - \{e_1, \ldots, e_m\}$ is not a triangle-star, then by Lemma 16, $G - \{e_1, \ldots, e_m\}$ admits an NZ-unoriented 5-flow. Since $e_1, \ldots, e_m$ are contained in different wheels, by Lemma 10, G admits an NZ-unoriented 6-flow. If $G - \{e_1, \ldots, e_m\}$ is a triangle-star, then H_i is a W_3 or a k-triangle-path, where $k \in \{1, 2\}$ for each $i \in \{1, 2, \ldots, t\}$ and all H_is have a common edge. Without loss of generality, we assume that H_i is W_3 for each $i \in \{1, 2, \ldots, m\}$. Clearly, others constitute a triangle-star. By types 1 and 2, we can deduce that $m = 1$ since each edge of W_3 can be an edge of cycle of this wheel. This case we have done.

If there exist an edge, say e_1, which is an edge of type 2, then by the maximization, each edge of cycle of this wheel is an edge of type 2. This means $m = 1$, and we have done. ∎

Acknowledgements

The first author was supported by NSFC No. 11501256, the second author was supported by NSFC No. 11301254, the Youth Backbone Teacher Foundation of

Henan's University (Grant No. 2015GGJS-115), Basic Research Foundation of Henan Educational Committee (No. 20ZX004) and State Scholarship Fund.

References

[1] S. Akbari, N. Ghareghani, G.B. Khosrovshahi and A. Mahmoody, *On zero-sum* 6-*flows of graphs*, Linear Algebra Appl. **430** (2009) 3047–3052. https://doi.org/10.1016/j.laa.2009.01.027

[2] S. Akbari, A. Daemi, O. Hatami, A. Javanmard and A. Mehrabian, *Zero-sum flows in regular graphs*, Graphs Combin. **26** (2010) 603–615. https://doi.org/10.1007/s00373-010-0946-5

[3] S. Akbari, A. Daemi, O. Hatami, A. Javanmard and A. Mehrabian, *Nowhere-zero unoriented flows in Hamiltonian graphs*, Ars Combin. **120** (2015) 51–63.

[4] S. Akbari, N. Ghareghani, G.B. Khosrovshahi and S. Zare, *A note on zero-sum* 5-*flows in regular graphs*, Electron. J. Combin. **19** (2012) #P7. https://doi.org/10.37236/2145

[5] A. Bouchet, *Nowhere-zero integeral flows on a bidirected graph*, J. Combin. Theory Ser. B **34** (1983) 279–292. https://doi.org/10.1016/0095-8956(83)90041-2

[6] M. DeVos, *Flows in bidirected graphs*, Mathematics **43** (2013) 95–115.

[7] J. Edmonds, *Maximum matching and a polyhedron with* $0,1$-*vertices*, J. Res. Nat. Bur. Stand. **69B** (1965) 125–130. https://doi.org/10.6028/jres.069B.013

[8] F. Jaeger, *Nowhere-zero flow problems*, in: Selected topics in Graph Theory **3**, L.W. Beineke and R.J. Wilson (Ed(s)), (Academic Press, London, 1988) 70–95.

[9] M. Kano, *Factors of regular graphs*, J. Combin. Theory Ser. B **41** (1986) 27–36. https://doi.org/10.1016/0095-8956(86)90025-0

[10] B. Korte and J. Vygen, Combinatorial Optimization: Theory and Algorithms (Springer, Berlin, 2006). https://doi.org/10.1007/3-540-29297-7

[11] A. Raspaud and X. Zhu, *Circular flow on signed graphs*, J. Combin. Theory Ser. B **101** (2011) 464–479. https://doi.org/10.1016/j.jctb.2011.02.007

[12] W.T. Tutte, *On the imbedding of linear graphs in surfaces*, Proc. Lond. Math. Soc. (2) **51** (1949) 474–483. https://doi.org/10.1112/plms/s2-51.6.474

[13] O. Zyka, Nowhere-zero 30-flows on bidirected graphs, Thesis (Charles University, Praha, 1987).

[14] F. Yang and X. Li, *Zero-sum 6-flows in 5-regular graphs*, Bull. Malays. Math. Sci. Soc. **42** (2019) 1319–1327.
https://doi.org/10.1007/s40840-017-0547-z

Received 2 January 2019
Revised 7 January 2020
Accepted 7 January 2020