# NOWHERE-ZERO UNORIENTED 6-FLOWS ON CERTAIN TRIANGULAR GRAPHS 

Fan Yang<br>School of Physical and Mathematical Sciences<br>Nanjing University of Technolgy<br>Nanjing, Jiangsu 211816, China<br>e-mail: fanyang_just@163.com<br>\section*{Liangchen Li}<br>Department of Mathematics<br>Luoyang Normal University<br>Luoyang 471022, China<br>AND<br>Sizhong Zhou<br>Department of Science<br>Jiangsu University of Science and Technology Zhenjiang, Jiangsu 212003, China<br>e-mail: zsz_cumt@163.com


#### Abstract

A nowhere-zero unoriented flow of graph $G$ is an assignment of non-zero real numbers to the edges of $G$ such that the sum of the values of all edges incident with each vertex is zero. Let $k$ be a natural number. A nowherezero unoriented $k$-flow is a flow with values from the set $\{ \pm 1, \ldots, \pm(k-1)\}$, for short we call it NZ-unoriented $k$-flow. Let $H_{1}$ and $H_{2}$ be two graphs, $H_{1} \oplus H_{2}$ denote the 2-sum of $H_{1}$ and $H_{2}$, if $E\left(H_{1} \oplus H_{2}\right)=E\left(H_{1}\right) \cup E\left(H_{2}\right)$, $\left|V\left(H_{1}\right) \cap V\left(H_{2}\right)\right|=2$, and $\left|E\left(H_{1}\right) \cap E\left(H_{2}\right)\right|=1$. A triangle-path in a graph $G$ is a sequence of distinct triangles $T_{1}, T_{2}, \ldots, T_{m}$ in $G$ such that for $1 \leq i \leq m$, $\left|E\left(T_{i}\right) \cap E\left(T_{i+1}\right)\right|=1$ and $E\left(T_{i}\right) \cap E\left(T_{j}\right)=\emptyset$ if $j>i+1$. A triangle-star is a graph with triangles such that each triangle having one common edges with other triangles. Let $G$ be a graph which can be partitioned into some triangle-paths or wheels $H_{1}, H_{2}, \ldots, H_{t}$ such that $G=H_{1} \oplus H_{2} \oplus \cdots \oplus H_{t}$. In this paper, we prove that $G$ except a triangle-star admits an NZ-unoriented 6 -flow. Moreover, if each $H_{i}$ is a triangle-path, then $G$ except a triangle-star admits an NZ-unoriented 5-flow.


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## 1. Introduction

All graphs in this paper are finite and undirected without loops, possibly with parallel edges. A nowhere-zero $k$-flow in a graph with orientation is an assignment of an integer from $\{ \pm 1, \ldots, \pm(k-1)\}$ to each of its edges such that Kirchhoff's law is respected, that is, the total incoming flow is equal to the total outgoing flow at each vertex. As noted in [8], the existence of a nowhere-zero flow of a graph $G$ is independent of the choice of the orientation. Nowhere-zero flows in graphs were introduced by Tutte [12] in 1949. A great deal of research in the area has been motivated by Tutte's 5-Flow Conjecture which asserts that every 2-edgeconnected graph admits a nowhere-zero 5-flow. In 1983, Bouchet [5] generalized this concept to bidirected graphs. A bidirected graph $G$ is a graph with vertex set $V(G)$ and edge set $E(G)$ such that each edge is oriented as one of the four possibilities in Figure 1.


Figure 1. Orientations of edges in bidirected graph
An edge with orientation as (a) (respectively, (b)) is called an in-edge (respectively, out-edge). An edge that is neither an in-edge nor an out-edge is called an ordinary edge as in (c) or (d). An integer-valued function $f$ on $E(G)$ is a nowhere-zero bidirected $k$-flow if for every $e \in E(G), 0<|f(e)|<k$, and at every vertex $v$, the sum of values of $f$ on all coming-in edges incident with $v$ is equal to the sum of values of $f$ on all going-out edges incident with $v$. The following conjecture, posed by Bouchet [5] and known as Bouchet's 6-flow conjecture, is one of the most important problems on nowhere-zero integer flows in bidirected graphs.

Conjecture 1 [5]. If a bidirected graph admits a nowhere-zero $k$-flow for some positive integer $k$, then it admits a nowhere-zero 6-flow.

In [5], Bouchet showed that the value 6 in this conjecture is best possible. Raspaud and Zhu [11] proved that, if a 4-edge-connected bidirected graph admits
a nowhere-zero integer flow, then it admits a nowhere-zero 4 -flow, which is best possible. In [13], Zyka proved the result in Bouchet's Conjecture is true if 6 is replaced by 30 .

A nowhere-zero unoriented flow (also called zero-sum flow) in a graph is an assignment of non-zero integers to the edges of $G$ such that the total sum of the assignments of all edges incident with any vertex of $G$ is zero. A nowhere-zero unoriented $k$-flow in a graph $G$ is an unoriented flow with flow values from the set $\{ \pm 1, \ldots, \pm(k-1)\}$, for short we call it NZ-unoriented $k$-flow. The following conjecture is known as Zero-Sum Conjecture.

Conjecture 2 [1]. If a graph $G$ admits an NZ-unoriented flow, then it admits an NZ-unoriented 6 -flow.

Indeed an NZ-unoriented $k$-flow in $G$ is exactly a nowhere-zero $k$-flow in the bidirected graph with underlying graph $G$ such that each edge is an in-edge or out-edge. The following theorem shows the equivalence between Bouchet's Conjecture and Zero-Sum Conjecture.

Theorem 3 [2]. Bouchet's Conjecture and Zero-Sum Conjecture are equivalent.
There are many results about Zero-Sum Conjecture recently. Akbari, Daemi et al. [3] proved that Zero-Sum Conjecture is true for hamiltonian graphs, with 6 replaced by 12. Akbari et al. [2] proved that every $r$-regular graph $(r \geq 3)$ admits an NZ-unoriented 7-flow. Moreover, Akbari et al. [4] showed that every $r$-regular graph, where $r \geq 3$ and $r \neq 5$, admits an NZ-unoriented 5 -flow. Recently, Yang and Li [14] proved that every 5 -regular graph admits an NZ-unoriented 6 -flow.

In this paper we prove the existence of NZ-unoriented $k$-flow in certain triangular graphs.

Theorem 4. Let $G=H_{1} \oplus H_{2} \oplus \cdots \oplus H_{t}$, where $H_{i}$ is a triangle-path or a wheel. If $G$ is not a triangle-star, then $G$ admits an NZ-unoriented 6 -flow. Moreover, if each $H_{i}$ is a triangle-path, then $G$ admits a NZ-unoriented 5 -flow.

We will prove this result in Section 3. In Section 2, we establish some lemmas which are crucial to our result.

## 2. Preliminaries

Given a subgraph $H$ of $G$, we use $G / H$ to denote the graph obtained from $G$ by contracting all edges in $H$ and deleting the resulting loops. Let $n$-path be a path with $n$ edges. For a graph $G$ and $X \subseteq E(G)$, we use $G-X$ denote a graph obtained from $G$ by deleting edge set $X$. When $X=\{e\}$, we use $G-e$
for conveniences. For $u_{i} \in V(G)$ and each $i=1,2, G+u_{1} u_{2}$ means a new graph obtained from $G$ by adding edge $u_{1} u_{2}$.

Akbari, Ghareghani [1] et al. deduce the following two results which present a geometric interpretation for graphs having an NZ-unoriented flow.

Theorem 5. Suppose $G$ is not a bipartite graph. Then $G$ has an NZ-unoriented flow if and only if for any edge e of $G, G-e$ has no bipartite component.

Lemma 6. Let $G$ be a 2-edge-connected bipartite graph. Then $G$ has an NZunoriented 6 -flow.

A graph $G$ is called even (respectively, odd), if its number of edges is even (respectively, odd). A circuit in $G$ is a closed walk with no repeated edge. A cycle in this paper is a connected 2-regular graph. Two parallel edges form a cycle of length two. An $n$-cycle denotes a cycle with $n$ edges. The following easy lemma will be useful.

Lemma 7 [2]. Every even circuit admits an NZ-unoriented 2-flow.
Let $H_{1}$ and $H_{2}$ be two graphs, $H_{1} \oplus H_{2}$ denote the 2-sum of $H_{1}$ and $H_{2}$, if $E\left(H_{1}\right) \cup E\left(H_{2}\right)=E\left(H_{1} \oplus H_{2}\right),\left|V\left(H_{1}\right) \cap V\left(H_{2}\right)\right|=2$, and $\left|E\left(H_{1}\right) \cap E\left(H_{2}\right)\right|=1$. We define that $H_{1} \oplus H_{2} \oplus \cdots \oplus H_{t} \cong\left(H_{1} \oplus H_{2} \oplus \cdots \oplus H_{t-1}\right) \oplus H_{t}$. A trianglepath in a graph $G$ is a sequence of distinct triangles $T_{1}, T_{2}, \ldots, T_{m}$ such that for $1 \leq i \leq m,\left|E\left(T_{i}\right) \cap E\left(T_{i+1}\right)\right|=1$ and $E\left(T_{i}\right) \cap E\left(T_{j}\right)=\emptyset$ if $j>i+1$. For simplify, in the rest of this paper we use $T^{m}$ to denote a triangle-path with $m$ triangles. An $n$-triangle-star $H$ is a graph with $n$ triangles such that each triangle has one common edge with other triangles (see Figure 2). By definition, a triangle is a 1-triangle-star and $K_{4}^{-}$is a 2-triangle-star. A triangle-tree $G$ is a graph which can be partitioned into some triangle-paths $H_{1}, H_{2}, \ldots, H_{t}$ such that $G=H_{1} \oplus H_{2} \oplus \cdots \oplus H_{t}$. By the definition of triangle-tree, we know that a triangle-star is also a special triangle-tree with each $H_{i}$ as a triangle.


Figure 2. A 3-triangle-star with common edge $e$.

By Theorem 5, the following lemma is obvious.

Lemma 8. A triangle-star does not admit an NZ-unoriented flow.
Proof. Let $G$ be a $n$-triangle-star. If $n=1$, then $G$ is a triangle which is an odd cycle. Clearly, $G$ does not admit an NZ-unoriented flow. Assume that $n \geq 2$ and $e_{0}$ is the common edge of $G$. Then $G-e_{0}$ is a graph containing no odd cycles, that is a bipartite graph. By Theorem $5, G$ does not admit an NZ-unoriented flow.

Definition [11]. A barbell is a graph if it consists of two odd cycles that have exactly one common vertex, or two vertex-disjoint odd cycles together with a path with exactly one end-vertex on each of them and all internal vertices outside of them.

By applying Lemma 2.6 in [11], the following result is obvious.
Lemma 9. Any barbell admits an NZ-unoriented 3-flow that assigns 1 or -1 to the edges on the odd cycles and 2 or -2 to all other edges.

The following results are technical key to prove Theorem 4.
Lemma 10. Let $G$ admit an NZ-unoriented $k$-flow $(k \geq 4)$ and $u_{1}, u_{2} \in V(G)$. If there is a 3-path between $u_{1}$ and $u_{2}$ in $G$, then $G+u_{1} u_{2}$ admits a NZ-unoriented $(k+1)$-flow.

Proof. Let $f_{1}$ be an NZ-unoriented $k$-flow of $G$. Since there is a 3-path between $u_{1}$ and $u_{2}$, there exists two vertices, say $u_{3}, u_{4}$, such that $u_{2} u_{3}, u_{3} u_{4}, u_{4} u_{1} \in E(G)$. In this case, this 3 -path plus the new added edge $u_{1} u_{2}$ is a 4-cycle $u_{1} u_{2} u_{3} u_{4} u_{1}$ of $G+u_{1} u_{2}$. In the rest of the proof, $u_{1} u_{2}$ means the new added edge. By Lemma 7, $u_{1} u_{2} u_{3} u_{4} u_{1}$ admits an NZ-unoriented flow $f_{2}$ such that $f_{2}\left(u_{1} u_{4}\right)=f_{2}\left(u_{3} u_{2}\right)=a$, $f_{2}\left(u_{1} u_{2}\right)=f_{2}\left(u_{4} u_{3}\right)=-a$ and $a \in\{ \pm 1, \pm 2\}$. Let $f$ be a function on $G+u_{1} u_{2}$ as follows: $f(e)=f_{1}(e)$ if $e \in E(G) \backslash\left\{u_{1} u_{4}, u_{4} u_{3}, u_{2} u_{3}\right\} ; f(e)=f_{2}(e)$ if $e \in\left\{u_{1} u_{2}\right\}$; $f(e)=f_{1}(e)+f_{2}(e)$ if $e \in\left\{u_{1} u_{4}, u_{4} u_{3}, u_{2} u_{3}\right\}$. Clearly, if $f(e) \in\{ \pm 1, \pm 2, \ldots, \pm k\}$ for each $e \in\left\{u_{1} u_{4}, u_{4} u_{3}, u_{2} u_{3}\right\}$, then $f$ is an NZ-unoriented $(k+1)$-flow of $G+$ $u_{1} u_{2}$. In order to satisfy that $f(e) \in\{ \pm 1, \pm 2, \ldots, \pm k\}$, for $e \in\left\{u_{1} u_{4}, u_{4} u_{3}, u_{2} u_{3}\right\}$, we have the next observation (see the following table).

| $f_{1}(e)$ | $f_{2}(e)$ | $f(e)=f_{1}(e)+f_{2}(e)$ |
| :---: | :---: | :---: |
| 1 | $S_{1}=S \backslash\{-1\}$ |  |
| -1 | $S_{1}^{-}=S \backslash\{1\}$ |  |
| $y n n$ | $\pm 1, \ldots, \pm k$ |  |
| $k-1,-2$ |  |  |
| $-(k-1), 2$ | $S_{2}^{-}=S \backslash\{-2\}$ |  |
| $\pm 3, \ldots, \pm(k-2)$ | $S=\{ \pm 1, \pm 2\}$ |  |

Observation. (a) If $f_{1}(e)=1($ or -1$)$, then $f_{2}(e) \in S_{1}=\{1, \pm 2\} \quad$ or $f_{2}(e) \in$ $S_{1}^{-}=\{-1, \pm 2\}$ );
(b) If $f_{1}(e) \in\{k-1,-2\}$ (or $\{2,-(k-1)\}$ ), then $f_{2}(e) \in S_{2}=\{ \pm 1,-2\}$ (or $\left.f_{2}(e) \in S_{2}^{-}=\{ \pm 1,2\}\right) ;$
(c) If $f_{1}(e) \in\{ \pm 3, \ldots, \pm(k-2)\}$, then $f_{2}(e) \in S=\{ \pm 1, \pm 2\}$.

By the above observation, we have that $\{ \pm 1\} \subseteq S_{2}, S_{2}^{-}, S$ and $1 \in S_{1},-1 \in$ $S_{1}^{-}$. If there is one edge in $\left\{u_{1} u_{4}, u_{2} u_{3}, u_{3} u_{4}\right\}$, say $u_{1} u_{4}$, such that the value of $f_{1}$ on this edges is in the set $\{ \pm 2, \pm 3, \ldots, \pm(k-1)\}$, then we only need to prove the case that the values of other two edges are not equal and are in the set $\{ \pm 1\}$. Since $\pm 2 \subseteq S_{1}, S_{1}^{-}, S$, we only need to prove the case that $f_{1}\left(u_{1} u_{4}\right) \in\{ \pm 2, \pm(k-1)\}$. If $f_{1}\left(u_{1} u_{4}\right) \in\{2,-(k-1)\}$ (or $\{-2, k-1\}$ ), then $a=2$ (or $a=-2$ ), we have done, $f$ is an NZ-unoriented $(k+1)$-flow.

Thus suppose $f_{1}\left(u_{1} u_{4}\right), f_{1}\left(u_{2} u_{3}\right), f_{1}\left(u_{3} u_{4}\right) \in\{ \pm 1\}$. Then in this case, $\{ \pm 2\} \subseteq$ $S_{1}, S_{1}^{-}$. Thus $a \in\{ \pm 2\}$, we have done, $f$ is an NZ-unoriented ( $k+1$ )-flow.

Let $G$ and $H$ be two different graphs with $E(G) \cap E(H)=\emptyset$. In this paper, $G \cup H$ denote a graph with vertex-set $V(G) \cup V(H)$ and edge-set $E(G) \cup E(H)$.

Lemma 11. Let $G$ admit an NZ-unoriented $k$-flow $(k \geq 4)$ and let $T$ be a triangle such that $E(G) \cap E(T)=\emptyset$ and $u_{1} \in V(G) \cap V(T)$. If there is a triangle in $G$ containing $u_{1}$ as a vertex, then $G \cup T$ admits a NZ-unoriented $(k+1)$-flow.

Proof. Let $f_{1}$ be an NZ-unoriented $k$-flow of $G$. Without loss of generality, we assume that $T_{1}=u_{1} e_{1} u_{2} e_{2} u_{3} e_{3} u_{1}$ is a triangle in $G$ which contains $u_{1}$. In this case, $T_{1} \cup T$ is an even circuit.

By Lemma 7, $T_{1} \cup T$ admits an NZ-unoriented flow $f_{2}$ such that $f_{2}\left(e_{1}\right)=$ $f_{2}\left(e_{3}\right)=a, f_{2}\left(e_{2}\right)=-a$ and $a \in\{ \pm 1, \pm 2\}$. Let $f$ be a function on $G \cup T$ as follows: $f(e)=f_{1}(e)$ if $e \in E(G) \backslash\left\{e_{1}, e_{2}, e_{3}\right\} ; f(e)=f_{2}(e)$ if $e \in E(T)$; $f(e)=f_{1}(e)+f_{2}(e)$ if $e \in\left\{e_{1}, e_{2}, e_{3}\right\}$. Clearly, if $f(e) \in\{ \pm 1, \pm 2, \ldots, \pm k\}$ for each $e \in\left\{e_{1}, e_{2}, e_{3}\right\}$, then $f$ is an NZ-unoriented $(k+1)$-flow of $G \cup T$. By the similar discussion in Lemma 10, we can find a proper $a$ such that $f$ is a NZ-unoriented $(k+1)$-flow of $G \cup T$.

Lemma 12. Let $G$ admit an NZ-unoriented $k$-flow and let $H$ be at-triangle-star. Assume that $E(G) \cap E(H)=\left\{e_{0}\right\}$. If $e_{0}$ is an edge of a triangle of $G$, then $G \oplus H$ admits an NZ-unoriented $(k+1)$-flow such that there is at most one edge with value $k$ or $-k$ for each $k \geq 4$. Moreover, if $t$ is odd, lemma holds for $k \geq 3$.

Proof. Let $f_{1}$ be an NZ-unoriented $k$-flow of $G$ and without loss of generality, we assume that $T=u_{1} e_{1} u_{2} e_{2} u_{3} e_{0} u_{1}$ is a triangle of $G$ containing $e_{0}$. Clearly, $t \geq 1$.

Case 1. $t=1$. In this case, $H$ is a triangle. Without loss of generality, we assume that $H$ is $u_{1} e_{0} u_{3} e_{3} v e_{4} u_{1}$. Then $e_{1} e_{2} e_{3} e_{4}$ is a 4 -cycle of $G \oplus H$. By Lemma 7, $e_{1} e_{2} e_{3} e_{4}$ admits a NZ-unoriented $k$-flow $f_{2}$ such that $f_{2}\left(e_{1}\right)=f_{2}\left(e_{3}\right)=$
$a, f_{2}\left(e_{2}\right)=f_{2}\left(e_{4}\right)=-a$ and $a \in\{ \pm 1, \pm 2\}$. Let $f$ be a function on $G \oplus H$ as follows: $f(e)=f_{1}(e)$ if $e \in E(G) \backslash\left\{e_{1}, e_{2}\right\} ; f(e)=f_{2}(e)$ if $e \in\left\{e_{3}, e_{4}\right\}$; $f(e)=f_{1}(e)+f_{2}(e)$ if $e \in\left\{e_{1}, e_{2}\right\}$. Clearly, if $f\left(e_{1}\right), f\left(e_{2}\right) \in\{ \pm 1, \pm 2, \ldots, \pm k\}$, then $f$ is an NZ-unoriented $(k+1)$-flow of $G \oplus H$. Next we prove that there exists $f_{2}$ on $e_{1} e_{2} e_{3} e_{4}$ such that $f\left(e_{1}\right), f\left(e_{2}\right) \in\{ \pm 1, \pm 2, \ldots, \pm k\}$.

By reversing the value of $f_{1}$ on each edge of $G$, we can assume that $f_{1}\left(e_{1}\right) \in$ $\{1,2, \ldots, k-1\}$.

When $f_{1}\left(e_{1}\right)=1$. If $f_{1}\left(e_{2}\right) \in\{-(k-1), 1\}$, then let $a=-2$ and $f$ is a NZunoriented $k$-flow of $G \oplus H$ if $k \geq 4$. If $k=3$ and $f_{1}\left(e_{2}\right)=-(k-1)=-2$, then let $a=1$ and $f$ is an NZ-unoriented $(k+1)$-flow of $G \oplus H$ just with $f\left(e_{2}\right)=-3=-k$. If $k=3$ and $f_{1}\left(e_{2}\right)=1$, then let $a=-2$ and $f$ is an NZ-unoriented $(k+1)$-flow of $G \oplus H$ just with $f\left(e_{2}\right)=3=k$. If $f_{1}\left(e_{2}\right) \notin\{-(k-1), 1\}$, then let $a=1$ and $f$ is an NZ-unoriented $k$-flow of $G \oplus H$ for $k \geq 3$.

When $f_{1}\left(e_{1}\right) \in\{2,3, \ldots, k-2\}$. If $f_{1}\left(e_{2}\right) \in\{-(k-1), 1\}$, then $a=-1$ and $f$ is an NZ-unoriented $k$-flow of $G \oplus H$ for $k \geq 3$. If $f_{1}\left(e_{2}\right) \notin\{-(k-1), 1\}$, then $a=1$ and $f$ is an NZ-unoriented $k$-flow of $G \oplus H$ for $k \geq 3$.

When $f_{1}\left(e_{1}\right)=k-1$. If $f_{1}\left(e_{2}\right)=-1$, then $a=-2$ and $f$ is an NZ-unoriented $k$-flow of $G \oplus H$ for $k \geq 4$. If $k=3$, then let $a=1$ and $f$ is a NZ-unoriented $(k+1)$ flow of $G \oplus H$ just with $f\left(e_{1}\right)=3=k$. If $f_{1}\left(e_{2}\right)=k-1$, then $a=1$ and $f$ is an NZ-unoriented $(k+1)$-flow of $G \oplus H$ just with $f\left(e_{1}\right)=k$. If $f_{1}\left(e_{2}\right) \notin\{-1, k-1\}$, then $a=-1$ and $f$ is an NZ-unoriented $k$-flow of $G \oplus H$.

Case 2. $t \geq 3$ and $t$ is odd. In this case, $G \oplus H=(G \oplus T) \cup C$, where $T$ is a triangle of $H$ containing the edge $e_{0}$ and $C=H-E(T)$ is an even circuit. By Case $1, G \oplus T$ admits an NZ-unoriented ( $k+1$ )-flow for $k \geq 3$ with at most one edge having value $k$ or $-k$. By Lemma $7, G \oplus H=(G \oplus T) \cup C$ admits an NZ-unoriented $(k+1)$-flow with at most one edge having value $k$ or $-k$ for each $k \geq 3$.

By Case 1 and 2 , if $t$ is odd, then $G \oplus H$ admits an NZ-unoriented ( $k+1$ )-flow such that there is at most one edge with value $k$ or $-k$ for each $k \geq 3$.

Case 3. $t \geq 2$ and $t$ is even. Without loss of generality, we assume that $e_{1}$ is the common edge of $H$. If $e_{1}=e_{0}$, then $G \oplus H \cong G \cup C$, where $C=H-e_{1}$ is an even circuit. Since $G$ admits an NZ-unoriented $k$-flow, by Lemma 7, $G \oplus H$ admits an NZ-unoriented $k$-flow for each $k \geq 3$. Thus we assume that $e_{1} \neq e_{0}$. In this case, $G \oplus H=(G \oplus C)+e_{1}$, where $C=H-e_{1}$ is an even circuit. By Lemma 7, $C$ admits an NZ-unoriented 2-flow, say $f_{2}$. Without loss of generality, we assume that $f_{2}\left(e_{0}\right)=1$. In this case, if $f_{1}\left(e_{0}\right) \in\{1,-2,-3, \ldots,-(k-1)\}$, then the combination of $f_{1}$ and $f_{2}$ is also an NZ-unoriented $k$-flow of $G \oplus C$. Hence we have done, because $f_{1}\left(e_{0}\right)$ can be an element in the set $\{1,-2,-3, \ldots,-(k-1)\}$ by reversing the value of $f_{1}$ on each edge of $G$. By Lemma $10, G \oplus H=(G \oplus C)+e_{1}$ admits an NZ-unoriented $(k+1)$-flow for each $k \geq 4$.

Lemma 13. Let $G=G_{1} \oplus G_{2}$, where $G_{i}$ admits an $N Z$-unoriented $k$-flow for each $i \in\{1,2\}$.
(i) If $k \geq 5$ and there exists $i \in\{1,2\}$ such that $G_{i}$ has a NZ-unoriented $k$ flow with value on $E\left(G_{1}\right) \cap E\left(G_{2}\right)$ in the set $\{ \pm 1, \pm 2\}$, then $G$ admits an $N Z$-unoriented $k$-flow.
(ii) If $k \geq 3$ and there exists $i \in\{1,2\}$ such that $G_{i}$ has a NZ-unoriented $k$ flow with value on $E\left(G_{1}\right) \cap E\left(G_{2}\right)$ in the set $\{ \pm 1\}$, then $G$ admits an NZunoriented $k$-flow.

Proof. We only prove (i), since (ii) can be proved similarly.
Since $G=G_{1} \oplus G_{2},\left|E\left(G_{1}\right) \cap E\left(G_{2}\right)\right|=1$. Without loss of generality, we assume that $E\left(G_{1}\right) \cap E\left(G_{2}\right)=\left\{e_{0}\right\}$. By the assumption of lemma, we can assume that $f_{i}$ is an NZ-unoriented $k$-flow of $G_{i}$ for each $i \in\{1,2\}$ satisfying $f_{1}\left(e_{0}\right) \in\{ \pm 1, \pm 2\}$. Define a flow $f$ on $G$ as follows.

$$
f(e)= \begin{cases}f_{1}(e)+f_{2}(e), & \text { if } e=e_{0}  \tag{1}\\ f_{i}(e), & \text { if } e \in E\left(G_{i}\right) \backslash\left\{e_{0}\right\}, \text { for } i \in\{1,2\}\end{cases}
$$

We only need to discuss cases that $f_{1}\left(e_{0}\right) \in\{1,2\}$ by reversing values of $f_{1}$. When $f_{1}\left(e_{0}\right)=1$. In this case, if $f_{2}\left(e_{0}\right) \in\{1,-2,-3, \ldots,-(k-1)\}$, then $G$ is an NZ-unoriented $k$-flow. By reversing value of flow on $G_{2}$, we can get a desired flow of $G_{2}$ such that $f_{2}\left(e_{0}\right) \in\{1,-2,-3, \ldots,-(k-1)\}$, hence $f$ is an NZ-unoriented $k$-flow of $G$.

When $f_{1}\left(e_{0}\right)=2$. In this case, if $f_{2}\left(e_{0}\right) \in\{1,2,-3,-4, \ldots,-(k-1)\}$, then $G$ is an NZ-unoriented $k$-flow. By reversing value of flow on $G_{2}$, we can get a desired flow of $G_{2}$ such that $f_{2}\left(e_{0}\right) \in\{1,2,-3,-4, \ldots,-(k-1)\}$, hence $f$ is an NZ-unoriented $k$-flow of $G$.

Next we discuss the existence of NZ-unoriented flow on triangle-paths and wheels.

Lemma 14. Triangle-path $T^{m}(m \geq 3)$ admits an NZ-unoriented $k$-flow, where

$$
k= \begin{cases}3, & \text { if } m \equiv 0(\bmod 3)  \tag{2}\\ 4, & \text { otherwise }\end{cases}
$$

Moreover, for $m \equiv 1(\bmod 3), T^{m}$ admits an unoriented 3 -flow such that there is at most one edge with value 0 , where this edge is an arbitrary edge of the second triangle and not the edge in the first triangle.

Proof. Without loss of generality, assume that $T^{m}$ is a triangle-path $T_{1} \oplus T_{2} \oplus$ $\cdots \oplus T_{m}$. Suppose $m \equiv 0(\bmod 3)$. If $m=3$, then $T^{3}$ admits an NZ-unoriented 3flow shown in Figure 3. By induction hypothesis, we suppose that if $m=3(t-1)$


Figure 3. NZ-unoriented 3-flow of graph $T^{3}$
$(t>1), T^{m}$ admits an NZ-unoriented 3 -flow. Next we assume that $m=3 t$. By the definition of triangle-path, $T^{m}=T^{m-3} \oplus T^{3}$. By Figure 3, there is an NZ-unoriented 3-flow of $T^{3}$ such that the value on edge $E\left(T^{m-3}\right) \cap E\left(T^{3}\right)$ is in the set $\{ \pm 1\}$. By Lemma 13(ii), $T^{m}$ admits an NZ-unoriented 3-flow.

Suppose $m=3 t+1 \equiv 1(\bmod 3)$. In this case, $T^{m}=T_{1} \oplus T^{3 t}$. By the above discussion, $T^{3 t}$ admits an NZ-unoriented 3 -flow. By Lemma $12, T^{m}$ admits NZunoriented 4-flow.

Next we prove that $T^{m}=T_{1} \oplus T^{3 t}$ admits an unoriented 3-flow such that at most one edge with value 0 , where this edge is an arbitrary edge of the second triangle and not the edge in the first triangle. Without loss of generality, we assume that $e_{1}, e_{2} \in E\left(T_{1}\right), e_{3}, e_{4}, e_{5} \in T^{3 t}$, see Figure 4. Clearly, we can get an NZ-unoriented flow $f_{1}$ of $T^{3 t}$ such that $f_{1}\left(e_{4}\right)=1, f_{1}\left(e_{3}\right)=f_{1}\left(e_{5}\right)=-1$. If $e_{5} \in E\left(T_{1}\right)$ (see Figure $4(\mathrm{a})$ ), then assign $-1,1,-1,1$ on edges $e_{1}, e_{2}, e_{3}, e_{4}$, respectively. Then combining with $f_{1}$, we get an NZ-unoriented 3-flow of $T^{m}$. If $e_{4} \in E\left(T_{1}\right)$ (see Figure $4(\mathrm{~b})$ ), then assign $-1,1,-1,1$ (or $1,-1,1,-1$ ) on edges $e_{1}, e_{2}, e_{5}, e_{3}$, respectively. Then combining with $f_{1}$, we get a unoriented 3 -flow of $T^{m}$ such that there is just edge $e_{3}$ (or $e_{5}$ ) with value 0 . We know that $e_{3}$ and $e_{5}$ are edges in $T_{2}$ but not edges in $T_{1}$.

Suppose $m=3 t+2 \equiv 2(\bmod 3)$. In this case, $T^{m}=T_{1} \oplus T_{2} \oplus T^{3 t}=K_{4}^{-} \oplus T^{3 t}$. By the above discussion, $T^{3 t}$ admits an NZ-unoriented 3-flow with value on edge of $T_{3}$ in the set $\{ \pm 1\}$ as shown in Figure 3(a) (b). By the similar discussion in

(a) $e_{5} \in E\left(T_{1}\right)$

(b) $e_{4} \in E\left(T_{1}\right)$

Figure 4. Two types of graph $T^{m}$, where $m \equiv 1(\bmod 3)$.

Case 3 of Lemma 12, we can deduce that $T^{m}$ admits an NZ-unoriented 4 -flow.
Suppose that a wheel denotes a graph consisting of a vertex $v$ and a cycle $C$ such that $v$ is adjacent to all vertices of $C$, where $v$ is called the center of this wheel. A $m$-wheel is a wheel such that the cycle is $m$-cycle. In this paper, a wheel means a $m$-wheel with $m \geq 3$.

Lemma 15. Let $W_{m}$ be a $m$-wheel $(m \geq 3)$. Then $W_{m}$ admits an NZ-unoriented $k$-flow,

$$
k= \begin{cases}3, & \text { if } m \equiv 0(\bmod 3),  \tag{3}\\ 4, & \text { if } m \equiv 1(\bmod 3), \\ 5, & \text { otherwise }\end{cases}
$$

Proof. Clearly, $W_{3} \cong K_{4}$. For any given edge $e_{0} \in E\left(K_{4}\right)$, we can find a perfect matching of $K_{4}$ containing $e_{0}$. Let $\left\{e_{0}, e_{1}\right\}$ be a perfect matching of $K_{4}$. Thus define $f \rightarrow E\left(K_{4}\right)$ as follows: $f\left(e_{0}\right)=f\left(e_{1}\right)=2$ (or -2 ), and the other edges $e$ of $K_{4}, f(e)=-1$ (or 1 ). Clearly, $f$ is an NZ-unoriented 3 -flow of $K_{4}$. Suppose $m \geq 4 . W_{m}$ is a graph obtained from $m$ triangles. We assume that $W_{m}$ contains triangles $T_{1}, T_{2}, \ldots, T_{m}$, where $V\left(T_{i}\right)=\left\{u_{i}, u_{i+1}, v\right\}$ for each $i \in\{1,2, \ldots, m\}$ $(\bmod m)$.

If $m=3 t(t \geq 2)$, then define a function $f_{1}$ on $E\left(T_{1} \oplus T_{2} \oplus T_{3}\right)$ as follows: $f_{1}\left(v u_{1}\right)=-2, f_{1}\left(v u_{2}\right)=-1, f_{1}\left(v u_{3}\right)=2, f_{1}\left(v u_{4}\right)=1$. Then $f_{1}\left(u_{1} u_{2}\right)=2$, $f_{1}\left(u_{2} u_{3}\right)=-1, f_{1}\left(u_{3} u_{4}\right)=-1$. Then $f_{1}$ is a NZ-unoriented 3-flow on $E\left(T_{1} \oplus T_{2} \oplus\right.$ $\left.T_{3}\right)$. Then similarly define an NZ-unoriented 3-flow $f_{j}$ on $E\left(T_{3 j-2} \oplus T_{3 j-1} \oplus T_{3 j}\right)$, where $j \in\{1,2, \ldots, t\}$ such that $f_{j}\left(v u_{3 j-2}\right)=-2, f_{j}\left(v u_{3 j-1}\right)=-1, f_{j}\left(v u_{3 j}\right)=2$, $f_{j}\left(v u_{3 j+1}\right)=1$. Then $f_{j}\left(u_{3 j-2} u_{3 j-1}\right)=2, f_{j}\left(u_{3 j-1} u_{3 j}\right)=-1, f_{j}\left(u_{3 j} u_{3 j+1}\right)=$ -1 . Let $f$ be a function of $W_{m}$ as follows: if $e \in E\left(T_{3 j-2} \oplus T_{3 j-1} \oplus T_{3 j}\right)$ and $e \notin\left\{v u_{3 j-2}, v u_{3 j+1}\right\}$, then $f(e)=f_{j}(e)$ for each $j \in\{1,2, \ldots, t\}$; if $e \in E\left(T_{3 j-2} \oplus\right.$ $\left.T_{3 j-1} \oplus T_{3 j}\right) \cap E\left(T_{3 j+1} \oplus T_{3 j+2} \oplus T_{3 j+3}\right)$ for each $j \in\{1,2, \ldots, t-1, t\}(\bmod 3 t)$, then $f(e)=f_{j}(e)+f_{j+1}(e)=1-2=-1$. In this case, $f$ is an NZ-unoriented 3 -flow of $W_{m}$.

If $m=3 t+1(t \geq 1)$, then $W_{m}$ is a graph obtained from $T^{3 t}$ by adding one edge. Let $T^{3 t}=T_{1} \oplus T_{2} \oplus \cdots \oplus T_{3 t}$. Then $W_{m}=T^{3 t}+u_{1} u_{3 t+1}$. By Lemma 14, $T^{3 t}$ admits an NZ-unoriented 3-flow $f_{1}$ with $f_{1}\left(u_{1} u_{2}\right)=2, f_{1}\left(u_{2} v\right)=-1$ and $f_{1}\left(v u_{3 t+1}\right)=1$. We know that $u_{1} u_{2} v u_{3 t+1} u_{1}$ is a 4 -cycle. Assign $1,-1,1,-1$ on edges of this cycle starting from $u_{1} u_{2}$. By this way, the combination of these two flows is an NZ-unoriented 4-flow of $W_{m}$.

If $m=3 t+2(t \geq 1)$, then $W_{m}$ is a graph obtained from $T^{3 t+1}$ by adding one edge. Then $W_{m}=T^{3 t+1}+u_{1} u_{3 t+2}$. By Lemma 14 and Lemma $10, W_{m}$ admits an NZ-unoriented 5 -flow.

## 3. Proof of Theorem 4

Lemma 16. A triangle-tree except triangle-star admits a nowhere-zero unoriented 5 -flow.

Proof. We prove this lemma by induction on $|V(G)|$. Since $G$ is not a trianglestar, $n \geq 5$. If $n=5$, then $G$ is a $T^{3}$, by Lemma $14, G$ admits a NZ-unoriented 3 -flow, so NZ-unoriented 5 -flow.

By induction hypothesis, we assume that $G$ admits an NZ-unoriented 5-flow if $G$ is a triangle-tree except triangle-star with less than $n$ vertices $(n \geq 6)$.

Suppose $n \geq 6$ and our theorem holds for $G$ with the number of vertices less than $n$ vertices. Now we prove our theorem holds for $|V(G)|=n$.

Since $G$ is a triangle-tree with $n \geq 6, G$ contains at least three triangles. Since $G$ is not a triangle-star, there exist three triangles constituting a $T^{3}$ which is an induced subgraph of $G$ with one 2 -vertex. Without loss of generality, we assume that $T^{3}=T_{1} \oplus T_{2} \oplus T_{3}$ and $T_{1}$ has one vertex with degree two in $G$. Without loss of generality, we assume that $u$ is the 2 -vertex of $T_{1}$, the other two vertices are $u_{1}, u_{2}$ (see Figure 5). Without loss of generality, we assume that $V\left(T_{2}\right)=\left\{u_{1}, u_{2}, u_{3}\right\}, V\left(T_{3}\right)=\left\{u_{2}, u_{3}, u_{4}\right\}$. In this case, $G=T^{3} \oplus G_{1} \oplus G_{2} \oplus$ $G_{3} \oplus G_{4} \oplus G_{5}$, where $G_{i}$ is a triangle-tree with less then $n$ vertices, and may be
an empty graph. Note that $E\left(G_{i}\right) \cap E\left(T^{3}\right)=e_{i}$ and $E\left(G_{i}\right) \cap E\left(G_{j}\right)=\emptyset$ for $i \neq j$, where $i, j \in\{1,2,3,4,5\}$.


Figure 5. $G$ contains an induced subgraph $T^{3}$ with a 2-vertex $u$.

Claim 1. If $G_{i}$ is not a triangle-star for each $i \in\{1,2,3,4,5\}$, then $G$ admits an NZ-unoriented 5 -flow.

Proof. In this case, by induction hypothesis, each $G_{i}$ admits a NZ-unoriented 5 -flow, say $f_{i}$, respectively. (If $G_{i}$ is an empty graph, then $G_{i}$ is not needed to discuss in this case.) By Lemma $14, T^{3}$ admits an NZ-unoriented 3 -flow, which means the value of each edge is in $\{ \pm 1, \pm 2\}$. Since $E\left(G_{i}\right) \cap E\left(G_{j}\right)=\emptyset$, $E\left(G_{i}\right) \cap E\left(T^{3}\right)=e_{i}$ for each $i \neq j \in\{1,2,3,4,5\}$, by Lemma 13(i), $G$ admits an NZ-unoriented 5 -flow.

By Claim 1, we assume that there is at least one $G_{i}$, such that $G_{i}$ is a triangle-star.

Claim 2. If $G_{i}$ is a 2-triangle-star with the common edge $e_{i}$ or a k-triangle-star with $k \geq 3$ for some $i \in\{1,2,3,4,5\}$, then $G$ admits an NZ-unoriented 5 -flow.

Proof. In this case, $G_{i}$ contains a 4-cycle, say $C$. Hence $G=C \cup H$, where $H$ is a triangle-tree containing $T^{3}$. By induction hypothesis, $H$ admits an NZunoriented 5-flow. By Lemma 7, $C$ admits an NZ-unoriented 2-flow. Thus $G$ admits an NZ-unoriented 5-flow.

Claim 3. If $G_{1}$ is a $k$-triangle-star, then $G$ admits a NZ-unoriented 5-flow.
Proof. By Claim 2, we only need to prove cases that $k=1$ or $k=2$ and the common edge of $G_{1}$ is not $e_{1}$. If $k=1$, then $G_{1} \oplus T_{1}-e_{1}$ is a 4-cycle, say $C$. Thus $G=C \cup H$, and $H$ is a triangle-tree containing $T_{2} \oplus T_{3}$. If $H$ is not a triangle-star, then by induction hypothesis, $H$ admits an NZ-unoriented 5-flow. By Lemma 7, $C$ admits a NZ-unoriented 2-flow. Thus $G$ admits an NZ-unoriented 5-flow. If $H$ is a triangle-star, then $H$ is a 2-triangle-star or a 3 -triangle-star by Claim 2. This means $G=T_{1} \oplus T_{2} \oplus T_{3} \oplus G_{1}$ or $G=T_{1} \oplus T_{2} \oplus T_{3} \oplus G_{1} \oplus G_{3}$, where $G_{1}, G_{3}$ are triangles. By Lemma $14, T_{1} \oplus T_{2} \oplus T_{3}$ admits an NZ-unoriented 3 -flow. By Lemma 12, $G=T_{1} \oplus T_{2} \oplus T_{3} \oplus G_{1}$ admits an NZ-unoriented 4-flow. By Lemma

12, $G=T_{1} \oplus T_{2} \oplus T_{3} \oplus G_{1} \oplus G_{3}$ admits a NZ-unoriented 5-flow. Hence, in either case, $G$ admits an NZ-unoriented 5 -flow.

If $k=2$ and the common edge of $G_{1}$ is not $e_{1}$, then $G_{1} \oplus T_{1}$ is a $T^{3}$. Thus $G=T^{3} \oplus H$, where $H$ is a triangle-tree containing $T_{2} \oplus T_{3}$. If $H$ is not a triangle-star, then by induction hypothesis, $H$ admits an NZ-unoriented 5 -flow. By Lemma 14 and Lemma 13(i), $G$ admits an NZ-unoriented 5 -flow. If $H$ is a triangle-star, then $H$ is a 2 -triangle-star or a 3 -triangle-star by Claim 2. By Lemma 14, $T^{3}$ admits an NZ-unoriented 3-flow. By Lemma $12, T^{3} \oplus H$ admits an NZ-unoriented 5 -flow.

Claim 4. If $G_{2}$ is a $k$-triangle-star, then $G$ admits a NZ-unoriented 5 -flow.
Proof. By Claim 2, $k=1$ or $k=2$ and the common edge of $G_{2}$ is not $e_{2}$. If $k=1$, then $T_{1} \oplus T_{2} \oplus G_{2}$ is a $T^{3}$. Thus $G=T_{1} \oplus T_{2} \oplus G_{2} \oplus H \oplus G_{1}=T^{3} \oplus H \oplus G_{1}$, where $H$ is a triangle-tree with less than $n$ vertices. By Claim 3, $G_{1}$ is either an empty graph or a triangle-tree except a triangle-star. If $G_{1}$ is a triangle-tree except a triangle-star, then by induction hypothesis, $G_{1}$ admits an NZ-unoriented 5-flow.

If $H$ is not a triangle-star, then by induction hypothesis, $H$ admits a NZunoriented 5-flow. By Lemma 14 and $13(\mathrm{i}), G=T^{3} \oplus H \oplus G_{1}$ admits an NZunoriented 5 -flow whatever $G_{1}$ is a triangle-tree except triangle-star or an empty graph. If $H$ is a triangle-star, by Claim $2, H$ is a 1 -triangle-star or 2 -trianglestar with common edge $e_{4}$ or $e_{5}$. Then $G_{2} \oplus T_{2} \oplus H$ is a $T^{3}$ or a $T^{4}$. Thus $G=T^{3} \oplus T_{1} \oplus G_{1}$ or $G=T^{4} \oplus T_{1} \oplus G_{1}$. If $G_{1}$ is an empty graph, then either $G=T^{3} \oplus T_{1}$ or $G=T^{4} \oplus T_{1}$, both of which admit an NZ-unoriented 5 -flow by Lemma 14 and Lemma 12. Thus we assume that $G_{1}$ is not an empty graph. Clearly, $T_{1} \oplus G_{1}$ is a triangle-tree except a triangle-star. By induction hypothesis, $T_{1} \oplus G_{1}$ admits an NZ-unoriented 5 -flow. By Lemma 14 and Lemma 13(i), $G=T^{3} \oplus T_{1} \oplus G_{1}$ admits an NZ-unoriented 5-flow. By Lemma 14, either $T^{4}$ has an NZ-unoriented 3 -flow or $T^{4}$ admits an unoriented 3 -flow just with edge $e_{1}$ having value zero. If $T^{4}$ has an NZ-unoriented 3-flow, by Lemma 13(i), $G=T^{4} \oplus T_{1} \oplus G_{1}$ admits an NZ-unoriented 5 -flow. If $T^{4}$ admits a unoriented 3flow with edge $e_{1}$ having value zero, then combining this flow with NZ-unoriented 5-flow of $T_{1} \oplus G_{1}, G=T^{4} \oplus T_{1} \oplus G_{1}$ admits an NZ-unoriented 5 -flow.

Thus we assume that $k=2$ and the common edge of $G_{2}$ is not $e_{2}$. In this case, $T_{2} \oplus G_{2}$ is a $T^{3}$. Thus $G=T_{1} \oplus G_{1} \oplus T^{3} \oplus H$, where $H$ is a triangle-tree with less than $n$ vertices.

If $H$ is not a triangle-star, then by induction hypothesis, $H$ admits a NZunoriented 5 -flow. When $G_{1}$ is an empty graph. By Lemma $14, T_{1} \oplus G_{1} \oplus T^{3}=T^{4}$ admits an NZ-unoriented 3 -flow or an unoriented 3 -flow such that the value just on edge $e_{3}$ is zero. In either case, by Lemma 13(i), $G$ admits an NZ-unoriented 5 -flow. When $G_{1}$ is a triangle-tree except a triangle-star, $T_{1} \oplus G_{1}$ is a triangle-tree
except triangle-star, $T_{1} \oplus G_{1}$ admits an NZ-unoriented 5-flow. By Lemma 14 and Lemma 13(i), $G=T_{1} \oplus G_{1} \oplus T^{3} \oplus H$ admits a NZ-unoriented 5 -flow.

If $H$ is a triangle-star, then by Lemma 14 and Lemma 12, $G=T_{1} \oplus G_{1} \oplus T^{3} \oplus H$ admits an NZ-unoriented 5 -flow when $G_{1}$ is an empty graph. Next we assume that $G_{1}$ is not an empty graph. In this case, by Claim $2, H$ is a 1-triangle-star or a 2-triangle-star. Then $T^{3} \oplus H=G_{2} \oplus T_{2} \oplus H$ is $T^{4}$ or $T^{5}$. If $T^{3} \oplus H=T^{4}$, then by Lemma 14, $T^{3} \oplus H$ admits an NZ-unoriented 3-flow or a unoriented 3-flow just with edge $e_{1}$ having value zero. By Lemma $13, G=T_{1} \oplus G_{1} \oplus T^{3} \oplus H$ admits an NZ-unoriented 5-flow. If $T^{3} \oplus H=T^{5}$, then in this case $T^{3} \oplus H=G_{2} \oplus T_{2} \oplus T_{3} \oplus G_{i}$, where $G_{i}$ is a triangle and $i=4$ or $i=5$. By Lemma 14, $G_{2} \oplus T_{2} \oplus T_{3}=T^{4}$ admits an NZ-unoriented 3-flow or an unoriented 3-flow just with edge $e_{1}$ having value zero. Then Lemma 12, $T^{5}=G_{2} \oplus T_{2} \oplus T_{3} \oplus G_{i}$ admits an NZ-unoriented 4-flow $f_{1}$ with $f_{1}\left(e_{1}\right) \in\{ \pm 1, \pm 2\}$ or an unoriented 4-flow just with edge $e_{1}$ having value zero. By Lemma 13(i), $G=T_{1} \oplus G_{1} \oplus T^{5}$ admits an NZ-unoriented 5-flow.

Claim 5. If $G_{3}$ is a $k$-triangle-star, then $G$ admits a NZ-unoriented 5-flow.
Proof. By Claim 2, $k=1$ or $k=2$ and the common edge of $G_{3}$ is not $e_{3}$. If both $G_{4}$ and $G_{5}$ are empty graphs, then we can discuss this case similarly as Claim 3. Hence we can assume that at least one of $G_{4}$ and $G_{5}$ is not an empty graph. By Claims 3 and $4, G_{i}$ is either a triangle-tree except triangle-star or an empty graph for each $i \in\{1,2\}$. If $k=1$, then $T_{1} \oplus T_{2} \oplus G_{3}$ is a $T^{3}$. Then $G=T^{3} \oplus G_{1} \oplus G_{2} \oplus H$, where $H$ is a triangle-tree containing $T_{3}, G_{4}$ and $G_{5}$. If $H$ is not a triangle-star, then by induction hypothesis, $H$ admits an NZ-unoriented 5-flow. By Lemma 14 and Lemma $13(\mathrm{i}), G$ admits an NZ-unoriented 5 -flow. Thus we assume $H$ is a triangle-star. By Claim 2, $H$ is a 2-triangle-star. In this case, $G_{3} \oplus H$ is a $T^{3}$ and $G=G_{3} \oplus H \oplus L=T^{3} \oplus L$, where $L$ contains $G_{1}, G_{2}, T_{1}, T_{2}$. If $L$ is a triangle-tree except a triangle-star, then by induction hypothesis and Lemma 14, Lemma 13(i), $G$ admits an NZ-unoriented 5-flow. If $L$ is a triangle-star, then $L$ is a 2-triangle-star. By Lemma 14 and Lemma 12, $G$ admits an NZ-unoriented 5 -flow.

If $k=2$, then $T_{1} \oplus T_{2} \oplus G_{3}$ is a $T^{4}$. Then $G=T^{4} \oplus G_{1} \oplus G_{2} \oplus H$, where $H$ is a triangle-tree containing $T_{3}, G_{4}$ and $G_{5}$. If $H$ is not a triangle-star, then by induction hypothesis, $H$ admits a NZ-unoriented 5 -flow. By Lemma $14, T_{1} \oplus$ $T_{2} \oplus G_{3}$ admits a NZ-unoriented 3-flow or an unoriented 3-flow just with edge $e_{3}$ having value zero. By Lemma $13(\mathrm{i}), G$ admits an NZ-unoriented 5 -flow. Thus we assume $H$ is a triangle-star. By Claim $2, H$ is a 2 -triangle-star. In this case, $G_{3} \oplus H$ is a $T^{4}$ and $G=G_{3} \oplus H \oplus L=T^{4} \oplus L$, where $L$ contains $G_{1}, G_{2}, T_{1}, T_{2}$. If $L$ is a triangle-tree except a triangle-star, then by induction hypothesis, $L$ admits an NZ-unoriented 5-flow. By Lemma 14, $T^{4}$ admits an NZ-unoriented 3-flow or an unoriented 3 -flow with edge $e_{3}$ having value zero. By Lemma 13(i), $G$ admits a NZ-unoriented 5-flow. If $L$ is a triangle-star, by Lemma 14 and Lemma 12, G
admits an NZ-unoriented 5-flow.
Claim 6. If $G_{4}$ is a $k$-triangle-star, then $G$ admits a NZ-unoriented 5 -flow.
Proof. By Claim 2, $k=1$ or $k=2$ and the common edge of $G_{4}$ is not $e_{4}$. By Claims $3,4,5, G_{i}$ is an empty graph or triangle-tree except a triangle-star for $i \in\{1,2,3\}$. If $k=1$, then $T_{2} \oplus T_{3} \oplus G_{4}$ is a $T^{3}$. Then $G=T^{3} \oplus\left(G_{1} \oplus T_{1}\right) \oplus$ $G_{2} \oplus G_{3} \oplus G_{5}$.

Suppose first $G_{5}$ is either an empty graph or a triangle-tree except a trianglestar. If $G_{1}$ is not an empty graph, then $G$ admits an NZ-unoriented 5 -flow by induction hypothesis, Lemma 14 and Lemma $13(\mathrm{i})$. If $G_{1}$ is an empty graph, then $T_{1} \oplus T_{2} \oplus T_{3} \oplus G_{4}$ is a $T^{4}$. If $G_{2}, G_{3}, G_{5}$ are empty graphs, then $G=T^{4}$, which admits an NZ-unoriented 4 -flow by Lemma 14. If there exists one $G_{i}$ such that $G_{i}$ is not empty for some $i \in\{2,3,5\}$, then by Lemma $14, T^{4}$ admits an NZ-unoriented 3 -flow or an unoriented 3 -flow just with edge $e_{i}$ having value zero. By Lemma 13(i), $G$ admits an NZ-unoriented 5 -flow.

Suppose $G_{5}$ is a triangle-star. By Claim 2, $G_{5}$ is a 1-triangle-star or a 2 -triangle-star. In this case, $G=G_{4} \oplus T_{3} \oplus G_{5} \oplus H$, where $H$ is a triangle-tree containing $T_{1}, T_{2}, G_{1}, G_{2}, G_{3}$. Clearly, $G_{4} \oplus T_{3} \oplus G_{5}$ is a $T^{3}$ or a $T^{4}$. If $H$ is not triangle-star, then by induction hypothesis, $H$ admits an NZ-unoriented 5 -flow. By Lemma 14, $G_{4} \oplus T_{3} \oplus G_{5}$ admits an NZ-unoriented 3-flow or an unoriented 3flow just with edge $e_{3}$ having value zero. Then $G$ admits an NZ-unoriented 5 -flow by Lemma $13(\mathrm{i})$. Thus we can assume that $H$ is a triangle-star. By Lemma 14 and Lemma 12, $G$ admits an NZ-unoriented 5 -flow.

If $k=2$ and the common edge of $G_{4}$ is not $e_{4}$, then $T_{3} \oplus G_{4}$ is a $T^{3}$. Then $G=T^{3} \oplus H \oplus G_{5}$, where $H$ contains $T_{1}, T_{2}, G_{1}, G_{2}, G_{3}$. If $H, G_{5}$ are triangletrees except triangle-star, then by induction hypothesis, $H$ and $G_{5}$ admit NZunoriented 5 -flow, respectively. By Lemma 14 and Lemma 13(i), $G$ admits an NZ-unoriented 5 -flow. If $G_{5}$ is a triangle-star, then by Claim $2, G_{5}$ is a 1 -trianglestar or a 2 -triangle-star with common edge which is not $e_{5}$. In either case, by Lemmas 14 and $12, T^{3} \oplus G_{5}$ admits an NZ-unoriented 4 -flow with the value on $e_{3}$ in $\{ \pm 1, \pm 2\}$ or an unoriented 4 -flow just with edge $e_{3}$ having value 0 . (If $G_{5}$ is a 1-triangle-star, then $T^{3} \oplus G_{5}$ is a $T^{4}$, by Lemma $14, T^{3} \oplus G_{5}$ admits an NZ-unoriented 3 -flow or an unoriented 3 -flow just with edge $e_{3}$ having value 0 , we have done. If $G_{5}$ is a 2 -triangle-star, then $T^{3} \oplus G_{5}$ is a $T^{5}$, by Lemma $14, T^{4}$ admits an NZ-unoriented 3 -flow or an unoriented 3 -flow just with edge $e_{3}$ having value 0 . By Lemma 12, $T^{5}$ admits an NZ-unoriented 4 -flow with the value on $e_{3}$ in $\{ \pm 1, \pm 2\}$ or an unoriented 4 -flow just with edge $e_{3}$ having value 0 .) When $H$ is not a triangle-star. By induction hypothesis and Lemma 13(i), $G$ admits an NZ-unoriented 5 -flow. When $H$ is a triangle-star, by Lemma 14, $T^{3} \oplus G_{5}$ admits an NZ-unoriented 4 -flow. By Lemma 12, $G$ admits an NZ-unoriented 5 -flow. Next we only need to prove the case that $G_{5}$ is not a triangle-star and $H$ is a
triangle-star. By Claim 2, $H$ is a 2-triangle-star. This means $H \oplus T_{3} \oplus G_{4}$ is a $T^{5}$. By Lemma 14, $H \oplus T_{3} \oplus G_{4}$ admits an NZ-unoriented 4 -flow with value on edge $e_{5}$ in $\{ \pm 1, \pm 2\}$ or an unoriented 4 -flow just with edge $e_{5}$ having value 0 . By induction hypothesis and Lemma 13(i), $G$ admits an NZ-unoriented 5 -flow.

Claim 7. If $G_{5}$ is a $k$-triangle-star, then $G$ admits a NZ-unoriented 5 -flow.
Proof. By Claim 2, $k=1$ or $k=2$ and the common edge of $G_{5}$ is not $e_{5}$. If $k=1$, then $T_{2} \oplus T_{3} \oplus G_{5}$ is a $T^{3}$. Then $G=T^{3} \oplus\left(G_{1} \oplus T_{1}\right) \oplus G_{2} \oplus G_{3} \oplus G_{4}$. By Claims 3, 4, 5, 6, $G_{i}$ is not a triangle-star, for each $i \in\{1,2,3,4\}$.

If $G_{1}$ is not empty graph, then $G$ admits an NZ-unoriented 5 -flow by induction hypothesis, Lemma 14 and Lemma 13(i). If $G_{1}$ is an empty graph, then $T_{1} \oplus$ $T_{2} \oplus T_{3} \oplus G_{5}$ is a $T^{4}$. If $G_{2}, G_{3}, G_{4}$ are empty, then $G=T^{4}$, which admits an NZ-unoriented 4 -flow by Lemma 14. If there exists one $G_{i}$ such that $G_{i}$ is not empty for some $i \in\{2,3,4\}$, then by Lemma $14, T^{4}$ admits an NZ-unoriented 3 -flow or an unoriented 3 -flow just with edge $e_{i}$ having value zero. By Lemma 13(i), $G$ admits a NZ-unoriented 5 -flow.

If $k=2$ and the common edge of $G_{5}$ is not $e_{5}$, then $T_{3} \oplus G_{5}$ is a $T^{3}$. Then $G=T^{3} \oplus H \oplus G_{4}$, where $H$ contains $T_{1}, T_{2}, G_{1}, G_{2}, G_{3}$. If $H$ is a triangle-tree except triangle-star, then by induction hypothesis, $H$ admits an NZ-unoriented 5 -flow. By Lemma 14 and Lemma 13(i), $G$ admits an NZ-unoriented 5 -flow. If $H$ is a triangle-star, then $H$ is a 2 -triangle-star by Claim 2. In this case, $T_{3} \oplus G_{5} \oplus H$ is a $T^{5}$. If $G_{4}$ is empty, then $G$ is a $T^{5}$. Hence $G$ admits an NZunoriented 4 -flow by Lemma 14 . Thus $G_{4}$ is a triangle-tree except triangle-star. By induction hypothesis, $G_{4}$ admits an NZ-unoriented 5 -flow. By Lemma 14, $T_{3} \oplus G_{5} \oplus H$ admits an unoriented 4 -flow just with edge $e_{4}$ having value zero or an NZ-unoriented 4 -flow with value on edge $e_{4}$ in the set $\{ \pm 1, \pm 2\}$. Thus $T_{3} \oplus G_{5} \oplus H \oplus G_{4}$ admits a NZ-unoriented 5 -flow.

By Claims 3, 4, 5, 6, 7, we can assume that $G_{i}$ is an empty graph or a triangle-tree except a triangle-star for $i \in\{1,2,3,4,5\}$. By induction hypothesis, Lemma 14, Lemma 13(i), $G$ admits an NZ-unoriented 5 -flow.

In the rest of this paper, we assume that $G$ is not a triangle-star and can be partitioned into some triangle-paths or wheels $H_{1}, H_{2}, \ldots, H_{t}$ such that $G=$ $H_{1} \oplus H_{2} \oplus \cdots \oplus H_{t}$, where $H_{i}$ is a triangle-path or a wheel, where $i \in\{1,2, \ldots, t\}$ and $t \geq 1$.

Proof of Theorem 4. Let $m$ be the number of wheels in $H_{1}, H_{2}, \ldots, H_{t}$. If $m=0$, then $G$ is a triangle-tree except a triangle-star. By Lemma 16, $G$ admits a NZ-unoriented 5 -flow, hence an NZ-unoriented 6 -flow. If $m=1$, then without loss of generality, we assume that $H_{j}$ is a wheel for some $1 \leq j \leq t$. By the definition of $G, H_{j}$ and $H_{1} \oplus H_{2} \oplus \cdots \oplus H_{j-1}$ just have one common edge. Since $H_{j}$ is a wheel, $H_{j}$ has at least three edges which is not adjacent to its center. Then
we can choose an edge, say $e_{0}$, of $H_{j}$ such that $e_{0}$ is not adjacent to the center of $H_{j}$ and is not an edge of $H_{1} \oplus H_{2} \oplus \cdots \oplus H_{j-1}$. Then $H_{j}-e_{0}$ is a triangle-path with at least two triangles. If $e_{0}$ is not an edge of $H_{j+1} \oplus H_{j+2} \oplus \cdots \oplus H_{t}$, then $G=T+e_{0}$, where $T=H_{1} \oplus H_{2} \oplus \cdots \oplus H_{j-1} \oplus\left(H_{j}-e_{0}\right) \oplus H_{j+1} \oplus \cdots \oplus H_{t}$. If $T$ is not a triangle-star, then $T$ is a triangle-tree. By Lemma $16, T$ admits an NZ-unoriented 5 -flow. By Lemma 10, $G=T+e$ admits an NZ-unoriented 6 -flow. If $T$ is a triangle-star, then $H_{j}$ is 3 -wheel and other $H_{i}$ s are triangle-star with a common edge. In this case, $G=W_{3} \oplus S$, where $S$ is a $k$-triangle-star and $W_{3}$ is a 3-wheel, $E\left(W_{3}\right) \cap E(S)$ is the common edge of $S$. If $k=0$, then $G=W_{3}$. By Lemma $15, G$ admits an NZ-unoriented 3 -flow, hence an NZ-unoriented 6 -flow. Thus $k \geq 1$. By Lemma 15 and Lemma 12, $G$ admits an NZ-unoriented 5 -flow, hence NZ-unoriented 6 -flow.

If $e_{0}$ is also an edge of $H_{j+1} \oplus H_{j+2} \oplus \cdots \oplus H_{t}$, then set $e_{0}=u v$ and $G-\{u, v\}$ contains at least two connected components such that one component, say $H$, contains the center of $H_{j}$. Let $G_{1}$ be the subgraph comprised of $H,\{u, v\}$ and $E(H,\{u, v\}), G_{2}=G-E\left(G_{1}\right)$. Clearly, $G=G_{1} \cup G_{2}$ and $e_{0} \in E\left(G_{2}\right)$, $E\left(G_{1}\right) \cap E\left(G_{2}\right)=\emptyset$. By the definition of $G_{1}$ and $G_{2}, G_{i}$ is a triangle-tree for each $i \in\{1,2\}$.

If $G_{i}$ is not a triangle-star for each $i \in\{1,2\}$, then by Lemma $16, G_{i}$ admits an NZ-unoriented 5 -flow, say $f_{i}$, for each $i \in\{1,2\}$. Since $E\left(G_{1}\right) \cap E\left(G_{2}\right)=\emptyset$, we can define $f$ on $E(G)$ as follows. Let $f(e)=f_{1}(e)$ when $e \in E\left(G_{1}\right), f(e)=f_{2}(e)$ when $e \in E\left(G_{2}\right)$. Clearly, $f$ is an NZ-unoriented 5 -flow of $G$, hence an NZunoriented 6 -flow.

Next we assume that there exists one $G_{i}$ such that $G_{i}$ is a triangle-star. Without loss of generality, we assume that $G_{1}$ is not a triangle-star and $G_{2}$ is a $k$-triangle-star. In this case, $G_{1}$ is a triangle-tree except a triangle-star and without loss of generality, we assume that the common edge of $G_{2}$ is $e_{1}$. (The proofs are the same whatever $e_{1}$ is $e_{0}$ or not.) By Lemma 16, $G_{1}$ admits an NZ-unoriented 5 -flow. If $k=1$, then by Lemma 11, $G$ admits an NZ-unoriented 6 -flow. If $k \geq 2$ and $k$ is even, then $G_{2}-e_{1}$ is an even circuit. By Lemma $7, G_{2}-e_{1}$ admits a NZ-unoriented 2-flow, hence the combination of flows on $G_{1}$ and $G_{2}-e_{1}$ is an NZ-unoriented 5 -flow of $G-e_{1}=G_{1} \cup\left(G_{2}-e_{1}\right)$. Thus $G$ admits a NZunoriented 6 -flow by Lemma 10 . If $k \geq 2$ and $k$ is odd, then $G_{2}-E(T)$ is an even circuit, where $T$ is an arbitrary triangle of $G_{2}$. By Lemma $7, G_{2}-E(T)$ admits an NZ-unoriented 2-flow, hence the combination of flows on $G_{1}$ and $G_{2}-E(T)$ is an NZ-unoriented 5 -flow of $G-E(T)$. Thus $G$ admits an NZ-unoriented 6 -flow by Lemma 11 .

If $G_{i}$ is a triangle-star for each $i \in\{1,2\}$, then without loss of generality, we assume that $G_{i}$ is a $k_{i}$-triangle-star. Since $H_{j}-e_{0}$ is contained in $G_{1}, k_{1} \geq 2$ and $H_{j}$ is a 3 -wheel. In this case, $G=S_{1} \oplus W_{3} \oplus G_{2}$, where $S_{1}$ is a ( $k_{1}-2$ )-trianglestar and $E\left(S_{1}\right) \cap E\left(G_{2}\right)=\emptyset$. If $k_{1}=2$, then $G=W_{3} \oplus G_{2}$. By Lemma 15
and Lemma 12, $G$ admits an NZ-unoriented 5-flow. If $k_{1} \geq 3$, by Lemma 15 and Lemma 12, $S_{1} \oplus W_{3}$ admits an NZ-unoriented 5 -flow. By Lemma 12, $G=$ $S_{1} \oplus W_{3} \oplus G_{2}$ admits an NZ-unoriented 6-flow.

Thus by induction hypothesis, theorem holds for less then $m$ wheels in $\left\{H_{1}, H_{2}, \ldots, H_{t}\right\}$. Next we prove the case that there are $m$ wheels in $\left\{H_{1}, H_{2}\right.$, $\left.\ldots, H_{t}\right\}$.

If there exists an edge on cycle of a wheel such that this edge is also an edge of another connected subgraph except a triangle-star, that is, $G=G_{1} \oplus G_{2}$ and $E\left(G_{1}\right) \cap E\left(G_{2}\right)=\left\{e_{0}\right\}$, where $G_{1}$ contains a wheel which $e_{0}$ is an edge of the cycle of this wheel, $G_{2}$ is not a triangle-star. In this case, we can choose $G_{1}$ such that $e_{0}$ is not a common edge in $G_{1}$. Then $G=\left(G_{1}-e_{0}\right) \cup G_{2}$. If $G_{1}-e_{0}$ is not a triangle-star, then by induction hypothesis, $G_{1}-e_{0}$ and $G_{2}$ admit an NZ-unoriented 6-flow. Then $G$ admits an NZ-unoriented 6 -flow. If $G_{1}-e_{0}$ is a triangle-star, then $G_{1}=W_{3} \oplus S$, where $S$ is a triangle-star and $E(S) \cap E\left(G_{2}\right)=\emptyset$. By Lemma 15 and Lemma 12, $G_{1}$ admits an NZ-unoriented 4-flow with $e_{0}$ having value in the set $\{ \pm 1, \pm 2\}$. By Lemma $13(\mathrm{i}), G$ admits an NZ-unoriented 6 -flow. Then we can assume that each edge of a cycle of wheel is not contained in other subgraph of $G$ or is an edge of another connected subgraph which is a trianglestar. This means edges of cycle of wheel can be divided into two types.

Type 1. edge is contained in just one $H_{i}$, where $H_{i}$ is a wheel;
Type 2. edge is contained in one $H_{i}$ and other $H_{j} \mathrm{~s}$, where $H_{i}$ is a wheel, other $H_{j}$ s constitute a triangle-star.

For each wheel of $G$, we can choose one edge from cycles of wheel, say $e_{1}, e_{2}, \ldots, \ldots, e_{m}$, such that the number of edges of type 1 is maximal and $e_{i}, e_{j}$ are different edges from different wheels. If all $e_{i} \mathrm{~s}$ are edges of type 1, then $G-\left\{e_{1}, \ldots, e_{m}\right\}$ is a triangle-tree. If $G-\left\{e_{1}, \ldots, e_{m}\right\}$ is not a triangle-star, then by Lemma $16, G-\left\{e_{1}, \ldots, e_{m}\right\}$ admits an NZ-unoriented 5 -flow. Since $e_{1}, \ldots, e_{m}$ are contained in different wheels, by Lemma $10, G$ admits an NZ-unoriented 6flow. If $G-\left\{e_{1}, \ldots, e_{m}\right\}$ is a triangle-star, then $H_{i}$ is a $W_{3}$ or a $k$-triangle-path, where $k \in\{1,2\}$ for each $i \in\{1,2, \ldots, t\}$ and all $H_{i}$ s have a common edge. Without loss of generality, we assume that $H_{i}$ is $W_{3}$ for each $i \in\{1,2, \ldots, m\}$. Clearly, others constitute a triangle-star. By types 1 and 2, we can deduce that $m=1$ since each edge of $W_{3}$ can be an edge of cycle of this wheel. This case we have done.

If there exist an edge, say $e_{1}$, which is an edge of type 2 , then by the maximization, each edge of cycle of this wheel is an edge of type 2 . This means $m=1$, and we have done.

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## References

[1] S. Akbari, N. Ghareghani, G.B. Khosrovshahi and A. Mahmoody, On zero-sum 6flows of graphs, Linear Algebra Appl. 430 (2009) 3047-3052. https://doi.org/10.1016/j.laa.2009.01.027
[2] S. Akbari, A. Daemi, O. Hatami, A. Javanmard and A. Mehrabian, Zero-sum flows in regular graphs, Graphs Combin. 26 (2010) 603-615. https://doi.org/10.1007/s00373-010-0946-5
[3] S. Akbari, A. Daemi, O. Hatami, A. Javanmard and A. Mehrabian, Nowhere-zero unoriented flows in Hamiltonian graphs, Ars Combin. 120 (2015) 51-63.
[4] S. Akbari, N. Ghareghani, G.B. Khosrovshahi and S. Zare, A note on zero-sum 5flows in regular graphs, Electron. J. Combin. 19 (2012) \#P7. https://doi.org/10.37236/2145
[5] A. Bouchet, Nowhere-zero integeral flows on a bidirected graph, J. Combin. Theory Ser. B 34 (1983) 279-292. https://doi.org/10.1016/0095-8956(83)90041-2
[6] M. DeVos, Flows in bidirected graphs, Mathematics 43 (2013) 95-115.
[7] J. Edmonds, Maximum matching and a polyhedron with 0, 1-vertices, J. Res. Nat. Bur. Stand. 69B (1965) 125-130. https://doi.org/10.6028/jres.069B.013
[8] F. Jaeger, Nowhere-zero flow problems, in: Selected topics in Graph Theory 3, L.W. Beineke and R.J. Wilson (Ed(s)), (Academic Press, London, 1988) 70-95.
[9] M. Kano, Factors of regular graphs, J. Combin. Theory Ser. B 41 (1986) 27-36. https://doi.org/10.1016/0095-8956(86)90025-0
[10] B. Korte and J. Vygen, Combinatorial Optimization: Theory and Algorithms (Springer, Berlin, 2006).
https://doi.org/10.1007/3-540-29297-7
[11] A. Raspaud and X. Zhu, Circular flow on signed graphs, J. Combin. Theory Ser. B 101 (2011) 464-479.
https://doi.org/10.1016/j.jctb.2011.02.007
[12] W.T. Tutte, On the imbedding of linear graphs in surfaces, Proc. Lond. Math. Soc. (2) 51 (1949) 474-483. https://doi.org/10.1112/plms/s2-51.6.474
[13] O. Zyka, Nowhere-zero 30-flows on bidirected graphs, Thesis (Charles University, Praha, 1987).
[14] F. Yang and X. Li, Zero-sum 6-flows in 5-regular graphs, Bull. Malays. Math. Sci. Soc. 42 (2019) 1319-1327.
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