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ANTIMAGIC LABELING OF SOME BIREGULAR BIPARTITE GRAPHS

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Abstract

An antimagic labeling of a graph $G = (V, E)$ is a one-to-one mapping from E to $\{1, 2, \ldots, |E|\}$ such that distinct vertices receive different label sums from the edges incident to them. G is called antimagic if it admits an antimagic labeling. It was conjectured that every connected graph other than K_2 is antimagic. The conjecture remains open though it was verified for several classes of graphs such as regular graphs. A bipartite graph is called (k, k')-biregular, if each vertex of one of its parts has the degree k, while each vertex of the other parts has the degree k'. This paper shows the following results. (1) Each connected $(2, k)$-biregular $(k \geq 3)$ bipartite graph is antimagic; (2) Each (k, pk)-biregular $(k \geq 3, p \geq 2)$ bipartite graph is antimagic; (3) Each $(k, k^2 + y)$-biregular $(k \geq 3, y \geq 1)$ bipartite graph is antimagic.

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1. Introduction

Let $G = (V, E)$ be a graph. Suppose f is a one-to-one mapping from E to $\{1, 2, \ldots, |E|\}$. For each vertex v in V, the *vertex sum* $\varphi_f(v)$ at v under f is defined as $\varphi_f(v) = \sum_{e \in E(v)} f(e)$, where $E(v)$ is the set of edges incident to v. If $\varphi_f(u) \neq \varphi_f(v)$ for any vertex pair $u, v \in V$, then f is called an *antimagic labeling* of G. A graph G is called *antimagic* if G admits an antimagic labeling. The antimagic labeling of graphs was introduced by Hartsfield and Ringel [8] in 1989 (also in [9]), who verified the antimagicnesses of paths, 2-regular graphs and complete graphs. Moreover, they put forth the following conjecture.

Conjecture 1 [9]**.** *Every connected graph other than K_2 is antimagic.*

The conjecture has received much attention, but remains open. It was proved by Alon *et al.* [1] that there is an absolute constant c such that graphs with minimum degree $\delta(G) \geq c \log |V|$ are antimagic, and graphs with maximum degree at least $|V| - 2$ and complete bipartite graphs except K_2 are antimagic. And then graphs of large linear size were shown to be antimagic [6]. For regular graphs, the antimagicnesses of k-regular ($k \geq 2$) bipartite graphs [3], cubic graphs [12], odd degree regular graphs [4], and finally even regular graphs [2] were verified, respectively. For more results on antimagic labeling such as those about trees, one can refer to [5, 10, 11, 13, 14, 17] and the survey of Gallian [7].

A bipartite graph is called (k, k')-*biregular*, if each vertex in one of its two parts has the degree k, while each vertex in the other part has the degree k'. This paper shows the following results. (1) Each connected $(2, k)$-biregular ($k \geq 3$) bipartite graph is antimagic; (2) Each (k, pk)-biregular ($k \geq 3, p \geq 2$) bipartite graph is antimagic; (3) Each $(k, k^2 + y)$-biregular ($k \geq 3, y \geq 1$) bipartite graph is antimagic. The first result is shown in Section 2, where we treat each connected $(2, k)$-biregular ($k \geq 3$) bipartite graph as the subdivision graph of a connected k-regular graph. A *subdivision graph* G_s of a graph G, is obtained from G by replacing each edge with a path of length two. The second and the third results are shown in Section 3, based on an extended version of Hall's matching theorem [15, 16].

2. Connected $(2, k)$-Biregular ($k \geq 3$) Bipartite Graph

With respect to a given labeling, two vertices are *in conflict* if they have a common vertex sum. When we have labeled a subset of the edges, we call the resulting sum at each vertex a *partial vertex-sum*. For short, we denote by $[i, j]$ the integer set $\{i, i + 1, \ldots, j\}$ for integers i and j (where $i < j$).

Theorem 2. *The subdivision graph G_s of every connected k-regular ($k \geq 3$) graph G is antimagic.*

Proof. Choose an arbitrary vertex v^* in G as a root. Let α be the longest distance of a vertex from v^* in G. Suppose $i \in [1, \alpha]$. Denote by V_i the sets of vertices at distance exactly i from v^*, by $G[V_i]$ the subgraph induced by V_i, and by $G[V_{i-1}; V_i]$ (here we suppose $V_0 = \{v^*\}$) the induced bipartite subgraph with parts V_{i-1} and V_i, respectively. For $v \in V_i$, let $\sigma(v)$ be an arbitrary edge in $G[V_{i-1}; V_i]$ which is incident to v. Let $\sigma(V_i) = \{\sigma(v) \mid v \in V_i\}$ and $G_\sigma[V_{i-1}; V_i] = G[V_{i-1}; V_i] \setminus \sigma(V_i)$.

Now subdivide G into G_s. Then every vertex in V_i is at distance exactly $2i$ from v^* in G_s. Denote by S_i, U_i and W_i the newly added vertex sets on the edges of $G[V_i]$, $G_\sigma[V_{i-1}; V_i]$ and $\sigma(V_i)$, respectively, when subdividing G into G_s. Let $X = \bigcup_{i=1}^{\alpha} X_i$ for $X = V, S, U, W$. For a vertex $v \in V_i$, let $w(v)$ be the vertex in W_i which is adjacent to v. For every vertex $x \in (S_i \cup U_i \cup W_i)$, let $\underline{e}^x$ and $\overline{e}^x$ be the two edges incident to x. If $x \in (U_i \cup W_i)$, we suppose $\underline{e}^x$ is incident to some vertex in V_i, while $\overline{e}^x$ is incident to some vertex in V_{i-1}. For $X = S, U, W$, let $\underline{E}_i^X = \{\underline{e}^x \mid x \in X_i\}$, $\overline{E}_i^X = \{\overline{e}^x | x \in X_i\}$ and $E_i^X = \underline{E}_i^X \cup \overline{E}_i^X$.

Respect to a labeling f on $E(G_s)$, if $v \in V_i$, we denote the partial sum at v (omitting the label on $\underline{e}^{w(v)}$) by $p(v) = \sum_{e \in E(v) \setminus \{\underline{e}^{w(v)}\}} f(e) = \varphi_f(v) - f(\underline{e}^{w(v)})$. Let $p(v^*) = \varphi_f(v^*) - f(e^*)$ where e^* is the edge in $E(v^*)$ which receives the greatest label among $E(v^*)$.

Note that $V(G_s) = V \cup S \cup U \cup W \cup \{v^*\}$. To show G_s is antimagic, we will construct a labeling f which satisfies the following conditions.

(1) The vertex sums in X_i are all odd and pairwise different, for $X \in \{S, U, W\}$ and $i \in [1, \alpha]$.

(2) The vertex sums in V_i are all even and pairwise different for $i \in [1, \alpha]$.

(3) The vertex sums in $(S_i \cup U_i \cup W_i)$ are smaller than those in $(S_{i-1} \cup U_{i-1} \cup W_{i-1})$ for $i \in [2, \alpha]$.

(4) The vertex sums in S_i are smaller than those in U_i, while the later ones are smaller than those in W_i for $i \in [1, \alpha]$.

(5) The vertex sums in V_i are smaller than those in V_{i-1} for $i \in [2, \alpha]$.

(6) The vertex sum at v^* is greater than those in V_1 and those in W_1.

Conditions (1) and (2) make sure there is no conflict between V and $(S \cup U \cup W)$. Conditions (1), (3), (4) make sure there is no conflict inside $(S \cup U \cup W)$. Conditions (2) and (5) make sure there is no conflict inside V. Conditions (3), (4), (5) and (6) make sure there is no conflict between v^* and any other vertex in G_s. So these conditions imply that f is antimagic.

Note that $E(G_s) = \bigcup_{i=1}^{\alpha} (E_i^S \cup E_i^U \cup E_i^W)$. We will label $E(G_s)$ in the order E_α^S, $(E_\alpha^U \cup E_\alpha^W)$, $E_{\alpha-1}^S$, $(E_{\alpha-1}^U \cup E_{\alpha-1}^W), \ldots, E_1^S$, $(E_1^U \cup E_1^W)$, using the smallest unused labels on each edge set when we come to it. This label assignment immediately implies that (3) holds, and that the vertex sums in S_i are smaller than those in $(U_i \cup W_i)$ for $i \in [1, \alpha]$.

Suppose $i \in [1, \alpha]$ in the following. Note that $|E_i^X| = 2|X_i|$, for $X = S, U, W$.

(I) The labeling of E^S. We first label $\underline{E}_i^S$ arbitrarily using the $|S_i|$ odd labels from the $2|S_i|$ assigned labels for E_i^S. Secondly let $f(\overline{e}^s) = f(\underline{e}^s) + 1$ for each $s \in S_i$. Then the vertex sums in S_i are odd and pairwise different.

(II) The labeling of $\left(E^U \cup E^W\right)$. If $|U_i|$ is odd, then $i \in [2, \alpha]$, since U_1 is an empty set. We will label $(E_i^U \cup E_i^W)$ in the order $\underline{E}_i^U$, $\overline{E}_i^U$, E_i^W using the smallest unused assigned labels on each edge subset when we come to it. This sub-assignment (based on our global assignment), gives that $p(v) < p(v')$ for arbitrary $v \in V_i$ and $v' \in V_{i-1}$, which implies $\varphi_f(v) = p(v) + f(vw(v)) < p(v') + f(v'w(v')) = \varphi_f(v')$, since $f(vw(v)) < f(v'w(v'))$ by our global assignment. So (5) holds for those i with $|U_i|$ being odd. It gives that the vertex sums in U_i are smaller than those in W_i. So (4) holds for those i with $|U_i|$ being odd. We first label $\underline{E}_i^U$ arbitrarily using its assigned labels. Secondly let $f(\overline{e}^u) = f(\underline{e}^u) + |U_i|$ for each $u \in U_i$. This gives that the vertex sums in U_i are odd and pairwise different. Third, suppose $V_i = \left\{v_1, v_2, \ldots, v_{|V_i|}\right\}$ where $p(v_1) \leq p(v_2) \leq \cdots \leq p(v_{|V_i|})$. For $r \in [1, |V_i|]$, label $\underline{e}^{w(v_r)}$ with the r-th smallest label among the odd (even) assigned labels for E_i^W, when $p(v_r)$ is odd (even). This implies that the vertex sums in V_i are even and pairwise different. So (2) holds for those i with $|U_i|$ being odd. Fourth, let $f(\overline{e}^w) = f(\underline{e}^w) + 1$ when $f(\underline{e}^w)$ is odd, while $f(\overline{e}^w) = f(\underline{e}^w) - 1$ when $f(\underline{e}^w)$ is even. This implies that vertex sums in W_i are odd and pairwise different. So (1) holds for those i with $|U_i|$ being odd.

If $|U_i|$ is even ($|U_i|$ may equal to 0), then $i \in [1, \alpha]$. We will label edges in E_i^U using the smallest $(2|U_i| + 1)$ assigned labels for $E_i^U \cup E_i^W$ except the $(|U_i| + 1)$-th smallest one (denoted by $\xi_{|U_i|+1}$). We first label the edges of $\underline{E}_i^U$ arbitrarily using the $|U_i|$ smallest assigned labels. This gives that $p(v) < p(v')$ for arbitrary $v \in V_i$ and $v' \in V_{i-1}$. And then, if $i \neq 1$, one has $\varphi_f(v) = p(v) + f(vw(v)) < p(v') + f(v'w(v')) = \varphi_f(v')$ for arbitrary $v \in V_i$ and $v' \in V_{i-1}$, since $f(vw(v)) < f(v'w(v'))$ by our global assignment. So (5) also holds for those i $(i \neq 1)$ with $|U_i|$ being even. Secondly let $f(\overline{e}^u) = f(\underline{e}^u) + |U_i| + 1$ for each $u \in U_i$. This implies that the vertex sums in U_i are odd and pairwise different. It also implies that the vertex sums in U_i are smaller than those in W_i, since any pair of the rest assigned labels left for W_i has a sum greater than any vertex sum in U_i. So (4) also holds for those i with $|U_i|$ being even. Note that, $\xi_{|U_i|+1}$ and $(\xi_{|U_i|+1} + |U_i| + 1)$ have distinct parity, and so far, they are the smallest two unused assigned labels for W_i. Third, suppose $|V_i| = \left\{v_1, v_2, \ldots, v_{|V_i|}\right\}$ where $p(v_1) \leq p(v_2) \leq \cdots \leq p\left(v_{|V_i|}\right)$. For $r \in [1, |V_i|]$, label $\underline{e}^{w(v_r)}$ with the r-th smallest label among the rest odd (even) assigned labels, if $p(v_r)$ is odd (even). This implies that the vertex sums in V_i are even and pairwise different. So (2) also holds for those i with $|U_i|$ being even. And note that either $\xi_{|U_i|+1}$ or $(\xi_{|U_i|+1} + |U_i| + 1)$ is assigned to $w(v_1)$ by our labeling way. Fourth, let $f\left(\overline{e}^{w(v_1)}\right) = \xi_{|U_i|+1}$ if $f\left(\underline{e}^{w(v_1)}\right) = \xi_{|U_i|+1} + |U_i| + 1$, while $f\left(\overline{e}^{w(v_1)}\right) = \xi_{|U_i|+1} + |U_i| + 1$ if

$f\left(\underline{e}^{w(v_1)}\right) = \xi_{|U_i|+1}$, so that $\left\{f\left(\underline{e}^{w(v_1)}\right), f\left(\overline{e}^{w(v_1)}\right)\right\} = \left\{\xi_{|U_i|+1}, \xi_{|U_i|+1} + |U_i| + 1\right\}$. And for $r \in [2, |V_i|]$, let $f\left(\overline{e}^{w(v_r)}\right) = f\left(\underline{e}^{w(v_r)}\right) + 1$ if $f\left(\underline{e}^{w(v_r)}\right)$ is odd, while $f\left(\overline{e}^{w(v_r)}\right) = f\left(\underline{e}^{w(v_r)}\right) - 1$ if $f\left(\underline{e}^{w}\right)$ is even. This implies that vertex sums in W_i are odd and pairwise different. So (1) also holds for those i with $|U_i|$ being even.

For (6), note that the process of the labeling of $E(v^*) = \overline{E}_1^W$ is discussed in the case when $|U_i|$ is even (since $U_1 = \emptyset$ and $|U_1| = 0$). Recall that, $|E_1^W| = 2k$ and E_1^W are assigned with the greatest $2k$ labels, i.e., those labels in $L_{2k} = \{|E(G_s)|, |E(G_s)| - 1, \ldots, |E(G_s)| - 2k + 1\}$. More precisely, $\overline{E}_1^W = E(v^*)$ are assigned with the labels in $\{i_1, i_2, \ldots, i_k\} \subseteq L_{2k}$ where either $i_j = |E(G_s)| - 2j + 1$ or $i_j = |E(G_s)| - 2j + 2$ for $j = 1, 2, \ldots, k$. So $p(v^*) \geq p(v_1) + 1 + 3 + \cdots + (2k-3) > p(v_1) + 3$ for arbitrary $v_1 \in V_1$ (recall that $k \geq 3$). Then $\varphi_f(v^*) = p(v^*) + f(e^*) \geq p(v^*) + |E(G_s)| - 1 > p(v_1) + |E(G_s)| + 2 > p(v_1) + |E(G_s)| \geq p(v_1) + f(v_1 w(v_1)) = \varphi_f(v_1)$ for each $v_1 \in V_1$. On the other hand, $\varphi_f(v^*) \geq (|E(G_s)| - 1) + (|E(G_s)| - 3) + (|E(G_s)| - 5) = 3|E(G_s)| - 9$, since $k \geq 3$. Thus, each vertex in W_1 receives a sum at most $(2|E(G_s)| - 1)$. So $\varphi_f(v^*) \geq 3|E(G_s)| - 9 > 2|E(G_s)| - 1 \geq \varphi_f(w_1)$ for each $w_1 \in W_1$ (one has $|E(G_s)| \geq 12$, because $k \geq 3$). So (6) holds.

Thus, G_s is antimagic. This completes our proof. ■

It is interesting to consider the case when G is k-regular ($k \geq 3$) but disconnected. In the proof of Theorem 2, suppose G has m edges. Then G_s has m 2-vertices. Note that the total sum of all the vertex sums is even, since each label contributes to the total sum twice. Thus, each 2-vertex contributes an odd value to the total sum, while each k-vertex other than v^* contributes an even value, under our labeling way in the proof of Theorem 2. Thus, $\varphi_f(v^*)$ is odd if and only if m is odd.

Theorem 3. *Let G be an disconnected k-regular ($k \geq 3$) graph, which has at most one connected component with an odd number of edges. Then G_s is antimagic.*

Proof. Suppose G consists of the connected components $H_1, H_2, \ldots, H_\beta$ ($\beta \geq 2$), where H_i has an even number of edges for each $i \in [1, \beta - 1]$. We can label $E(G_s)$ in the order $E((H_1)_s), E((H_2)_s), \ldots, E((H_\beta)_s)$ using the smallest unused labels on each edge set when we come to it. Next, we label each connected component of G_s in the same way to that in Theorem 2, choosing a root for each component of G. Then there is no conflict among each $(H_i)_s$ for $i \in [1, \beta]$. Each 2-vertex receives an odd sum, while each k-vertex other than the root of $(H_\beta)_s$ receives an even sum. Each 2-vertex in $(H_i)_s$ receives a smaller sum than each 2-vertex in $(H_j)_s$, while each k-vertex in $(H_i)_s$ receives a smaller sum than each k-vertex in $(H_j)_s$, whenever $i < j \leq \beta$ holds. And the root vertex in $(H_\beta)_s$ receives a greater sum than those of any other vertex in G_s. So we obtain an antimagic labeling. ■

Sice $m = \frac{nk}{2}$, for each k-regular graph with n vertices and m edges, we have the following corollary.

Corollary 4. *Let G be an disconnected k-regular $(k \geq 3)$ graph. Then G_s is antimagic if one of the following holds.*

(1) *$k = 4t$ $(t \geq 1)$;*

(2) *k is even and at most one of the connected components of G has an odd number of vertices;*

(3) *At most one of the connected components of G has a number of vertices which is not a multiple of 4.*

3. (k, pk)-BIREGULAR $(k \geq 3, p \geq 2)$ BIPARTITE GRAPH

For a bipartite graph $G(A, B)$, a *complete p-claw matching CM from A to B* is a set of edges of G that induce a subgraph $G[CM]$ such that each vertex of A in G is also a vertex in $G[CM]$ and each component of $G[CM]$ is a copy of $K_{1,p}$ where the vertex of degree p is in A, while the vertices of degree 1 are in B. For $A_0 \subseteq A$, denote by $N(A_0)$ the set of vertices in B each of which has a neighbor in A_0. Let $E_1, E_2, \ldots, E_k \subseteq E(G)$ be disjoint edge sets. If $E_1 \cup E_2 \cup \cdots \cup E_k = E(G)$, then we say G *decomposes into* $E_1, E_2, \ldots, E_k$.

Lemma 5 (An extended version of Hall's theorem, [15, 16])**.** *A bipartite graph $G[A, B]$ admits a complete p-claw matching from A to B, if and only if $p|A_0| \leq |N(A_0)|$ for every subset A_0 of A.*

Lemma 6. *Let $G[A, B]$ be a (k, pk)-biregular $(k \geq 3, p \geq 2)$ bipartite graph where the degree of each vertex in A is kp, while each vertex in B has degree k. Then G decomposes into k complete p-claw matchings from A to B.*

Proof. Let $A_0 \subseteq A$. Let $G[A_0, N(A_0)]$ be the graph induced by $A_0 \cup N(A_0)$. Then each vertex of A_0 in $G[A_0, N(A_0)]$ has the degree kp, while each vertex of $N(A_0)$ in $G[A_0, N(A_0)]$ has the degree at most k. So there are exactly $kp|A_0|$ edges in $G[A_0, N(A_0)]$. On the other hand, suppose $|N(A_0)| < p|A_0|$. Then the number of edges in $G[A_0, N(A_0)]$ is less than $k \cdot p|A_0|$, a contradiction. So $|N(A_0)| \geq p|A_0|$. By Lemma 5, there exists a complete p-claw matching CM_1 from A to B in $G[A, B]$. Then $G_1 = G[A, B] - CM_1$ is a $(k-1, p(k-1))$-biregular bipartite graph. So we can use Lemma 5 repeatedly until we obtain a $(1, p)$-biregular bipartite graph G_{k-1} which is also a complete p-claw matching from A to B. Thus, $G[A, B]$ decomposes into k complete p-claw matchings from A to B. ∎

Lemma 7. *Let $I = [i+1, i+2q]$. Then, there exist partitions P_1 (when q is odd) and $\{P_2, P_3, P_4\}$ (when q is even) of I, such that under P_j, $j \in [1, 4]$, I is departed into q parts where each part has 2 integers, integers in $[i + (x-1)q + 1, i + xq]$ $(x \in [1, 2])$ are in distinct parts and the following conditions are satisfied.*

(1) *Under P_1, the q parts have distinct sums which attain all the values in* $[(2i+2q+1)-(q-1)/2, (2i+2q+1)+(q-1)/2]$;

(2) *Under P_2, $q/2$ parts have distinct sums which attain all the values in* $[(2i+2q+1)-(q/2-1), 2i+2q+1]$, *while the other $q/2$ parts have distinct sums which attain all the values in* $[2i+2q+1, (2i+2q+1)+(q/2-1)]$;

(3) *Under P_3, the $(q-1)$ parts have distinct sums which attain all the values in* $[(2i+2q+2)-(q/2-1), (2i+2q+2)+(q/2-1)]$ *and the other part has the sum $2i+q+2$;*

(4) *Under P_4, the $(q-1)$ parts have distinct sums which attain all the values in* $[(2i+2q)-(q/2-1), (2i+2q)+(q/2-1)]$ *and the other part has the sum $2i+3q$.*

Proof. It is sufficient to show the case when $i=0$.

(1) If q is odd, let $\{2j-1, -j+(3q+1)/2+1\}$ be in the same partition for $j \in [1,(q+1)/2]$, and let $\{2j, -j+2q+1\}$ be in the same partition for $j \in [1,(q-1)/2]$, which is the desired partition P_1.

(2) If q is even, let $\{2j, -j+3q/2+1\}$ be in the same partition and let $\{2j-1, -j+2q+1\}$ be in the same partition for $j \in [1, q/2]$, which is the desired partition P_2.

(3) If q is even, let $\{2j, -j+3q/2+2\}$ be in the same partition for $j \in [1, q/2]$, let $\{2j+1, -j+2q+1\}$ be in the same partition for $j \in [1, q/2-1]$, and let $\{1, q+1\}$ be in the same partition, which is the desired partition P_3.

(4) If q is even, let $\{2j-1, -j+3q/2+1\}$ be in the same partition for $j \in [1, q/2]$, let $\{2j, -j+2q\}$ be in the same partition for $j \in [1, q/2-1]$, and let $\{q, 2q\}$ be in the same partition, which is the desired partition P_4. ■

Lemma 8. *Let $I=[i+1, i+zq]$ $(z \geq 3)$. Then, there exist partitions P_1 (when z is even or q is odd) and $\{P_2, P_3\}$ (when z is odd and q is even) of I, such that under P_j, $j \in [1,3]$, I is departed into q parts where each part has z integers, integers in $[i+(x-1)q+1, i+xq]$ $(x \in [1,z])$ are in distinct parts and the following conditions are satisfied.*

(1) *Under P_1, the q parts have the same sum $(2i+zq+1)z/2$;*

(2) *Under P_2, $q/2$ parts have the same sum $(2i+zq+1)z/2+1/2$ and the other $q/2$ parts have the same sum $(2i+zq+1)z/2-1/2$;*

(3) *Under P_3, $(q-1)$ parts have the same sum $(2i+zq+1)z/2+3/2$ and the other part has the sum $(2i+zq+1)z/2-3q/2+3/2$.*

Proof. It is sufficient to show the case when $i=0$.

(1) If z is even, let $\{(j-1)q+l | j \in [1, z/2]\} \cup \{jq-l+1 | j \in [z/2+1, z]\}$ be in the partition for $l \in [1,q]$, which is the desired partition P_1 and (1) holds in this case.

If z is odd (then $(z-3)$ is even) and q is odd, we first assign the $(z-3)q$ integers in $[2q+1,(z-1)q]$ to the q parts (suppose $I_1, I_2, \ldots, I_q$ are the q parts) such that these q parts receive the same partial sum $(zq+q+1)(z-3)/2$. We can do this since $(z-3)$ is even. Second, assign $[(z-1)q+l]$ to I_l for $l \in [1,q]$ such that the q parts have distinct partial sums and attain all values in $[(qz+q+1)(z-3)/2+(z-1)q+1,(qz+q+1)(z-3)/2+zq]$. Third, partition $[1,2q]$ into q parts (denoted by $I'_1, I'_2, \ldots, I'_q$) which have distinct sums which attain all the values in $[(2q+1)-(q-1)/2,(2q+1)+(q-1)/2]$. We can do this owing to the partition in Lemma 7(1). Then assign I'_{i_l} to I_l if the sum of I'_{i_l} equals to $[(2q+1)+(q-1)/2-l+1]$ for $l \in [1,q]$. Then the final sum of I_l equals to $[(qz+q+1)(z-3)/2]+[(z-1)q+l]+[(2q+1)+(q-1)/2-l+1]=(qz+1)z/2$ for each $l \in [1,q]$. So (1) also holds in this case.

(2) If z is odd and q is even, we first partition $[2q+1,zq]$ into q parts $I_1, I_2, \ldots, I_q$ which have distinct partial sums and attain all values in $[(qz+q+1)(z-3)/2+(z-1)q+1,(qz+q+1)(z-3)/2+zq]$. We can do this owing to the discussion in (1). Then partition $[1,2q]$ into q parts (denoted by $I'_1, I'_2, \ldots, I'_q$) such that $q/2$ parts have distinct sums which attain all the values in $[(2q+1)-(q/2-1),2q+1]$, while the other $q/2$ parts have distinct sums which attain all the values in $[2q+1,(2q+1)+(q/2-1)]$. We can do this owing to the partition in Lemma 7(2). Denote by $I'_{i_{q/2,1}}$ and $I'_{i_{q/2,2}}$ the two parts each of which admits the sum $(2q+1)$. Then assign I'_{i_l} to I_l if the sum of I'_{i_l} equals to $[(2q+1)+(q/2-1)-l+1]$ for $l \in [1,q/2-1]$. Assign $I'_{i_{q/2,1}}$ to $I_{q/2}$, while assign $I'_{i_{q/2,2}}$ to $I_{q/2+1}$. And assign I'_{i_l} to I_l if the sum of I'_{i_l} equals to $(2q+1)+(q/2-1)-l+2$ for $l \in [q/2+2,q]$. Then for $l \in [1,q/2-1]$ the final sum of I_l equals to $[(qz+q+1)(z-3)/2]+[(z-1)q+l]+[(2q+1)+(q/2-1)-l+1]=(qz+1)z/2-1/2$. The final sum of $I_{q/2}$ equals to $[(qz+q+1)(z-3)/2]+[(z-1)q+q/2]+[(2q+1)]=(qz+1)z/2-1/2$, while the final sum of $I_{q/2+1}$ equals to $[(qz+q+1)(z-3)/2]+[(z-1)q+q/2+1]+[(2q+1)]=(qz+1)z/2+1/2$. Thus, for $l \in [q/2+2,q]$ the final sum of I_l equals to $[(qz+q+1)(z-3)/2]+[(z-1)q+l]+[(2q+1)+(q/2-1)-l+2]=(qz+1)z/2+1/2$. So (2) holds.

(3) If z is odd and q is even, we first partition $[2q+1,zq]$ into q parts $I_1, I_2, \ldots, I_q$ which have distinct partial sums and attain all values in $[(qz+q+1)(z-3)/2+(z-1)q+1,(qz+q+1)(z-3)/2+zq]$. We can do this owing to the discussion in (1). Then partition $[1,2q]$ into q parts (denoted by $I'_1, I'_2, \ldots, I'_q$) such that the $(q-1)$ parts have distinct sums which attain all the values in $[(2q+2)-(q/2-1),(2q+2)+(q/2-1)]$ and the other part has the sum $(q+2)$. We can do this owing to the partition in Lemma 7(3). Denote by I'_{i_1} the part with the sum $(q+2)$. Then assign I'_{i_1} to I_1, and assign I'_{i_l} to I_l if the sum of I'_{i_l} equals to $[(2q+2)+(q/2-1)-l+2]$ for $l \in [2,q]$. Then the final sum of I_1 equals to $[(qz+q+1)(z-3)/2]+[(z-1)q+l]+[q+2]=(qz+1)z/2-3q/2+3/2$, and

for $l \in [2, q]$, the final sum of I_l equals to $[(qz+q+1)(z-3)/2] + [(z-1)q+l] + [(2q+2)+(q/2-1)-l+2] = (qz+1)z/2+3/2$. So (3) holds. ■

Theorem 9. *Every (k,pk)-biregular $(k \geq 3, p \geq 2)$ bipartite graph is antimagic.*

Proof. Let $G[A, B]$ be a (k, pk)-biregular $(k \geq 3, p \geq 2)$ bipartite graph, where each vertex in A has the degree pk, while each vertex in B has the degree k. Suppose $|A| = n$ $(n \geq k)$ and $|B| = pn$. Let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_{pn}\}$. By Lemma 6, G decomposes into k complete p-claw matchings CM_1, $CM_2, \ldots, CM_k$ from A to B. Denote by $CM_i(V_0)$ $(i \in [1, k])$ the edges in CM_i which are incident to some vertex in V_0 for $V_0 \subseteq V(G)$.

Step 1. Label $\left(\bigcup_{i=1}^{k-1} CM_i\right)$ with $[1, (k-1)pn]$.

First, label CM_{k-1} with $[(k-2)pn+1, (k-1)pn]$, i.e., $[(k-2)pn+1, (k-2)pn+pn]$ such that the following conditions are satisfied.

(1.1) Within CM_{k-1}, vertices in A have the same partial sum $[(2k-3)pn+1]p/2$ if p is even or n is odd. We can do this owing to the partition in Lemma 8(1).

(1.2) Within CM_{k-1}, $n/2$ vertices in A have the same partial sum $[(2k-3)pn+1]p/2+1/2$ and the other $n/2$ vertices in A have the same partial sum $[(2k-3)pn+1]p/2-1/2$ if p is odd and n is even. We can do this owing to the partition in Lemma 8(2).

Second, based on the labeling to CM_{k-1}, for each $i \in [1, k-2]$, label CM_i with $[(i-1)pn+1, ipn]$, such that the following conditions are satisfied.

(1.3) Within $\left(\bigcup_{i=1}^{k-1} CM_i\right)$, the vertices in B have the same partial sum $[(k-1)pn+1](k-1)/2$ if $(k-1)$ even or pn is odd. We can do this owing to the partition in Lemma 8(1).

(1.4) Within $\left(\bigcup_{i=1}^{k-1} CM_i\right)$, $(pn-1)$ vertices in B have the same partial sum $[(k-1)pn+1](k-1)/2+3/2$ while the other vertex (denoted by b_0) has the partial sum $[(k-1)np+1](k-1)/2+3/2-3pn/2$ if $(k-1)$ is odd and pn is even. We can do this owing to the partition in Lemma 8(3).

Note that, (1.3) implies the vertices in B will receive distinct final vertex sums, when $(k-1)$ is even or pn is odd, if we label the rest edges CM_k using the rest labels $[(k-1)pn+1, kpn]$. Thus in (1.4), the partial sum of b_0 is at least $3pn/2$ smaller than those of the vertices in $(B \setminus \{b_0\})$. So the final vertex sum of b_0 will still be smaller than those of the vertices in $(B \setminus \{b_0\})$, if we label CM_k with $[(k-1)pn+1, kpn]$. Hence, the final vertex sums of in $(B \setminus \{b_0\})$ will be pairwise different. That is, all vertices in B will also receive distinct final vertex sums when $(k-1)$ is odd and pn is even.

Step 2. Label CM_k with $[(k-1)pn+1, kpn]$, i.e., $[(k-1)pn+1, (k-1)pn+pn]$.

Suppose $f_1(a_{t_1}) \leq f_1(a_{t_2}) \leq \cdots \leq f_1(a_{t_n})$ where $f_1(a_{t_j})$ is the partial vertex sum of a_{t_j} within $\left(\bigcup_{i=1}^{k-1} CM_i\right)$ for $j \in [1, n]$.

(2.1) If p is odd (then $(p-1)$ is even) or n is odd, let $\sigma(a)$ be an edge in $CM_k(a)$ for each $a \in A$. Label $\big[CM_k \setminus \big(\bigcup_{a\in A}\{\sigma(a)\}\big)\big]$ with $[(k-1)pn+1, (k-1)pn+(p-1)n]$ such that, within $\big[CM_k \setminus \big(\bigcup_{a\in A}\{\sigma(a)\}\big)\big]$, the vertices in A have the same partial sum $[(2k-1)pn-n+1](p-1)/2$. We can do this owing to the partition in Lemma 8(1). Next label $\sigma(a_{t_j})$ with $(kpn-n+j)$ for $j \in [1,n]$. Then the vertex sums in A are pairwise different.

(2.2) If p is even (then $(p-2)$ is also even) and n is even, let $\sigma_1(a)$ and $\sigma_2(a)$ be two distinct edges in $CM_k(a)$ for each $a \in A$. Label $\big[CM_k \setminus \big(\bigcup_{a\in A}\{\sigma_1(a), \sigma_2(a)\}\big)\big]$ using the labels in $[(k-1)pn+1, (k-1)pn+(p-2)n]$ such that, within $\big[CM_k \setminus \big(\bigcup_{a\in A}\{\sigma_1(a), \sigma_2(a)\}\big)\big]$, the vertices in A have the same partial sum $[(2k-1)pn-2n+1](p-2)/2$. We can do this owing to the partition in Lemma 8(1). Then label $\big(\bigcup_{a\in A}\{\sigma_1(a), \sigma_2(a)\}\big)$ with $[(kpn-2n)+1, (kpn-2n)+2n]$ such that $f(\sigma_1(a_{t_j})) + f(\sigma_2(a_{t_j})) = 2kpn - 5n/2 + j$ for $j \in [1, n-1]$ while $f(\sigma_1(a_{t_n})) + f(\sigma_2(a_{t_n})) = 2kpn - n$. We can do this owing to the partition in Lemma 7(4). Then the vertex sums in A are also pairwise different.

Recall that, owing to Step 1 (1.3) and (1.4), for each $b \in B$, one has

$$\varphi_f(b) \leq \frac{[(k-1)pn+1](k-1)}{2} + \frac{3}{2} + kpn.$$

On the other hand, owing to the labeling way in Step 1 (1.3) and (1.4), the labels assigned to CM_i are those in $[(i-1)pn+1, ipn]$ for $i \in [1, k-2]$. Let $a \in A$. Then the sum of the labels in $CM_i(a)$ is at least $\sum_{j=1}^{p}[(i-1)pn+j]$ for $i \in [1, k-2]$. So the sum of the labels in $\big(\bigcup_{i=1}^{k-2} CM_i(a)\big)$ is at least $\sum_{i=1}^{k-2}\sum_{j=1}^{p}[(i-1)pn+j]$. Recall that, owing to Step 1 (1.1) and (1.2), the sum of labels in $CM_{k-1}(a)$ is at least $[(2k-3)pn+1]p/2 - 1/2$. Next recall that, owing to Step 2 (2.1), the sum of labels in $CM_k(a)$ is at least $[(2k-1)pn-n+1](p-1)/2 + (kpn-n+1)$ if p is odd (then $(p-1)$ is even) or n is odd, while owing to Step 2 (2.2), the sum of labels in $CM_k(a)$ is at least $[(2k-1)pn-2n+1](p-2)/2 + (2kpn-5n/2+1)$ if p is even (then $(p-2)$ is also even) and n is even. Thus, the later lower bound is 1/2 smaller than the first lower bound. So

$$\begin{aligned}\varphi_f(a) \geq & \sum_{i=1}^{k-2}\sum_{j=1}^{p}[(i-1)pn+j] + \left\{\frac{[(2k-3)pn+1]p}{2} - \frac{1}{2}\right\} \\ & + \left\{\frac{[(2k-1)pn-2n+1](p-2)}{2} + \left(2kpn - \frac{5n}{2} + 1\right)\right\}.\end{aligned}$$

Then for each $a \in A$ and $b \in B$, one has

$$\begin{aligned}\varphi_f(a) - \varphi_f(b) \geq & \frac{1}{2}\Bigg[\left(\frac{1}{2}k-1\right)p^2kn + (k-3)p^2 + k^2\left(\frac{1}{2}p-1\right)pn \\ & + (p-1)(np+k) + \left(p^2-1\right)n + \left(p^2-3\right)\Bigg] > 0,\end{aligned}$$

since $k \geq 3$ and $p \geq 2$.

Thus, we obtain an antimagic labeling. This completes our proof. ■

Theorem 10. *Every* $(k, k^2 + y)$*-biregular* $(k \geq 3, y \geq 1)$ *bipartite graph is antimagic.*

Proof. Let $G[A, B]$ be a (k, k')-biregular $(k' = k^2 + y)$ bipartite graph, where each vertex in A has the degree k', while each vertex in B has the degree k. Suppose $|A| = k\eta$ and $|B| = k'\eta$ where η may be not an integer. It is sufficient to consider the case when $k' = kp + r$ for some integers p and r satisfying $p \geq k$ and $1 \leq r \leq k - 1$ (note that $r\eta$ is an integer since $k\eta$ and $k'\eta$ are integers). Let $A = \{a_1, a_2, \ldots, a_{k\eta}\}$ and $B = \{b_1, b_2, \ldots, b_{k'\eta}\}$. For $A_0 \subseteq A$, the graph $G[A_0, N(A_0)]$ has $k'|A_0|$ edges, since each vertex of A_0 in $G[A_0, N(A_0)]$ has the degree k'. On the other hand, suppose $|N(A_0)| < p|A_0|$. Then the number of edges in $G[A_0, N(A_0)]$ is at most $k|N(A_0)| < pk|A_0| < k'|A_0|$, since each vertex of $N(A_0)$ in $G[A_0, N(A_0)]$ has the degree at most k, a contradiction. So $|N(A_0)| \geq p|A_0|$. So, by Lemma 5, G admits a complete p-claw matching CM from A to B. Suppose $B = B_1 \cup B_2$ where $B_1 = V(CM) \cap B$ and $B_2 = B \setminus B_1$. Then $|B_1| = kp\eta$ and $|B_2| = r\eta$. Let $\sigma(b)$ be an edge incident to b for each $b \in B_2$, and let $\sigma(B_2) = \{\sigma(b) | b \in B_2\}$.

Step 1. Label $(E(G) - CM - \sigma(B_2))$ with $[1, (k-1)k'\eta]$.

(1.1) If $(k-1)$ is even or $k'\eta$ is odd, label $(E(G) - CM - \sigma(B_2))$ with $[1, (k-1)k'\eta]$ such that, within $(E(G) - CM - \sigma(B_2))$, the vertices in B have the same partial sum $((k-1)k'\eta + 1)(k-1)/2$. We can do this owing to the partition in Lemma 8(1).

(1.2) If $(k-1)$ is odd and $k'\eta$ is even, label $(E(G) - CM - \sigma(B_2))$ with $[1, (k-1)k'\eta]$ such that, within $(E(G) - CM - \sigma(B_2))$, the vertices in B have the same partial sum $[(k-1)k'\eta + 1](k-1)/2 + 3/2$ except one (denoted by b_0) which equals to $[(k-1)k'\eta + 1](k-1)/2 - 3k'\eta/2 + 3/2$. We can do this owing to the partition in Lemma 8(3).

Note that (1.1) implies the final vertex sums in B will be pairwise different when $(k-1)$ is even or $k'\eta$ is odd, if we label the rest edges $(CM \cup \sigma(B_2))$ with the rest labels $[(k-1)k'\eta + 1, kk'\eta]$. Then in (1.2), the partial sum of b_0 is at least $3k'\eta/2$ smaller than those of the vertices in $(B \setminus \{b_0\})$. So the final vertex sum of b_0 will be smaller than those of the vertices in $(B \setminus \{b_0\})$, if we label $(CM \cup \sigma(B_2))$ with $[(k-1)k'\eta + 1, kk'\eta]$. Next, the final vertex sums of in $(B \setminus \{b_0\})$ will be pairwise different. That is, vertices in B will also receive distinct final vertex sums, when $(k-1)$ is odd and $k'\eta$ is even.

Step 2. Label $\sigma(B_2)$ with $[(k-1)k'\eta + 1, (k-1)k'\eta + r\eta]$ arbitrarily.

Step 3. Label CM with $[(k-1)k'\eta+r\eta+1, kk'\eta]$, i.e., $[(kk'\eta-pk\eta)+1, (kk'\eta-pk\eta)+pk\eta]$. Suppose $f_1(a_{t_1}) \le f_1(a_{t_2}) \le \cdots \le f_1(a_{t_{k\eta}})$, where $f_1(a_{t_j})$ is the partial vertex sum of a_{t_j} after Steps 1 and 2, for $j \in [1, k\eta]$.

(3.1) If p is odd (then $(p-1)$ is even) or $k\eta$ is odd, let $\sigma(a)$ be an edge in $CM(a)$ for each $a \in A$. Label $\big[CM \setminus \big(\bigcup_{a\in A}\{\sigma(a)\}\big)\big]$ with $[(kk'\eta-pk\eta)+1, (kk'\eta-pk\eta)+(p-1)k\eta]$ such that, within $\big[CM \setminus \big(\bigcup_{a\in A}\{\sigma(a)\}\big)\big]$, vertices in A have the same vertex sum $[2(kk'\eta-pk\eta)+(p-1)k\eta+1](p-1)/2$. We can do this owing to the partition in Lemma 8(1). And label $\sigma(a_{t_j})$ with $(kk'\eta-k\eta+j)$ for $j\in[1,k\eta]$. Then vertex sums in A are pairwise different.

(3.2) If p is even (then $(p-2)$ is also even) and $k\eta$ is even, let $\sigma_1(a)$ and $\sigma_2(a)$ be two distinct edges in $CM(a)$ for each $a\in A$. Label $\big[CM \setminus \big(\bigcup_{a\in A}\{\sigma_1(a),\sigma_2(a)\}\big)\big]$ with $[(kk'\eta-pk\eta)+1, (kk'\eta-pk\eta)+(p-2)k\eta]$ such that, within $\big[CM\setminus \big(\bigcup_{a\in A}\{\sigma_1(a),\sigma_2(a)\}\big)\big]$, vertices in A have the same vertex sum $[2(kk'\eta-pk\eta)+(p-2)k\eta+1](p-2)/2$. We can do this owing to the partition in Lemma 8(1). Next, label $\big(\bigcup_{a\in A}\{\sigma_1(a),\sigma_2(a)\}\big)$ with $[(kk'\eta-2k\eta)+1, (kk'\eta-2k\eta)+2k\eta]$ such that $f(\sigma_1(a_{t_j}))+f(\sigma_2(a_{t_j})) = 2kk'\eta-5k\eta/2+j$ for $j\in[1,k\eta-1]$, while $f(\sigma_1(a_{t_{k\eta}}))+f(\sigma_2(a_{t_{k\eta}})) = 2kk'\eta-k\eta$. We can do this owing to the partition in Lemma 7(4). Then the vertex sums in A are also pairwise different.

Recall that, owing to Step 1 (1.1) and (1.2), for each $b\in B$, one has

$$\varphi_f(b) \le \frac{((k-1)k'\eta+1)(k-1)}{2} + \frac{3}{2} + kk'p.$$

On the other hand, let $a\in A$. Recall that, owing to Step 3 (3.1) the sum of the labels in $CM(a)$ is at least $\{[2(kk'\eta-pk\eta)+(p-1)k\eta+1](p-1)/2\}+(kk'\eta-k\eta+1)$ if p is odd (then $(p-1)$ is even) or $k\eta$ is odd. Then owing to Step 3 (3.2), the sum of the labels in $CM(a)$ is at least $\{[2(kk'\eta-pk\eta)+(p-2)k\eta+1](p-2)/2\}+(2kk'\eta-5k\eta/2+1)$ if p is even (then $(p-2)$ is also even) and $k\eta$ is even. And the later lower bound is $1/2$ smaller than the first lower bound. So

$$\varphi_f(a) > \frac{[2(kk'\eta-pk\eta)+(p-2)k\eta+1](p-2)}{2} + \left(2kk'\eta-\frac{5k\eta}{2}+1\right).$$

Then for each $a\in A$ and $b\in B$ one has

$$\begin{aligned}\varphi_f(a)-\varphi_f(b) > {} & \frac{1}{2}\big[(p-k)kk'\eta+(k'-2p-1)pk\eta+(p^2-3)k\eta \\ & + (pk+k-k')\eta+(\eta-1)k+(p-2)\big] > 0,\end{aligned}$$

since $p\ge k\ge 3$ and $2p+1<k'=pk+r\le pk+k$.

Thus, we obtain an antimagic labeling. This completes our proof. ∎

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