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DOMINATION NUMBER OF GRAPHS WITH MINIMUM DEGREE FIVE

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Abstract

We prove that for every graph G on n vertices and with minimum degree five, the domination number $\gamma(G)$ cannot exceed $n/3$. The proof combines an algorithmic approach and the discharging method. Using the same technique, we provide a shorter proof for the known upper bound $4n/11$ on the domination number of graphs of minimum degree four.

Keywords: dominating set, domination number, discharging method.

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1. Introduction

In this paper we study the minimum dominating sets in graphs of given order n and minimum degree δ. For the case of $\delta = 5$, we improve the previous best upper bound $0.344\,n$ by proving that the domination number γ is at most $n/3$. For graphs of $\delta = 4$, the relation $\gamma \leq 4n/11$ was proved by Sohn and Xudong [22] in 2009. Using a different approach, we provide a simpler proof for this theorem.

Standard definitions. In a simple graph G, the vertex set is denoted by $V(G)$ and the edge set by $E(G)$. For a vertex $v \in V(G)$, its *closed neighborhood* $N[v]$ contains v and its neighbors. For a set $S \subseteq V(G)$, we use the analogous notation $N[S] = \bigcup_{v \in S} N[v]$. The *degree* of a vertex v is denoted by $d(v)$, while $\delta(G)$ and $\Delta(G)$, respectively, stand for the *minimum* and *maximum vertex degree* in G. A set $D \subseteq V(G)$ is a *dominating set* if $N[D] = V(G)$. The minimum cardinality of a dominating set is the *domination number* $\gamma(G)$ of the graph. An earlier general survey on domination theory is [11], while two new directions were initiated recently in [6] and [5].

General upper bounds on $\gamma(G)$ in terms of the order and minimum degree. The first general upper bound on $\gamma(G)$ in terms of the order n and the minimum degree δ was given by Arnautov [2] and, independently, by Payan [20]:

$$\gamma(G) \le \frac{n}{\delta+1}\sum_{j=1}^{\delta+1}\frac{1}{j}\,. \tag{1}$$

We remark that a bit stronger general results were later published by Clark *et al.* [9] and Biró *et al.* [3]. On the other hand, already (1) implies the upper bound

$$\gamma(G) \le n\left(\frac{1+\ln(\delta+1)}{\delta+1}\right). \tag{2}$$

It was proved by Alon [1] that (2) is asymptotically sharp when $\delta \to \infty$.

Upper bounds for graphs of small minimum degrees. There are several ways to show that $\gamma(G) \le n/2$ holds if $\delta(G) = 1$ (see [19] for the first proof). Blank [4], and later independently McCuaig and Shepherd [18] proved that $\gamma(G) \le 2n/5$ is true if G is connected, $\delta(G) = 2$, and $n \ge 8$. [1] For graphs G with $\delta(G) = 3$, Reed [21] proved the famous result that $\gamma(G) \le 3n/8$. He also presented a connected cubic graph on 8 vertices for which the upper bound is tight.

In the same paper [21], Reed provided the conjecture that the upper bound can be improved to $\lceil n/3 \rceil$ once the connected cubic graph has an appropriately large order. It was disproved by Kostochka and Stodolsky [14] by constructing an infinite sequence of connected cubic graphs such that all of them have $\gamma(G) \ge \left(\frac{1}{3}+\frac{1}{69}\right)n$. Later, in [15], the same authors proved that $\gamma(G) \le \frac{4}{11}n = \left(\frac{1}{3}+\frac{1}{33}\right)n$ holds for every connected cubic graph of order $n > 8$. However, it seems a challenging and difficult problem to close the small gap between $\frac{1}{3}+\frac{1}{69}$ and $\frac{1}{3}+\frac{1}{33}$.

For graphs of minimum degree 4, the best known upper bound is $\gamma(G) \le \frac{4}{11}\,n$ that was established by Sohn and Xudong [22]. For the case of $\delta(G) = 5$, Xing, Sun, and Chen [23] proved $\gamma(G) \le \frac{5}{14}\,n$ which was improved to $\gamma(G) \le \frac{2671}{7766}\,n < 0.344\,n$ by the authors of [7]. It was also shown in [7] that for graphs of minimum degree 6, the domination number is strictly smaller than $n/3$. Note that similar upper bounds involving the girth and other parameters of the graph can be found in many papers, e.g. in [10, 12, 16, 17], while results for plane triangulations and maximal outerplanar graphs were established in [13] and [8].

Our approach. In the seminal paper [21] of Reed, the upper bound $3n/8$ was proved by considering a vertex-disjoint path cover with specific properties. Later,

[1] There are seven small graphs, the cycle C_4 and six graphs with $n = 7$ and $\delta = 2$, which do not satisfy $\gamma(G) \le 2n/5$.

the same method (with updated conditions and thorough analysis) was used in [15, 22, 23] to establish results on cubic graphs and on graphs of minimum degree 4 and 5. In [7], we introduced a different algorithmic method that resulted in improvement for all cases with $5 \leq \delta \leq 50$. Here, we combine the latter approach with a discharging process. This allows us to prove that already graphs of minimum degree 5 satisfy $\gamma(G) \leq n/3$.

Residual graph. Given a graph G and a set $D \subseteq V(G)$, the *residual graph* G_D is obtained from G by assigning colors to the vertices and deleting some edges according to the following definitions.

- A vertex v is *white* if $v \notin N[D]$.
- A vertex v is *blue* if $v \in N[D]$ and $N[v] \not\subseteq N[D]$.
- A vertex v is *red* if $N[v] \subseteq N[D]$.
- G_D contains only those edges from G that are incident to at least one white vertex.

In G_D, we refer to the set of white, blue, and red vertices, respectively, by the notations W, B, and R. It is clear by definitions that $D \subseteq R$ and $W \cup B \cup R = V(G)$ hold. The *white-degree* $d_W(v)$ of a vertex v is the number of its white neighbors in G_D. Analogously, we sometimes refer to the *blue-degree* $d_B(v)$ of a vertex. The maximum of white-degrees over the sets of white and blue vertices, respectively, are denoted by $\Delta_W(W)$ and $\Delta_W(B)$.

Observation 1. *Let G be a graph and $D \subseteq V(G)$. The following statements are true for the residual graph G_D.*

(i) *If $v \in W$, then G_D contains all edges which are incident with v in G and, in particular, $N[v] \cap R = \emptyset$ and $d_W(v) + d_B(v) = d(v)$ hold.*

(ii) *If $v \in B$, then $d_W(v) = |W \cap N[v]| < d(v)$ and $d_B(v) = 0$.*

(iii) *If $v \in R$, then v is an isolated vertex in G_D.*

(iv) *If $\delta(G) = d$ and v is a white vertex with $d_W(v) = \ell < d$, then $d_B(v) \geq d - \ell$ holds in G_D.*

(v) *D is a dominating set of G if and only if $R = V(G)$ (or equivalently, $W = \emptyset$) in G_D.*

(vi) *If $D \subseteq D' \subseteq V(G)$ and a vertex v is red in G_D, it remains red in $G_{D'}$; if v is blue in G_D, then it is either blue or red in $G_{D'}$.*

Structure of the paper. In the next section we prove the improved upper bound $n/3$ on the domination number of graphs with minimum degree 5. In Section 3 we consider graphs of minumum degree 4 and show an alternative proof for the theorem $\gamma \leq 4n/11$.

2. Graphs of Minimum Degree 5

Theorem 2. *For every graph G on n vertices and with minimum degree 5, the domination number satisfies $\gamma(G) \le \frac{n}{3}$.*

Proof. Consider a graph G and a subset D of the vertex set $V = V(G)$. Let W, B, and R denote the set of white, blue, and red vertices respectively, in the residual graph G_D. Further, for the sets of blue vertices that have at least 5 white neighbors, or exactly 4, 3, 2, 1 white neighbors, we use the notations B_5, B_4, B_3, B_2, and B_1 respectively. A vertex is a blue leaf if it belongs to B_1. In the proof, a residual graph G_D is associated with the following value:

$$f(G_D) = 35|W| + 23|B_5| + 21|B_4| + 19|B_3| + 17|B_2| + 14|B_1|.$$

By Observation 1(v), $f(G_D)$ equals zero if and only if D is a dominating set in G. If G and D are fixed and A is a subset of $V \setminus D$, we define

$$\mathrm{s}(A) = f(G_D) - f(G_{D\cup A})$$

that is the decrease in the value of f when D is extended by the vertices of A. We define the following property for G_D:

Property 1. There exists a nonempty set $A \subseteq V \setminus D$ such that $\mathrm{s}(A) \ge 105\,|A|$.

Our goal is to prove that every graph G with $\delta(G) = 5$ and every $D \subseteq V$ with $f(G_D) > 0$ satisfy Property 1. Once we do it, Theorem 2 will follow easily. In the continuation, we suppose that a graph G with minimum degree 5 and a set D with $f(G_D) > 0$ do not satisfy Property 1 and prove, by a series of claims, that this assumption leads to a contradiction.

Claim A. *In G_D, every white vertex v has at most two white neighbors, and every blue vertex u has at most three white neighbors.*

Proof. First suppose that there is vertex $v \in W$ with $d_W(v) \ge 6$. Choosing $A = \{v\}$, the white vertex v becomes red in $G_{D\cup A}$ that decreases f by 35. The white neighbors of v become blue or red which decreases f by at least $6\cdot(35-23)$. Hence, we have $\mathrm{s}(A) \ge 35 + 72 = 107 > 105\,|A|$ complying with Property 1. This contradicts our assumption on G_D and implies that $\Delta_W(W) \le 5$.

Now, suppose that $\Delta_W(W) = 5$ in G_D. Let v be a white vertex with $d_W(v) = 5$ and consider $A = \{v\}$. In $G_{D\cup A}$, the vertex v becomes red and its white neighbors become blue (or red). Since each neighbor u had at most 5 white neighbors in G_D and at least one of them, namely v, becomes red, u may have at most 4 white neighbors in $G_{D\cup A}$. Therefore, $\mathrm{s}(A) \ge 35 + 5\cdot(35-21) = 105\,|A|$ holds which is a contradiction again.

If $\Delta_W(W) \le 4$ and $\Delta_W(B) \ge 6$, let v be a blue vertex with $d_W(v) \ge 6$ and define $A = \{v\}$ again. In G_D, the vertex v belongs to B_5, while we have $v \in R$ in

$G_{D\cup A}$ which causes a decrease of 23 in the value of f. Each white neighbor u of v has at most four white neighbors in G_D and, therefore, $u \in B_4 \cup B_3 \cup B_2 \cup B_1 \cup R$ in $G_{D\cup A}$. Hence, we have $\mathrm{s}(A) \geq 23+6(35-21) = 107 > 105\,|A|$, a contradiction to our assumption. Note that in the continuation, where we suppose $\Delta_W(B) \leq 5$, if a blue vertex loses ℓ white neighbors in a step, it causes a decrease of at least 2ℓ in the value of f.

Assume that $\Delta_W(W) = 4$ and $\Delta_W(B) \leq 5$ and let v be a white vertex with $d_W(v) = 4$ in G_D. Set $A = \{v\}$ and consider the decrease $\mathrm{s}(A)$. As v turns to be red, this contributes by 35 to $\mathrm{s}(A)$. The four white neighbors become blue (or red) and each of them has at most 3 white neighbors in $G_{D\cup A}$. Hence, the contribution to $\mathrm{s}(A)$ is at least $4(35 - 19)$. Further, we have $d_W(u) \leq 4$ for each white vertex u from $N[v]$. This implies, by Observation 1(iv), that u has at least one blue neighbor in G_D the white-degree of which is smaller in $G_{D\cup A}$ than in G_D. Even if some blue vertices from $N[N[v]]$ have more than one neighbor from $N[v]$, it remains true that the sum of the white-degrees over $B \cap N[N[v]]$ decreases by at least $d_W(v)+1 = 5$. We may conclude $\mathrm{s}(A) \geq 35+4(35-19)+5\cdot 2 = 109 > 105\,|A|$.

Assume that $\Delta_W(W) \leq 3$ and $\Delta_W(B) = 5$ hold in G_D and v is a blue vertex with $d_W(v) = 5$. Let $A = \{v\}$ and consider the decrease $\mathrm{s}(A)$. Since v belongs to B_5 in G_D and to R in $G_{D\cup A}$, this change contributes by 23 to $\mathrm{s}(A)$. The five white neighbors of u become blue or red and belong to $B_3 \cup B_2 \cup B_1 \cup R$ in $G_{D\cup A}$. The contribution to $\mathrm{s}(A)$ is not smaller than $5(35 - 19)$. By Observation 1(iv) and by $\Delta_W(W) \leq 3$, each white vertex has at least two blue neighbors in G_D. That is, each white neighbor has at least one blue neighbor that is different from v. As the five white vertices from $N(v)$ turn blue (or red) in $G_{D\cup A}$, the sum of the white-degrees over $B \cap (N[N[v]] \setminus \{v\})$ decreases by at least 5. We infer that $\mathrm{s}(A) \geq 23 + 5(35 - 19) + 5 \cdot 2 = 113 > 105\,|A|$ which is a contradiction again.

The next case which we consider is $\Delta_W(W) = 3$ and $\Delta_W(B) \leq 4$. Let v be a white vertex with $d_W(v) = 3$ and estimate the value of $\mathrm{s}(A)$ for $A = \{v\}$. When D is replaced by $D \cup A$, vertex v is recolored red, the three white neighbors of v become blue or red and belong to $B_2 \cup B_1 \cup R$ in $G_{D\cup A}$. Additionally, each of the three white neighbors and also v itself has at least two blue neighbors. The decrease in their white-degrees contributes to $\mathrm{s}(A)$ by at least $4\cdot 2\cdot 2$. Consequently, we have $\mathrm{s}(A) \geq 35 + 3(35 - 17) + 16 = 105\,|A|$ that is a contradiction.

The last case is when $\Delta_W(W) \leq 2$ and $\Delta_W(B) = 4$. We assume that v is a vertex from B_4 in G_D. Let $A = \{v\}$ and observe that v is recolored red and the white neighbors of v belong to $B_2 \cup B_1 \cup R$ in $G_{D\cup A}$. Since now we have $\Delta_W(W) \leq 2$ in G_D, each white vertex has at least three blue neighbors. Therefore, each white neighbor of v has at least two blue neighbors which are different from v. We conclude that $\mathrm{s}(A) \geq 21+4(35-17)+4\cdot 2\cdot 2 = 109 > 105\,|A|$. This contradiction finishes the proof of Claim A. $\square$

From now on we may suppose that $\Delta_W(W) \leq 2$ and $\Delta_W(B) \leq 3$ holds in the counterexample G_D. This implies that the graph $G_D[W]$, which is induced by the white vertices of G_D, contains only paths and cycles as components. Before performing a discharging, we prove some further properties of G_D.

Claim B. *In $G_D[W]$, each component is a path P_1, P_2 or a cycle C_4, C_5, C_7 or C_{10}.*

Proof. First, suppose that $P_j \colon v_1 \cdots v_j$ is a path component on $j \geq 3$ vertices in $G_D[W]$. Let us choose $A = \{v_2\}$. In $G_{D\cup A}$ not only v_2 but also v_1 becomes red, while v_3 turns to be either a blue leaf or a red vertex. These changes contribute to $\mathrm{s}(A)$ by at least $2 \cdot 35 + (35 - 14)$. By Observation 1(iv), v_1, v_2, and v_3, respectively, have at least 4, 3, 3 blue neighbors in G_D. The decrease in their white-degrees contributes to $\mathrm{s}(A)$ by at least 20. We may infer that $\mathrm{s}(A) \geq 70 + 21 + 20 = 111 > 105\,|A|$, a contradiction to our assumption.

We now prove that no cycle of length $3k$ occurs in $G_D[W]$. Assuming that a cycle $C_{3k} \colon v_1 \cdots v_{3k} v_1$ exists, all vertices of it can be dominated by the k-element set $A = \bigcup_{i=1}^{k} \{v_{3i}\}$. Then, in $G_{D\cup A}$, all the $3k$ vertices are red and, by Observation 1(iv), the sum of the white-degrees of the blue neighbors decreases by at least $3{\cdot}3k$. Consequently, we get the contradiction $w(A) \geq 35 \cdot 3k + 2 \cdot 9k = 123k > 105\,|A|$.

Similarly, if we suppose the existence of a cycle $C_{3k+2} \colon v_1 \cdots v_{3k+2} v_1$ with $k \geq 2$ and define $A = \left(\bigcup_{i=1}^{k} \{v_{3i}\}\right) \cup \{v_{3k+2}\}$, the set A dominates all vertices. Since $k \geq 2$, the relation $\mathrm{s}(A) \geq 35 \cdot (3k+2) + 2 \cdot 3 \cdot (3k+2) = 123k + 82 > 105(k+1) = 105\,|A|$ clearly holds and gives the contradiction.

In the last case, consider a cycle $C_{3k+1} \colon v_1 \cdots v_{3k+1} v_1$ with $k \geq 4$ and set $A = \left(\bigcup_{i=1}^{k} \{v_{3i}\}\right) \cup \{v_{3k+1}\}$. In $G_{D\cup A}$, every vertex from the cycle is red and, as before, one can prove that $\mathrm{s}(A) \geq 35 \cdot (3k+1) + 2 \cdot 3 \cdot (3k+1) = 123k + 41 > 105(k+1) = 105\,|A|$. This contradiction finishes the proof of Claim B. $\square$

For $i = 0, 1, 2$, we will use the notation W_i for the set of white vertices having exactly i white neighbors in G_D. Note that W_0 consists of the vertices of the components of $G_D[W]$ which are isomorphic to P_1, while W_1 and W_2, respectively, contain the vertices from the P_2-components and the cycles of $G_D[W]$.

Claim C. *No vertex from B_3 is adjacent to a vertex from W_0 in G_D.*

Proof. In contrary, suppose that a vertex $v \in B_3$ has a neighbor u from W_0. Let $A = \{v\}$ and denote by u_1 and u_2 the further two white neighbors of v. In $G_{D\cup A}$, we have $v, u \in R$ and $u_1, u_2 \in B_2 \cup B_1 \cup R$. This contributes to $\mathrm{s}(A)$ by at least $19 + 35 + 2(35 - 17) = 90$. By Observation 1(iv), the neighbors u, u_1 and u_2 have, respectively, at least 4, 2, 2 blue neighbors which are different from v. As follows, $\mathrm{s}(A) \geq 90 + 2 \cdot 8 = 106 > 105\,|A|$ must be true but this contradicts our assumption on G_D. $\square$

We call a vertex from B_2 *special*, if it is adjacent to a vertex from W_0.

Claim D. *No special vertex is adjacent to two vertices from W_0.*

Proof. Suppose that a vertex $v \in B_2$ is adjacent to two vertices, say u_1 and u_2 from W_0. Then, we set $A = \{v\}$ and observe that all the three vertices v, u_1 and u_2 are red in $G_{D\cup A}$. By Claim C, all the blue neighbors of u_1 and u_2 are from $B_2 \cup B_1$ in G_D and, therefore, when the white-degree of these neighbors decreases by ℓ, the value of f falls by at least $(17-14)\ell = 3\ell$. Since, by Observation 1(iv), each of u_1 and u_2 has at least four blue neighbors, we have $\mathrm{s}(A) \geq 17+2\cdot 35+3\cdot 8 = 111 > 105\,|A|$. This contradiction proves the claim. □

Claim E. *No special vertex is adjacent to a vertex from a C_4 or C_7.*

Proof. Suppose first that a special vertex $v \in B_2$ is adjacent to u_1 which is from a 4-cycle component $C_4\colon u_1u_2u_3u_4u_1$ in G_D. The other neighbor of v is u_0 which is from W_0. Let $A = \{v, u_3\}$ and observe that all the six vertices v, u_0, u_1, u_2, u_3 and u_4 are red in $G_{D\cup A}$. In G_D, the white vertex u_0 has at least four blue neighbors which are different from v and, by Claim C, each of them belongs to $B_2 \cup B_1$; u_1 has at least two neighbors from $(B_3 \cup B_2 \cup B_1) \setminus \{v\}$; each of u_2, u_3 and u_4 has at least three neighbors from $(B_3 \cup B_2 \cup B_1) \setminus \{v\}$. Therefore, $\mathrm{s}(A) \geq 17 + 5 \cdot 35 + 4 \cdot 3 + 11 \cdot 2 = 226 > 105\,|A|$, a contradiction.

The argumentation is similar if we suppose that a special vertex v is adjacent to u_0 from W_0 and to a vertex u_1 from the 7-cycle $u_1 \cdots u_7u_1$. Here we set $A = \{v, u_3, u_6\}$ and observe that $\mathrm{s}(A) \geq 17+8\cdot 35+4\cdot 3+20\cdot 2 = 349 > 105\,|A|$ that contradicts our assumption on G_D. □

Claim F. *If v_1 and v_2 are two adjacent vertices from W_1, then at most one of them may have a special blue neighbor.*

Proof. Assume to the contrary that v_1 is adjacent to the special vertex u_1, and v_2 is adjacent to the special vertex u_2. Denote the other neighbors of u_1 and u_2 by x_1 and x_2, respectively. Hence, $v_1, v_2 \in W_1$, $u_1, u_2 \in B_2$ and $x_1, x_2 \in W_0$ hold in G_D. Consider the set $A = \{u_1, u_2\}$ and observe that all the six vertices become red in $G_{D\cup A}$. Further, for $i = 1, 2$, vertex x_i has at least four neighbors from $(B_2 \cup B_1) \setminus \{u_i\}$ and v_i has at least three neighbors from $(B_3 \cup B_2 \cup B_1) \setminus \{u_i\}$. Thus, $\mathrm{s}(A) \geq 2 \cdot 17 + 4 \cdot 35 + 8 \cdot 3 + 6 \cdot 2 = 210 = 105\,|A|$ and this contradiction proves the claim. □

Having Claims A–F in hand, we are ready to prove that every G_D (where D is not a dominating set) satisfies Property 1. The last step of this proof is based on a discharging.

Discharging. First, we assign charges to the (non-red) vertices of G_D so that every white vertex gets 35, and every vertex from B_3, B_2, and B_1 gets 19, 17, and 14, respectively. Note that the sum of the charges equals $f(G_D)$. Then, every blue vertex, except the special ones, distributes its charge equally among the white neighbors. The exact rules are the following.

- Every vertex from B_3 gives $19/3$ to each white neighbor.
- Every non-special vertex from B_2 gives $17/2$ to each white neighbor.
- Every special vertex gives 14 to its neighbor from W_0, and gives 3 to the other neighbor.
- Every vertex from B_1 gives 14 to its neighbor.

After the discharging, every vertex from a P_1-component of G_D has a charge of at least $35+5\cdot 14 = 105$. By Claim F, every P_2-component has at least four non-special blue neighbors and, therefore, its charge is at least $2\cdot 35+4\cdot 3+4\cdot 19/3 = 321/3$. By Claim E, every C_4-component has at least $4\cdot 35+12\cdot 19/3 = 216$ and every C_7-component has at least $7\cdot 35+21\cdot 19/3 = 378$ as a charge. Finally, every C_5-component has $5\cdot 35+15\cdot 3 = 220$, and every C_{10}-component has $10\cdot 35+30\cdot 3 = 440$ after the discharging. Let the number of P_1-, P_2-, C_4- C_5-, C_7-, and C_{10}-components of $G_D[W]$ be denoted by p_1, p_2, c_4, c_5, c_7, and c_{10}, respectively, and let A be a minimum dominating set in $G_D[W]$. Then,

$$|A| = p_1 + p_2 + 2\,c_4 + 2\,c_5 + 3\,c_7 + 4\,c_{10}.$$

As $D \cup A$ is a dominating set in the graph G, we have $f(G_{D\cup A}) = 0$. Thus, $\mathrm{s}(A) = f(G_D)$, and the discharging shows the following lower bound:

$$\begin{aligned}\mathrm{s}(A) = f(G_D) &\geq 105\,p_1 + \frac{321}{3}\,p_2 + 216\,c_4 + 220\,c_5 + 378\,c_7 + 440\,c_{10}\\ &\geq 105\,(p_1 + p_2 + 2\,c_4 + 2\,c_5 + 3\,c_7 + 4\,c_{10}) = 105\,|A|.\end{aligned}$$

As it contradicts our assumption on G_D, we infer that every graph G with minimum degree 5 and every $D \subseteq V(G)$ with $f(G_D) > 0$ satisfy Property 1.

To finish the proof of Theorem 2, we first observe that $f(G_\emptyset) = 35\,n$. Then, by Property 1, there exists a nonempty set A_1 such that $f(G_{A_1}) \leq f(G_\emptyset) - 105\,|A_1|$. Applying this iteratively, at the end we obtain a dominating set $D = A_1 \cup \cdots \cup A_j$ such that

$$f(G_D) = 0 \leq f(G_\emptyset) - 105|D| = 35\,n - 105|D|,$$

and we may conclude

$$\gamma(G) \leq |D| \leq \frac{35\,n}{105} = \frac{n}{3}. \qquad \blacksquare$$

In a graph G, a set $X \subseteq V(G)$ is a 2-*packing*, if any two distinct vertices from X are at a distance of at least 3. The proof of Theorem 2 directly corresponds to an algorithm that outputs a dominating set of cardinality at most $n/3$. If G is 5-regular and X is a 2-packing in it, we may start the algorithmic process with choosing the vertices of X one by one. Hence, we conclude the following.

Corollary 1. *If G is a 5-regular graph on n vertices and $X \subseteq V(G)$ is a 2-packing in G, then X can be extended to a dominating set D of cardinality at most $n/3$.*

3. Graphs of Minimum Degree 4

In this section, we apply the previous approach for graphs of minimum degree four and get a shorter alternative proof for the following theorem which was first proved by Sohn and Xudong [22] in 2009.

Theorem 3. *For every graph G on n vertices and with minimum degree 4, the domination number satisfies $\gamma(G) \leq \frac{4n}{11}$.*

Proof. Consider a graph G of minimum degree 4 and let D be a subset of $V = V(G)$. Let W, B, and R denote the set of white, blue, and red vertices in G_D. The set of blue vertices that have at least 4 white neighbors is denoted by B_4 while, for $i = 1, 2, 3$, B_i stands for the set of blue vertices that have exactly i white neighbors. In the proof, a residual graph G_D is associated with the following value:

$$g(G_D) = 16|W| + 10|B_4| + 9|B_3| + 8|B_2| + 7|B_1|.$$

For a set $A \subseteq V \setminus D$, we use the notation

$$\mathrm{s}(A) = g(G_D) - g(G_{D\cup A})$$

and define the following property for G_D:

Property 2. There exists a nonempty set $A \subseteq V \setminus D$ such that $\mathrm{s}(A) \geq 44\,|A|$.

We now suppose for a contradiction that a residual graph G_D with $\delta(G) = 4$ and $g(G_D) > 0$ does not satisfy Property 2. We prove several claims for G_D and then get the final contradiction via performing a discharging.

Claim G. $\Delta_W(W) \leq 2$ *and* $\Delta_W(B) \leq 3$ *hold.*

Proof. All the following cases can be excluded.

Case 1. $\Delta_W(W) \geq 5$. Choose a white vertex v with $d_W(v) \geq 5$ and let $A = \{v\}$. In $G_{D\cup A}$, the white vertex v becomes red and its white neighbors become blue or red. This gives $\mathrm{s}(A) \geq 16 + 5 \cdot (16 - 10) = 46 > 44\,|A|$ which contradicts our assumption that G_D does not satisfy Property 2.

Case 2. $\Delta_W(W) = 4$. Consider a white vertex v with $d_W(v) = 4$ and set $A = \{v\}$. In $G_{D\cup A}$, the vertex v becomes red and its white neighbors become blue or red. Since each white neighbor u had at most four white neighbors in G_D, u may have at most three white neighbors in $G_{D\cup A}$. Therefore, $\mathrm{s}(A) \geq 16 + 4 \cdot (16 - 9) = 44\,|A|$, a contradiction.

Case 3. $\Delta_W(W) \leq 3$ and $\Delta_W(B) \geq 5$. Let v be a blue vertex with $d_W(v) \geq 5$ and define $A = \{v\}$ again. In G_D, the vertex v belongs to B_4, while we have

$v \in R$ in $G_{D\cup A}$. Further, since $\Delta_W(W) \le 3$, each white neighbor u of v has at most three white neighbors in G_D and $u \in B_3 \cup B_2 \cup B_1 \cup R$ in $G_{D\cup A}$. As follows, $\mathrm{s}(A) \ge 10 + 5(16-9) = 45 > 44\,|A|$ that is a contradiction to our assumption.

Case 4. $\Delta_W(W) = 3$ and $\Delta_W(B) \le 4$. First remark that, by the condition $\Delta_W(B) \le 4$, if a blue vertex loses ℓ white neighbors in a step, then $g(G_D)$ decreases by at least ℓ. Select a white vertex v with $d_W(v) = 3$ and let $A = \{v\}$. In $G_{D\cup A}$, vertex v becomes red and its three white neighbors become blue or red having at most 2 white neighbors. By Observation 1(iv), each of v and its white neighbors has at least one blue neighbor in G_D. Thus, we get $\mathrm{s}(A) \ge 16 + 3(16-8) + 4 \cdot 1 = 44\,|A|$ which is a contradiction.

Case 5. $\Delta_W(W) \le 2$ and $\Delta_W(B) = 4$. Here, we choose a vertex v from B_4 and define $A = \{v\}$. First, observe that v belongs to B_4 in G_D and to R in $G_{D\cup A}$. In G_D, v has four white neighbors which become blue or red and belong to $B_2 \cup B_1 \cup R$ in $G_{D\cup A}$. By Observation 1(iv) and by $\Delta_W(W) \le 2$, each white neighbor has at least one blue neighbor that is different from v. Therefore, $\mathrm{s}(A) \ge 10 + 4(16-8) + 4 \cdot 1 = 46 > 44\,|A|$ that is a contradiction again. This finishes the proof of the claim. □

In the continuation, we suppose that $\Delta_W(W) \le 2$ and $\Delta_W(B) \le 3$ hold in the counterexample G_D and, therefore, the graph $G_D[W]$, which is induced by the white vertices of G_D, consists of components which are paths and cycles. We prove some further properties for G_D.

Claim H. *In $G_D[W]$, each component is a path P_1, P_2 or a cycle C_4 or C_7.*

Proof. Assume that there is a path component $P_j\colon v_1 \cdots v_j$ of order $j \ge 3$ in $G_D[W]$. We set $A = \{v_2\}$ and observe that both v_1 and v_2 become red and v_3 belongs to $B_1 \cup R$ in $G_{D\cup A}$. This contributes to $\mathrm{s}(A)$ by at least $2 \cdot 16 + (16-7)$. By Observation 1(iv), v_1, v_2, and v_3, respectively, have at least 3, 2, 2 blue neighbors in G_D. The decrease in their white-degrees contributes to $\mathrm{s}(A)$ by at least $7 \cdot 1$. Then, we get $\mathrm{s}(A) \ge 32 + 9 + 7 = 48 > 44\,|A|$, a contradiction.

Now, assume that a cycle $C_{3k}\colon v_1 \cdots v_{3k} v_1$ exists in $G_D[W]$ and set $A = \bigcup_{i=1}^{k} \{v_{3i}\}$. In $G_{D\cup A}$, all the $3k$ vertices of the cycle are recolored red and, by Observation 1(iv), the sum of the white-degrees of the blue vertices decreases by at least $2 \cdot 3k$. Consequently, we get the contradiction $w(A) \ge 16 \cdot 3k + 6k = 54k > 44\,|A|$. A similar argumentation can be given if the cycle is $C_{3k+2}\colon v_1 \cdots v_{3k+2} v_1$, where $k \ge 1$, and $A = \left(\bigcup_{i=1}^{k} \{v_{3i}\}\right) \cup \{v_{3k+2}\}$. Here, $|A| = k+1$ and we get $\mathrm{s}(A) \ge 16 \cdot (3k+2) + 2 \cdot (3k+2) = 54k + 36 > 44k + 44 = 44\,|A|$ that is a contradiction. For the case when the cycle is of order $3k+1$, we suppose $k \ge 3$ and obtain a contradiction as follows. Let $C_{3k+1}\colon v_1 \cdots v_{3k+1} v_1$ and let A be the $(k+1)$-element dominating set $\left(\bigcup_{i=1}^{k} \{v_{3i}\}\right) \cup \{v_{3k+2}\}$. We get $\mathrm{s}(A) \ge$

$16 \cdot (3k+1) + 2 \cdot (3k+1) = 54k + 18 > 44k + 44 = 44\,|A|$ since $k \geq 3$ is supposed. This finishes the proof of Claim H. □

Claim I. *No vertex from B_3 is adjacent to any vertices from W_0 in G_D.*

Proof. Assume for a contradiction that a vertex $v \in B_3$ has a neighbor u_0 from W_0. Let $A = \{v\}$ and denote by u_1 and u_2 the further two white neighbors of v. In $G_{D \cup A}$, $v, u_0 \in R$ and $u_1, u_2 \in B_2 \cup B_1 \cup R$. This change contributes to $\mathrm{s}(A)$ by at least $9 + 16 + 2(16 - 8) = 41$. By Observation 1(iv), the neighbors u_0, u_1 and u_2 have, respectively, at least 3, 1, 1 blue neighbors which are different from v. Therefore, $\mathrm{s}(A) \geq 41 + 5 \cdot 1 = 46 > 44\,|A|$ should be true but this contradicts our assumption on G_D. □

As follows, the vertices from W_0 may be adjacent only to some vertices from $B_2 \cup B_1$. We call a vertex from B_2 *special*, if it is adjacent to a vertex from W_0.

Claim J. *No special vertex is adjacent to two vertices from W_0.*

Proof. Suppose that a vertex $v \in B_2$ is adjacent to two vertices, say u_1 and u_2 from W_0. We set $A = \{v\}$ and observe that all the three vertices v, u_1 and u_2 are red in $G_{D \cup A}$. By Observation 1(iv), each of u_1 and u_2 has at least three blue neighbors different from v. This yields $\mathrm{s}(A) \geq 8 + 2 \cdot 16 + 6 \cdot 1 = 46 > 44\,|A|$ that contradicts our assumption on G_D. □

Claim K. *No special vertex is adjacent to a vertex from a C_4 or C_7.*

Proof. If a special vertex v is adjacent to a vertex u_0 from W_0 and to a vertex u_1 from a 4-cycle component $C_4\colon u_1u_2u_3u_4u_1$ of $G_D[W]$, then we set $A = \{v, u_3\}$ and observe that v, u_0, u_1, u_2, u_3 and u_4 turn red in $G_{D \cup A}$. In G_D, the vertices u_0, u_1, u_2, u_3 and u_4, respectively, have at least 3, 1, 2, 2, 2 neighbors from $(B_3 \cup B_2 \cup B_1) \setminus \{v\}$. Thus, $\mathrm{s}(A) \geq 8 + 5 \cdot 16 + 10 \cdot 1 = 98 > 44\,|A|$, a contradiction. Similarly, if we suppose that a special vertex v is adjacent to u_0 from W_0 and to a vertex u_1 from the 7-cycle $u_1 \cdots u_7u_1$, we set $A = \{v, u_3, u_6\}$ and conclude that $\mathrm{s}(A) \geq 8 + 8 \cdot 16 + 16 \cdot 1 = 152 > 44\,|A|$ that contradicts our assumption on G_D. □

Claim L. *If v_1 and v_2 are two adjacent vertices from W_1, then at most one of them may have a special blue neighbor.*

Proof. Assume to the contrary that $v_1u_1, v_2u_2 \in E(G)$ such that u_1, and u_2 are special vertices in G_D, and let x_1 and x_2 be the further white neighbors of u_1 and u_2. Hence, we have $v_1, v_2 \in W_1$, $u_1, u_2 \in B_2$, and $x_1, x_2 \in W_0$ in G_D. Consider the set $A = \{u_1, u_2\}$ and observe that all the six vertices v_1, v_2, u_1, u_2, x_1, x_2 become red in $G_{D \cup A}$. For $i = 1, 2$, by Claim I and Observation 1(iv), the vertex x_i has at least three neighbors from $(B_2 \cup B_1) \setminus \{v\}$ and v_i has at least two neighbors from $(B_3 \cup B_2 \cup B_1) \setminus \{v\}$. This implies the contradiction $\mathrm{s}(A) \geq 2 \cdot 8 + 4 \cdot 16 + 10 \cdot 1 = 90 > 44\,|A|$. □

Discharging. Applying Claims G–L, we now perform a discharging and prove that G_D satisfies Property 1. We assign charges to the (non-red) vertices of G_D so that every white vertex gets 16, and every vertex from B_3, B_2, and B_1 gets 9, 8, and 7, respectively. We remark that the sum of these charges equals $g(G_D)$. Then, every blue vertex, except the special ones, distributes its charge equally among the white neighbors as follows:

- Every vertex from B_3 gives 3 to each white neighbor.
- Every non-special vertex from B_2 gives 4 to each white neighbor.
- Every special vertex gives 7 to its neighbor from W_0, and gives 1 to the other neighbor.
- Every vertex from B_1 gives 7 to its neighbor.

After the discharging, every vertex from a P_1-component of $G_D[W]$ has a charge of at least $16+4\cdot 7=44$. By Claim L, every P_2-component has at least three non-special blue neighbors and, therefore, its charge is at least $2\cdot 16+3\cdot 1+3\cdot 3=44$. By Claim K, every C_4-component has at least $4\cdot 16+8\cdot 3=88$ and every C_7-component has at least $7\cdot 16+14\cdot 3=154$ as a charge. Let the number of P_1-, P_2-, C_4-, and C_7-components of $G[W]$ be denoted by p_1, p_2, c_4, and c_7, respectively, and let A be a minimum dominating set in $G[W]$. Then,

$$|A| = p_1 + p_2 + 2\,c_4 + 3\,c_7.$$

As $D \cup A$ is a dominating set in the graph G, we have $g(G_{D\cup A}) = 0$. Thus, $\mathrm{s}(A) = g(G_D)$, and the discharging proves the following lower bound:

$$\begin{aligned} \mathrm{s}(A) = g(G_D) &\geq 44\,p_1 + 44\,p_2 + 88\,c_4 + 154\,c_7 \\ &\geq 44\,(p_1 + p_2 + 2\,c_4 + 3\,c_7) = 44\,|A|. \end{aligned}$$

As it contradicts our assumption on G_D, we infer that every graph G with minimum degree 4 and every $D \subseteq V(G)$ with $g(G_D) > 0$ satisfy Property 2.

To prove Theorem 3, we observe that $g(G_\emptyset) = 16\,n$ and, by Property 2, there exists a set A_1 such that $g(G_{A_1}) \leq g(G_\emptyset) - 44\,|A_1|$. As G_{A_1} also satisfies Property 2, we may continue the process if $g(G_{A_1}) > 0$, and at the end we obtain a dominating set $D = A_1 \cup \cdots \cup A_j$ such that

$$g(G_D) = 0 \leq g(G_\emptyset) - 44\,|D| = 16\,n - 44\,|D|.$$

Consequently,

$$\gamma(G) \leq |D| \leq \frac{16}{44}\,n = \frac{4}{11}\,n$$

holds for every graph G of minimum degree 4. ∎

4. Concluding Remarks

Theorem 2 shows that $\gamma(G) \leq n/3$ holds for every graph with minimum degree at least 5. However, I do not believe that this upper bound is tight over the class of graphs with $\delta(G) \geq 5$. Examples with $\gamma/n > 1/4$ can possibly be found among larger graphs via computer search or large constructions, but it seems that $\delta(G) \geq 5$ and $n \leq 12$ together implies $\gamma(G) \leq n/4$ that is quite far from the proved $n/3$-upper bound.

Unfortunately, Theorem 3 does not seem sharp either. However, here we have 4-regular examples where the quotient γ/n equals $1/3$ that is relatively close to the proved upper bound $4/11$. The smallest such 4-regular graph is $G = K_6 - M$ that is obtained from the complete graph K_6 by the deletion of a perfect matching. Then, we have $\gamma(G) = 2 = n/3$. One may guess that this is the sharp upper bound for graphs of minimum degree 4 or, at least, it is true under the following stronger condition:

Conjecture 1. *There exists a constant n_0 such that for every connected 4-regular graph G of order $n > n_0$, we have $\gamma(G) \leq \frac{n}{3}$.*

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