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# THE TURÁN NUMBER FOR $4 \cdot S_\ell$[1]

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**Abstract**

The *Turán number* of a graph $H$, denoted by $ex(n,H)$, is the maximum number of edges of an $n$-vertex simple graph having no $H$ as a subgraph. Let $S_\ell$ denote the star on $\ell+1$ vertices, and let $k \cdot S_\ell$ denote $k$ disjoint copies of $S_\ell$. Erdős and Gallai determined the value $ex(n, k \cdot S_1)$ for all positive integers $k$ and $n$. Yuan and Zhang determined the value $ex(n, k \cdot S_2)$ and characterized all extremal graphs for all positive integers $k$ and $n$. Recently, Lan *et al.* determined the value $ex(n, 2 \cdot S_3)$ for all positive integers $n$, and Li and Yin determined the values $ex(n, k \cdot S_\ell)$ for $k = 2, 3$ and all positive integers $\ell$ and $n$. In this paper, we further determine the value $ex(n, 4 \cdot S_\ell)$ for all positive integers $\ell$ and almost all $n$, improving one of the results of Lidický *et al.*

**Keywords:** Turán number, disjoint copies, $k \cdot S_\ell$.

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## 1. Introduction

Graphs in this paper are finite and simple. Terms and notation not defined here are from [1]. Let $S_\ell$ denote the *star* on $\ell + 1$ vertices and let $P_\ell$ denote the *path* on $\ell$ vertices. For a graph $G$ and a vertex $v \in V(G)$, the degree of $v$ in $G$ is the number of edges incident to $v$, is denoted by $d_G(v)$, and the set of neighbors of $v$

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in $G$ is denoted by $N_G(v)$. Moreover, we define $N_G[v] = N_G(v) \cup \{v\}$. The vertex with degree $\ell$ in $S_\ell$ is called the *center* of $S_\ell$. For a set $S$ by $|S|$ we denote the cardinality of $S$. Clearly, $d_G(v) = |N_G(v)|$. For graphs $G$ and $H$, $G \cup H$ denotes the disjoint union of $G$ and $H$, $p \cdot G$ denotes the disjoint union of $p$ copies of $G$, and $G \vee H$ denotes the join of $G$ and $H$, that is, the graph obtained from $G \cup H$ by joining each vertex of $G$ to each vertex of $H$. For $S \subseteq V(G)$, the subgraph of $G$ induced by $S$ is denoted by $G[S]$.

The *Turán number* $ex(n, H)$ of the graph $H$ is the maximum number of edges of an $n$-vertex simple graph having no $H$ as a subgraph. Let $H_{ex}(n, H)$ denote a graph on $n$ vertices with $ex(n, H)$ edges not containing $H$. We call this graph an *extremal graph for* $H$. Let $T_r(n)$ denote the complete $r$-partite graph on $n$ vertices in which all parts are as equal in size as possible. Turán [9] determined the value $ex(n, K_{r+1})$ and showed that $T_r(n)$ is the unique extremal graph for $K_{r+1}$, where $K_{r+1}$ is the complete graph on $r + 1$ vertices. Turán's theorem is regarded as the basis of a significant branch of graph theory known as *extremal graph theory*. It was shown by Simonovits [8] that if $n$ is sufficiently large, then $K_{p-1} \vee T_r(n-p+1)$ is the unique extremal graph for $p \cdot K_{r+1}$. Gorgol [3] further considered the Turán number for $p$ disjoint copies of any connected graph $T$ on $t$ vertices and gave a lower bound for $ex(n, p \cdot T)$ by simply counting the number of edges of the graphs $H_{ex}(n - pt + 1, T) \cup K_{pt-1}$ and $H_{ex}(n - p + 1, T) \vee K_{p-1}$ which do not contain $p \cdot T$.

**Theorem 1** [3]**.** *Let $T$ be an arbitrary connected graph on $t$ vertices, $p$ be an arbitrary positive integer and $n$ be an integer such that $n \geq pt$. Then* $ex(n, p\cdot T) \geq \max\left\{ex(n - pt + 1, T) + \binom{pt-1}{2}, ex(n - p + 1, T) + (p - 1)n - \binom{p}{2}\right\}$.

Lidický *et al.* [7] investigated the Turán number of a star forest (a forest whose connected components are stars), and determined the value $ex(n, F)$ for sufficiently large $n$, where $F = S_{d_1} \cup S_{d_2} \cup \cdots \cup S_{d_k}$ and $d_1 \geq d_2 \geq \cdots \geq d_k$. Lidický *et al.* [7] also pointed out that they make no attempt to minimize the bound on $n$ in their proof. Yin and Rao [10] improved the result of Lidický *et al.* by determining the value $ex(n, k\cdot S_\ell)$ for $n \geq \frac{1}{2}\ell^2 k(k-1)+k-2+\max\{\ell k, \ell^2+2\ell\}$. Lan *et al.* [4] further improved these results by determining the value $ex(n, k \cdot S_\ell)$ for $n \geq k(\ell^2 + \ell + 1) - \frac{\ell}{2}(\ell - 3)$. However, there are very few cases when the Turán number $ex(n, k \cdot S_\ell)$ is known exactly for all positive integers $k$, $\ell$ and $n$. Erdős and Gallai [2] determined $ex(n, k \cdot S_1)$ for all positive integers $k$ and $n$. Yuan and Zhang [11] determined $ex(n, k\cdot S_2)$ (i.e., $ex(n, k\cdot P_3)$) and characterized all extremal graphs for all positive integers $k$ and $n$. Lan *et al.* [4] determined $ex(n, 2 \cdot S_3)$ for all positive integers $n$. Li and Yin [6] determined $ex(n, k \cdot S_\ell)$ for $k = 2, 3$ and all positive integers $\ell$ and $n$. Recently, Lan *et al.* [5] studied the degree powers for forbidding star forests, which is a classical generalization of the Turán number for star forests.

**Theorem 2** [2].

$$ex(n, k \cdot S_1) = \begin{cases} \binom{n}{2}, & \text{if } n < 2k, \\ \binom{2k-1}{2}, & \text{if } 2k \le n < \frac{5k}{2} - 1, \\ \binom{k-1}{2} + (n-k+1)(k-1), & \text{if } n \ge \frac{5k}{2} - 1. \end{cases}$$

**Theorem 3** [11].

$$ex(n, k \cdot S_2) = \begin{cases} \binom{n}{2}, & \text{if } n < 3k, \\ \binom{3k-1}{2} + \left\lfloor \frac{n-3k+1}{2} \right\rfloor, & \text{if } 3k \le n < 5k - 1, \\ \binom{k-1}{2} + (n-k+1)(k-1) + \left\lfloor \frac{n-k+1}{2} \right\rfloor, & \text{if } n \ge 5k - 1. \end{cases}$$

*Furthermore, all extremal graphs for $k \cdot S_2$ are characterized.*

**Theorem 4** [4].

$$ex(n, 2 \cdot S_3) = \begin{cases} \binom{n}{2}, & \text{if } n < 8, \\ n + 14, & \text{if } 8 \le n < 16, \\ 2(n-1), & \text{if } n \ge 16. \end{cases}$$

**Theorem 5** [6].

$$ex(n, 2 \cdot S_\ell) = \begin{cases} \binom{n}{2}, & \text{if } n < 2(\ell+1), \\ \left\lfloor \frac{(\ell-1)n + (2\ell+1)(\ell+1)}{2} \right\rfloor, & \text{if } 2(\ell+1) \le n < (\ell+1)^2, \\ \left\lfloor \frac{(\ell+1)n - (\ell+1)}{2} \right\rfloor, & \text{if } n \ge (\ell+1)^2. \end{cases}$$

**Theorem 6** [6].

$$ex(n, 3 \cdot S_\ell) = \begin{cases} \binom{n}{2}, & \text{if } n < 3(\ell+1), \\ \left\lfloor \frac{(\ell-1)n + (3\ell+2)(2\ell+2)}{2} \right\rfloor, & \text{if } 3(\ell+1) \le n < \frac{3\ell^2+6\ell+4}{2}, \\ \left\lfloor \frac{(\ell+3)n - 2(\ell+2)}{2} \right\rfloor, & \text{if } n \ge \frac{3\ell^2+6\ell+4}{2}. \end{cases}$$

In this paper, we further determine the Turán number $ex(n, 4 \cdot S_\ell)$ for all positive integers $\ell$ and almost all $n$.

**Theorem 7.**

$$ex(n, 4 \cdot S_\ell) = \begin{cases} \binom{n}{2}, & \text{if } n < 4(\ell+1), \\ \left\lfloor \frac{(\ell-1)n + (4\ell+3)(3\ell+3)}{2} \right\rfloor, & \text{if } 5(\ell+1) \le n < 2\ell^2 + 4\ell + 3, \\ \left\lfloor \frac{(\ell+5)n - 3(\ell+3)}{2} \right\rfloor, & \text{if } n \ge 2\ell^2 + 4\ell + 3. \end{cases}$$

## 2. Proof of Theorem 7

For $\ell = 1$ and 2, Theorem 7 follows from Theorems 2–3 (the case $k = 4$). Assume $\ell \geq 3$. Note that the extremal graph $K_n$ gives the lower and upper bounds for $ex(n, 4 \cdot S_\ell)$ in the case $n \leq 4\ell + 3$. Thus, we consider only the case $n \geq 5(\ell + 1)$. Denote $f(\ell, n) = \max\left\{\left\lfloor\frac{(\ell-1)n+(4\ell+3)(3\ell+3)}{2}\right\rfloor, \left\lfloor\frac{(\ell+5)n-3(\ell+3)}{2}\right\rfloor\right\}$. Clearly,

$$f(\ell, n) = \begin{cases} \left\lfloor\frac{(\ell-1)n+(4\ell+3)(3\ell+3)}{2}\right\rfloor, & \text{if } 5(\ell+1) \leq n < 2\ell^2 + 4\ell + 3, \\ \left\lfloor\frac{(\ell+5)n-3(\ell+3)}{2}\right\rfloor, & \text{if } n \geq 2\ell^2 + 4\ell + 3. \end{cases}$$

The lower bound $ex(n, 4 \cdot S_\ell) \geq f(\ell, n)$ follows from $ex(n, S_\ell) = \left\lfloor\frac{n(\ell-1)}{2}\right\rfloor$ and Theorem 1. To show the upper bound, we assume that $G$ is a graph on $n \geq 5(\ell + 1)$ vertices with $e(G) \geq f(\ell, n) + 1$ and $G$ contains no $4 \cdot S_\ell$ as a subgraph. The degree sequence of $G$ is denoted by $(d_1, d_2, \ldots, d_n)$, where $d_1 \geq d_2 \geq \cdots \geq d_n$. By $\left\lfloor\frac{(\ell+5)n-3(\ell+3)}{2}\right\rfloor = \left\lfloor\frac{(\ell+3)n+2n-3(\ell+3)}{2}\right\rfloor \geq \left\lfloor\frac{(\ell+3)n+2\times4(\ell+1)-3(\ell+3)}{2}\right\rfloor \geq \left\lfloor\frac{(\ell+3)n-2(\ell+2)}{2}\right\rfloor$, we can see that

$$e(G) > \max\left\{\left\lfloor\frac{(\ell-1)n + (3\ell+2)(2\ell+2)}{2}\right\rfloor, \left\lfloor\frac{(\ell+3)n - 2(\ell+2)}{2}\right\rfloor\right\}.$$

It follows from Theorem 6 that $G$ contains three disjoint copies of $S_\ell$, denoted $F_1$, $F_2$ and $F_3$. For convenience, we let $V(F_i) = \{v_{i0}, v_{i1}, \ldots, v_{i\ell}\}$ and $E(F_i) = \{v_{i0}v_{i1}, v_{i0}v_{i2}, \ldots, v_{i0}v_{i\ell}\}$, for $i = 1, 2, 3$. Denote $H = G \backslash (V(F_1) \cup V(F_2) \cup V(F_3))$, and $H' = G[V(F_1) \cup V(F_2) \cup V(F_3)]$. We first have the following Claims 1–4.

**Claim 1.** $d_3 \geq 2\ell + 3$.

***Proof.*** Note that $G - S_\ell$ contains no $3 \cdot S_\ell$. Let $m_0$ be the number of edges incident to $S_\ell$ in $G$. Thus, we have

$$m_0 = e(G) - e(G - S_\ell) \geq e(G) - ex(n - \ell - 1, 3 \cdot S_\ell).$$

If $n \geq 2\ell^2 + 4\ell + 3$, then $n - \ell - 1 \geq 2\ell^2 + 3\ell + 2 \geq \frac{3\ell^2+6\ell+4}{2}$. By Theorem 6, we have

$$\begin{aligned} m_0 &\geq \left\lfloor\frac{(\ell+5)n-3(\ell+3)}{2}\right\rfloor + 1 - \left\lfloor\frac{(\ell+3)(n-\ell-1)-2(\ell+2)}{2}\right\rfloor \\ &\geq \frac{(\ell+5)n-3(\ell+3)-1}{2} + 1 - \frac{(\ell+3)(n-\ell-1)-2(\ell+2)}{2} \\ &= \frac{2n+\ell^2+3\ell-1}{2} \geq \frac{2(2\ell^2+4\ell+3)+\ell^2+3\ell-1}{2} = \frac{5(\ell+1)^2+\ell}{2}. \end{aligned}$$

Assume $5(\ell + 1) \leq n < 2\ell^2 + 4\ell + 3$, that is, $4(\ell + 1) \leq n - \ell - 1 < 2\ell^2 + 3\ell + 2$.

If $4(\ell+1) \le n-\ell-1 < \frac{3\ell^2+6\ell+4}{2}$, by Theorem 6, then we have

$$\begin{aligned} m_0 &\ge \left\lfloor \tfrac{(\ell-1)n+(4\ell+3)(3\ell+3)}{2} \right\rfloor + 1 - \left\lfloor \tfrac{(\ell-1)(n-\ell-1)+(3\ell+2)(2\ell+2)}{2} \right\rfloor \\ &\ge \tfrac{(\ell-1)n+(4\ell+3)(3\ell+3)-1}{2} + 1 - \tfrac{(\ell-1)(n-\ell-1)+(3\ell+2)(2\ell+2)}{2} \\ &= \tfrac{7\ell^2+11\ell+5}{2} \ge \tfrac{5(\ell+1)^2+\ell}{2}. \end{aligned}$$

If $\frac{3\ell^2+6\ell+4}{2} \le n-\ell-1 < 2\ell^2+3\ell+2$, by Theorem 6, then we have

$$\begin{aligned} m_0 &\ge \left\lfloor \tfrac{(\ell-1)n+(4\ell+3)(3\ell+3)}{2} \right\rfloor + 1 - \left\lfloor \tfrac{(\ell+3)(n-\ell-1)-2(\ell+2)}{2} \right\rfloor \\ &\ge \tfrac{(\ell-1)n+(4\ell+3)(3\ell+3)-1}{2} + 1 - \tfrac{(\ell+3)(n-\ell-1)-2(\ell+2)}{2} \\ &= \tfrac{13\ell^2+27\ell+17-4n}{2} \ge \tfrac{13\ell^2+27\ell+17-4(2\ell^2+4\ell+3)}{2} = \tfrac{5(\ell+1)^2+\ell}{2}. \end{aligned}$$

Hence each $S_\ell$ must contain a vertex of degree at least

$$\frac{m_0}{\ell+1} \ge \frac{\frac{5(\ell+1)^2+\ell}{2}}{(\ell+1)} \ge 2\ell+3.$$

This implies that $G$ contains three vertices of degree at least $2\ell+3$, which proves Claim 1. □

**Claim 2.** $d_4 \ge \ell+3$.

***Proof.*** If $d_4 \le \ell+2$, then $e(G) \le \left\lfloor \frac{3(n-1)+(\ell+2)(n-3)}{2} \right\rfloor = \left\lfloor \frac{(\ell+5)n-3(\ell+3)}{2} \right\rfloor < f(\ell,n)+1$, a contradiction, which proves Claim 2. □

**Claim 3.** *If $1 \le |N_H(v_{i0})| \le \ell$ for some $i \in \{1,2,3\}$, then $|N_H(v_{ij})| \le \ell$ for all $j \in \{1,\dots,\ell\}$.*

***Proof.*** Assume $|N_H(v_{ij})| \ge \ell+1$ for some $j \in \{1,\dots,\ell\}$. Let $v \in N_H(v_{i0})$; we can find an $S_\ell$ in $G[(V(F_i) \setminus \{v_{ij}\}) \cup \{v\}]$ whose center is $v_{i0}$. By $|N_H(v_{ij}) \setminus \{v\}| \ge \ell+1-1 = \ell$, we can find another $S_\ell$ in $G[N_H[v_{ij}] \setminus \{v\}]$ whose center is $v_{ij}$. Therefore, $G$ contains $4 \cdot S_\ell$, a contradiction. This proves Claim 3. □

**Claim 4.** *If $|N_H(v_{i0})| \ge \ell+1$ for some $i \in \{1,2,3\}$, then $|N_H(v_{ij})| \le \ell-1$ for all $j \in \{1,\dots,\ell\}$.*

***Proof.*** If $|N_H(v_{ij})| \ge \ell$ for some $j \in \{1,\dots,\ell\}$, then we can find an $S_\ell$ in $G[N_H[v_{ij}]]$ whose center is $v_{ij}$. This $S_\ell$ is denoted by $F$. Let $v \in N_H(v_{i0}) \setminus V(F)$; we can find another $S_\ell$ in $G[(V(F_i) \setminus \{v_{ij}\}) \cup \{v\}]$ whose center is $v_{i0}$. Therefore, $G$ contains $4 \cdot S_\ell$, a contradiction. This proves Claim 4. □

We consider the following two cases in terms of the value of $d_1$.

*Case* 1. $d_1 \geq 4\ell+3$. If $d_2 \geq 3\ell+3$, by Claims 1–2, then $G$ contains $4 \cdot S_\ell$. Hence $d_2 \leq 3\ell+2$. By Claim 1, we may take $v_{i0}$ to be the vertex with degree $d_i$, for $i = 1, 2, 3$. Denote $H_1 = G \setminus V(F_3)$.

**Claim 5.** *$|N_{H_1}(v_{3j})| \leq 2\ell+2$ for all $j \in \{1, \ldots, \ell\}$.*

***Proof.*** Assume $N_{H_1}(v_{3j}) \geq 2\ell+3$ for some $j \in \{1, \ldots, \ell\}$. By Claim 1, $|N_G(v_{30}) \setminus (\{v_{31}, \ldots, v_{3\ell}\} \cup V(F_2) \cup \{v_{10}\})| \geq d_3 - \ell - (\ell+1) - 1 \geq 2\ell+3-(2\ell+2) = 1$. Let $v \in N_G(v_{30}) \setminus (\{v_{31}, \ldots, v_{3\ell}\} \cup V(F_2) \cup \{v_{10}\})$, we can find the first $S_\ell$ (denoted $F$) in $G[(V(F_3) \setminus \{v_{3j}\}) \cup \{v\}]$ whose center is $v_{30}$. By $|N_{H_1}(v_{3j}) \setminus (V(F_2) \cup \{v_{10}, v\})| \geq 2\ell+3-(\ell+1+1+1) = \ell$; we can find the second $S_\ell$ (denoted $F'$) in $G[N_{H_1}[v_{3j}] \setminus (V(F_2) \cup \{v_{10}, v\})]$ whose center is $v_{3j}$. By $|N_G(v_{10}) \setminus (V(F_2) \cup V(F) \cup V(F'))| \geq d_1 - 3(\ell+1) \geq 4\ell+3-3\ell-3 = \ell$, we can find the third $S_\ell$ in $G[N_G[v_{10}] \setminus (V(F_2) \cup V(F) \cup V(F'))]$ whose center is $v_{10}$. Thus $G$ contains $4 \cdot S_\ell$ if we view $F_2$ as the fourth $S_\ell$, a contradiction which proves Claim 5. □

Now by $|N_{H_1}(v_{30})| = |N_G(v_{30}) \setminus \{v_{31}, \ldots, v_{3\ell}\}| = d_3 - \ell \leq 3\ell+2-\ell = 2\ell+2$ and Claim 5, we have

$$\begin{aligned} e(H_1) &= e(G) - e(G[V(F_3)]) - |N_{H_1}(v_{30})| - \sum_{j=1}^{\ell} |N_{H_1}(v_{3j})| \\ &\geq e(G) - \tfrac{(\ell+1)\ell}{2} - (2\ell+2) - (2\ell+2)\ell = e(G) - \tfrac{5\ell^2+9\ell+4}{2}. \end{aligned}$$

If $5(\ell+1) \leq n < 2\ell^2+4\ell+3$, i.e., $4(\ell+1) \leq n-\ell-1 < 2\ell^2+3\ell+2$, then

$$\begin{aligned} e(H_1) &\geq \left\lfloor \tfrac{(\ell-1)n+(4\ell+3)(3\ell+3)}{2} \right\rfloor + 1 - \tfrac{5\ell^2+9\ell+4}{2} \\ &\geq \tfrac{(\ell-1)n+(4\ell+3)(3\ell+3)-1}{2} + 1 - \tfrac{5\ell^2+9\ell+4}{2} = \tfrac{(\ell-1)n+7\ell^2+12\ell+6}{2}. \end{aligned}$$

However, since $H_1$ contains no $3 \cdot S_\ell$, we have that if $4(\ell+1) \leq n-\ell-1 < \frac{3\ell^2+6\ell+4}{2}$, by Theorem 6, then $e(H_1) \leq ex(n-\ell-1, 3 \cdot S_\ell) = \left\lfloor \frac{(\ell-1)(n-\ell-1)+(3\ell+2)(2\ell+2)}{2} \right\rfloor = \left\lfloor \frac{(\ell-1)n+5\ell^2+10\ell+5}{2} \right\rfloor$, a contradiction; and if $\frac{3\ell^2+6\ell+4}{2} \leq n-\ell-1 < 2\ell^2+3\ell+2$, by Theorem 6, then

$$\begin{aligned} e(H_1) &\leq ex(n-\ell-1, 3 \cdot S_\ell) = \left\lfloor \tfrac{(\ell+3)(n-\ell-1)-2(\ell+2)}{2} \right\rfloor = \left\lfloor \tfrac{(\ell-1)n+4n-\ell^2-6\ell-7}{2} \right\rfloor \\ &\leq \left\lfloor \tfrac{(\ell-1)n+4(2\ell^2+4\ell+3)-\ell^2-6\ell-7}{2} \right\rfloor = \left\lfloor \tfrac{(\ell-1)n+7\ell^2+10\ell+5}{2} \right\rfloor, \end{aligned}$$

a contradiction.

If $n \geq 2\ell^2+4\ell+3$, i.e., $n-\ell-1 \geq 2\ell^2+3\ell+2$ ($\geq \frac{3\ell^2+6\ell+4}{2}$), then

$$\begin{aligned} e(H_1) &\geq \left\lfloor \tfrac{(\ell+5)n-3(\ell+3)}{2} \right\rfloor + 1 - \tfrac{5\ell^2+9\ell+4}{2} \geq \tfrac{(\ell+5)n-3(\ell+3)-1}{2} + 1 - \tfrac{5\ell^2+9\ell+4}{2} \\ &= \tfrac{(\ell+3)n+2n-5\ell^2-12\ell-12}{2} \geq \tfrac{(\ell+3)n+2(2\ell^2+4\ell+3)-5\ell^2-12\ell-12}{2} = \tfrac{(\ell+3)n-\ell^2-4\ell-6}{2}. \end{aligned}$$

However, $e(H_1) \le ex(n-\ell-1, 3\cdot S_\ell) = \left\lfloor \frac{(\ell+3)(n-\ell-1)-2(\ell+2)}{2} \right\rfloor = \left\lfloor \frac{(\ell+3)n-\ell^2-6\ell-7}{2} \right\rfloor$, a contradiction.

*Case* 2. $d_1 \le 4\ell+2$.

*Case* 2.1. $d_3 \ge 3\ell+3$. Let $v_{i0}$ be the vertex with degree $d_i$ for $i = 1, 2, 3$, and let $\{v_{31}, \ldots, v_{3\ell}\} \subseteq N_G(v_{30})$, $\{v_{21}, \ldots, v_{2\ell}\} \subseteq N_G(v_{20}) \setminus \{v_{30}, v_{31}, \ldots, v_{3\ell}\}$ and

$$\{v_{11}, \ldots, v_{1\ell}\} \subseteq N_G(v_{10}) \setminus \{v_{20}, v_{21}, \ldots, v_{2\ell}, v_{30}, v_{31}, \ldots, v_{3\ell}\}.$$

We take $F_i$ to be the graph with $V(F_i) = \{v_{i0}, v_{i1}, \ldots, v_{i\ell}\}$ and $E(F_i) = \{v_{i0}v_{i1}, v_{i0}v_{i2}, \ldots, v_{i0}v_{i\ell}\}$ for $i = 1, 2, 3$. Then $F_i$ is the $S_\ell$ whose center is $v_{i0}$ for $i = 1, 2, 3$. Moreover, $|N_H(v_{i0})| \ge d_3 - (3\ell+2) \ge 1$ for all $i \in \{1, 2, 3\}$. Let $I = \{i \,|\, i \in \{1, 2, 3\}$ and $1 \le |N_H(v_{i0})| \le \ell\}$, $J = \{1, 2, 3\} \setminus I$, $A = \bigcup_{i\in I} V(F_i)$, $B = \bigcup_{i\in J} V(F_i)$, $B_1 = \{v | v \in B \setminus \{v_{10}, v_{20}, v_{30}\}$ and $1 \le |N_H(v)| \le \ell-1\}$ and $B_2 = B \setminus (B_1 \cup \{v_{10}, v_{20}, v_{30}\})$. Clearly, $|A| = (\ell+1)|I|$, $|I| + |J| = 3$ and $|B_1| + |B_2| = \ell|J|$. By Claim 4, $|N_H(v)| = 0$ for $v \in B_2$.

**Claim 6.** *If $v \in B_1$, then $d_{H'}(v) \le 3\ell+1$, where $H' = G[V(F_1) \cup V(F_2) \cup V(F_3)]$.*

***Proof.*** We may assume $v = v_{ij}$ for some $i \in J$ and some $j \in \{1, \ldots, \ell\}$. If $d_{H'}(v_{ij}) = 3\ell+2$, let $u \in N_H(v_{ij})$, then we can find an $S_\ell$ in $G[\{u\} \cup (V(F_i) \setminus \{v_{i0}\})]$ whose center is $v_{ij}$. By $|N_H(v_{i0}) \setminus \{u\}| \ge \ell+1-1 = \ell$, we can find another $S_\ell$ in $G[N_H[v_{i0}] \setminus \{u\}]$ whose center is $v_{i0}$. Therefore, $G$ contains $4 \cdot S_\ell$, a contradiction. This proves Claim 6. □

Now by $|N_H(v_{i0})| \le |N_G(v_{i0}) \setminus \{v_{i1}, \ldots, v_{i\ell}\}| \le d_1 - \ell \le 4\ell+2-\ell = 3\ell+2$ for $i \in J$, $\ell \ge 3$ and Claims 3, 4 and 6, we have

$$\begin{aligned}
e(H) &= e(G) - e(H') - \sum_{i=1}^{3}\sum_{j=0}^{\ell} |N_H(v_{ij})| \\
&= e(G) - \frac{\sum_{v\in A} d_{H'}(v) + \sum_{i\in J} d_{H'}(v_{i0}) + \sum_{v\in B_1} d_{H'}(v) + \sum_{v\in B_2} d_{H'}(v)}{2} \\
&\quad - \sum_{v\in A} |N_H(v)| - \sum_{i\in J} |N_H(v_{i0})| - \sum_{v\in B_1} |N_H(v)| - \sum_{v\in B_2} |N_H(v)| \\
&\ge e(G) - \frac{(3\ell+2)|A| + \sum_{i\in J}(d_1 - |N_H(v_{i0})|) + (3\ell+1)|B_1| + (3\ell+2)|B_2|}{2} \\
&\quad - -\ell|A| - \sum_{i\in J} |N_H(v_{i0})| - (\ell-1)|B_1| \\
&= e(G) - \frac{(5\ell+2)|A| + \sum_{i\in J}(d_1 + |N_H(v_{i0})|) + (5\ell-1)|B_1| + (3\ell+2)|B_2|}{2} \\
&\ge e(G) - \frac{(5\ell+2)(\ell+1)|I| + (4\ell+2+3\ell+2)|J| + (5\ell-1)(|B_1|+|B_2|)}{2} \\
&= e(G) - \frac{(5\ell^2+7\ell+2)|I| + (5\ell^2+6\ell+4)|J|}{2} \\
&\ge e(G) - \frac{(5\ell^2+7\ell+2)|I| + (5\ell^2+7\ell+2)|J|}{2} = e(G) - \frac{15\ell^2+21\ell+6}{2}.
\end{aligned}$$

If $5(\ell+1) \le n < 2\ell^2+4\ell+3$, then

$$\begin{aligned} e(H) &\ge \left\lfloor \frac{(\ell-1)n+(4\ell+3)(3\ell+3)}{2} \right\rfloor + 1 - \tfrac{1}{2}(15\ell^2+21\ell+6) \\ &\ge \frac{(\ell-1)n+(4\ell+3)(3\ell+3)-1}{2} + 1 - \tfrac{1}{2}(15\ell^2+21\ell+6) = \frac{(\ell-1)n-3\ell^2+4}{2}. \end{aligned}$$

However, since $H$ contains no $S_\ell$, by $ex(n, S_\ell) = \left\lfloor \frac{n(\ell-1)}{2} \right\rfloor$, then $e(H) \le ex(n - 3\ell - 3, S_\ell) = \left\lfloor \frac{(n-3\ell-3)(\ell-1)}{2} \right\rfloor = \left\lfloor \frac{(\ell-1)n-3\ell^2+3}{2} \right\rfloor$, a contradiction.

If $n \ge 2\ell^2+4\ell+3$, then

$$\begin{aligned} e(H) &\ge \left\lfloor \frac{(\ell+5)n-3(\ell+3)}{2} \right\rfloor + 1 - \tfrac{1}{2}(15\ell^2+21\ell+6) \\ &\ge \frac{(\ell+5)n-3(\ell+3)-1}{2} + 1 - \tfrac{1}{2}(15\ell^2+21\ell+6) \\ &= \frac{(\ell-1)n+6n-15\ell^2-24\ell-14}{2} \\ &\ge \frac{(\ell-1)n+6(2\ell^2+4\ell+3)-15\ell^2-24\ell-14}{2} = \frac{(\ell-1)n-3\ell^2+4}{2}. \end{aligned}$$

However, $e(H) \le ex(n-3\ell-3, S_\ell) = \left\lfloor \frac{(\ell-1)n-3\ell^2+3}{2} \right\rfloor$, a contradiction.

*Case* 2.2. $d_3 \le 3\ell+2$. If $d_1 \ge 3\ell+3$, by Claim 1, we take $F_1$, $F_2$ and $F_3$ to be the same as Case 2.1. Clearly, $d_G(v) \le d_3 \le 3\ell+2$ for all $v \in V(H') \setminus \{v_{10}, v_{20}, v_{30}\}$. This implies that $d_H(v) \le 3\ell+1$ for all $v \in V(H') \setminus \{v_{10}, v_{20}, v_{30}\}$. Let $I = \{i \,|\, i \in \{1,2,3\}$ and $|N_H(v_{i0})| \ge \ell+1\}$, $J = \{1,2,3\} \setminus I$, $A = \bigcup_{i \in I} V(F_i)$, $A_1 = A \setminus \{v_{10}, v_{20}, v_{30}\}$, $B = \bigcup_{i \in J} V(F_i)$, $B_1 = \{v | v \in B \setminus \{v_{10}, v_{20}, v_{30}\}$ and $|N_H(v)| \ge 2\ell-1\}$ and $B_2 = B \setminus (B_1 \cup \{v_{10}, v_{20}, v_{30}\})$. Clearly, $|A_1| = \ell|I|$, $|B_2| = \ell|J| - |B_1|$ and $|I| + |J| = 3$.

**Claim 7.** *If $|N_H(v_{i0})| = 0$ for some $i \in \{1,2,3\}$, and $|N_H(v_{ij})| \ge 2\ell-1$ for some $j \in \{1, \dots, \ell\}$, then $|N_H(v_{ij'})| \le \ell-2$ for all $j' \in \{1, \dots, \ell\} \setminus \{j\}$.*

***Proof.*** If $|N_H(v_{ij'})| \ge \ell-1$ for some $j' \in \{1, \dots, \ell\} \setminus \{j\}$, let $\{u_1, \dots, u_{\ell-1}\} \subseteq N_H(v_{ij'})$, then we can find an $S_\ell$ in $G[\{u_1, \dots, u_{\ell-1}\} \cup \{v_{i0}, v_{ij'}\}]$ whose center is $v_{ij'}$. By $|N_H(v_{ij}) \setminus \{u_1, \dots, u_{\ell-1}\}| \ge 2\ell-1-(\ell-1) = \ell$, we can find another $S_\ell$ in $G[N_H[v_{ij}] \setminus \{u_1, \dots, u_{\ell-1}\}]$ whose center is $v_{ij}$. Therefore, $G$ contains $4 \cdot S_\ell$, a contradiction. This proves Claim 7. □

**Claim 8.** $|B_1| \le |J|$.

***Proof.*** Let $i \in J$. If $1 \le |N_H(v_{i0})| \le \ell$, by Claim 3, then $|N_H(v_{ij})| \le \ell$ for all $j \in \{1, \dots, \ell\}$, implying that $|N_H(v)| < 2\ell-1$ for all $v \in V(F_i)$. If $|N_H(v_{i0})| = 0$, by Claim 7, then $F_i$ contains at most one vertex, say $v$, with $|N_H(v)| \ge 2\ell-1$. Thus $|B_1| \le |J|$. This proves Claim 8. □

Now by $|N_H(v_{i0})| \le |N_G(v_{i0}) \setminus \{v_{i1}, \dots, v_{i\ell}\}| \le d_1 - \ell \le 3\ell + 2$ for $i \in I$, $\ell \ge 3$ and Claims 4 and 8, we have

$$\begin{aligned}
e(H) &= e(G) - e(H') - \sum_{i=1}^{3}\sum_{j=0}^{\ell} |N_H(v_{ij})| \\
&= e(G) - \frac{\sum_{i\in I} d_{H'}(v_{i0}) + \sum_{v\in A_1} d_{H'}(v) + \sum_{i\in J} d_{H'}(v_{i0}) + \sum_{v\in B_1} d_{H'}(v) + \sum_{v\in B_2} d_{H'}(v)}{2} \\
&- \sum_{i\in I} |N_H(v_{i0})| - \sum_{v\in A_1} |N_H(v)| - \sum_{i\in J} |N_H(v_{i0})| - \sum_{v\in B_1} |N_H(v)| - \sum_{v\in B_2} |N_H(v)| \\
&\ge e(G) - \frac{1}{2}\Bigg( \sum_{i\in I}(d_1 - |N_H(v_{i0})|) + \sum_{v\in A_1}(d_G(v) - |N_H(v)|) + (3\ell+2)|J| \\
&+ \sum_{v\in B_1}(d_G(v) - |N_H(v)|) + \sum_{v\in B_2}(d_G(v) - |N_H(v)|)\Bigg) \\
&- \sum_{i\in I} |N_H(v_{i0})| - \sum_{v\in A_1} |N_H(v)| - \ell|J| - \sum_{v\in B_1} |N_H(v)| - \sum_{v\in B_2} |N_H(v)| \\
&= e(G) - \frac{1}{2}\Bigg( \sum_{i\in I}(d_1 + |N_H(v_{i0})|) + \sum_{v\in A_1}(d_G(v) + |N_H(v)|) + (5\ell+2)|J| \\
&+ \sum_{v\in B_1}(d_G(v) + |N_H(v)|) + \sum_{v\in B_2}(d_G(v) + |N_H(v)|)\Bigg) \\
&\ge e(G) - \frac{1}{2}\big((4\ell+2+3\ell+2)|I| + (3\ell+2+\ell-1)|A_1| + (5\ell+2)|J| \\
&+ (3\ell+2+3\ell+1)|B_1| + (3\ell+2+2\ell-2)|B_2|\big) \\
&= e(G) - \frac{(4\ell^2+8\ell+4)|I| + (5\ell^2+5\ell+2)|J| + (\ell+3)|B_1|}{2} \\
&\ge e(G) - \frac{(4\ell^2+8\ell+4)|I| + (5\ell^2+5\ell+2)|J| + (\ell+3)|J|}{2} = e(G) - \frac{(4\ell^2+8\ell+4)|I| + (5\ell^2+6\ell+5)|J|}{2} \\
&\ge e(G) - \frac{(5\ell^2+7\ell+2)|I| + (5\ell^2+7\ell+2)|J|}{2} \ge e(G) - \frac{15\ell^2+21\ell+6}{2}.
\end{aligned}$$

If $5(\ell+1) \le n < 2\ell^2+4\ell+3$, then $e(H) \ge \left\lfloor \frac{(\ell-1)n+(4\ell+3)(3\ell+3)}{2} \right\rfloor + 1 - \frac{1}{2}(15\ell^2 + 21\ell+6) \ge \frac{(\ell-1)n-3\ell^2+4}{2}$. However, $e(H) \le ex(n-3\ell-3, S_\ell) = \left\lfloor \frac{(\ell-1)n-3\ell^2+3}{2} \right\rfloor$, a contradiction. If $n \ge 2\ell^2+4\ell+3$, then $e(H) \ge \left\lfloor \frac{(\ell+5)n-3(\ell+3)}{2} \right\rfloor + 1 - \frac{1}{2}(15\ell^2 + 21\ell+6) \ge \frac{(\ell-1)n-3\ell^2+4}{2}$. However, $e(H) \le ex(n-3\ell-3, S_\ell) = \left\lfloor \frac{(\ell-1)n-3\ell^2+3}{2} \right\rfloor$, a contradiction.

Thus, we have proved that every graph $G$ on $n \ge 5(\ell+1)$ vertices with $e(G) \ge f(\ell,n)+1$ contains $4 \cdot S_\ell$ as a subgraph. In other words, $ex(n, 4\cdot S_\ell) \le f(\ell, n)$. The proof of Theorem 7 is completed. ■

**Remark.** The general case $ex(n, k\cdot S_\ell)$ seems to be much more challenging. The method presented here cannot be used to determine $ex(n, k\cdot S_\ell)$ for all positive integers $k$, $\ell$ and $n$. The proofs of Claims 2–4 can be adapted to the general $k$, but the proofs of the remaining parts cannot be extended to the general case $k$.

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