

## INDEPENDENCE NUMBER AND PACKING COLORING OF GENERALIZED MYCIELSKI GRAPHS

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### Abstract

For a positive integer  $k \geq 1$ , a graph  $G$  with vertex set  $V$  is said to be  $k$ -packing colorable if there exists a mapping  $f : V \mapsto \{1, 2, \dots, k\}$  such that any two distinct vertices  $x$  and  $y$  with the same color  $f(x) = f(y)$  are at distance at least  $f(x) + 1$ . The packing chromatic number of a graph  $G$ , denoted by  $\chi_\rho(G)$ , is the smallest integer  $k$  such that  $G$  is  $k$ -packing colorable.

In this work, we study both independence and packing colorings in the  $m$ -generalized Mycielskian of a graph  $G$ , denoted  $\mu_m(G)$ . We first give an explicit formula for  $\alpha(\mu_m(G))$  when  $m$  is odd and bounds when  $m$  is even. We then use these results to give exact values of  $\alpha(\mu_m(K_n))$  for any  $m$  and  $n$ . Next, we give bounds on the packing chromatic number,  $\chi_\rho$ , of  $\mu_m(G)$ . We also prove the existence of large planar graphs whose packing chromatic number is 4. The rest of the paper is focused on packing chromatic numbers of the Mycielskian of paths and cycles.

**Keywords:** independence number, packing chromatic number, Mycielskians, generalized Mycielskians.

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## 1. INTRODUCTION

All considered graphs in this paper are finite, simple, and undirected. For a graph  $G$ , we denote by  $V(G)$  its set of vertices and by  $E(G)$  its set of edges.

If  $G$  is a graph and  $i$  a positive integer, then  $A \subseteq V(G)$  is an  $i$ -packing if vertices of  $A$  are pairwise at distance more than  $i$ . The  $i$ -packing number  $\eta_i(G)$  is the maximum size of an  $i$ -packing of  $G$ . In particular, a 1-packing is simply an independent set and  $\eta_1(G)$  is the independence number  $\alpha(G)$ .

A  $k$ -packing coloring of a graph  $G$  is a mapping  $c : V(G) \rightarrow \{1, 2, \dots, k\}$  such that for any  $i \in \{1, 2, \dots, k\}$  the subset  $c^{-1}(i)$  is an  $i$ -packing, in this case the graph  $G$  is said to be  $k$ -packing colorable. The packing chromatic number  $\chi_\rho(G)$  of a graph  $G$  is the smallest integer  $k$  such that  $G$  is  $k$ -packing colorable, if there is no such integer  $k$ , then we set  $\chi_\rho(G) = \infty$ .

The concept of packing coloring was introduced under the name *broadcast coloring* [20]. The current name was given in [6] and studied by a number of authors [1–11, 13, 14, 16, 17, 19, 23–25, 32–36]. Its related decision problem is  $\mathcal{NP}$ -complete even when restricted to trees [15].

The boundedness of the packing chromatic number of graphs was the main subject of most papers investigating this parameter for several classes of graphs. In the seminal paper [20], a natural upper bound of  $\chi_\rho$  was presented as follows.

**Proposition 1.1** [20]. *If  $G$  is a graph with order  $n$ , then  $\chi_\rho(G) \leq n - \alpha(G) + 1$ , with equality if  $\text{diam}(G) = 2$ .*

The *generalized Mycielski graphs* are a transformation that generalize the famous Mycielski construction developed by Mycielski [30] in search of triangle free graphs with large chromatic number. Given a graph  $G$  and an integer  $m \geq 1$ , one can transform  $G$  into a new graph  $\mu_m(G)$ , the generalized Mycielskian of  $G$ . The Mycielskians and the generalized Mycielskians were investigated from many points of view [12, 18, 21, 22, 26–29, 31].

In this paper, we present a formula for  $\alpha(\mu_m(G))$ , the independence number of the  $m$ -generalized Mycielskian of any connected graph  $G$  with order  $n \geq 2$  when  $m$  is odd and we present sharp bounds when  $m$  is even. We present also tight bounds of  $\alpha(\mu_m(G))$  for an arbitrary  $m \geq 1$  in terms of  $n$ ,  $m$  and  $\alpha(G)$ . The particular case of  $m = 1$  was investigated in [10].

We prove using the generalized Mycielski transformation the existence of a family of large planar graphs with small finite packing chromatic number, namely 4. We establish first bounds for the packing chromatic number of generalized Mycielskians. We present also bounds for the packing chromatic number of the Mycielskian of paths and cycles.

2. PRELIMINARIES AND NOTATIONS

Let  $G$  be a graph with vertex set  $V^0 = \{v_1^0, v_2^0, \dots, v_n^0\}$  and edge set  $E^0$ . Given an integer  $m \geq 1$ , the  $m$ -generalized Mycielskian (or the  $m$ -Mycielskian for short) of  $G$ , denoted by  $\mu_m(G)$ , is the graph with vertex set

$$V(\mu_m(G)) = V^0 \cup V^1 \cup \dots \cup V^m \cup \{z\},$$

where  $V^i = \{v_j^i : v_j^0 \in V^0\}$  is the  $i^{th}$  distinct copy of  $V^0$  for all  $i \in \{1, 2, \dots, m\}$ , and edge set

$$E(\mu_m(G)) = E^0 \cup \left( \bigcup_{i=0}^{m-1} \{v_j^i v_{j'}^{i+1} : v_j^0 v_{j'}^0 \in E^0\} \right) \cup \{v_j^m z : v_j^m \in V^m\}.$$

A vertex  $v_j^i$  from  $V^i$  is called the  $i^{th}$  copy of the vertex  $v_j^0$ . The vertices  $v_j^i$  and  $v_j^0$  are called *twins*. The Mycielskian of  $G$  is simply  $\mu_1(G)$ .

Let  $G$  be a graph and let  $X$  be a subset of  $V(G)$ . A vertex  $u$  is adjacent to  $X$  if there exists a vertex  $v$  from  $X$  such that  $uv \in E(G)$ . Two subsets  $X$  and  $Y$  of  $V(G)$  are adjacent if some vertex in  $X$  is adjacent to some vertex in  $Y$ . The *open neighborhood* of a subset  $X$ , denoted by  $N(X)$ , is the set of all adjacent vertices to  $X$ , and the *closed neighborhood* of  $X$ , denoted by  $N[X]$ , is the set  $X \cup N(X)$ . We denote by  $\overline{N}(X)$  the set  $V(G) \setminus N[X]$ . We denote by  $\mathbb{I}(G)$  the family of all independent sets of  $G$ .

3. INDEPENDENCE NUMBER OF GENERALIZED MYCIELSKIANS

In this section we present first a formula for the independence number of  $m$ -Mycielskians when  $m$  is odd, and bounds when  $m$  is even. We give then bounds for this parameter in terms of the order of  $G$ ,  $\alpha(G)$  and  $m$ . We show that all obtained bounds are sharp.

**Theorem 3.1.** *If  $G$  is a connected graph of order  $n \geq 2$ , then for all odd  $m \geq 1$*

$$\alpha(\mu_m(G)) = \max \left\{ \phi_m(G), \frac{(m+1)n}{2} \right\}$$

where  $\phi_m(G) = \max_{S \in \mathbb{I}(G)} \left\{ (m+1)|S| + \frac{m+1}{2} |\overline{N}(S)| \right\}$ .

**Proof.** Let  $S$  be a maximum independent set of  $\mu_m(G)$  and let  $V^i$  be the  $i^{th}$  copy of  $V^0$  in  $\mu_m(G)$ . For each  $i \in \{0, 1, \dots, m\}$  we set  $S_i = S \cap V^i$ . We denote by  $S_i(j)$  the set of  $j^{th}$  twin vertices of all vertices in  $S_i$  with  $S_i(i) = S_i$ . Note that  $S_i(j)$  is a subset of  $V^j$  and that  $|S_i(j)| = |S_i|$ . We first make the following observations.

**Observation 1.** *The subset  $S_0$  (which can be empty) is an independent set of  $G$ , so  $0 \leq |S_0| \leq \alpha(G) \leq n - 1$ .*

**Observation 2.** *For all  $i \in \{0, 1, \dots, m - 1\}$ ,  $S_i(0) \cap S_{i+1}(0) \in \{\emptyset\} \cup \mathbb{I}(G)$ .*

Note that  $S_i(0) \cap S_{i+1}(0)$  can be empty. If not, let  $x^0$  and  $y^0$  be two adjacent vertices from  $S_i(0) \cap S_{i+1}(0)$ , so  $x^0 \in S_i(0)$  and  $y^0 \in S_{i+1}(0)$ . Since  $x^0$  and  $y^0$  are adjacent, their twins  $x^i$  from  $S^i$  and  $y^{i+1}$  from  $S_{i+1}$  are adjacent, which is a contradiction since  $S_i$  and  $S_{i+1}$  are subsets of the independent set  $S$  of  $\mu_m(G)$ .

**Observation 3.** *If  $S_i(0) \cap S_{i+1}(0) \neq \emptyset$  for some  $i \in \{1, 2, \dots, m - 1\}$ , then*

$$|S_i(0) \Delta S_{i+1}(0)| \leq |\overline{N}(S_i(0) \cap S_{i+1}(0))|,$$

where  $\Delta$  is the symmetric difference of two sets.

Indeed, an arbitrary vertex  $x^0$  from  $S_i(0) \cap S_{i+1}(0)$  cannot be adjacent to any vertex from  $S_i(0)$  or  $S_{i+1}(0)$ , so in particular  $x^0$  cannot be adjacent to  $S_i(0) \setminus S_{i+1}(0)$  or  $S_{i+1}(0) \setminus S_i(0)$ . This means that  $x^0$  cannot be adjacent to  $S_i(0) \Delta S_{i+1}(0)$ . Consequently, the subsets  $S_i(0) \Delta S_{i+1}(0)$  and  $S_i(0) \cap S_{i+1}(0)$  are not adjacent. It follows that  $S_i(0) \Delta S_{i+1}(0) \subset \overline{N}(S_i(0) \cap S_{i+1}(0))$ .

**Claim 3.2.** *For all  $i \in \{0, 1, \dots, m - 1\}$ ,*

$$|S_i| + |S_{i+1}| \leq \max \left\{ n, \max_{S \in \mathbb{I}(G)} \{2|S| + |\overline{N}(S)|\} \right\}.$$

Let  $i \in \{0, 1, \dots, m - 1\}$ , so

- if  $S_i(0) \cap S_{i+1}(0) = \emptyset$ , then  $|S_i| + |S_{i+1}| = |S_i(0)| + |S_{i+1}(0)| = |S_i(0) \cup S_{i+1}(0)| \leq |V^0| = n$ ,
- if  $S_i(0) \cap S_{i+1}(0) \neq \emptyset$ , it is well known that  $|S_i(0)| + |S_{i+1}(0)| = 2|S_i(0) \cap S_{i+1}(0)| + |S_i(0) \Delta S_{i+1}(0)|$ , then by Observation 3

$$(1) \quad |S_i| + |S_{i+1}| \leq 2|S_i(0) \cap S_{i+1}(0)| + |\overline{N}(S_i(0) \cap S_{i+1}(0))|.$$

Since  $S_i(0) \cap S_{i+1}(0) \in \mathbb{I}(G)$  (Observation 2), the inequality (1) gives

$$|S_i| + |S_{i+1}| \leq \max_{S \in \mathbb{I}(G)} \{2|S| + |\overline{N}(S)|\}.$$

Consequently

$$|S_i| + |S_{i+1}| \leq \max \left\{ n, \max_{S \in \mathbb{I}(G)} \{2|S| + |\overline{N}(S)|\} \right\}.$$

We next discuss the following cases on  $z$ .

Case 1. If  $z \notin S$ , then  $|S| = \sum_{i=0}^m |S_i| = \sum_{i=0}^{\frac{m-1}{2}} (|S_{2i}| + |S_{2i+1}|)$ , Claim 3.2 gives

$$\begin{aligned} |S| &\leq \frac{m+1}{2} \max \left\{ n, \max_{S \in \mathcal{I}(G)} \{2|S| + |\overline{N}(S)|\} \right\} \\ &= \max \left\{ \frac{m+1}{2} n, \frac{m+1}{2} \max_{S \in \mathcal{I}(G)} \{2|S| + |\overline{N}(S)|\} \right\} \\ &= \max \left\{ \frac{m+1}{2} n, \max_{S \in \mathcal{I}(G)} \left\{ (m+1)|S| + \frac{m+1}{2} |\overline{N}(S)| \right\} \right\} \\ &= \max \left\{ \frac{(m+1)n}{2}, \phi_m(G) \right\}. \end{aligned}$$

Case 2. If  $z \in S$ , then  $S_m = \emptyset$  by the fact that  $z$  is adjacent to all vertices of  $V^m$ . It follows that  $|S| = 1 + \sum_{i=0}^{m-1} |S_i|$ . Thus

$$\begin{aligned} |S| &= 1 + \sum_{i=0}^{m-1} |S_i| = 1 + |S_0| + \sum_{i=0}^{\frac{m-3}{2}} (|S_{2i+1}| + |S_{2i+2}|) \\ &\leq 1 + (n-1) + \frac{m-1}{2} \max \left\{ n, \max_{S \in \mathcal{I}(G)} \{2|S| + |\overline{N}(S)|\} \right\} \\ &= n + \frac{m-1}{2} \max \left\{ n, \max_{S \in \mathcal{I}(G)} \{2|S| + |\overline{N}(S)|\} \right\} \\ &\leq \max \left\{ n, \max_{S \in \mathcal{I}(G)} \{2|S| + |\overline{N}(S)|\} \right\} + \frac{m-1}{2} \max \left\{ n, \max_{S \in \mathcal{I}(G)} \{2|S| + |\overline{N}(S)|\} \right\} \\ &= \max \left\{ \frac{(m+1)n}{2}, \phi_m(G) \right\}. \end{aligned}$$

On the other hand, for any arbitrary independent set  $S$  of  $G$  we set  $S = S^0$ ,  $X^0 = \overline{N}(S)$ , and for each  $i \in \{1, 2, \dots, m\}$  we denote by  $S^i$  and  $X^i$  the  $i^{th}$  copies of  $S^0$  and  $X^0$  in  $\mu_m(G)$ , respectively. Note first the following observations.

- (O1):  $I = \bigcup_{i=0}^m S^i$  is an independent set of  $\mu_m(G)$ .
- (O2): The subsets  $S^0$  and  $X^0$  are not adjacent. Hence, for  $0 \leq i \leq m$  and  $1 \leq j \leq m$ ,  $S^i \cup X^j$  is an independent set of  $\mu_m(G)$ .
- (O3): Since  $X^0$  can contain adjacent vertices,  $X^0$  can be adjacent to  $X^1$ , and generally, any two consecutive copies  $X^i$  and  $X^{i+1}$  can be adjacent. But no non-consecutive copies  $X^i$  and  $X^{i+2}$  can be adjacent. In other words,  $X^i \cup X^{i+2}$  is an independent set of  $\mu_m(G)$ . Consequently, the subset  $I_o = \bigcup_{i=0}^{\frac{m-1}{2}} X^{2i+1}$  is an independent set of  $\mu_m(G)$ .

- (O4): By the construction of  $\mu_m(G)$ , every non-consecutive copies  $V^i$  and  $V^{i+2}$  are not adjacent. Therefore, the subset  $V_o = \bigcup_{i=0}^{\frac{m-1}{2}} V^{2i+1}$  is an independent set of  $\mu_m(G)$ .
- (O5): By Observations (O1), (O2) and (O3) the subset  $I \cup I_o$  is an independent set of  $\mu_m(G)$ .

From Observation (O4), we have  $V_o \in \mathbb{I}(\mu_m(G))$ . Thus

$$\alpha(\mu_m(G)) \geq |V_o| = \frac{(m+1)n}{2}.$$

Furthermore, from Observation (O5), it follows that  $I \cup I_o \in \mathbb{I}(\mu_m(G))$ . As all copies of  $S^0$  and  $X^0$  have the same cardinality respectively, we obtain

$$\alpha(\mu_m(G)) \geq |I \cup I_o| = (m+1)|S^0| + \frac{m+1}{2}|X^0|.$$

Thus,  $\alpha(\mu_m(G)) \geq \phi_m(G)$ . It follows that

$$\alpha(\mu_m(G)) \geq \max \left\{ \phi_m(G), \frac{(m+1)n}{2} \right\}. \quad \blacksquare$$

**Remark 3.3.** Let  $S$  be a subset of  $V(G)$ . If we suppose that  $S = \emptyset$ , then we get  $\overline{N}(S) = G$ . Hence

$$(m+1)|S| + \frac{m+1}{2} |\overline{N}(S)| = \frac{m+1}{2} n,$$

where  $V(G) = n$ . Thus, Theorem 3.1 becomes

$$\alpha(\mu_m(G)) = \max_{S \in \mathbb{I}(G) \cup \{\emptyset\}} \left\{ (m+1)|S| + \frac{m+1}{2} |\overline{N}(S)| \right\}$$

which we will use from now on.

The result of Theorem 3.1 is a generalization of the formula proved by Brešar *et al.* in [10] for Mycielskians.

**Corollary 3.4** [10]. *If  $G$  is a connected graph, then*

$$\alpha(\mu_1(G)) = \max_{S \in \mathbb{I}(G) \cup \{\emptyset\}} \{2|S| + |\overline{N}(S)|\}.$$

Contrarily to the odd case, determining a formula for the independence number of  $m$ -Mycielskians for some even  $m$  was not feasible, but we present upper and lower bounds of it.

**Theorem 3.5.** *If  $G$  is a connected graph of order  $n \geq 2$ , then for all even  $m \geq 1$*

$$\max \left\{ \phi'_m(G), \frac{mn}{2} + \alpha(G) \right\} \leq \alpha(\mu_m(G)) \leq \max \left\{ \phi''_m(G), \frac{mn}{2} + \alpha(G) \right\},$$

where for any  $S \in \mathbb{I}(G)$ ,  $|I_S| = \max \{ |I| : I \in \mathbb{I}(G), I \subset \overline{N}(S) \}$  and

$$\phi'_m(G) = \max_{S \in \mathbb{I}(G)} \left\{ (m+1)|S| + \frac{m}{2} |\overline{N}(S)| + |I_S| \right\}$$

and

$$\phi''_m(G) = \max_{S \in \mathbb{I}(G)} \left\{ m|S| + \frac{m}{2} |\overline{N}(S)| + \alpha(G) \right\}.$$

**Proof.** Let  $m \geq 2$  be an even integer. For the lower bound, let  $S$  be an arbitrary independent set of  $G$  and let  $I_S \in \mathbb{I}(G)$  such that  $I_S \subset \overline{N}(S)$  and  $|I_S| = \max \{ |I| : I \in \mathbb{I}(G), I \subset \overline{N}(S) \}$ . We shall keep in this proof the same notations and Observations (O1) and (O2) as in the proof of Theorem 3.1 with these additional similar observations.

(O3'):  $I_e = \bigcup_{i=1}^{\frac{m}{2}} X^{2i}$  is an independent set of  $\mu_m(G)$ .

(O4'):  $V_e = \bigcup_{i=1}^{\frac{m}{2}} V^{2i}$  is an independent set of  $\mu_m(G)$ .

(O5'):  $V^0$  and  $V^2$  are not adjacent. Thus, any maximum independent set  $S_G$  of  $G$  is not adjacent to  $V_e$ . It follows by Observation (O4') that  $V_e \cup S_G \in \mathbb{I}(\mu_m(G))$ .

Let  $I = \bigcup_{i=0}^m S^i$  be the set defined in the proof of Theorem 3.1.  $I_S$  is not adjacent to  $I$  since  $I_S$  is not adjacent to  $S$ . It is obvious that  $I_S$  is not adjacent to  $I_e$ . Consequently, by Observations (O1), (O2) and (O3'), we get  $I \cup I_e \cup I_S \in \mathbb{I}(\mu_m(G))$ , thus  $\alpha(\mu_m(G)) \geq |I \cup I_e \cup I_S| = (m+1)|S| + \frac{m}{2} |\overline{N}(S)| + |I_S|$ . Observation (O5') implies that  $\alpha(\mu_m(G)) \geq |V_e| + \alpha(G) = \frac{mn}{2} + \alpha(G)$ . Hence  $\alpha(\mu_m(G)) \geq \max \left\{ \phi'_m(G), \frac{mn}{2} + \alpha(G) \right\}$ .

In the other hand, let  $S$  be a maximum independent set of  $\mu_m(G)$ .

*Case 1.* If  $z \notin S$ , then  $|S| = \sum_{i=0}^m |S_i| = |S_0| + \sum_{i=0}^{\frac{m}{2}-1} (|S_{2i+1}| + |S_{2i+2}|)$ . Claim 3.2 gives

$$\begin{aligned} \sum_{i=0}^{\frac{m}{2}-1} (|S_{2i+1}| + |S_{2i+2}|) &\leq \frac{m}{2} \max \left\{ n, \max_{S \in \mathbb{I}(G)} \{ 2|S| + |\overline{N}(S)| \} \right\} \\ &= \max \left\{ \frac{m}{2} n, \frac{m}{2} \max_{S \in \mathbb{I}(G)} \{ 2|S| + |\overline{N}(S)| \} \right\} \\ &= \max \left\{ \frac{mn}{2}, \max_{S \in \mathbb{I}(G)} \left\{ m|S| + \frac{m}{2} |\overline{N}(S)| \right\} \right\}. \end{aligned}$$

As  $|S_0| \leq \alpha(G)$ , we get

$$\begin{aligned} \alpha(\mu_m(G)) = |S| &\leq \alpha(G) + \max \left\{ \frac{mn}{2}, \max_{S \in \mathbb{I}(G)} \left\{ m|S| + \frac{m}{2} |\overline{N}(S)| \right\} \right\} \\ &= \max \left\{ \frac{mn}{2} + \alpha(G), \max_{S \in \mathbb{I}(G)} \left\{ m|S| + \frac{m}{2} |\overline{N}(S)| + \alpha(G) \right\} \right\} \\ &= \max \left\{ \frac{mn}{2} + \alpha(G), \phi_m''(G) \right\}. \end{aligned}$$

*Case 2.* If  $z \in S$ , then  $|S| = 1 + \sum_{i=0}^{m-1} |S_i| = 1 + \sum_{i=0}^{\frac{m}{2}-1} (|S_{2i}| + |S_{2i+1}|)$ .  
 Since

$$\begin{aligned} \sum_{i=0}^{\frac{m}{2}-1} (|S_{2i}| + |S_{2i+1}|) &\leq \frac{m}{2} \max \left\{ n, \max_{S \in \mathbb{I}(G)} \{2|S| + |\overline{N}(S)|\} \right\} \\ &= \max \left\{ \frac{mn}{2}, \max_{S \in \mathbb{I}(G)} \left\{ m|S| + \frac{m}{2} |\overline{N}(S)| \right\} \right\} \end{aligned}$$

and  $1 \leq \alpha(G)$ , we obtain

$$\begin{aligned} \alpha(\mu_m(G)) = |S| &\leq \alpha(G) + \max \left\{ \frac{mn}{2}, \max_{S \in \mathbb{I}(G)} \left\{ m|S| + \frac{m}{2} |\overline{N}(S)| \right\} \right\} \\ &= \max \left\{ \frac{mn}{2} + \alpha(G), \phi_m''(G) \right\}. \end{aligned}$$

■

**Remark 3.6.** Remark that  $|\overline{N}(\emptyset)| = n$  and  $|I_\emptyset| = \alpha(G)$ , hence

$$(m + 1)|\emptyset| + \frac{m}{2} |\overline{N}(\emptyset)| + |I_\emptyset| = \frac{mn}{2} + \alpha(G)$$

and

$$m|\emptyset| + \frac{m}{2} |\overline{N}(\emptyset)| + \alpha(G) = \frac{mn}{2} + \alpha(G).$$

Thus, according to Theorem 3.5, we obtain

$$\phi_m'(G) \leq \alpha(\mu_m(G)) \leq \phi_m''(G)$$

where  $\phi_m'(G)$  and  $\phi_m''(G)$  are redefined by

$$\phi_m'(G) = \max_{S \in \mathbb{I}(G) \cup \{\emptyset\}} \left\{ (m + 1)|S| + \frac{m}{2} |\overline{N}(S)| + |I_S| \right\}$$

and

$$\phi_m''(G) = \max_{S \in \mathbb{I}(G) \cup \{\emptyset\}} \left\{ m|S| + \frac{m}{2} |\overline{N}(S)| + \alpha(G) \right\}.$$



The following corollary presents bounds for the independence number of the  $m$ -Mycielskian of any connected graph  $G$  in terms of the order of  $G$ ,  $m$  and  $\alpha(G)$ .

**Corollary 3.7.** *If  $G$  is a connected graph of order  $n$ , then for all  $m \geq 1$*

$$(m + 1)\alpha(G) \leq \alpha(\mu_m(G)) \leq \begin{cases} \frac{m}{2}(\alpha(G) + n - 1) + \alpha(G) & \text{if } m \text{ is even,} \\ \frac{m + 1}{2}(\alpha(G) + n - 1) & \text{if } m \text{ is odd.} \end{cases}$$

**Proof.** Let  $G$  be a connected graph of order  $n$  and let  $m \geq 1$ . As  $\alpha(\mu_m(G)) \geq \phi'_m(G)$ , we have  $\alpha(\mu_m(G)) \geq (m + 1)|S| + \frac{m}{2} |\overline{N}(S)| + |I_S|$  for all  $S \in \mathbb{I}(G) \cup \{\emptyset\}$ . If  $S$  is a maximum independent set, then  $\alpha(\mu_m(G)) \geq (m + 1)\alpha(G)$ .

For the upper bounds, let  $S$  be an independent set of  $G$ . Note that  $|\overline{N}(S)| \leq n - |S| - 1$  and  $n \leq \alpha(G) + n - 1$ .

- If  $m$  is even, then  $m|S| + \frac{m}{2} |\overline{N}(S)| + \alpha(G) \leq \frac{m}{2}(\alpha(G) + n - 1) + \alpha(G)$ .  $S = \emptyset$  implies  $\frac{mn}{2} + \alpha(G) \leq \frac{m}{2}(\alpha(G) + n - 1) + \alpha(G)$ . Consequently,  $\phi''_m(G) \leq \frac{m}{2}(\alpha(G) + n - 1) + \alpha(G)$ .
- If  $m$  is odd, then  $(m + 1)|S| + \frac{m+1}{2} |\overline{N}(S)| \leq \frac{m+1}{2}(\alpha(G) + n - 1)$ .  $S = \emptyset$  implies  $\frac{(m+1)n}{2} \leq \frac{m+1}{2}(\alpha(G) + n - 1)$ . Thus, Theorem 3.1 implies that  $\alpha(\mu_m(G)) \leq \frac{m+1}{2}(\alpha(G) + n - 1)$ .

Finally,

$$(m + 1)\alpha(G) \leq \alpha(\mu_m(G)) \leq \begin{cases} \frac{m}{2}(\alpha(G) + n - 1) + \alpha(G) & \text{if } m \text{ is even,} \\ \frac{m + 1}{2}(\alpha(G) + n - 1) & \text{if } m \text{ is odd.} \quad \blacksquare \end{cases}$$

Corollary 3.7 is again a generalization of bounds found by Brešar *et al.* in [10] for Mycielskians.

**Corollary 3.8** [10]. *If  $G$  is a connected graph, then*

$$2\alpha(G) \leq \alpha(\mu_1(G)) \leq |V(G)| + \alpha(G) - 1.$$

Both bounds in Corollary 3.7 coincide for a star graph  $K_{1,n}$ . Indeed, since  $\alpha(K_{1,n}) = n$  and  $|V(K_{1,n})| = n + 1$ , then for any  $m \geq 1$

$$(m + 1)n \leq \alpha(\mu_m(K_{1,n})) \leq \begin{cases} \frac{m}{2}(n + n + 1 - 1) + n & \text{if } m \text{ is even,} \\ \frac{m + 1}{2}(n + n + 1 - 1) & \text{if } m \text{ is odd.} \end{cases}$$

This means that  $(m + 1)n \leq \alpha(\mu_m(K_{1,n})) \leq (m + 1)n$ . Thus,  $\alpha(\mu_m(K_{1,n})) = (m + 1)n$ .

**Remark 3.9.** The bounds of  $\alpha(\mu_m(G))$  in Corollary 3.7 are another representation of bounds established independently in [28] for the cover number  $\tau(\mu_m(G))$  in terms of  $m$  and the cover number  $\tau(G)$ .

In the next corollary, we present other sharp bounds for the independence number of  $m$ -Mycielskians.

**Corollary 3.10.** *If  $G$  is a connected graph with order  $n \geq 2$ , then for all  $m \geq 1$*

$$\left\lceil \frac{m}{2} \right\rceil n + \varepsilon_m(G) \leq \alpha(\mu_m(G)) \leq (n - 1)(m + 1)$$

where  $\varepsilon_m(G) = \begin{cases} \alpha(G) & \text{if } m \text{ is even,} \\ 0 & \text{if } m \text{ is odd.} \end{cases}$

**Proof.** Let  $G$  be a connected graph and let  $n \geq 2$  and  $m \geq 1$ .

By combining Theorem 3.1 and Theorem 3.5, we get

$$\alpha(\mu_m(G)) \geq \begin{cases} \frac{mn}{2} + \alpha(G) & \text{if } m \text{ is even,} \\ \frac{(m + 1)n}{2} & \text{if } m \text{ is odd.} \end{cases}$$

On the other hand, let  $S$  be an arbitrary independent set of graph  $G$ . Note that  $|\overline{N}(S)| = n - |S| - |N(S)|$ ,  $1 \leq |N(S)| \leq n - 1$ ,  $1 \leq |S| \leq n - 1$  and  $\alpha(G) \leq n - 1$ .

*Case 1.* If  $m$  is even, then

$$\begin{aligned} m|S| + \frac{m}{2}|\overline{N}(S)| + \alpha(G) &= \frac{m}{2}|S| + \frac{mn}{2} - \frac{m}{2}|N(S)| + \alpha(G) \\ &\leq \frac{m}{2}(n - 1) + \frac{mn}{2} - \frac{m}{2} + n - 1 = (m + 1)(n - 1). \end{aligned}$$

Thus,  $\alpha(\mu_m(G)) \leq \phi_m''(G) \leq (m + 1)(n - 1)$ .

*Case 2.* If  $m$  is odd, then

$$\begin{aligned} (m + 1)|S| + \frac{m + 1}{2}|\overline{N}(S)| &= \frac{m + 1}{2}|S| + \frac{(m + 1)n}{2} - \frac{m + 1}{2}|N(S)| \\ &\leq \frac{m + 1}{2}(n - 1) + \frac{(m + 1)n}{2} - \frac{m + 1}{2} \\ &= (m + 1)(n - 1). \end{aligned}$$

By Theorem 3.1, we get  $\alpha(\mu_m(G)) \leq (m + 1)(n - 1)$ . ■

Note that the upper bound of Corollary 3.10 is tight for star graphs since  $((n + 1) - 1)(m + 1) = n(m + 1) = \alpha(\mu_m(K_{1,n}))$ .

For a complete graph  $K_n$ , the upper bound of Corollary 3.7 coincide with the lower bound of Corollary 3.10.

**Corollary 3.11.** *For all  $m \geq 1$  and  $n \geq 2$ ,*

$$\alpha(\mu_m(K_n)) = \begin{cases} \frac{mn}{2} + 1 & \text{if } m \text{ is even,} \\ \frac{(m+1)n}{2} & \text{if } m \text{ is odd.} \end{cases}$$

4. PACKING COLORING OF GENERALIZED MYCIELSKIANS

The following lemmas help to establish bounds of  $\chi_\rho(\mu_m(G))$ , the packing chromatic number of the  $m$ -Mycielskian of a graph  $G$ .

**Lemma 4.1** [20]. *For  $n \geq 3$ , if  $n$  is 3 or a multiple of four, then  $\chi_\rho(C_n) = 3$ , otherwise  $\chi_\rho(C_n) = 4$ .*

**Lemma 4.2** [20]. *If  $H$  is a subgraph of a graph  $G$ , then  $\chi_\rho(H) \leq \chi_\rho(G)$ .*

**Theorem 4.3.** *If  $G$  is a connected graph of order  $n \geq 2$ , then for all  $m \geq 1$*

$$4 \leq \chi_\rho(\mu_m(G)) \leq \begin{cases} \min \left\{ \frac{mn}{2} + n + 2 - \alpha(G), (m+1)(n - \alpha(G)) + 2 \right\} & \text{if } m \text{ is even,} \\ \min \left\{ \frac{n(m+1)}{2} + 2, (m+1)(n - \alpha(G)) + 2 \right\} & \text{if } m \text{ is odd.} \end{cases}$$

Moreover, if the diameter of  $G$  is 2, then

$$\chi_\rho(\mu_m(G)) \leq \begin{cases} \min \left\{ \frac{mn}{2} + \chi_\rho(G) + 1, (m+1)\chi_\rho(G) + 1 - m \right\} & \text{if } m \text{ is even,} \\ \min \left\{ \frac{n(m+1)}{2} + 2, (m+1)\chi_\rho(G) + 1 - m \right\} & \text{if } m \text{ is odd.} \end{cases}$$

**Proof.** Let  $S_G$  be a maximum independent set of  $G$  and let  $S^1, S^2, \dots, S^m$  be its copies in  $\mu_m(G)$ . We set  $A = S_G \cup S^1 \cup S^2 \cup \dots \cup S^m$ ,  $B = S_G \cup V^2 \cup V^4 \cup \dots \cup V^m$  when  $m$  is even and  $C = V^1 \cup V^3 \cup \dots \cup V^m$  when  $m$  is odd. It is straightforward to see that the  $A$ ,  $B$  and  $C$  are independent sets of  $\mu_m(G)$ . Thus,  $\alpha(\mu_m(G)) \geq \max(|A|, |B|)$  if  $m$  is even and  $\alpha(\mu_m(G)) \geq \max(|A|, |C|)$  if  $m$  is odd. Hence, by Proposition 1.1, we have  $\chi_\rho(\mu_m(G)) \leq \min \{|V(\mu_m(G))| - |A| + 1, |V(\mu_m(G))| - |B| + 1\}$  if  $m$  is even and  $\chi_\rho(\mu_m(G)) \leq \min \{|V(\mu_m(G))| - |A| + 1, |V(\mu_m(G))| - |C| + 1\}$  if  $m$  is odd. As  $|V(\mu_m(G))| = n(m+1) + 1$ ,  $|A| = (m+1)\alpha(G)$ ,  $|B| = \frac{mn}{2} + \alpha(G)$  and  $|C| = \frac{m+1}{2}n + \alpha(G)$ , we get

$$\chi_\rho(\mu_m(G)) \leq \begin{cases} \min \left\{ \frac{mn}{2} + n + 2 - \alpha(G), (m+1)(n - \alpha(G)) + 2 \right\} & \text{if } m \text{ is even,} \\ \min \left\{ \frac{n(m+1)}{2} + 2, (m+1)(n - \alpha(G)) + 2 \right\} & \text{if } m \text{ is odd.} \end{cases}$$

On the other hand,  $\mu_m(K_2)$  is an induced subgraph of  $\mu_m(G)$  which is isomorphic to the cycle  $C_{2m+3}$ . According to Lemma 4.1,  $\chi_\rho(\mu_m(K_2)) = 4$ . It follows by Lemma 4.2 that  $\chi_\rho(\mu_m(G)) \geq 4$ .

If the diameter of  $G$  is 2, then by Proposition 1.1,  $\chi_\rho(G) = n - \alpha(G) + 1$ . Thus,

$$(m + 1)(\chi_\rho(G) - 1) + 2 = (m + 1)\chi_\rho(G) + 1 - m$$

and

$$\frac{mn}{2} + n + 2 - \alpha(G) = \frac{mn}{2} + \chi_\rho(G) + 1. \quad \blacksquare$$

For  $m > 1$ , we do not know if the upper bound of Theorem 4.3 is sharp for some classes of graphs. For instance, when  $G$  is the complete graph  $K_n$  and  $m = 2$ , one can easily prove that  $\chi_\rho(\mu_2(K_n)) \leq 2n < \min \{2n + 1, 3(n - 1) + 2\}$ . The exact value of  $\chi_\rho(\mu_m(K_n))$  seems to be a challenging question.

Brešar *et al.* [10] established an upper bound for the packing chromatic number of the Mycielskian of a connected graph. The authors showed that this bound is sharp for the star and the complete graph.

**Corollary 4.4** [10]. *If  $G$  is a connected graph of order  $n \geq 2$ , then*

$$\chi_\rho(\mu_1(G)) \leq \min \{n + 2, 2(n - \alpha(G) + 1)\}.$$

It is known that the packing chromatic number of a graph can become quite large, even unbounded, for simple classes of graphs. However, in the following theorem we show that there exists a family of large planar graphs whose packing chromatic number is 4.

**Theorem 4.5.** *For every  $n \geq 2$  and  $m \geq 1$ , there exists a planar graph  $G_{m,n}$  such that  $\chi_\rho(G_{m,n}) = 4$ .*

**Proof.** Let  $n \geq 2$ ,  $m \geq 1$  and let  $G_{m,n}$  be the  $m$ -Mycielskian of the star  $K_{1,n}$ . Figure 1 shows how can the graph  $G_{m,n}$  be drawn as a planar graph. Moreover, according to Theorem 4.3, we have

$$\chi_\rho(G_{m,n}) \geq 4.$$

Set  $V(K_{1,n}) = \{x_0^0, x_1^0, \dots, x_n^0\}$ , where  $x_0^0$  is the central vertex of  $K_{1,n}$ . Let  $V^i = \{x_0^i, x_1^i, \dots, x_n^i\}$  be the  $i^{th}$  copy of  $V(K_{1,n})$ .

Let  $F$  be the subgraph of  $K_{1,n}$  induced by the set of vertices  $\{x_0^0, x_1^0\}$  and let  $H$  be the subgraph of  $G_{m,n}$  induced by  $\mu_m(F)$  which is isomorphic to the cycle  $C_{2m+3}$ . It is straightforward to check that  $H$  is the cycle

$$x_1^m x_0^{m-1} x_1^{m-2} x_0^{m-3} \dots x_0^1 x_1^0 x_0^1 x_1^2 x_0^3 \dots x_1^{m-1} x_0^m x_1^m$$

such that  $|V(H)| = 2m + 3$ . As  $W^0 = \{x_2^0, x_3^0, \dots, x_n^0\}$  is an independent set of  $K_{1,n}$ , then  $W = W^0 \cup W^1 \cup \dots \cup W^m$  is also an independent set of  $\mu_m(K_{1,n})$  where  $W^i$  is the  $i^{th}$  copy of  $W^0$ . Moreover,  $V(H) \cup W = V(G_{m,n})$  and  $V(H) \cap W = \emptyset$ . Let  $c$  be a 4-packing coloring of  $G_{m,n}$  such that all vertices of the subset  $W$  receive the color 1. Since  $|V(H)| = 2m + 3$ , we distinguish two cases.

- If  $|V(H)| \equiv 1[4]$ , then  $c$  provides a coloring for  $H$  by the pattern

$$1, 2, 1, 3, 1, 2, 1, 3, \dots, 1, 2, 1, 3, 4$$

where the coloring starts by the vertex  $x_1^m$  and ends by  $x_0^m$  and  $z$ , i.e.,  $c(x_1^m) = 1$ ,  $c(x_0^m) = 3$  and  $c(z) = 4$ .

- If  $|V(H)| \equiv 3[4]$ , then  $c$  provides a coloring for  $H$  by the pattern

$$1, 2, 1, 3, 1, 2, 1, 3, \dots, 1, 2, 1, 3, 1, 2, 4$$

where the coloring starts by the vertex  $x_1^m$  and ends by  $x_0^m$  and  $z$ , i.e.,  $c(x_1^m) = 1$ ,  $c(x_0^m) = 2$  and  $c(z) = 4$ .

Note that the used patterns assigns the color 1 to all vertices  $x_i^i$  in  $H$  for all  $i \in \{0, 1, \dots, m\}$  which are not adjacent to any vertex from  $W$ , also colored 1. ■

### 5. PACKING CHROMATIC NUMBER OF $\mu_1(P_n)$ AND $\mu_1(C_n)$

It was shown in [18] (Theorem 9), that  $\eta_2(\mu_1(G)) = \eta_2(G)$  for every graph  $G$  without isolated vertices. We present an analogous result for the maximum 3-packing of connected graphs.

**Lemma 5.1.** *For every connected graph  $G$ ,  $\eta_3(\mu_1(G)) = \eta_3(G)$ .*

**Proof.** If  $S$  is a maximum 3-packing of  $G$ , then  $S$  is a 3-packing of  $\mu_1(G)$ . Thus,  $\eta_3(\mu_1(G)) \geq \eta_3(G)$ .

Let  $V^1$  be the set of copies of  $V(G)$ . Let  $S$  be a maximum 3-packing of  $\mu_1(G)$ . If  $z \in S$ , then  $S = \{z\}$  by the fact  $ecc(z) = 2$  ( $ecc(z)$  is the distance between  $z$  and the farthest vertex from  $z$  in  $\mu_1(G)$ ). Thus,  $\eta_3(\mu_1(G)) = 1 \leq \eta_3(G)$ . Otherwise, since  $d_{\mu_1(G)}(x^1, y^1) = 2$  for all vertices  $x^1$  and  $y^1$  from  $V^1$ ,  $S$  contains at most one vertex from  $V^1$ . If  $S$  contain no vertex from  $V^1$ , then  $S$  is a 3-packing subset of  $V(G)$ , hence  $\eta_3(\mu_1(G)) \leq \eta_3(G)$ . If  $S$  contains one vertex  $x^1$  from  $V^1$ , let  $S'$  be the set  $(S \cap V(G)) \cup \{x\}$ . In other words,  $S'$  is the same as  $S$  with replacing the vertex  $x^1$  by its twin vertex  $x$  from  $V(G)$ .

Assume that  $x \in S$ , since  $G$  is connected, there is a vertex  $y \in V(G)$  that is adjacent with  $x$ . By Mycielski construction, vertices  $x^1$  and  $y$  are adjacent. Thus,  $x^1yx$  is a path in  $\mu_1(G)$ , which is a contradiction since  $d_G(x, x^1) > 3$ .

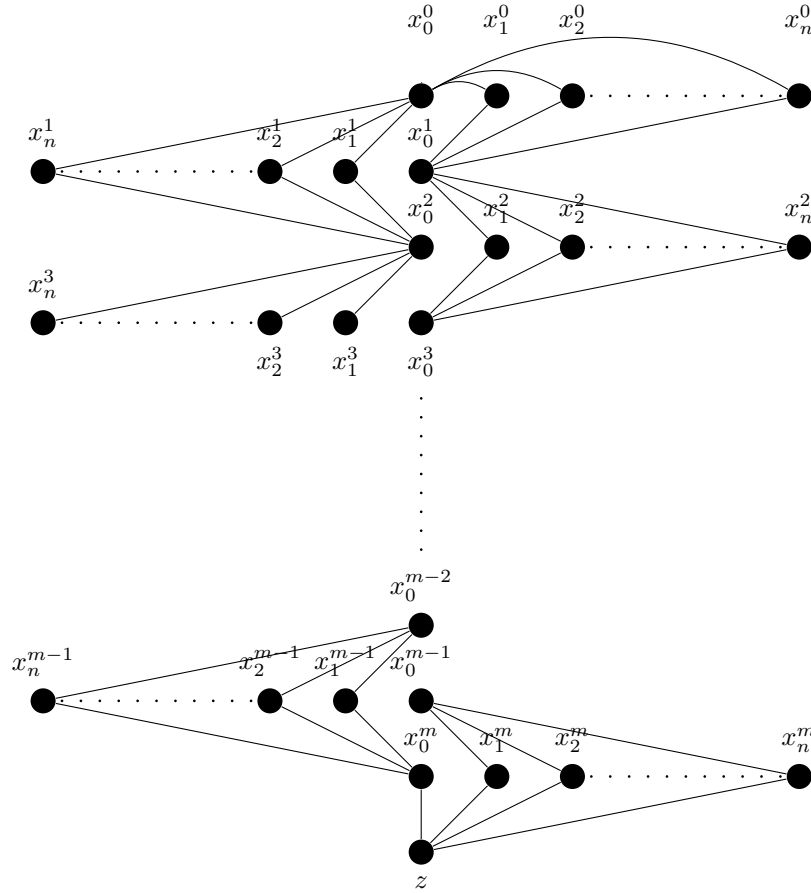


Figure 1. An  $m$ -generalized Mycielskian of the star  $K_{1,n}$ .

Since  $x \notin S$ , we have  $|S| = |S'|$ . For  $S'$  to be a 3-packing of  $G$ , it is sufficient to show that  $d_G(x, y) > 3$  for all  $y$  from  $S \cap V(G)$ . Assume that  $d_G(x, y) = 3$  (and similarly for  $d_G(x, y) \in \{1, 2\}$ ) for some  $y$  from  $S \cap V(G)$ . Let  $xuvy$  be the path joining  $x$  and  $y$  in  $G$ . So  $x^1uvy$  is a path joining  $x^1$  and  $y$  in  $\mu_1(G)$ , thus  $d_{\mu_1(G)}(x^1, y) = 3$ , which is a contradiction since  $S$  is a 3-packing of  $\mu_1(G)$ . ■

**Proposition 5.2.** For every connected graph  $G$ ,

$$\chi_\rho(\mu_1(G)) \geq 4 + 2n - \alpha(\mu_1(G)) - \eta_2(G) - \eta_3(G).$$

**Proof.** The diameter of a Mycielskian is at most four [18], then only colors 1, 2 and 3 can be repeated in any packing coloring of  $\mu_1(G)$ .

A packing coloring  $c$  of  $\mu_1(G)$  assigns the colors 1, 2 and 3 to at most  $\alpha(\mu_1(G))$ ,  $\eta_2(\mu_1(G))$  and  $\eta_3(\mu_1(G))$  vertices respectively, and distinct colors to

the remaining vertices. As  $\eta_2(\mu_1(G)) = \eta_2(G)$  and  $\eta_3(\mu_1(G)) = \eta_3(G)$ , Lemma 5.1 gives

$$\begin{aligned} \chi_\rho(\mu_1(G)) &\geq 3 + 2n + 1 - \alpha(\mu_1(G)) - \eta_2(G) - \eta_3(G) \\ &= 4 + 2n - \alpha(\mu_1(G)) - \eta_2(G) - \eta_3(G). \end{aligned}$$

■

Note that the lower bound of Proposition 5.2 is not always a good lower bound and can take negative values for some graphs by the diameter fact and that the maximum 2-packing and 3-packing can have some vertices in common. But it can be useful to determine some exact values of  $\chi_\rho(\mu_1(P_n))$  and  $\chi_\rho(\mu_1(C_n))$  as we will see.

We establish now bounds for the packing chromatic number of the Mycielskian of paths and cycles.

**Lemma 5.3.** *For all  $n \geq 2$ ,*

1.  $\alpha(\mu_1(P_n)) = 2 \lceil \frac{n}{2} \rceil$ ,
2.  $\eta_2(P_n) = \lceil \frac{n}{3} \rceil$  and  $\eta_3(P_n) = \lceil \frac{n}{4} \rceil$ .

**Proof.** Let  $n \geq 2$ .

1. Let  $S \in \mathbb{I}(P_n) \cup \{\emptyset\}$  be such that  $|S| = k$  and set  $\beta_S = 2|S| + |\overline{N}(S)|$ . Note that

- if  $k = \alpha(P_n)$ , then  $\beta_S = 2\alpha(P_n)$ ;
- for a fixed value of  $k$  in  $\{0, 1, \dots, \alpha(P_n) - 1\}$ , the number  $|\overline{N}(S)|$  varies in  $\{n - 2k, n - 2k - 1, \dots, n - 3k\}$ .

So by Corollary 3.4

$$\begin{aligned} \alpha(\mu_1(P_n)) &= \max_{S \in \mathbb{I}(G) \cup \{\emptyset\}} \{2|S| + |\overline{N}(S)|\} \\ &= \max_{0 \leq k \leq \alpha(P_n) - 1} \left\{ 2\alpha(P_n), \max_{2k \leq i \leq 3k} \{2k + (n - i)\} \right\} \\ &= \max_{0 \leq k \leq \alpha(P_n) - 1} \left\{ 2 \lceil \frac{n}{2} \rceil, 2k + n - 2k \right\} = \max \left\{ 2 \lceil \frac{n}{2} \rceil, n \right\} = 2 \lceil \frac{n}{2} \rceil. \end{aligned}$$

2. Let  $V(P_n) = \{x_1, x_2, \dots, x_n\}$ . The sets of vertices  $A = \{x_i : i \equiv 1 \pmod{3}, 1 \leq i \leq \lceil \frac{n}{3} \rceil\}$  and  $B = \{x_i : i \equiv 1 \pmod{4}, 1 \leq i \leq \lceil \frac{n}{4} \rceil\}$  are the best 2-packing and 3-packing of  $P_n$ , respectively. So  $\eta_2(P_n) \geq |A| = \lceil \frac{n}{3} \rceil$  and  $\eta_3(P_n) \geq |B| = \lceil \frac{n}{4} \rceil$ . No better sets can be obtained. To show that  $\eta_2(P_n) \leq \lceil \frac{n}{3} \rceil$ , we assume that there exists a 2-packing  $A' = \{x_{i_1}, x_{i_2}, \dots, x_{i_r}\}$  of  $P_n$  with cardinality  $r = \lceil \frac{n}{3} \rceil + 1$ . Without loss of generality, assume that vertices of  $A'$  are consecutive, which means that  $|i_j - i_{j+1}| = 3$  for all  $j \in \{1, \dots, r - 1\}$ . Note that  $|i_1 - i_r| = \sum_{j=1}^{r-1} |i_j - i_{j+1}| = 3(r - 1) \leq \text{diam}(P_n) = n - 1$ . We have the following two cases.

- If  $n = 3k$ , then  $r = k + 1$ . So  $|i_1 - i_r| = 3k = n$ , which is a contradiction to  $|i_1 - i_r| \leq n - 1$ .
- If  $n \in \{3k + 1, 3k + 2\}$ , then  $r = k + 2$ . So  $|i_1 - i_r| = 3(k + 1) \in \{n + 1, n + 2\}$ , which is also a contradiction to  $|i_1 - i_r| \leq n - 1$ .

A similar reasoning can be done to prove  $\eta_3(P_n) \leq \lceil \frac{n}{4} \rceil$  by discussing cases on  $n \in \{4k, 4k + 1, 4k + 2, 4k + 3\}$ . ■

**Theorem 5.4.** *For every positive integer  $n \geq 2$  with  $n \equiv r \pmod{9}$ ,*

$$4 + 2n - 2 \left\lceil \frac{n}{2} \right\rceil - \left\lceil \frac{n}{3} \right\rceil - \left\lceil \frac{n}{4} \right\rceil \leq \chi_\rho(\mu_1(P_n)) \leq \begin{cases} 4 + n - \lceil \frac{n}{3} \rceil - 2 \lceil \frac{n}{9} \rceil & \text{if } r \in \{0, 1\}, \\ 3 + n - \lceil \frac{n}{3} \rceil - 2 \lceil \frac{n}{9} \rceil & \text{if } r \in \{2, 3, 4, 5\}, \\ 2 + n - \lceil \frac{n}{3} \rceil - 2 \lceil \frac{n}{9} \rceil & \text{if } r \in \{6, 7, 8\}. \end{cases}$$

**Proof.** Set  $V = V(P_n) = \{x_1, \dots, x_n\}$ ,  $E(P_n) = \{x_i x_{i+1} : 1 \leq i \leq n - 1\}$  and  $V^1 = \{x_1^1, \dots, x_n^1\}$  the copy of  $V$  in  $\mu_1(P_n)$ . Let  $z$  be the root vertex of  $\mu_1(P_n)$ .

The lower bound is a consequence of Proposition 5.2 and the Lemma 5.3.

For the upper bound, let  $f$  be a coloring of  $\mu_1(P_n)$  defined as follows.

$$\begin{cases} f(x_i^1) = 1, & i \in \{1, \dots, n\}, \\ f(x_i) = 2, & \text{if } i \equiv 1 \pmod{3} \text{ and } i \in \{1, \dots, n\}, \\ f(x_i) = 3, & \text{if } (i \equiv 2 \pmod{9} \text{ or } i \equiv 6 \pmod{9}) \text{ and } i \in \{1, \dots, n\}, \\ f \text{ assigns distinct colors greater than 3 for the remaining vertices.} \end{cases}$$

We prove that  $f$  is a packing coloring of  $\mu_1(P_n)$ . Let  $u$  and  $v$  be two distinct vertices of  $V(\mu_1(P_n))$ .

*Case 1.*  $f(u) = f(v) = 1$ . Only vertices from  $V^1$  are colored 1, so  $(u, v) \in V^1 \times V^1$ . Since vertices of  $V^1$  are not adjacent by Mycielski construction, then  $d_{\mu_1(P_n)}(u, v) > 1$ .

*Case 2.*  $f(u) = f(v) = 2$ . Vertices colored 2 are from  $V$ , so  $(u, v) \in V \times V$ . Let  $u = x_i$  and  $v = x_j$  ( $i \neq j$ ), so  $i \equiv 1 \pmod{3}$  and  $j \equiv 1 \pmod{3}$ . The distance  $d_{P_n}(u, v)$  equals  $|i - j|$  which is congruent to 0 modulo 3. Since  $i \neq j$ , then by Mycielski construction  $d_{\mu_1(P_n)}(u, v) = \min\{|i - j|, 4\} > 2$ . Note that this case covers the subset of vertices  $\{x_i : i \equiv d \pmod{9}, d \in \{1, 4, 7\}\}$ .

*Case 3.*  $f(u) = f(v) = 3$ . In this case again  $(u, v) \in V \times V$ . Let  $u = x_i$  and  $v = x_j$ , ( $i \neq j$ ). If  $(i \equiv 2 \pmod{9} \text{ and } j \equiv 2 \pmod{9})$  or  $(i \equiv 6 \pmod{9} \text{ and } j \equiv 6 \pmod{9})$ , then  $d_{P_n}(u, v) \equiv 0 \pmod{9}$ , so by Mycielski construction  $d_{\mu_1(P_n)}(u, v) = 4 > 3$ . If  $i \equiv 2 \pmod{9}$  and  $j \equiv 6 \pmod{9}$ , then  $d_{P_n}(u, v) \equiv 4 \pmod{9}$ , so again  $d_{\mu_1(P_n)}(u, v) = 4 > 3$ .



*Case 4.*  $f(u) = f(v) > 3$ . This case corresponds to the set  $A$  of remaining vertices after assigning colors 1, 2 and 3 as defined in the coloring  $f$ . Since they are colored distinctly, then the property of packing coloring is preserved.

Let  $k$  be the number of colors used by  $f$  and let  $V_i = f^{-1}(i)$  be the set of vertices colored  $i$  in  $\mu_1(P_n)$ ,  $i \in \{1, \dots, k\}$ . We have  $k = 3 + |\mu_1(P_n)| - (|V_1| + |V_2| + |V_3|)$ , which is  $4 + 2n - (|V_1| + |V_2| + |V_3|)$ . Since  $V^1$  contains all vertices colored 1,  $|V_1| = |V^1| = n$ . We have  $V_2 = \{x_i : i \equiv 1 \pmod{3}\}$ , so in each three consecutive vertices from  $V$ , there exists only one vertex colored 2. Then  $|V_2| = \lceil \frac{n}{3} \rceil$ . The number  $k$  equals now  $4 + n - \lceil \frac{n}{3} \rceil - |V_3|$ . Since  $V_3 = \{x_i : i \equiv 2 \pmod{9}\} \cup \{x_i : i \equiv 6 \pmod{9}\}$ , then each nine consecutive vertices from  $V$  contains exactly two vertices colored 3. So the number  $|V_3|$  depends on the value of the positive integer  $r$  where  $n \equiv r \pmod{9}$ . We discuss the following cases.

- If  $r \in \{0, 1\}$ , then  $|V_3| = 2 \lfloor \frac{n}{9} \rfloor$ . So  $k = 4 + n - \lceil \frac{n}{3} \rceil - 2 \lfloor \frac{n}{9} \rfloor$ .
- If  $r \in \{2, 3, 4, 5\}$ , then  $|V_3| = 2 \lfloor \frac{n}{9} \rfloor + 1$ . So  $k = 3 + n - \lceil \frac{n}{3} \rceil - 2 \lfloor \frac{n}{9} \rfloor$ .
- If  $r \in \{6, 7, 8\}$ , then  $|V_3| = 2 \lfloor \frac{n}{9} \rfloor + 2$ . So  $k = 2 + n - \lceil \frac{n}{3} \rceil - 2 \lfloor \frac{n}{9} \rfloor$ .

Since  $\chi_\rho(\mu_1(P_n)) \leq k$ , this concludes the proof. ■

Bellow is the illustration of the coloring  $f$  of  $\mu_1(P_n)$  given in the last proof. Vertices of  $V$  are colored periodically (each nine vertices are separated by ||) and  $\diamond$  indicates distinct colors (greater than 3).

$$\begin{aligned}
 V : & 2 - 3 - \diamond - 2 - \diamond - 3 - 2 - \diamond - \diamond || 2 - 3 - \diamond - 2 - \diamond - 3 - 2 - \diamond - \diamond || \dots \\
 V^1 : & 1 - \dots \\
 & \qquad \qquad \qquad \diamond z
 \end{aligned}$$

Both bounds of Theorem 5.4 coincide for  $n \in \{2, 3, 4, 8, 12, 16, 20, 24\}$  where  $\chi_\rho(P_n) \in \{4, 5, 5, 7, 9, 10, 12, 14\}$ , respectively.

**Lemma 5.5.** *For all  $n \geq 3$ ,*

1.  $\alpha(\mu_1(C_n)) = n$ ,
2.  $\eta_2(C_n) = \lfloor \frac{n}{3} \rfloor$  and  $\eta_3(C_n) = \begin{cases} 1 & \text{if } n = 3, \\ \lfloor \frac{n}{4} \rfloor & \text{if } n > 3. \end{cases}$

**Proof.** Let  $n \geq 3$ .

1. Let  $S \in \mathbb{I}(C_n) \cup \{\emptyset\}$  such that  $|S| = k$  and set  $\beta_S = 2|S| + |\overline{N}(S)|$ . Note that

- if  $k = 0$ , then  $\beta_S = n$ ;
- if  $k = \alpha(C_n)$ , then  $\beta_S = 2\alpha(C_n)$ ;

- for a fixed value of  $k$  in  $\{1, \dots, \alpha(C_n) - 1\}$ , the number  $|\overline{N}(S)|$  varies in  $\{n - 2k - 1, \dots, n - 3k\}$ .

So by Corollary 3.4

$$\begin{aligned} \alpha(\mu_1(C_n)) &= \max_{S \in \mathbb{I}(G) \cup \{\emptyset\}} \{2|S| + |\overline{N}(S)|\} \\ &= \max_{1 \leq k \leq \alpha(C_n) - 1} \left\{ n, 2\alpha(C_n), \max_{2k+1 \leq i \leq 3k} \{2k + (n - i)\} \right\} \\ &= \max_{1 \leq k \leq \alpha(C_n) - 1} \left\{ n, 2 \left\lfloor \frac{n}{2} \right\rfloor, 2k + n - 2k - 1 \right\} \\ &= \max \left\{ n, 2 \left\lfloor \frac{n}{2} \right\rfloor, n - 1 \right\} = n. \end{aligned}$$

2. If  $n = 3$ , a single vertex is a 2-packing. Assume now that  $n > 3$  and let  $V(C_n) = \{x_1, x_2, \dots, x_n\}$ . The sets of vertices  $A = \{x_i : i \equiv 1 \pmod{3}, 1 \leq i \leq \lfloor \frac{n}{3} \rfloor\}$  and  $B = \{x_i : i \equiv 1 \pmod{4}, 1 \leq i \leq \lfloor \frac{n}{4} \rfloor\}$  are the best maximum 2-packing and 3-packing of  $C_n$ , respectively, with  $|A| = \lfloor \frac{n}{3} \rfloor$  and  $|B| = \lfloor \frac{n}{4} \rfloor$ . No better sets can be obtained.

Indeed, suppose that there exists a 2-packing  $A' = \{x_{i_1}, x_{i_2}, \dots, x_{i_r}\}$  of  $C_n$  with cardinality  $r = \lfloor \frac{n}{3} \rfloor + 1$ . Without loss of generality, assume that vertices of  $A'$  are consecutive, which means that  $d(x_j, x_{j+1}) = 3$  for all  $j \in \{i_1, i_2, \dots, i_r - 1\}$ . Let  $P$  and  $P'$  be the two simple directed paths  $x_{i_1} \cdots x_{i_2} \cdots x_{i_3} \cdots x_{i_r}$  (of length  $3(r - 1)$ ) and  $x_{i_r} \cdots x_{i_1}$ , respectively. Since  $A'$  is a 2-packing, the length of  $P'$  is greater than or equal to 3. Note that the sum of both lengths of  $P$  and  $P'$  equals  $n$ . It follows that  $n \geq 3 + 3(r - 1) = 3r$ . In each case of  $n \in \{3k, 3k + 1, 3k + 2\}$ , we have  $3r = 3(k + 1) > n$ , which is a contradiction to  $3r \leq n$ . A similar reasoning can be done to prove that a 3-packing of cardinality  $\lfloor \frac{n}{4} \rfloor + 1$  cannot be obtained for  $C_n$ . ■

It was proved in [18] that for any graph  $G$  without isolated vertices

$$diam(\mu_1(G)) = \min \{ \max \{ 2, diam(G) \}, 4 \}.$$

So since  $diam(C_3) = 2$ , we have  $diam(\mu_1(C_3)) = 2$ . Consequently,  $\chi_\rho(\mu_1(C_3)) = |V(\mu_1(C_3))| - \alpha(\mu_1(C_3)) + 1$  by Proposition 1.1. Thus,  $\chi_\rho(\mu_1(C_3)) = 7 - 3 + 1 = 5$ .

**Theorem 5.6.** *For every positive integer  $n \geq 4$ ,*

$$4 + n - \left\lfloor \frac{n}{3} \right\rfloor - \left\lfloor \frac{n}{4} \right\rfloor \leq \chi_\rho(\mu_1(C_n)) \leq \begin{cases} 4 + n - \left\lfloor \frac{n}{3} \right\rfloor - 2 \left\lfloor \frac{n}{9} \right\rfloor & \text{if } r \in \{0, 1, 2, 3\}, \\ 3 + n - \left\lfloor \frac{n}{3} \right\rfloor - 2 \left\lfloor \frac{n}{9} \right\rfloor & \text{if } r \in \{4, 5, 6, 7\}, \\ 2 + n - \left\lfloor \frac{n}{3} \right\rfloor - 2 \left\lfloor \frac{n}{9} \right\rfloor & \text{if } r = 8, \end{cases}$$

where  $n \equiv r \pmod{9}$ .

**Proof.** The lower bound is a consequence of Proposition 5.2 and Lemma 5.5.

Let  $n \geq 4$ . Set  $V = V(C_n) = \{x_1, \dots, x_n\}$  and  $V^1$  its copy in  $\mu_1(C_n)$ . Let  $r$  be a positive integer such that  $n \equiv r \pmod{9}$ . Let  $f$  be a coloring of  $\mu_1(C_n)$  such that

$$\begin{cases} f(x_i^1) = 1, & i \in \{1, \dots, n\}, \\ f(x_i) = 2, & \text{if } i \equiv 1 \pmod{3} \text{ and } i \in \{1, \dots, n-2\}, \\ f(x_i) = 3, & \text{if } (i \equiv 2 \pmod{9} \text{ or } i \equiv 6 \pmod{9}) \text{ and } i \in \{1, \dots, n-2\}, \\ \text{distinct colors greater than 3 for the remaining vertices.} \end{cases}$$

The coloring  $f$  is similar to the one of the Mycielskian of a path in the previous theorem with an adjustment at the last  $r$  vertices where the vertices  $x_n$  and  $x_{n-1}$  cannot receive colors 2 and 3 since  $x_1$  and  $x_2$  are colored by 2 and 3 respectively. By the same reasoning of path case in the proof of Theorem 5.4, one can see that  $f$  is also a packing coloring of  $\mu_1(C_n)$ . It remains to determine the number  $k$  of colors used by  $f$ . Again,  $k = 4 + 2n - (|V_1| + |V_2| + |V_3|)$ . Note that  $f$  assigns the color 1 to all vertices of the independent set  $V^1$ . So  $|V_1| = n$ . Hence  $k = 4 + n - (|V_2| + |V_3|)$ .

For the set  $V$ , the coloring  $f$  is illustrated as follows.

- If  $r = 0$ ,  $f$  uses periodically the following pattern

$$2 - 3 - \diamond - 2 - \diamond - 3 - 2 - \diamond - \diamond$$

where  $\diamond$  represents distinct colors greater than 3.

- If  $1 \leq r \leq 8$ ,  $f$  uses periodically the pattern of the case  $r = 0$  for each nine consecutive vertices (starting by  $x_1$ ) while the coloring of the last  $r$  vertices is illustrated by

$$\begin{aligned} \text{case } r = 1 & : \diamond \\ \text{case } r = 2 & : \diamond - \diamond \\ \text{case } r = 3 & : 2 - \diamond - \diamond \\ \text{case } r = 4 & : 2 - 3 - \diamond - \diamond \\ \text{case } r = 5 & : 2 - 3 - \diamond - \diamond - \diamond \\ \text{case } r = 6 & : 2 - 3 - \diamond - 2 - \diamond - \diamond \\ \text{case } r = 7 & : 2 - 3 - \diamond - 2 - \diamond - \diamond - \diamond \\ \text{case } r = 8 & : 2 - 3 - \diamond - 2 - \diamond - 3 - \diamond - \diamond. \end{aligned}$$

It is easy to see that the number of vertices from  $V$  colored 2 is  $\lfloor \frac{n}{3} \rfloor$ . So  $k = 4 + n - \lfloor \frac{n}{3} \rfloor - |V_3|$ . Note also that each pattern of nice vertices from  $V$  contains two vertices with color 3. So  $|V_3| = 2 \lfloor \frac{n}{9} \rfloor + j$ , where  $j$  is the number of vertices colored 3 in the last  $r$  vertices. We have the following cases.

- If  $r \in \{0, 1, 2, 3\}$ , then  $j = 0$ , and  $k = 4 + n - \lfloor \frac{n}{3} \rfloor - 2 \lfloor \frac{n}{9} \rfloor$ .
- If  $r \in \{4, 5, 6, 7\}$ , then  $j = 1$ , and  $k = 3 + n - \lfloor \frac{n}{3} \rfloor - 2 \lfloor \frac{n}{9} \rfloor$ .
- If  $r = 8$ , then  $j = 2$ , and  $k = 2 + n - \lfloor \frac{n}{3} \rfloor - 2 \lfloor \frac{n}{9} \rfloor$ .

This concludes the proof. ■

Both bounds of Theorem 5.6 coincide for

$$n \in \{4, \dots, 11, 13, 14, 15, 17, 18, 19, 22, 23, 26, 27, 31, 35\}.$$

Using the computer, we found a packing coloring to the Mycielskian of paths and cycles with a number of colors equal to the upper bound of Theorem 5.4 and Theorem 5.6, respectively. So we may ask the following two questions.

**Question 5.7.** *Is the inequality of the upper bound in Theorem 5.4 an equality for every positive integer  $n \geq 5$ ?*

**Question 5.8.** *Is the inequality of the upper bound in Theorem 5.6 an equality for every positive integer  $n \geq 5$ ?*

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