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DECOMPOSING 10-REGULAR GRAPHS INTO PATHS OF LENGTH 5

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Abstract

Let G be a 10-regular graph which does not contain any 4-cycles. In this paper, we prove that G can be decomposed into paths of length 5, such that every vertex is a terminal of exactly two paths.

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1. Introduction

Graphs in this paper are simple. Let G and H be graphs. We say that G has an H-*decomposition* $\mathcal{D} = \{H_1, H_2, \ldots, H_n\}$, if any two elements of $\mathcal{D}$ are edge-disjoint subgraphs of G, H_i ($1 \leq i \leq n$) is isomorphic to H and $E(G) = \bigcup_{i=1}^{n} H_i$. For convenience, we use P_m and C_m to denote the path and cycle with m edges, respectively. For a positive integer r, an r-*factor* of G is a spanning subgraph F of G such that $d_F(v) = r$ for each vertex v of G. A decomposition $\mathcal{F}$ of G is an r-*factorization* if every element of $\mathcal{F}$ is an r-factor, any two elements of $\mathcal{F}$ are edge-disjoint subgraphs of G, and $E(G)$ can be covered by $\mathcal{F}$.

Graham and Häggkvist [6] posed the following conjecture.

Conjecture 1 (Graham-Häggkvist [6]). *Let T be a tree with l edges. If G is a $2l$-regular graph, then G admits a T-decomposition.*

In the same paper, Häggkvist proved that Conjecture 1 is true when the girth of G is at least the diameter of T. In the past several decades, this conjecture interested many researchers and many related results were presented. Fink [5] stated that if T is any tree having n edges ($n \geq 1$), then the n-cube Q_n can be decomposed into 2^{n-1} edge-disjoint induced subgraphs, each of which is isomorphic to T. Erde [4] confirmed that if n is odd and $k \leq n$ such that $k|n2^{n-1}$, then Q_n can be decomposed into paths of length k. In [7], Jacobson, Truszczyński and Tuza proved that: (1) there is a wide class of r-regular bipartite graphs that can be decomposed into any tree of size r; (2) every r-regular bipartite graph can be decomposed into any double star of size r; (3) every 4-regular bipartite graph can be decomposed into paths of length 4. As one corollary of main result in [8], Jao, Kostochka and West confirmed Conjecture 1 for a $2l$-regular graph which has a 2-factorization such that every cycle consisting of edges from distinct 2-factors has length greater than the diameter of T. El-Zanati *et al.* [3] verified Conjecture 1 when T is a double-star, and further they proved that the double-star $S_{k,k-1}$ can decompose every $2k$-regular graph which contains a perfect matching.

It is natural to ask whether Conjecture 1 holds if T is a path. Unfortunately, there is no definitive answer for general graphs. Kouider and Lonc [9] proposed a strengthening of Conjecture 1 in the case where T is a path, and it is still open. We say a path decomposition $\mathcal{D}$ is *balanced* if each vertex is a terminal of exactly two paths of $\mathcal{D}$.

Conjecture 2 (Kouider and Lonc [9]). *Let l be a positive integer. If G is a $2l$-regular graph, then G admits a balanced P_l-decomposition.*

By Petersen's Factorization Theorem (see Theorem 3.1), Botler *et al.* [1] proposed an equivalent form of Conjecture 2.

Conjecture 3 (Botler *et al.* [1]). *Let m and l be positive integers. Then every $2ml$-regular graph admits a balanced P_l-decomposition.*

In the same paper, they proved that if $m \geq \lfloor (l-2)/(g-2) \rfloor$, then every $2ml$-regular graph with girth at least g admits a P_l-decomposition. Furthermore, every $2ml$-regular graph with girth at least $l-1$ admits a P_l-decomposition for $m \geq 1$. By controlling the girth, Kouider and Lonc [9] confirmed Conjecture 2 for a $2l$-regular graph G with girth at least $(l+3)/2$.

Theorem 4 (Kouider and Lonc [9]). *If $l \leq 2g-3$, then every $2l$-regular graph G of girth g has a balanced P_l-decomposition.*

By Theorem 4, Conjecture 2 is true for $l = 1, 2$ and 3. When $l = 4$ or 5, every $2l$-regular graph G without triangles has a balanced P_l-decomposition. For later use, we will present a short proof when $l = 3$ in Conjecture 2 in Section 3. Based on analysis of the structure of the graph, Botler and Talon [2] used a different method from that in [9] to confirm the conjecture for $l = 4$.

Theorem 5 (Botler and Talon [2]). *If G is an 8-regular graph, then G admits a balanced P_4-decomposition.*

Motivated by Theorem 5, we want to solve the case $l = 5$. However, the structure of a P_5-decomposition in a 10-regular graph is more complex than the structure of a P_4-decomposition in an 8-regular graph. Thus we consider P_5-decompositions of 10-regular graphs which contain no 4-cycles, and get the main result of this paper.

Theorem 6. *Let G be a 10-regular graph. If G does not contain any 4-cycles, then G admits a balanced P_5-decomposition.*

2. Notations and Terminologies

A *trail* $T = x_0x_1 \cdots x_l$ is a graph for whose $V(T) = \{x_i \mid 0 \leq i \leq l\}$, $E(T) = \left\{x_ix_{i+1} \mid 0 \leq i \leq l-1\right\}$ and $x_ix_{i+1} \neq x_jx_{j+1}$, for every $i \neq j$. Denote the vertices x_0 and x_l as the *terminal* vertices of T, x_1 and x_{l-1} as the *preterminal* vertices of T. If a trail has l edges, then we call it an *l-trail*. If a set of edge-disjoint trails $\mathcal{B}$ of a graph G is such that $\bigcup_{B\in\mathcal{B}} E(B) = E(G)$, then $\mathcal{B}$ is a decomposition of G into trails. If every trail of $\mathcal{B}$ has length l, then $\mathcal{B}$ is a decomposition into l-trails (or an l-trail decomposition). For a trail decomposition $\mathcal{B}$ of G, if every vertex of G is a terminal of exactly two trails of $\mathcal{B}$, then $\mathcal{B}$ is called balanced. If every trail of $\mathcal{B}$ is a path, then $\mathcal{B}$ is a decomposition into paths (or a path decomposition). We use $\tau(\mathcal{B})$ to denote the number of elements of $\mathcal{B}$ that are cycles.

A *tour* of a connected graph G is a closed walk that traverses each edge of G at least once, and an *Eulerian tour* one that traverses each edge exactly once. A graph is *Eulerian* if it admits an Eulerian tour. Since an Eulerian tour traverses each edge exactly once, $d(v)$ is even for every vertex $v \in V(G)$. On the other hand, if G is a connected graph and every vertex has even degree, then G has an Eulerian tour by Fleury's Algorithm. A graph in which each vertex has even degree is called an *even graph*. Therefore, a graph is Eulerian if and only if is even and connected. An orientation O of a subset E' of $E(G)$ is an attribution of a direction to each edge of E'. If an edge xy is directed from x to y in O, we say that xy leaves x and enters y. For a vertex v of G, let $d^+_O(v)$ (respectively, $d^-_O(v)$) be the number of edges leaving (respectively, entering) v with respect to O. If O is an orientation of G and every vertex v has $d^+_O(v) = d^-_O(v)$, then O is an *Eulerian orientation* of G. It is easy to see that G is even if it has an Eulerian orientation. If G is even, then each of its components has an Eulerian tour. We can get an Eulerian orientation of G by assigning each edge of G an orientation in such a way that the Eulerian tour of each component of G is a directed Eulerian tour. Thus a graph has an Eulerian orientation if and only if it is even.

3. Proof of Main Theorem

First of all, we will present Petersen's Factorization Theorem [10].

Theorem 7 (Petersen's 2-Factorization Theorem [10]). *Every $2k$-regular graph admits a 2-factorization.*

Nextly, we will introduce a approach used in [2] to get a trial decomposition. By adjusting this decomposition, we finally get the desired result.

Let G be an r-regular graph ($r \geq 6$ and is even), $\mathcal{F}$ be a 2-factorization of G given by Theorem 7. By combining the elements of $\mathcal{F}$, we obtain a decomposition of G into an $(r-4)$-factor and a 4-factor, say F_1 and F_2, respectively. Let O be an Eulerian orientation of F_2. Suppose F_1 has a balanced $P_{(r-4)/2}$-decomposition $\mathcal{D}$. So every vertex v of G is a terminal of exactly two paths in $\mathcal{D}$. Note that $d^+_O(v) = 2$ for every vertex v of F_2. Thus, we can extend every path $P = x_1x_2 \cdots x_{(r-4)/2+1}$ in $\mathcal{D}$ to a $(\mathcal{D}, O)$-*extension* $Q_P = x_0x_1 \cdots x_{(r-4)/2+2}$ such that x_0x_1 and $x_{(r-4)/2+1}x_{(r-4)/2+2}$ are two edges in F_2 leaving x_1 and $x_{(r-4)/2+1}$, respectively, and further every edge of F_2 is used exactly once. Therefore, $\{Q_P \mid P \in \mathcal{D}\}$ is a decomposition into $(\mathcal{D}, O)$-extensions of G, which may not be a decomposition into paths, just into trails. Obviously, each decomposition into $(\mathcal{D}, O)$-extensions is balanced.

In this paper, we focus on the path decompositions of a 10-regular graph which does not contain any 4-cycles. Let F_1 be a 6-factor of G, F_2 be a 4-factor such that $F_1 \cup F_2 = G$, O be an Eulerian orientation of F_2. By Theorem 4, F_1 has a balanced P_3-decomposition $\mathcal{D}$. Following the method above, we first obtain a decomposition into $(\mathcal{D}, O)$-extensions of G from $\mathcal{D}$, and then adjust this trail decomposition to a path decomposition of G.

Let G be a 6-regular graph. We present a brief proof that G has a balanced P_3-decomposition.

Lemma 8. *If G is a 6-regular graph, then G admits a balanced P_3-decomposition $\mathcal{D}$ and every vertex of G is a preterminal of exactly two paths in $\mathcal{D}$.*

Proof. Let $\mathcal{F}$ be a 2-factorization of G given by Theorem 7. By combining the elements of $\mathcal{F}$, we obtain a decomposition of G into a 2-factor and a 4-factor, say F_3 and F_4, respectively. Obviously, F_3 has a balanced P_1-decomposition, denoted by $\mathcal{D}_1$. Because every vertex of F_4 has even degree, there is an Eulerian orientation O on F_4. Let $\mathcal{D}$ be a decomposition of G into $(\mathcal{D}_1, O)$-extensions which minimizes $\tau(\mathcal{D})$. If every element in $\mathcal{D}$ is a P_3, then we are done. Suppose there is a triangle $C = x_0x_1x_2x_3$ in $\mathcal{D}$, $x_0 = x_3$, $x_1x_2 \in \mathcal{D}_1$. There is an element $T = y_0y_1y_2y_3$ of $\mathcal{D}$ such that $y_1 = x_1$, $y_1y_2 \in \mathcal{D}_1$, $y_1y_2 \neq x_1x_2$ and $T \neq C$. Let $C' = y_0x_1x_2x_3$ and $T' = x_0y_1y_2y_3$. Obviously, C' is a path of length 3. Because, G is simple, y_1, y_2 and y_3 are distinct vertices, $x_0 \neq y_1$ and $x_0 \neq y_2$. If T'

is a triangle, then $y_3 = x_0 = x_3$ and $d^-_O(x_0) = 3$, which is a contradiction to the assumption before that O is an Eulerian orientation on F_4. Hence, T' is a path of length 3. Let $\mathcal{D}' = (\mathcal{D} - \{T, C\}) \cup \{T', C'\}$. $\mathcal{D}'$ is a decomposition of G into $(\mathcal{D}_1, O)$-extensions and $\tau(\mathcal{D}') \leq \tau(\mathcal{D}) - 1$, which is a contradiction to the minimality of $\tau(\mathcal{D})$. Therefore, $\mathcal{D}$ is a balanced P_3-decomposition of G. By the construction of $\mathcal{D}$, we can find that every vertex of G is a preterminal of exactly two paths in $\mathcal{D}$. ■

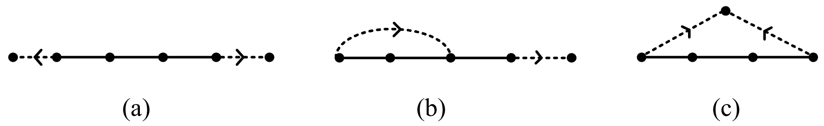

Figure 1. Extensions.

Now, let G be a 10-regular graph without a C_4, F_1 be a 6-factor of G, F_2 be a 4-factor such that $F_1 \cup F_2 = G$. Let O be an Eulerian orientation of F_2 and D_1 be a balanced P_3-decomposition of F_1, and further, $\mathcal{T} = \{Q_P \mid P \in \mathcal{D}_1\}$ be a decomposition into $(\mathcal{D}_1, O)$-extensions of G. Let $T = x_0x_1x_2x_3x_4x_5 \in \mathcal{T}$. Because $\mathcal{D}_1$ is a balanced P_3-decomposition of F_1 and G does not contain any C_4, we have x_1, x_2, x_3 and x_4 are distinct vertices, $x_0 \neq x_4$, $x_5 \neq x_1$ and it is impossible that both $x_0 = x_3$ and $x_5 = x_2$ hold. Hence, if T is a trail of $\mathcal{T}$, then exactly one of the following holds: (a) T is a path of length 5; (b) T is a trail of length 5 which contains a triangle; (c) T is a cycle of length 5 (see Figure 1). In the figures throughout this section, we illustrate the edges of F_1 as straight edges, and the edges of F_2 as dashed edges. The next result shows that every 10-regular graph admits a decomposition into $(\mathcal{D}_1, O)$-extensions which are not cycles.

Lemma 9. *Let G be a $2l$-regular graph, F_1 be a $2(l-2)$-factor of G, $F_2 = G \setminus E(F_1)$ and O be an Eulerian orientation of F_2. If there is a balanced $P_{(l-2)}$-decomposition $\mathcal{D}_1$ of F_1, then G admits a decomposition into $(\mathcal{D}_1, O)$-extensions which are not cycles.*

Proof. Let G, F_1, F_2, $\mathcal{D}_1$, and O be as in the statement above. Now, let $\mathcal{D}$ be a decomposition of G into $(\mathcal{D}_1, O)$-extensions which minimizes $\tau(\mathcal{D})$.

Suppose, for contradiction, that $\tau(\mathcal{D}) > 0$. Let $T = x_0x_1x_2 \cdots x_{l-1}x_l$ be a cycle of length l in $\mathcal{D}$, where $L_1 = x_1x_2 \cdots x_{l-1} \in \mathcal{D}_1$ and $x_0 = x_l$. Note that $\mathcal{D}_1$ is balanced. Let $L_2 = y_1y_2 \cdots y_{l-1}$ be the element of $\mathcal{D}_1$ such that $L_2 \neq L_1$ and $y_1 = x_1$. Suppose $Q = y_0y_1y_2 \cdots y_{l-1}y_l$ is the $(\mathcal{D}_1, O)$-extension of L_2 in $\mathcal{D}$. Let $T' = y_0x_1x_2 \cdots x_{l-1}x_l$ and $Q' = x_0y_1y_2 \cdots y_{l-1}y_l$. Clearly, T' and Q' are $(\mathcal{D}_1, O)$-extensions. Because G is simple, $y_0 \neq x_l$. Hence, T' is not a cycle. Moreover, if Q' is a cycle, then the edges x_0x_1, $x_{l-1}x_l$, and $y_{l-1}y_l$ are directed towards x_0, which implies $d^-_O(x_0) \geq 3$, contrary to the fact that O is an Eulerian orientation of F_2. Therefore, $\mathcal{D}' = (\mathcal{D} - \{T, Q\}) \cup \{T', Q'\}$ is a decomposition into $(\mathcal{D}_1, O)$-extensions of G such that $\tau(\mathcal{D}') \leq \tau(\mathcal{D}) - 1$, which is a contradiction to the minimality of $\tau(\mathcal{D})$. This completes the proof of Lemma 9. ■

In the following, we will define a special Eulerian orientation, which is important for the proof of Theorem 6.

Definition 10. Let G be a 10-regular graph, F be a 6-factor of G, $\mathcal{D}$ be a balanced P_3-decomposition of F, $H = G \setminus E(F)$. We say that an Eulerian orientation O on H is good if the following holds. For each path $U = v_1v_2v_3$ of H and distinct vertices $x_2, x_3, y_2, y_4, z_2, z_4, v_1, v_2, v_3$, if there exists three elements $T_1 = x_1x_2x_3x_4, T_2 = y_1y_2y_3y_4, T_3 = z_1z_2z_3z_4 \in \mathcal{D}$ and $x_1 = v_1 = y_1$, $y_3 = v_2 = z_3$, $x_4 = v_3 = z_1$, then U is a directed path under orientation O, no mater which direction it goes (see Figure 2).

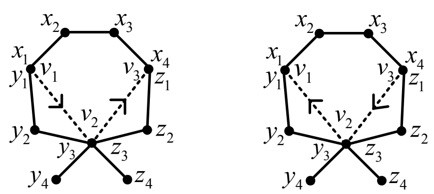

Figure 2. A good orientation on $U = v_1v_2v_3$.

Lemma 11. *Let G be a 10-regular graph without C_4, F be a 6-factor of G, and $H = G \setminus E(F)$. Then, there is a good Eulerian orientation on the edges of H.*

Proof. By Lemma 8, we assume that $\mathcal{D}$ is a balanced P_3-decomposition of F such that every vertex of G is a preterminal of exactly two paths in $\mathcal{D}$. Let $U = v_1v_2v_3$ be a path of length 2 in H. Because G is simple and does not contain any 4-cycles, if there exists three elements $T_1 = x_1x_2x_3x_4, T_2 = y_1y_2y_3y_4, T_3 = z_1z_2z_3z_4 \in \mathcal{D}$ and $x_1 = v_1 = y_1$, $y_3 = v_2 = z_3$, $x_4 = v_3 = z_1$, then x_2, x_3, y_2, y_4, z_2, z_4, v_1, v_2 and v_3 are distinct vertices. Hence, U and T_1, T_2, T_3 form a structure defined in Definition 10. In order to obtain a good Eulerian orientation on H, we need to construct a new even graph H' from H.

Let $\mathcal{U} = \left\{v_1^i v_2^i v_3^i \mid 1 \leq i \leq k\right\}$ be the set of all the paths of length 2 in H which are contained in the structure defined in Definition 10. Note that these paths in $\mathcal{U}$ are not necessarily edge-disjoint. We claim that $v_2^i \neq v_2^j$ for $U_i = v_1^i v_2^i v_3^i, U_j = v_1^j v_2^j v_3^j \in \mathcal{U}$ and $i \neq j$. If not, suppose that $v_2^i = v_2^j$. Without loss of generality, let U_i and three elements T_1, T_2, T_3 of $\mathcal{D}$ be contained in the structure depicted in Definition 10 such that $v_1^i, v_2^i \in V(T_1)$, $v_3^i, v_2^i \in V(T_2)$, $v_1^i, v_3^i \in V(T_3)$. If $|E(U_i) \cap E(U_j)| = 1$, then without loss of generality let $v_1^i = v_1^j$. This implies that there are two paths T_4 and T_5 of $\mathcal{D}$ (because G is simple, $E(U_i), E(U_j) \subseteq E(H)$ and $E(T_m) \subseteq E(F)$ $(1 \leq m \leq 5)$, $T_k \neq T_q, k \in \{4,5\}, q \in \{1,2,3\}$) together with U_j and T_1 form another structure defined in Definition 10, such that $v_1^j, v_2^j \in V(T_1)$, $v_3^j, v_2^j \in V(T_4)$, $v_1^j, v_3^j \in V(T_5)$. If $|E(U_i) \cap E(U_j)| = 0$, then this implies that there are three paths T_4, T_5 and T_6 of $\mathcal{D}$ (because G is simple, $E(U_i), E(U_j) \subseteq E(H)$ and $E(T_m) \subseteq E(F)$ $(1 \leq m \leq 6)$, $T_k \neq T_q, k \in \{4,5,6\}, q \in \{1,2,3\}$) together with U_j form another structure defined in Definition 10, such that $v_1^j, v_2^j \in V(T_4)$, $v_3^j, v_2^j \in V(T_5)$, $v_1^j, v_3^j \in V(T_6)$. In the two cases, v_2^i appears in at least three paths in $\mathcal{D}$ as their preterminal vertex, contrary to that v_2^i is the preterminal vertex of exactly two paths in $\mathcal{D}$. Thus, $v_2^i \neq v_2^j$ when $i \neq j$, as claimed. This means that for every vertex v of G, there is at most one $U_i \in \mathcal{U}$ such that edges incident with U_i is contained in the subgraph induced by $E_H(v)$ which is the set of v in H.

Now we can split edges of U_i in the following way: delete edges $v_1^i v_2^i$ and $v_2^i v_3^i$, add a new vertex z_i and two edges $v_1^i z_i$ and $z_i v_3^i$. By operating on all elements in $\mathcal{U}$ as described above, we can get a new graph H' from H. Let O' be an Eulerian orientation on H'. By identifying z_i and v_2^i $(1 \leq i \leq k)$ in H' and preserving the orientation of O' on all edges after identifying, we get an Eulerian orientation O on H. It is obviously that O is good. ∎

Now we are able to prove Theorem 6. For a 5-trail decomposition $\mathcal{B}$ of a 10-regular graph G, we use $\tau'(\mathcal{B})$ to denote the number of elements of $\mathcal{B}$ that are paths.

***Proof of Theorem* 6.** Let G be a 10-regular graph without C_4, F be a 6-factor of G, $\mathcal{D}$ be a balanced P_3-decomposition of F, $H = G \setminus E(F)$, and O be a good Eulerian orientation of H. By Lemma 9, G has a decomposition $\mathcal{B}$ into $(\mathcal{D}, O)$-extensions which are not cycles. Further, we may assume that $\tau'(\mathcal{B})$ is maximum. If $\tau'(\mathcal{B}) = |\mathcal{B}|$, then we are done. Suppose that $\tau'(\mathcal{B}) < |\mathcal{B}|$. Let $T \in \mathcal{B}$ be a trail containing a triangle.

Let $T = x_0x_1x_2x_3x_4x_5$, where $x_1x_2x_3x_4 \in \mathcal{D}$, $x_0 = x_3$. There is a trail $Q = y_0y_1y_2y_3y_4y_5 \in \mathcal{B}$ with $Q \neq T$ such that $y_1y_2y_3y_4 \in \mathcal{D}$ and $y_1 = x_1$. We put $T' = y_0x_1x_2x_3x_4x_5$, $Q' = x_0y_1y_2y_3y_4y_5$. Because G is simple and does not contain C_4, $y_0 \notin V(T)$, which implies that T' is a path. Moreover, $x_0 \neq y_3$ which follows from the fact that G does not contain C_4. Hence, if Q' contains a triangle only if Q contains the triangle $y_2y_3y_4y_5$. If Q' is not a cycle, then $\mathcal{B}' = (\mathcal{B} - \{T, Q\}) \cup \{T', Q'\}$ is a decomposition of G into $(\mathcal{D}, O)$-extensions with $\tau'(\mathcal{B}') \geq \tau'(\mathcal{B}) + 1$, which is a contradiction to the maximality of $\tau'(\mathcal{B})$. In the following, we assume Q' is a cycle.

Now, $y_5 = x_0 = x_3$. Note that G is simple and does not contain any 4-cycles. We have that $y_1 \neq y_4$, y_2 and y_3 are not equal to any one of $\{x_1, x_2, x_3, x_4, x_5\}$, y_4 is not equal to any one of $\{x_1, x_2, x_3, x_4\}$. In this case, y_4 and x_5 may be the same one. Let $R = z_0z_1z_2z_3z_4z_5$ be an element in $\mathcal{B}$ different from T and

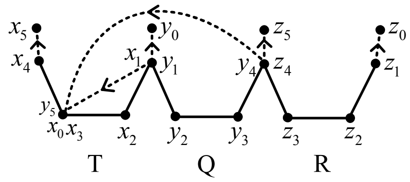

Figure 3. Q' is a cycle.

Q, where $z_1z_2z_3z_4 \in \mathcal{D}$ and $z_4 = y_4$ (see Figure 3). Let $Q'' = x_0y_1y_2y_3y_4z_5$, $R' = z_0z_1z_2z_3z_4y_5$. Because G is simple and does not contain any 4-cycles. We have that $z_5 \notin V(Q')$. Hence Q'' is a path. If R' is a cycle, we have $x_0 = x_3 = z_0$, $d^-_O(x_0) \geq 3$, contrary to the fact that O is an Eulerian orientation of H. Hence, R' is not a cycle. If R contains a triangle, then $\mathcal{B}' = (\mathcal{B} - \{T, Q, R\}) \cup \{T', Q'', R'\}$ is a decomposition of G into $(\mathcal{D}, O)$-extensions with $\tau'(\mathcal{B}') \geq \tau'(\mathcal{B}) + 1$, which is a contradiction to the maximality of $\tau'(\mathcal{B})$. In the following, we assume R is a path. Because G is simple and does not contain C_4, $y_5 \neq z_1, z_3$. If $y_5 = z_2$, then let $U = x_1x_0y_4$, $T_1 = y_1y_2y_3y_4$, $T_2 = x_1x_2x_3x_4$ and $T_3 = z_4z_3z_2z_1$. Now, we want to prove that U, T_1, T_2 and T_3 form the structure defined in the Definition 10. Note that $x_0 = x_3 = y_5 = z_2$, $x_1 = y_1$ and $y_4 = z_4$. Therefore, we should check $x_0, x_1, x_2, x_4, y_2, y_3, y_4, z_1$ and z_3 are distinct vertices of G. Because G is simple, x_0, x_1, x_2, x_4, y_4, z_1 and z_3 are distinct vertices, x_0, x_1, y_2, y_3 and y_4 are distinct vertices, $x_2 \neq y_2$ and $y_3 \neq z_3$. What remains is the following cases. If $z_1 = y_2$ (respectively, y_3), then $z_1x_1x_2x_3z_1$ (respectively, $z_1z_4z_3z_2z_1$) is a cycle of length 4, a contradiction. If $x_4 = y_2$ (respectively, y_3), then $y_2x_1x_2x_3y_2$ (respectively, $y_3y_2y_1x_3y_3$) is a cycle of length 4, a contradiction. If $y_2 = z_3$, then $y_2x_0x_2x_1y_2$ is a cycle of length 4, a contradiction. If $y_3 = x_2$ (respectively, z_1), then $y_3z_4z_3z_2x_2$ (respectively, $z_1z_4z_3z_2z_1$) is a cycle of length 4, also a contradiction. Hence, $x_0, x_1, x_2, x_4, y_2, y_3, y_4, z_1$ and z_3 are distinct vertices of G, and U, T_1, T_2 and T_3 form the structure defined in the Definition 10. But the orientation of $E(U)$ implies that O is not a good Eulerian orientation of H, a contradiction to our assumption. Hence, R' is a path and $\mathcal{B}' = (\mathcal{B} - \{T, Q, R\}) \cup \{T', Q'', R'\}$ is a decomposition of G into $(\mathcal{D}, O)$-extensions with $\tau'(\mathcal{B}') \geq \tau'(\mathcal{B}) + 1$, which is a contradiction to the maximality of $\tau'(\mathcal{B})$. This completes the proof of Theorem 6. ■

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