

DECOMPOSING 10-REGULAR GRAPHS INTO PATHS OF LENGTH 5

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Abstract

Let G be a 10-regular graph which does not contain any 4-cycles. In this paper, we prove that G can be decomposed into paths of length 5, such that every vertex is a terminal of exactly two paths.

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1. INTRODUCTION

Graphs in this paper are simple. Let G and H be graphs. We say that G has an H -decomposition $\mathcal{D} = \{H_1, H_2, \dots, H_n\}$, if any two elements of \mathcal{D} are edge-disjoint subgraphs of G , H_i ($1 \leq i \leq n$) is isomorphic to H and $E(G) = \bigcup_{i=1}^n H_i$. For convenience, we use P_m and C_m to denote the path and cycle with m edges, respectively. For a positive integer r , an r -factor of G is a spanning subgraph F of G such that $d_F(v) = r$ for each vertex v of G . A decomposition \mathcal{F} of G is an r -factorization if every element of \mathcal{F} is an r -factor, any two elements of \mathcal{F} are edge-disjoint subgraphs of G , and $E(G)$ can be covered by \mathcal{F} .

Graham and Häggkvist [6] posed the following conjecture.

Conjecture 1 (Graham-Häggkvist [6]). *Let T be a tree with l edges. If G is a $2l$ -regular graph, then G admits a T -decomposition.*

In the same paper, Häggkvist proved that Conjecture 1 is true when the girth of G is at least the diameter of T . In the past several decades, this conjecture interested many researchers and many related results were presented. Fink [5] stated that if T is any tree having n edges ($n \geq 1$), then the n -cube Q_n can be decomposed into 2^{n-1} edge-disjoint induced subgraphs, each of which is isomorphic to T . Erde [4] confirmed that if n is odd and $k \leq n$ such that $k|n2^{n-1}$, then Q_n can be decomposed into paths of length k . In [7], Jacobson, Truszczyński and Tuza proved that: (1) there is a wide class of r -regular bipartite graphs that can be decomposed into any tree of size r ; (2) every r -regular bipartite graph can be decomposed into any double star of size r ; (3) every 4-regular bipartite graph can be decomposed into paths of length 4. As one corollary of main result in [8], Jao, Kostochka and West confirmed Conjecture 1 for a $2l$ -regular graph which has a 2-factorization such that every cycle consisting of edges from distinct 2-factors has length greater than the diameter of T . El-Zanati *et al.* [3] verified Conjecture 1 when T is a double-star, and further they proved that the double-star $S_{k,k-1}$ can decompose every $2k$ -regular graph which contains a perfect matching.

It is natural to ask whether Conjecture 1 holds if T is a path. Unfortunately, there is no definitive answer for general graphs. Kouider and Lonc [9] proposed a strengthening of Conjecture 1 in the case where T is a path, and it is still open. We say a path decomposition \mathcal{D} is *balanced* if each vertex is a terminal of exactly two paths of \mathcal{D} .

Conjecture 2 (Kouider and Lonc [9]). *Let l be a positive integer. If G is a $2l$ -regular graph, then G admits a balanced P_l -decomposition.*

By Petersen's Factorization Theorem (see Theorem 3.1), Botler *et al.* [1] proposed an equivalent form of Conjecture 2.

Conjecture 3 (Botler *et al.* [1]). *Let m and l be positive integers. Then every $2ml$ -regular graph admits a balanced P_l -decomposition.*

In the same paper, they proved that if $m \geq \lfloor (l-2)/(g-2) \rfloor$, then every $2ml$ -regular graph with girth at least g admits a P_l -decomposition. Furthermore, every $2ml$ -regular graph with girth at least $l-1$ admits a P_l -decomposition for $m \geq 1$. By controlling the girth, Kouider and Lonc [9] confirmed Conjecture 2 for a $2l$ -regular graph G with girth at least $(l+3)/2$.

Theorem 4 (Kouider and Lonc [9]). *If $l \leq 2g-3$, then every $2l$ -regular graph G of girth g has a balanced P_l -decomposition.*

By Theorem 4, Conjecture 2 is true for $l = 1, 2$ and 3. When $l = 4$ or 5, every $2l$ -regular graph G without triangles has a balanced P_l -decomposition. For later use, we will present a short proof when $l = 3$ in Conjecture 2 in Section 3. Based on analysis of the structure of the graph, Botler and Talon [2] used a different method from that in [9] to confirm the conjecture for $l = 4$.

Theorem 5 (Botler and Talon [2]). *If G is an 8-regular graph, then G admits a balanced P_4 -decomposition.*

Motivated by Theorem 5, we want to solve the case $l = 5$. However, the structure of a P_5 -decomposition in a 10-regular graph is more complex than the structure of a P_4 -decomposition in an 8-regular graph. Thus we consider P_5 -decompositions of 10-regular graphs which contain no 4-cycles, and get the main result of this paper.

Theorem 6. *Let G be a 10-regular graph. If G does not contain any 4-cycles, then G admits a balanced P_5 -decomposition.*

2. NOTATIONS AND TERMINOLOGIES

A *trail* $T = x_0x_1 \cdots x_l$ is a graph for whose $V(T) = \{x_i \mid 0 \leq i \leq l\}$, $E(T) = \{x_ix_{i+1} \mid 0 \leq i \leq l-1\}$ and $x_ix_{i+1} \neq x_jx_{j+1}$, for every $i \neq j$. Denote the vertices x_0 and x_l as the *terminal* vertices of T , x_1 and x_{l-1} as the *preterminal* vertices of T . If a trail has l edges, then we call it an l -*trail*. If a set of edge-disjoint trails \mathcal{B} of a graph G is such that $\bigcup_{B \in \mathcal{B}} E(B) = E(G)$, then \mathcal{B} is a decomposition of G into trails. If every trail of \mathcal{B} has length l , then \mathcal{B} is a decomposition into l -trails (or an l -trail decomposition). For a trail decomposition \mathcal{B} of G , if every vertex of G is a terminal of exactly two trails of \mathcal{B} , then \mathcal{B} is called balanced. If every trail of \mathcal{B} is a path, then \mathcal{B} is a decomposition into paths (or a path decomposition). We use $\tau(\mathcal{B})$ to denote the number of elements of \mathcal{B} that are cycles.

A *tour* of a connected graph G is a closed walk that traverses each edge of G at least once, and an *Eulerian tour* one that traverses each edge exactly once. A graph is *Eulerian* if it admits an Eulerian tour. Since an Eulerian tour traverses each edge exactly once, $d(v)$ is even for every vertex $v \in V(G)$. On the other hand, if G is a connected graph and every vertex has even degree, then G has an Eulerian tour by Fleury's Algorithm. A graph in which each vertex has even degree is called an *even graph*. Therefore, a graph is Eulerian if and only if it is even and connected. An orientation O of a subset E' of $E(G)$ is an attribution of a direction to each edge of E' . If an edge xy is directed from x to y in O , we say that xy leaves x and enters y . For a vertex v of G , let $d_O^+(v)$ (respectively, $d_O^-(v)$) be the number of edges leaving (respectively, entering) v with respect to O . If O is an orientation of G and every vertex v has $d_O^+(v) = d_O^-(v)$, then O is an *Eulerian orientation* of G . It is easy to see that G is even if it has an Eulerian orientation. If G is even, then each of its components has an Eulerian tour. We can get an Eulerian orientation of G by assigning each edge of G an orientation in such a way that the Eulerian tour of each component of G is a directed Eulerian tour. Thus a graph has an Eulerian orientation if and only if it is even.

3. PROOF OF MAIN THEOREM

First of all, we will present Petersen's Factorization Theorem [10].

Theorem 7 (Petersen's 2-Factorization Theorem [10]). *Every $2k$ -regular graph admits a 2-factorization.*

Nextly, we will introduce a approach used in [2] to get a trail decomposition. By adjusting this decomposition, we finally get the desired result.

Let G be an r -regular graph ($r \geq 6$ and is even), \mathcal{F} be a 2-factorization of G given by Theorem 7. By combining the elements of \mathcal{F} , we obtain a decomposition of G into an $(r-4)$ -factor and a 4-factor, say F_1 and F_2 , respectively. Let O be an Eulerian orientation of F_2 . Suppose F_1 has a balanced $P_{(r-4)/2}$ -decomposition \mathcal{D} . So every vertex v of G is a terminal of exactly two paths in \mathcal{D} . Note that $d_O^+(v) = 2$ for every vertex v of F_2 . Thus, we can extend every path $P = x_1x_2 \cdots x_{(r-4)/2+1}$ in \mathcal{D} to a (\mathcal{D}, O) -extension $Q_P = x_0x_1 \cdots x_{(r-4)/2+2}$ such that x_0x_1 and $x_{(r-4)/2+1}x_{(r-4)/2+2}$ are two edges in F_2 leaving x_1 and $x_{(r-4)/2+1}$, respectively, and further every edge of F_2 is used exactly once. Therefore, $\{Q_P \mid P \in \mathcal{D}\}$ is a decomposition into (\mathcal{D}, O) -extensions of G , which may not be a decomposition into paths, just into trails. Obviously, each decomposition into (\mathcal{D}, O) -extensions is balanced.

In this paper, we focus on the path decompositions of a 10-regular graph which does not contain any 4-cycles. Let F_1 be a 6-factor of G , F_2 be a 4-factor such that $F_1 \cup F_2 = G$, O be an Eulerian orientation of F_2 . By Theorem 4, F_1 has a balanced P_3 -decomposition \mathcal{D} . Following the method above, we first obtain a decomposition into (\mathcal{D}, O) -extensions of G from \mathcal{D} , and then adjust this trail decomposition to a path decomposition of G .

Let G be a 6-regular graph. We present a brief proof that G has a balanced P_3 -decomposition.

Lemma 8. *If G is a 6-regular graph, then G admits a balanced P_3 -decomposition \mathcal{D} and every vertex of G is a preterminal of exactly two paths in \mathcal{D} .*

Proof. Let \mathcal{F} be a 2-factorization of G given by Theorem 7. By combining the elements of \mathcal{F} , we obtain a decomposition of G into a 2-factor and a 4-factor, say F_3 and F_4 , respectively. Obviously, F_3 has a balanced P_1 -decomposition, denoted by \mathcal{D}_1 . Because every vertex of F_4 has even degree, there is an Eulerian orientation O on F_4 . Let \mathcal{D} be a decomposition of G into (\mathcal{D}_1, O) -extensions which minimizes $\tau(\mathcal{D})$. If every element in \mathcal{D} is a P_3 , then we are done. Suppose there is a triangle $C = x_0x_1x_2x_3$ in \mathcal{D} , $x_0 = x_3$, $x_1x_2 \in \mathcal{D}_1$. There is an element $T = y_0y_1y_2y_3$ of \mathcal{D} such that $y_1 = x_1$, $y_1y_2 \in \mathcal{D}_1$, $y_1y_2 \neq x_1x_2$ and $T \neq C$. Let $C' = y_0x_1x_2x_3$ and $T' = x_0y_1y_2y_3$. Obviously, C' is a path of length 3. Because, G is simple, y_1 , y_2 and y_3 are distinct vertices, $x_0 \neq y_1$ and $x_0 \neq y_2$. If T'

is a triangle, then $y_3 = x_0 = x_3$ and $d_O^-(x_0) = 3$, which is a contradiction to the assumption before that O is an Eulerian orientation on F_4 . Hence, T' is a path of length 3. Let $\mathcal{D}' = (\mathcal{D} - \{T, C\}) \cup \{T', C'\}$. \mathcal{D}' is a decomposition of G into (\mathcal{D}_1, O) -extensions and $\tau(\mathcal{D}') \leq \tau(\mathcal{D}) - 1$, which is a contradiction to the minimality of $\tau(\mathcal{D})$. Therefore, \mathcal{D} is a balanced P_3 -decomposition of G . By the construction of \mathcal{D} , we can find that every vertex of G is a preterminal of exactly two paths in \mathcal{D} . ■

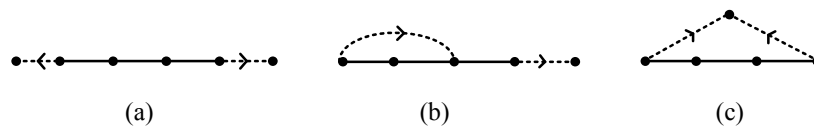


Figure 1. Extensions.

Now, let G be a 10-regular graph without a C_4 , F_1 be a 6-factor of G , F_2 be a 4-factor such that $F_1 \cup F_2 = G$. Let O be an Eulerian orientation of F_2 and \mathcal{D}_1 be a balanced P_3 -decomposition of F_1 , and further, $\mathcal{T} = \{Q_P \mid P \in \mathcal{D}_1\}$ be a decomposition into (\mathcal{D}_1, O) -extensions of G . Let $T = x_0x_1x_2x_3x_4x_5 \in \mathcal{T}$. Because \mathcal{D}_1 is a balanced P_3 -decomposition of F_1 and G does not contain any C_4 , we have x_1, x_2, x_3 and x_4 are distinct vertices, $x_0 \neq x_4$, $x_5 \neq x_1$ and it is impossible that both $x_0 = x_3$ and $x_5 = x_2$ hold. Hence, if T is a trail of \mathcal{T} , then exactly one of the following holds: (a) T is a path of length 5; (b) T is a trail of length 5 which contains a triangle; (c) T is a cycle of length 5 (see Figure 1). In the figures throughout this section, we illustrate the edges of F_1 as straight edges, and the edges of F_2 as dashed edges. The next result shows that every 10-regular graph admits a decomposition into (\mathcal{D}_1, O) -extensions which are not cycles.

Lemma 9. *Let G be a $2l$ -regular graph, F_1 be a $2(l-2)$ -factor of G , $F_2 = G \setminus E(F_1)$ and O be an Eulerian orientation of F_2 . If there is a balanced $P_{(l-2)}$ -decomposition \mathcal{D}_1 of F_1 , then G admits a decomposition into (\mathcal{D}_1, O) -extensions which are not cycles.*

Proof. Let $G, F_1, F_2, \mathcal{D}_1$, and O be as in the statement above. Now, let \mathcal{D} be a decomposition of G into (\mathcal{D}_1, O) -extensions which minimizes $\tau(\mathcal{D})$.

Suppose, for contradiction, that $\tau(\mathcal{D}) > 0$. Let $T = x_0x_1x_2 \cdots x_{l-1}x_l$ be a cycle of length l in \mathcal{D} , where $L_1 = x_1x_2 \cdots x_{l-1} \in \mathcal{D}_1$ and $x_0 = x_l$. Note that \mathcal{D}_1 is balanced. Let $L_2 = y_1y_2 \cdots y_{l-1}$ be the element of \mathcal{D}_1 such that $L_2 \neq L_1$ and $y_1 = x_1$. Suppose $Q = y_0y_1y_2 \cdots y_{l-1}y_l$ is the (\mathcal{D}_1, O) -extension of L_2 in \mathcal{D} . Let $T' = y_0x_1x_2 \cdots x_{l-1}x_l$ and $Q' = x_0y_1y_2 \cdots y_{l-1}y_l$. Clearly, T' and Q' are (\mathcal{D}_1, O) -extensions. Because G is simple, $y_0 \neq x_l$. Hence, T' is not a cycle. Moreover, if Q' is a cycle, then the edges $x_0x_1, x_{l-1}x_l$, and $y_{l-1}y_l$ are directed

towards x_0 , which implies $d_O^-(x_0) \geq 3$, contrary to the fact that O is an Eulerian orientation of F_2 . Therefore, $\mathcal{D}' = (\mathcal{D} - \{T, Q\}) \cup \{T', Q'\}$ is a decomposition into (\mathcal{D}_1, O) -extensions of G such that $\tau(\mathcal{D}') \leq \tau(\mathcal{D}) - 1$, which is a contradiction to the minimality of $\tau(\mathcal{D})$. This completes the proof of Lemma 9. ■

In the following, we will define a special Eulerian orientation, which is important for the proof of Theorem 6.

Definition 10. Let G be a 10-regular graph, F be a 6-factor of G , \mathcal{D} be a balanced P_3 -decomposition of F , $H = G \setminus E(F)$. We say that an Eulerian orientation O on H is good if the following holds. For each path $U = v_1v_2v_3$ of H and distinct vertices $x_2, x_3, y_2, y_4, z_2, z_4, v_1, v_2, v_3$, if there exists three elements $T_1 = x_1x_2x_3x_4, T_2 = y_1y_2y_3y_4, T_3 = z_1z_2z_3z_4 \in \mathcal{D}$ and $x_1 = v_1 = y_1, y_3 = v_2 = z_3, x_4 = v_3 = z_1$, then U is a directed path under orientation O , no matter which direction it goes (see Figure 2).

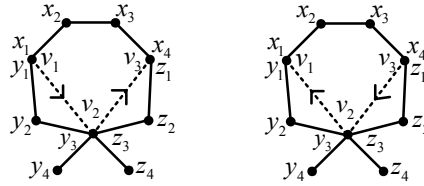


Figure 2. A good orientation on $U = v_1v_2v_3$.

Lemma 11. Let G be a 10-regular graph without C_4 , F be a 6-factor of G , and $H = G \setminus E(F)$. Then, there is a good Eulerian orientation on the edges of H .

Proof. By Lemma 8, we assume that \mathcal{D} is a balanced P_3 -decomposition of F such that every vertex of G is a preterminal of exactly two paths in \mathcal{D} . Let $U = v_1v_2v_3$ be a path of length 2 in H . Because G is simple and does not contain any 4-cycles, if there exists three elements $T_1 = x_1x_2x_3x_4, T_2 = y_1y_2y_3y_4, T_3 = z_1z_2z_3z_4 \in \mathcal{D}$ and $x_1 = v_1 = y_1, y_3 = v_2 = z_3, x_4 = v_3 = z_1$, then $x_2, x_3, y_2, y_4, z_2, z_4, v_1, v_2$ and v_3 are distinct vertices. Hence, U and T_1, T_2, T_3 form a structure defined in Definition 10. In order to obtain a good Eulerian orientation on H , we need to construct a new even graph H' from H .

Let $\mathcal{U} = \{v_1^i v_2^i v_3^i \mid 1 \leq i \leq k\}$ be the set of all the paths of length 2 in H which are contained in the structure defined in Definition 10. Note that these paths in \mathcal{U} are not necessarily edge-disjoint. We claim that $v_2^i \neq v_2^j$ for $U_i = v_1^i v_2^i v_3^i, U_j = v_1^j v_2^j v_3^j \in \mathcal{U}$ and $i \neq j$. If not, suppose that $v_2^i = v_2^j$. Without loss of generality, let U_i and three elements T_1, T_2, T_3 of \mathcal{D} be contained in the structure depicted in Definition 10 such that $v_1^i, v_2^i \in V(T_1), v_3^i, v_2^i \in V(T_2), v_1^i, v_3^i \in V(T_3)$. If $|E(U_i) \cap E(U_j)| = 1$, then without loss of generality let $v_1^i = v_1^j$. This implies that there

are two paths T_4 and T_5 of \mathcal{D} (because G is simple, $E(U_i), E(U_j) \subseteq E(H)$ and $E(T_m) \subseteq E(F)$ ($1 \leq m \leq 5$), $T_k \neq T_q, k \in \{4, 5\}, q \in \{1, 2, 3\}$) together with U_j and T_1 form another structure defined in Definition 10, such that $v_1^j, v_2^j \in V(T_1)$, $v_3^j, v_2^j \in V(T_4)$, $v_1^j, v_3^j \in V(T_5)$. If $|E(U_i) \cap E(U_j)| = 0$, then this implies that there are three paths T_4, T_5 and T_6 of \mathcal{D} (because G is simple, $E(U_i), E(U_j) \subseteq E(H)$ and $E(T_m) \subseteq E(F)$ ($1 \leq m \leq 6$), $T_k \neq T_q, k \in \{4, 5, 6\}, q \in \{1, 2, 3\}$) together with U_j form another structure defined in Definition 10, such that $v_1^j, v_2^j \in V(T_4)$, $v_3^j, v_2^j \in V(T_5)$, $v_1^j, v_3^j \in V(T_6)$. In the two cases, v_2^i appears in at least three paths in \mathcal{D} as their preterminal vertex, contrary to that v_2^i is the preterminal vertex of exactly two paths in \mathcal{D} . Thus, $v_2^i \neq v_2^j$ when $i \neq j$, as claimed. This means that for every vertex v of G , there is at most one $U_i \in \mathcal{U}$ such that edges incident with U_i is contained in the subgraph induced by $E_H(v)$ which is the set of v in H .

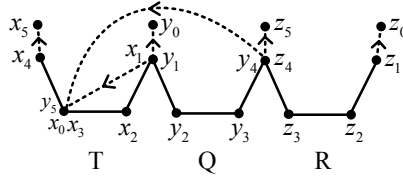
Now we can split edges of U_i in the following way: delete edges $v_1^i v_2^i$ and $v_2^i v_3^i$, add a new vertex z_i and two edges $v_1^i z_i$ and $z_i v_3^i$. By operating on all elements in \mathcal{U} as described above, we can get a new graph H' from H . Let O' be an Eulerian orientation on H' . By identifying z_i and v_2^i ($1 \leq i \leq k$) in H' and preserving the orientation of O' on all edges after identifying, we get an Eulerian orientation O on H . It is obviously that O is good. ■

Now we are able to prove Theorem 6. For a 5-trail decomposition \mathcal{B} of a 10-regular graph G , we use $\tau'(\mathcal{B})$ to denote the number of elements of \mathcal{B} that are paths.

Proof of Theorem 6. Let G be a 10-regular graph without C_4 , F be a 6-factor of G , \mathcal{D} be a balanced P_3 -decomposition of F , $H = G \setminus E(F)$, and O be a good Eulerian orientation of H . By Lemma 9, G has a decomposition \mathcal{B} into (\mathcal{D}, O) -extensions which are not cycles. Further, we may assume that $\tau'(\mathcal{B})$ is maximum. If $\tau'(\mathcal{B}) = |\mathcal{B}|$, then we are done. Suppose that $\tau'(\mathcal{B}) < |\mathcal{B}|$. Let $T \in \mathcal{B}$ be a trail containing a triangle.

Let $T = x_0 x_1 x_2 x_3 x_4 x_5$, where $x_1 x_2 x_3 x_4 \in \mathcal{D}$, $x_0 = x_3$. There is a trail $Q = y_0 y_1 y_2 y_3 y_4 y_5 \in \mathcal{B}$ with $Q \neq T$ such that $y_1 y_2 y_3 y_4 \in \mathcal{D}$ and $y_1 = x_1$. We put $T' = y_0 x_1 x_2 x_3 x_4 x_5$, $Q' = x_0 y_1 y_2 y_3 y_4 y_5$. Because G is simple and does not contain C_4 , $y_0 \notin V(T)$, which implies that T' is a path. Moreover, $x_0 \neq y_3$ which follows from the fact that G does not contain C_4 . Hence, if Q' contains a triangle only if Q contains the triangle $y_2 y_3 y_4 y_5$. If Q' is not a cycle, then $\mathcal{B}' = (\mathcal{B} - \{T, Q\}) \cup \{T', Q'\}$ is a decomposition of G into (\mathcal{D}, O) -extensions with $\tau'(\mathcal{B}') \geq \tau'(\mathcal{B}) + 1$, which is a contradiction to the maximality of $\tau'(\mathcal{B})$. In the following, we assume Q' is a cycle.

Now, $y_5 = x_0 = x_3$. Note that G is simple and does not contain any 4-cycles. We have that $y_1 \neq y_4$, y_2 and y_3 are not equal to any one of $\{x_1, x_2, x_3, x_4, x_5\}$, y_4 is not equal to any one of $\{x_1, x_2, x_3, x_4\}$. In this case, y_4 and x_5 may be the same one. Let $R = z_0 z_1 z_2 z_3 z_4 z_5$ be an element in \mathcal{B} different from T and



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