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ROMAN {2}-DOMINATION PROBLEM IN GRAPHS[1]

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Abstract

For a graph $G = (V, E)$, a Roman $\{2\}$-dominating function (R2DF) $f : V \to \{0, 1, 2\}$ has the property that for every vertex $v \in V$ with $f(v) = 0$, either there exists a neighbor $u \in N(v)$, with $f(u) = 2$, or at least two neighbors $x, y \in N(v)$ having $f(x) = f(y) = 1$. The weight of an R2DF f is the sum $f(V) = \sum_{v \in V} f(v)$, and the minimum weight of an R2DF on G is the Roman $\{2\}$-domination number $\gamma_{\{R2\}}(G)$. An R2DF is independent if the set of vertices having positive function values is an independent set. The independent Roman $\{2\}$-domination number $i_{\{R2\}}(G)$ is the minimum weight of an independent Roman $\{2\}$-dominating function on G. In this paper, we show that the decision problem associated with $\gamma_{\{R2\}}(G)$ is NP-complete even when restricted to split graphs. We design a linear time algorithm for computing the value of $i_{\{R2\}}(T)$ in any tree T, which answers an open problem raised by Rahmouni and Chellali [*Independent Roman* {2}-*domination in graphs*, Discrete Appl. Math. 236 (2018) 408–414]. Moreover, we present a linear time algorithm for computing the value of $\gamma_{\{R2\}}(G)$ in any block graph G, which is a generalization of trees.

Keywords: Roman {2}-domination, domination, algorithms.

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1. Introduction

Let $G = (V, E)$ be a simple graph. The *open neighborhood* $N_G(v)$ of a vertex v consists of the vertices adjacent to v and its *closed neighborhood* is $N_G[v] = N_G(v) \cup \{v\}$. $N_G^2[v] = \{u : d_G(u, v) \leq 2\}$, where $d_G(u, v)$ is the distance between u and v in graph G. For an edge $e = uv$, it is said that u (respectively, v) is incident to e, denoted by $u \in e$ (respectively, $v \in e$). A *Roman dominating function* (RDF) on graph G is a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The *weight of a Roman dominating function* f is the value $f(V) = \sum_{v \in V} f(v)$. The minimum weight of a Roman dominating function on a graph G is called the *Roman domination number* $\gamma_R(G)$ of G. Roman domination and its variations have been studied in a number of recent papers (see, for example, [1, 6, 9]).

Chellali, Haynes, Hedetniemi and McRae [4] introduced a variant of Roman dominating functions. For a graph $G = (V, E)$, a *Roman* $\{2\}$*-dominating function* (R2DF) $f : V \to \{0, 1, 2\}$ has the slightly different property that only for every vertex $v \in V$ with $f(v) = 0$, $f(N(v)) \geq 2$, that is, either there exists a neighbor $u \in N(v)$, with $f(u) = 2$, or at least two neighbors $x, y \in N(u)$ have $f(x) = f(y) = 1$. The *weight of a Roman* $\{2\}$*-dominating function* is the sum $f(V) = \sum_{v \in V} f(v)$, and the minimum weight of a Roman $\{2\}$-dominating function f is the *Roman* $\{2\}$*-domination number*, denoted $\gamma_{\{R2\}}(G)$. Roman $\{2\}$-domination is also called *Italian domination* by some scholars ([8]). Suppose that $f : V \to \{0, 1, 2\}$ is an R2DF on a graph $G = (V, E)$. Let $V_i = \{v : f(v) = i\}$, for $i \in \{0, 1, 2\}$. If $V_1 \cup V_2$ is an independent set, then f is called an *independent Roman* $\{2\}$*-dominating function* (IR2DF), which was introduced by Rahmouni and Chellali [11] in a recent paper. The minimum weight of an independent Roman $\{2\}$-dominating function f is the *independent Roman* $\{2\}$*-domination number*, denoted $i_{\{R2\}}(G)$. The authors in [4, 11] have showed that the associated decision problems for Roman $\{2\}$-domination and independent Roman $\{2\}$-domination are NP-complete for bipartite graphs. The authors in [4] have showed that $\gamma_{\{R2\}}(T)$ can be computed by a linear time algorithm for any tree T. In [11], the authors raised some interesting open problems, one of which is whether there is a linear time algorithm for computing $i_{\{R2\}}(T)$ for any tree T.

A graph $G = (V, E)$ is a *split graph* if V can be partitioned into C and I, where C is a clique and I is an independent set of G. Split graph is an important subclass of chordal graphs, and it turns out to be very important in the domination theory (see [2, 7]). A maximal connected induced subgraph without a cut-vertex is called a *block* of G. We use K_n to denote the complete graph of order n. A graph G is a *block graph* if every block in G is a complete graph. If every block of G is a K_2, then G is a tree. Hence, block graphs contain trees as its subclass. There are widely research on variations of domination in block graphs (see, for example, [3, 5, 10, 14]).

In this paper, we first show that the decision problem associated with $\gamma_{\{R2\}}(G)$ is NP-complete for split graphs. Then, we give a linear time algorithm for computing $i_{\{R2\}}(T)$ in any tree T. Moreover, we present a linear time algorithm for computing $\gamma_{\{R2\}}(G)$ in any block graph G.

2. Complexity Result

In this section, we consider the decision problem associated with Roman {2}-dominating functions.

ROMAN {2}-DOMINATING FUNCTION (R2D)

INSTANCE: A graph $G = (V, E)$ and a positive integer $k \leq |V|$.

QUESTION: Does G have a Roman {2}-dominating function of weight at most k?

A *vertex cover* of G is a subset $V' \subseteq V$ such that for each edge $uv \in E$, at least one of u and v belongs to V'. Vertex Cover (VC) problem is a well-known NP-complete problem. We show R2D problem is NP-complete by reducing the Vertex Cover (VC) to R2D.

VERTEX COVER (VC)

INSTANCE: A graph $G = (V, E)$ and a positive integer $k \leq |V|$.

QUESTION: Is there a vertex cover of size k or less for G?

Theorem 1. R2D *is NP-complete for split graphs.*

Proof. R2D is a member of NP, since we can check in polynomial time that a function $f : V \to \{0, 1, 2\}$ has weight at most k and is a Roman {2}-dominating function. The proof is given by reducing the VC problem in general graphs to the R2D problem in split graphs.

Let $G = (V, E)$ be a graph with $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$. Let $V^1 = \{v'_1, v'_2, \ldots, v'_n\}$. We construct the graph $G' = (V', E')$ with $V' = V^1 \cup V \cup E$, $E' = \{v_iv_j : v_i \neq v_j, v_i \in V, v_j \in V\} \cup \{v_iv'_i : i = 1, ..., n\} \cup \{ve : v \in e, e \in E\}$.

Notice that G' is a split graph whose vertex set V' is the disjoint union of the clique V and the independent set $V^1 \cup E$. It is clear that G' can be constructed in polynomial time from G.

If G has a vertex cover C of size at most k, let $f : V' \to \{0, 1, 2\}$ be a function defined as follows.

$$f(v) = \begin{cases} 2, & \text{if } v \in C, \\ 1, & \text{if } v \in V^1 \text{ and let } v' \text{ be a neighbor of } v \text{ such that } v' \in V \setminus C, \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that f is a Roman $\{2\}$-dominating function of G' with weight at most $2k + (n - k)$.

On the other hand, suppose that G' has a Roman $\{2\}$-dominating function of weight at most $2k + (n - k)$. Among all such functions, let $g = (V_0, V_1, V_2)$ be one chosen so that:

(C1) $|V^1 \cap V_2|$ is minimized;

(C2) subject to condition (C1): $|E \cap V_0|$ is maximized;

(C3) subject to conditions (C1) and (C2): $|V \cap V_1|$ is minimized;

(C4) subject to conditions (C1), (C2) and (C3): the weight of g is minimized.

We make the following remarks.

(i) No vertex in V^1 belongs to V_2. Indeed, suppose to the contrary that $g(v_i') = 2$ for some i. We reassign 0 to v_i' instead of 2 and reassign 2 to v_i. Then it provides an R2DF on G' of weight at most $2k + (n - k)$ but with less vertices of V^1 assigned 2, contradicting the condition (C1) in the choice of g.

(ii) No vertex in E belongs to V_2. Indeed, suppose that $g(e) = 2$ for some $e \in E$ and $v_j, v_k \in e$. By reassigning 0 to e instead of 2 and reassigning 2 to v_j instead of $g(v_j)$, we obtain an R2DF on G' of weight at most $2k + (n - k)$ but with more vertices of E assigned 0, contradicting the condition (C2) in the choice of g.

(iii) No vertex in E belongs to V_1. Suppose that $g(e) = 1$ for some $e \in E$ and $v_j, v_k \in e$. If $g(v_j') = 0$, then $g(v_j) = 2$ (by the definition of R2DF). By reassigning 0 to e instead of 1, we obtain an R2DF on G' of weight at most $2k + (n - k)$ but with more vertices of E assigned 0, contradicting the condition (C2) in the choice of g. Hence we may assume that $g(v_j') = 1$ (by (i)). Clearly we can reassign 2 to v_j instead of 0, 0 to v_j' instead of 1 and 0 to e instead of 1. We also obtain a R2DF on G' of weight at most $2k + (n - k)$ but with more vertices of E assigned 0, contradicting the condition (C2) in the choice of g.

(iv) No vertex in V belongs to V_1. Suppose to the contrary that $g(v_i) = 1$ for some i, then $g(v_i') = 1$ (by (i) and the definition of R2DF). We reassign 0 to v_i' instead of 1 and 2 to v_i instead of 1. It provides a R2DF on G' of weight at most

$2k + (n - k)$ but with less vertices of V assigned 1, contradicting the condition (C3) in the choice of g.

(v) If a vertex in V is assigned 2, then its neighbor in V^1 is assigned 0 by the condition (C4) in the choice of g.

(vi) If a vertex in V is assigned 0, then its neighbor in V^1 is assigned 1 by the definition of R2DF and (i).

Therefore, according to the previous items, we conclude that $V^1 \cap V_2 = \emptyset$, $E \subseteq V_0$, and $V \cap V_1 = \emptyset$. Hence $V_2 \subseteq V$. Let $C = \{v : g(v) = 2\}$. Since each vertex in $E \cup (V \setminus C)$ belongs to V_0 in G', it is clear that C is a vertex cover of G by the definition of R2DF. Then $g(V^1)+g(V)+g(E) = 2|C|+(n-|C|) \leq 2k+(n-k)$, implying that $|C| \leq k$. Consequently, C is a vertex cover for G of size at most k.

Since the vertex cover problem is NP-complete, the Roman {2}-domination problem is NP-complete for split graphs. ■

3. Independent Roman {2}-Domination in Trees

In this section, a linear time dynamic programming style algorithm is given to compute the exact value of the independent Roman {2}-dominating number in any tree. This algorithm is constructed using the methodology of Wimer [13].

A *rooted tree* is a pair (T, r) with T is a tree and r is a vertex of T. We call r is the root of tree T. A rooted tree (T, r) is trivial if $V(T) = r$. Given two rooted trees (T_1, r_1) and (T_2, r_2) with $V(T_1) \cap V(T_2) = \emptyset$, the *composition* of them is $(T_1, r_1) \circ (T_2, r_2) = (T, r_1)$ with $V(T) = V(T_1) \cup V(T_2)$ and $E(T) = E(T_1) \cup E(T_2) \cup \{r_1r_2\}$. It is clear that any rooted tree can be constructed recursively from trivial rooted trees using the defined composition.

Let $f : V(T) \to \{0, 1, 2\}$ be a function on T. Then f splits two functions f_1 and f_2 according to this decomposition. We express this as follows: $(T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2)$, where $r = r_1$, $f_i = f|_{T_i}$ is the function f restricted to the vertices of T_i, $i = 1, 2$. On the other hand, let $f_i : V(T_i) \to \{0, 1, 2\}$ be a function on T_i $(i = 1, 2)$. We can define the composition as follows: $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) = (T, f, r)$, where $V(T) = V(T_1) \cup V(T_2)$, $E(T) = E(T_1) \cup E(T_2) \cup \{r_1r_2\}$, $r = r_1$ and $f = f_1 \circ f_2 : V(T) \to \{0, 1, 2\}$ with $f(v) = f_i(v)$ if $v \in V(T_i)$, $i = 1, 2$. Before presenting the algorithm, let us give the following observation.

Observation 2. *Let f be an* IR2DF *of $T = T_1 \circ T_2$ and $f_i = f|_{T_i}$ $(i = 1, 2)$. If $f_i(r_i) \neq 0$, then f_i is an* IR2DF *of T_i. If $f_i(r_i) = 0$, then f_i restricted to the vertices of $T_i - r_i$ is an* IR2DF *of $T_i - r_i$.*

In order to construct an algorithm for computing the independent Roman {2}-domination number, we must characterize the possible tree-subset tuples (T, f, r). For this purpose, we introduce some additional notations as follows:

$\mathrm{IR2DF}(T) = \{f \ : \ f \text{ is an IR2DF of } T\}$,
$\mathrm{IR2DF}_r(T) = \{f \ : \ f \notin \mathrm{IR2DF}(T), \text{ but } f|_{T-r} \text{ is an IR2DF of } T - r\}$.

Then we consider the following five classes:

$A = \{(T, f, r) \ : \ f \in \mathrm{IR2DF}(T) \text{ and } f(r) = 2\}$,
$B = \{(T, f, r) \ : \ f \in \mathrm{IR2DF}(T) \text{ and } f(r) = 1\}$,
$C = \{(T, f, r) \ : \ f \in \mathrm{IR2DF}(T) \text{ and } f(r) = 0\}$,
$D = \{(T, f, r) \ : \ f \in \mathrm{IR2DF}_r(T) \text{ and } f(N[r]) = 1\}$,
$F = \{(T, f, r) \ : \ f \in \mathrm{IR2DF}_r(T) \text{ and } f(N[r]) = 0\}$.

Let $M, N \in \{A, B, C, D, F\}$. If $(T_1, f_1, r_1) \in M$ and $(T_2, f_2, r_2) \in N$, we use $M \circ N$ to denote the set of $(T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2)$. Let $(T, r) = (T_1, r_1) \circ (T_2, r_2)$ and $r = r_1$. Suppose that f_1 (respectively, f_2) is a function on T_1 (respectively, T_2). Define f as the function on T with $f|_{T_1} = f_1$ and $f|_{T_2} = f_2$. Next, we provide some lemmas.

Lemma 3. $A = (A \circ C) \cup (A \circ D) \cup (A \circ F)$.

Proof. It is clear that the following items are true.

(i) If $(T_1, f_1, r_1) \in A$ and $(T_2, f_2, r_2) \in C$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in A$.

(ii) If $(T_1, f_1, r_1) \in A$ and $(T_2, f_2, r_2) \in D$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in A$.

(iii) If $(T_1, f_1, r_1) \in A$ and $(T_2, f_2, r_2) \in F$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in A$.

Thus, $(A \circ C) \cup (A \circ D) \cup (A \circ F) \subseteq A$.

Now we prove that $A \subseteq (A \circ C) \cup (A \circ D) \cup (A \circ F)$. Let $(T, f, r) \in A$ and $(T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2)$, then $f_1(r_1) = f(r) = 2$. Since $f \in \mathrm{IR2DF}(T)$, then $f_1 \in \mathrm{IR2DF}(T_1)$. So $(T_1, f_1, r_1) \in A$. From the independence of $V_1 \cup V_2$, we have $f_2(r_2) = f(r_2) = 0$. If $f_2 \in \mathrm{IR2DF}(T_2)$, then we obtain $(T_2, f_2, r_2) \in C$. If $f_2 \notin \mathrm{IR2DF}(T_2)$, then $(T_2, f_2, r_2) \in D$ or F. Hence, we conclude that $A \subseteq (A \circ C) \cup (A \circ D) \cup (A \circ F)$. ■

Lemma 4. $B = (B \circ C) \cup (B \circ D)$.

Proof. It is easy to check the following items.

(i) If $(T_1, f_1, r_1) \in B$ and $(T_2, f_2, r_2) \in C$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B$.

(ii) If $(T_1, f_1, r_1) \in B$ and $(T_2, f_2, r_2) \in D$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B$.

So, $(B \circ C) \cup (B \circ D) \subseteq B$.

Next we need to show $B \subseteq (B \circ C) \cup (B \circ D)$. Let $(T, f, r) \in B$ and $(T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2)$, then $f_1(r_1) = f(r) = 1$. It is clear that $f_1 \in \mathrm{IR2DF}(T_1)$. So we conclude that $(T_1, f_1, r_1) \in B$. From the definition of IR2DF, we must have $f_2(r_2) = f(r_2) = 0$. If $f_2 \in \mathrm{IR2DF}(T_2)$, then we obtain $(T_2, f_2, r_2) \in C$. If $f_2 \notin \mathrm{IR2DF}(T_2)$, then $f_2(N_{T_2}[r_2]) = 1$ and $f_2|_{T_2 - r_2} \in \mathrm{IR2DF}(T_2 - r_2)$ using the fact that $(T, f, r) \in B$. Therefore, we have $f_2 \in \mathrm{IR2DF}_{r_2}(T_2)$, implying that $(T_2, f_2, r_2) \in D$. Hence, we deduce that $B \subseteq (B \circ C) \cup (B \circ D)$. ■

Lemma 5. $C = (C \circ A) \cup (C \circ B) \cup (C \circ C) \cup (D \circ A) \cup (D \circ B) \cup (F \circ A)$.

Proof. It is easy to check the following remarks by definitions.

(i) If $(T_1, f_1, r_1) \in C$ and $(T_2, f_2, r_2) \in A$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in C$.

(ii) If $(T_1, f_1, r_1) \in C$ and $(T_2, f_2, r_2) \in B$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in C$.

(iii) If $(T_1, f_1, r_1) \in C$ and $(T_2, f_2, r_2) \in C$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in C$.

(iv) If $(T_1, f_1, r_1) \in D$ and $(T_2, f_2, r_2) \in A$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in C$.

(v) If $(T_1, f_1, r_1) \in D$ and $(T_2, f_2, r_2) \in B$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in C$.

(vi) If $(T_1, f_1, r_1) \in F$ and $(T_2, f_2, r_2) \in A$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in C$.

Hence, we deduce that $(C \circ A) \cup (C \circ B) \cup (C \circ C) \cup (D \circ A) \cup (D \circ B) \cup (F \circ A) \subseteq C$.

Therefore, we need to prove $C \subseteq (C \circ A) \cup (C \circ B) \cup (C \circ C) \cup (D \circ A) \cup (D \circ B) \cup (F \circ A)$. Let $(T, f, r) \in C$ and $(T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2)$, then $f \in$ IR2DF(T) and $f_1(r_1) = f(r) = 0$. Consider the following cases.

Case 1. $f(r_2) = 2$. Since $f \in$ IR2DF(T), $f_2 \in$ IR2DF(T_2). Hence, $(T_2, f_2, r_2) \in A$. If $f_1 \in$ IR2DF(T_1), then we obtain that $(T_1, f_1, r_1) \in C$. If $f_1 \notin$ IR2DF(T_1), we have $(T_1, f_1, r_1) \in D$ or F.

Case 2. $f(r_2) = 1$. Since $f \in$ IR2DF(T), $f_2 \in$ IR2DF(T_2). So $(T_2, f_2, r_2) \in B$. If $f_1 \in$ IR2DF(T_1), then we deduce $(T_1, f_1, r_1) \in C$. If $f_1 \notin$ IR2DF(T_1), therefore, it implies that $(T_1, f_1, r_1) \in D$.

Case 3. $f(r_2) = 0$. It is clear that f_1 and f_2 are both IR2DF. Then we obtain that $(T_1, f_1, r_1) \in C$ and $(T_2, f_2, r_2) \in C$.

Hence, $C \subseteq (C \circ A) \cup (C \circ B) \cup (C \circ C) \cup (D \circ A) \cup (D \circ B) \cup (F \circ A)$. ■

Lemma 6. $D = (D \circ C) \cup (F \circ B)$.

Proof. It is easy to check the following remarks by definitions.

(i) If $(T_1, f_1, r_1) \in D$ and $(T_2, f_2, r_2) \in C$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in D$.

(ii) If $(T_1, f_1, r_1) \in F$ and $(T_2, f_2, r_2) \in B$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in D$.

Thus, $(D \circ C) \cup (F \circ B) \subseteq D$.

On the other hand, we show $D \subseteq (D \circ C) \cup (F \circ B)$. Let $(T, f, r) \in D$ and $(T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2)$. Then $f_1(r_1) = f(r) = 0$. By the definition of D, $f_2 \in$ IR2DF(T_2). Using the fact that $f(N_T[r_1]) = 1$, we deduce that $f(r_2) < 2$. Consider the following cases.

Case 1. $f(r_2) = 1$. It is clear that $(T_2, f_2, r_2) \in B$ because f_2 is an IR2DF of T_2. Since $f_1(N_{T_1}[r_1]) = 0$, we obtain $f_1|_{T_1 - r_1} \in$ IR2DF$(T_1 - r_1)$. Hence, we have $f_1 \in$ IR2DF$_{r_1}(T_1)$, implying that $(T_1, f_1, r_1) \in F$.

Case 2. $f(r_2) = 0$. Then f_2 is an IR2DF of T_2, implying that $(T_2, f_2, r_2) \in C$. Using the fact that $f(N_T[r_1]) = 1$ and $f(r_2) = 0$, we know $f_1(N_{T_1}[r_1]) = 1$. So $f_1 \in$ IR2DF$_{r_1}(T_1)$. It implies that $(T_1, f_1, r_1) \in D$. ■

Lemma 7. $F = F \circ C$.

Proof. If $(T_1, f_1, r_1) \in F$ and $(T_2, f_2, r_2) \in C$, then it is clear that $(T, f, r) \in F$. Hence, $(F \circ C) \subseteq F$.

On the other hand, let $(T, f, r) \in F$ and $(T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2)$. Then $f_1(r_1) = f(r) = 0$. By the definition of F, we deduce that $f(r_2) = 0$. Using the fact that $(T, f, r) \in F$, we have that $f_2 \in \text{IR2DF}(T_2)$. So $(T_2, f_2, r_2) \in C$. Notice that $(T, f, r) \in F$, we have $f_1(N_{T_1}[r_1]) = 0$, implying that $(T_1, f_1, r_1) \notin D$. We can easily check that $f_1 \in \text{IR2DF}_{r_1}(T_1)$. Hence, we have $(T_1, f_1, r_1) \in F$, implying that $F \subseteq (F \circ C)$. ∎

Let $T = (V, E)$ be a tree with n vertices. It is well known that the vertices of T have an ordering $v_1, v_2, \ldots, v_n$ such that for each $1 \leq i \leq n-1$, v_i is adjacent to exactly one vertex v_j with $j > i$ (see [12]). The ordering is called a *tree ordering* where the only neighbor v_j with $j > i$ is called the *father* of v_i and v_i is a *child* of v_j. For each $1 \leq i \leq n-1$, the father of v_i is denoted by $F(v_i) = v_j$.

For each vertex v_i ($1 \leq i \leq n$), define a vector $l[i, 1..5]$. Let T_{v_i} be a tree such that v_i is the root of T_{v_i}. For each rooted tree (T_{v_i}, v_i), let $f_{v_i} : V(T_{v_i}) \to \{0, 1, 2\}$ be a function on T_{v_i} and define $w(f_{v_i}) = f_{v_i}(V(T_{v_i}))$. In this case, for a tree, the only basis graph is a single vertex. Then, the vector $l[i, 1..5]$ is initialized by

$$[\min_{(T_{v_i}, f_{v_i}, v_i) \in A} w(f_{v_i}), \min_{(T_{v_i}, f_{v_i}, v_i) \in B} w(f_{v_i}), \min_{(T_{v_i}, f_{v_i}, v_i) \in C} w(f_{v_i}), \min_{(T_{v_i}, f_{v_i}, v_i) \in D} w(f_{v_i}), \min_{(T_{v_i}, f_{v_i}, v_i) \in F} w(f_{v_i})].$$

It means $l[i, 1..5] = [2, 1, \infty, \infty, 0]$, where $'\infty'$ means undefined. Now, we are ready to present the algorithm.

Algorithm 1: INDEPENDENT-ROMAN {2}-DOM-IN-TREE

Input: A tree $T = (V, E)$ with a tree ordering $v_1, v_2, \cdots, v_n$.
Output: The independent Roman {2}-domination number $i_{\{R2\}}(T)$.

if $T = K_1$ **then**
  return $i_{\{R2\}}(T) = 1$;
for $i := 1$ **to** n **do**
  initialize $l[i, 1..5]$ to $[2, 1, \infty, \infty, 0]$;
for $j := 1$ **to** $n - 1$ **do**
  $v_k = F(v_j)$;
  $l[k, 1] = \min\{l[k, 1] + l[j, 3], l[k, 1] + l[j, 4], l[k, 1] + l[j, 5]\}$;
  $l[k, 2] = \min\{l[k, 2] + l[j, 3], l[k, 2] + l[j, 4]\}$;
  $l[k, 3] = \min\{l[k, 3] + l[j, 1], l[k, 3] + l[j, 2], l[k, 3] + l[j, 3], l[k, 4] + l[j, 1],$
    $l[k, 4] + l[j, 2], l[k, 5] + l[j, 1]\}$;
  $l[k, 4] = \min\{l[k, 4] + l[j, 3], l[k, 5] + l[j, 2]\}$;
  $l[k, 5] = \min\{l[k, 5] + l[j, 3]\}$;
return $i_{\{R2\}}(T) = \min\{l[n, 1], l[n, 2], l[n, 3]\}$;

From the above argument, we can obtain the following theorem.

Theorem 8. *Algorithm INDEPENDENT-ROMAN* $\{2\}$*-DOM-IN-TREE can output the independent Roman* $\{2\}$*-domination number of any tree* $T = (V, E)$ *in linear time* $O(n)$*, where* $n = |V|$*.*

Proof. It is clear that the running time of Algorithm 1 is linear. We only need to show $i_{\{R2\}}(T) = \min\{l[n, 1], l[n, 2], l[n, 3]\}$. Suppose that $f \in \mathrm{IR2DF}(T)$. Then, $(T, f, r) \in A \cup B \cup C$. By the Algorithm 1 and Lemmas 3–7, we have $l[n, 1] = \min_{(T,f,r)\in A} f(V)$, $l[n, 2] = \min_{(T,f,r)\in B} f(V)$, and $l[n, 3] = \min_{(T,f,r)\in C} f(V)$. By the definition of $i_{\{R2\}}(T)$, we deduce that

$$i_{\{R2\}}(T) = \min_{(T,f,r)\in A\cup B\cup C} f(V) = \min\{l[n, 1], l[n, 2], l[n, 3]\}.$$

■

4. Roman {2}-Domination in Block Graph

Let $G(\ncong K_n)$ be a connected block graph. The *block-cutpoint graph* of G is a bipartite graph $T_G = (C \cup B, E)$ in which one partite set C consists of the cut-vertices of G, and the other B has a vertex h for each block H of G. Let $v \in C$ and $h \in B$. We include vh as an edge of T_G if and only if v is in H, where H is the block of G represented by h. Obviously, T_G is a tree and can be constructed from G in linear time (see [12]). In this section, we call each vertex in C a *C-vertex* and each vertex in B a *B-vertex*.

Let H be a block of G. Suppose that $S = \{v : v \in H \text{ and } v \text{ is a cut-vertex of } G\}$. We say H is a block of *type* 0 if $|H| = |S|$ and H is a block of *type* 1 if $|H| = |S| + 1$. If $|H| \geq |S| + 2$, we say H is a block of *type* 2. Let $f : V(G) \to \{0, 1, 2\}$ be a function of a block graph $G(\ncong K_n)$. $f_* : V(T_G) \to \mathbb{Z}$ is defined as follows:

$$f_*(v) = \begin{cases} f(v), & \text{if } v \text{ is a } C\text{-vertex}, \\ f(H) - f(S), & \text{if } v \text{ is a } B\text{-vertex } \textit{representing the block } H. \end{cases}$$

We say that f_* is the function induced by f. Now we present a key result on the relationship between f and f_*.

Theorem 9. *Let* $f : V(G) \to \{0, 1, 2\}$ *be a function of a connected block graph* G $(G \ncong K_n)$ *and* f_* *be the function induced by* f*. Then,* f *satisfies the following properties:*

(1) $f(v) = 0$ *or* 1 *if* $v \in H$ *is not a cut-vertex of* G*, where* H *is a block of type* 1 *of* G*.*

(2) *$f(v) = 0$ if $v \in H$ is not a cut-vertex of G, where H is a block of type* 2 *of G.*

(3) *f is an* R2DF *of G.*

if and only if f_* satisfies the following properties:

(a) *$f_*(v) = 0$ or* 1 *if v is a B-vertex and the block H represented by v is type* 1.

(b) *$f_*(v) = 0$ if v is a B-vertex and the block H represented by v is not type* 1.

(c) *If v is a C-vertex with $f_*(v) = 0$, then there exists either $u \in N^2_{T_G}(v)$ with $f_*(u) = 2$ or $u_1, u_2 \in N^2_{T_G}(v)$ with $f_*(u_1) = f_*(u_2) = 1$.*

(d) *If v is a B-vertex with $f_*(v) = 0$ and the block H represented by v is not type* 0*, then there exists either $u \in N_{T_G}(v)$ with $f_*(u) = 2$ or $u_1, u_2 \in N_{T_G}(v)$ with $f_*(u_1) = f_*(u_2) = 1$.*

Proof. If f satisfies the above properties, it is clear that f_* satisfies the above items (a), (b). Suppose that v is a C-vertex with $f_*(v) = 0$. By the definition of f_*, $f(v) = 0$. If there exists a vertex $u \in N_G(v)$ with $f(u) = 2$, then u is a cut-vertex of G, and hence $u \in N^2_{T_G}[v]$ with $f_*(u) = 2$. Otherwise, there exists at least two vertices $x, y \in N_G(v)$ having $f(x) = f(y) = 1$. If x and y are both cut-vertices of G, then we obtain $x, y \in N^2_{T_G}[v]$ having $f_*(x) = f_*(y) = 1$. If x is not a cut-vertex of G and H is the block containing x, we deduce that H is type 1 by the second property of f. It implies that $f_*(h) = 1$ and $vh \in E(T_G)$, where h is the B-vertex representing the block H. In this case, f_* also satisfies item (c). Suppose that v is a B-vertex with $f_*(v) = 0$ and the block H represented by v is not type 0. Let $S = \{u : u \in H \text{ and } u \text{ is a cut-vertex of } G\}$. By the definition of f_*, we know that $f(x) = 0$ for each $x \in H \setminus S$. Since f is an R2DF of G, then there exists either $u \in N_G(v)$ with $f(u) = 2$ or $u_1, u_2 \in N_G(v)$ such that $f(u_1) = f(u_2) = 1$. It is clear that u, u_1, u_2 are cut-vertices. It means that $f_*(u) = 2$ and $f_*(u_1) = f_*(u_2) = 1$. So f_* satisfies item (d).

On the other hand, if f_* satisfies the above properties, by the definition of f_*, it is easy to know that f satisfies items (1) and (2).

We now need to show that f is an R2DF of G. Suppose that v is a cut-vertex with $f(v) = 0$. Hence, $f_*(v) = f(v) = 0$. If there exists $u \in N^2_{T_G}[v]$ such that $f_*(u) = 2$, we deduce that u is a cut-vertex of G, $f(u) = 2$ and $u \in N_G(v)$. Otherwise, there exists $h_1, h_2 \in N^2_{T_G}[v]$ such that $f_*(h_1) = f_*(h_2) = 1$. If h_1 and h_2 are both C-vertex, then we have $h_1, h_2 \in N_G(v)$ and $f(h_1) = f(h_2) = 1$. If h_1 is a B-vertex and h_1 represent block H_1 in T_G. We deduce that H_1 is a block of type 1. Hence, there exists $v_1 \in H_1$ and v_1 is not a cut-vertex of G such that $f(v_1) = f_*(h_1) = 1$. Therefore, we obtain $f(N(v)) \geq 2$. Suppose that H is a block containing v and v is not a cut-vertex with $f(v) = 0$. Then $f_*(h) = f(v) = 0$, where h is the B-vertex representing the block H. As H is not type 0, there either exists $u \in N_{T_G}(h)$ such that $f_*(u) = 2$ or exists $u_1, u_2 \in N_{T_G}(h)$

such that $f_*(u_1) = f_*(u_2) = 1$. It is clear that u, u_1, u_2 are cut-vertices and $u, u_1, u_2 \in N_G(v)$. We also obtain $f(u) = f_*(u) = 2$ and $f(u_1) = f(u_2) = 1$. Therefore, we deduce $f(N(v)) \geq 2$. ■

Lemma 10. *There exists an* R2DF f *of* G *with weight* $\gamma_{\{R2\}}(G)$*, which satisfies the following properties:*

(1) $f(v) = 0$ *or* 1 *if* $v \in H$ *is not a cut-vertex of* G*, where* H *is a block of type* 1 *of* G*.*

(2) $f(v) = 0$ *if* $v \in H$ *is not a cut-vertex of* G*, where* H *is a block of type* 2 *of* G*.*

Proof. Let f be an R2DF of weight $\gamma_{\{R2\}}(G)$ and $u \in H$ be a cut-vertex of G, where H is not a block of type 0, $S = \{v : v \in H$ and v is a cut-vertex of $G\}$ and $f(u) = \max_{v_0 \in S} f(v_0)$. Suppose $v \in H$ is not a cut-vertex of G. If $f(v) = 2$, we can reassign 0 to v and 2 to u. Hence, $f(v) = 0$ or 1. Furthermore, if H is a block of type 2, we suppose that there exists a vertex $v \in H$ such that $f(v) = 1$. If $f(u) \geq 1$, then we can reassign 2 to u and 0 to v, a contradiction. Suppose that $f(u) = 0$, then there exists a vertex $w \in H$, such that w is not a cut-vertex and $f(w) \geq 1$. We reassign 2 to u and 0 to v, w, a contradiction. ■

Let f be an R2DF of block graph $G(\ncong K_n)$ and f_* be the function induced by f. We say f_* is an *induced Roman* {2}*-domination function* (R2DF$_*$) of T_G if it satisfies the four properties in Theorem 9. By Theorem 9 and Lemma 10, we can transform the Roman {2}-domination problem on block graph G into the induced Roman {2}-domination problem on tree T_G. Then, we can also use the method of tree composition and decomposition in Section 3. For convenience, $T_G = (C \cup B, E)$ is denoted by T and $v \in C$ (respectively, $v \in B$) is used to represent that v is a C-vertex (respectively, B-vertex) of T_G if there is no ambiguity.

Suppose that T is a tree rooted at r and $f : V(T) \to \{0, 1, 2\}$ is a function on T. T' is defined as a new tree rooted at r' and $f' : V(T') \to \{0, 1, 2\}$ is a function on T', where $V(T') = V(T) \cup \{r'\}$ and $E(T') = E(T) \cup \{rr'\}$, $f' :_T = f$.

In order to construct an algorithm for computing the Roman {2}-domination number, we must characterize the possible tree-subset tuples (T, f, r). For this purpose, we introduce some additional notations as follows:

R2DF$_*(T) = \{f : f$ is an R2DF$_*$ of $T\}$,

$F_1(T) = \{f : f \in$ R2DF$_*(T)$ with $f(r) = 1\}$,

$F_2(T) = \{f : f \in$ R2DF$_*(T)$ with $f(r) = 2\}$,

R2DF$_*(T^{+1}) = \{f : f \notin$ R2DF$_*(T)$, $f' \in F_1(T')$ and $f'|_T = f\}$,

R2DF$_*(T^{+2}) = \{f : f \notin$ R2DF$_*(T)$, $f' \in F_2(T')$ and $f'|_T = f\}-$R2DF$_*(T^{+1})$.

Then we consider the following eleven classes:

$A_1 = \{(T, f, r) : f \in \mathrm{R2DF}_*(T),\ r \in C \text{ and } f(r) = 2\}$,
$A_2 = \{(T, f, r) : f \in \mathrm{R2DF}_*(T),\ r \in C \text{ and } f(r) = 1\}$,
$A_3 = \{(T, f, r) : f \in \mathrm{R2DF}_*(T),\ r \in C \text{ and } f(r) = 0\}$,
$A_4 = \{(T, f, r) : f \in \mathrm{R2DF}_*(T^{+1}),\ r \in C\}$,
$A_5 = \{(T, f, r) : f \in \mathrm{R2DF}_*(T^{+2}),\ r \in C\}$,
$B_1 = \{(T, f, r) : f \in \mathrm{R2DF}_*(T),\ r \in B \text{ and } f(N[r]) \geq 2\}$,
$B_2 = \{(T, f, r) : f \in \mathrm{R2DF}_*(T),\ r \in B \text{ and } f(N[r]) = 1\}$,
$B_3 = \{(T, f, r) : f \in \mathrm{R2DF}_*(T),\ r \in B \text{ and } f(N[r]) = 0\}$,
$B_4 = \{(T, f, r) : f \in \mathrm{R2DF}_*(T^{+1}),\ r \in B \text{ and } f(N[r]) = 1\}$,
$B_5 = \{(T, f, r) : f \in \mathrm{R2DF}_*(T^{+1}),\ r \in B \text{ and } f(N[r]) = 0\}$,
$B_6 = \{(T, f, r) : f \in \mathrm{R2DF}_*(T^{+2}),\ r \in B\}$.

Let $(T, r) = (T_1, r_1) \circ (T_2, r_2)$ and $r = r_1$. Suppose that f_1 (respectively, f_2) is a function on T_1 (respectively, T_2). Define f as the function on T with $f|_{T_1} = f_1$ and $f|_{T_2} = f_2$. In order to give the algorithm, we present the following lemmas.

Lemma 11. $A_1 = (A_1 \circ B_1) \cup (A_1 \circ B_2) \cup (A_1 \circ B_3) \cup (A_1 \circ B_4) \cup (A_1 \circ B_5) \cup (A_1 \circ B_6)$.

Proof. For each $1 \leq i \leq 6$, if $(T_1, f_1, r_1) \in A_1$ and $(T_2, f_2, r_2) \in B_i$, it is clear that f is an $\mathrm{R2DF}_*$ of T, $r \in C$ and $f(r) = f(r_1) = 2$. We deduce that $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in A_1$. So $(A_1 \circ B_1) \cup (A_1 \circ B_2) \cup (A_1 \circ B_3) \cup (A_1 \circ B_4) \cup (A_1 \circ B_5) \cup (A_1 \circ B_6) \subseteq A_1$.

Now we prove that $A_1 \subseteq (A_1 \circ B_1) \cup (A_1 \circ B_2) \cup (A_1 \circ B_3) \cup (A_1 \circ B_4) \cup (A_1 \circ B_5) \cup (A_1 \circ B_6)$. Let $(T, f, r) \in A_1$ and $(T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2)$, then $f_1(r_1) = f(r) = 2$. Since $f \in \mathrm{R2DF}_*(T)$, $f_1 \in \mathrm{R2DF}_*(T_1)$ and $r_1 \in C$. So $(T_1, f_1, r_1) \in A_1$ and $r_2 \in B$. If $f_2 \in \mathrm{R2DF}_*(T_2)$, then we obtain $(T_2, f_2, r_2) \in B_1$, B_2 or B_3. If $f_2 \notin \mathrm{R2DF}_*(T_2)$, then $(T_2, f_2, r_2) \in B_4$, B_5 or B_6. Hence, we conclude that $A_1 \subseteq (A_1 \circ B_1) \cup (A_1 \circ B_2) \cup (A_1 \circ B_3) \cup (A_1 \circ B_4) \cup (A_1 \circ B_5) \cup (A_1 \circ B_6)$. ∎

Lemma 12. $A_2 = (A_2 \circ B_1) \cup (A_2 \circ B_2) \cup (A_2 \circ B_3) \cup (A_2 \circ B_4) \cup (A_2 \circ B_5)$.

Proof. For each $1 \leq i \leq 5$, if $(T_1, f_1, r_1) \in A_2$ and $(T_2, f_2, r_2) \in B_i$, it is clear that f is an $\mathrm{R2DF}_*$ of T, $r \in C$ and $f(r) = f(r_1) = 1$. We conclude that $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in A_2$, implying that $(A_2 \circ B_1) \cup (A_2 \circ B_2) \cup (A_2 \circ B_3) \cup (A_2 \circ B_4) \cup (A_2 \circ B_5) \subseteq A_2$.

Then we need to show $A_2 \subseteq (A_2 \circ B_1) \cup (A_2 \circ B_2) \cup (A_2 \circ B_3) \cup (A_2 \circ B_4) \cup (A_2 \circ B_5)$. Let $(T, f, r) \in A_2$ and $(T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2)$, then $f_1(r_1) = f(r) = 1$. It is clear that $(T_1, f_1, r_1) \in A_2$ and $r_2 \in B$. If f_2 is an $\mathrm{R2DF}_*$ of T_2, then we obtain $(T_2, f_2, r_2) \in B_1, B_2$ or B_3. If f_2 is not an $\mathrm{R2DF}_*$ of T_2, then $f_2(N_{T_2}[r_2]) \leq 1$ and $f_2 \in \mathrm{R2DF}_*(T_2^{+1})$ by using the fact that $(T, f, r) \in A_2$. Therefore, we have $(T_2, f_2, r_2) \in B_4$ or B_5. Hence, $A_2 \subseteq (A_2 \circ B_1) \cup (A_2 \circ B_2) \cup (A_2 \circ B_3) \cup (A_2 \circ B_4) \cup (A_2 \circ B_5)$. ■

Lemma 13. $A_3 = (A_3 \circ B_1) \cup (A_3 \circ B_2) \cup (A_3 \circ B_3) \cup (A_4 \circ B_1) \cup (A_4 \circ B_2) \cup (A_5 \circ B_1)$.

Proof. We make some remarks.

(i) For each $1 \leq i \leq 3$, if $(T_1, f_1, r_1) \in A_3$ and $(T_2, f_2, r_2) \in B_i$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in A_3$. Indeed, if $(T_1, f_1, r_1) \in A_3$ and $(T_2, f_2, r_2) \in B_i$, then f_1 is an R2DF$_*$ of T_1 and f_2 is an R2DF$_*$ of T_2. Hence, f is an R2DF$_*$ of T, $r \in C$ and $f(r) = 0$. So $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in A_3$.

(ii) For each $1 \leq i \leq 2$, if $(T_1, f_1, r_1) \in A_4$ and $(T_2, f_2, r_2) \in B_i$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in A_3$. Indeed, if $(T_1, f_1, r_1) \in A_4$, then we have that $f_1 \in$ R2DF$_*(T_1^{+1})$, $r \in C$, $f(r) = 0$ and $f(N^2_{T_1}[r]) = 1$. By the definition of B_i, we obtain $f(N^2_T[r]) \geq 2$ and $f \in$ R2DF$_*(T)$. Hence, $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in A_3$.

(iii) If $(T_1, f_1, r_1) \in A_5$ and $(T_2, f_2, r_2) \in B_1$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in A_3$. Indeed, if $(T_1, f_1, r_1) \in A_5$, then we have that $f_1 \in$ R2DF$_*(T_1^{+2})$, $r \in C$, $f(r) = 0$ and $f(N^2_{T_1}[r]) = 0$. By the definition of B_1, we obtain $f(N^2_T[r]) \geq 2$ and $f \in$ R2DF$_*(T)$. It means that $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in A_3$. Hence, $(A_3 \circ B_1) \cup (A_3 \circ B_2) \cup (A_3 \circ B_3) \cup (A_4 \circ B_1) \cup (A_4 \circ B_2) \cup (A_5 \circ B_1) \subseteq A_3$.

Therefore, we need to prove $A_3 \subseteq (A_3 \circ B_1) \cup (A_3 \circ B_2) \cup (A_3 \circ B_3) \cup (A_4 \circ B_1) \cup (A_4 \circ B_2) \cup (A_5 \circ B_1)$. Let $(T, f, r) \in A_3$ and $(T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2)$, then we have that $f_1(r_1) = f(r) = 0$, $r_1 \in C$ and $f_2 \in$ R2DF$_*(T_2)$. So $r_2 \in B$. If $f_1 \in$ R2DF$_*(T_1)$, then we obtain $(T_1, f_1, r_1) \in A_3$, implying that $(T_2, f_2, r_2) \in B_1, B_2$ or B_3. Suppose that $f_1 \notin$ R2DF$_*(T_1)$. Consider the following cases.

Case 1. $f_1(N^2_{T_1}[r_1]) = 1$. Then we obtain $f_1 \in$ R2DF$_*(T_1^{+1})$, implying that $(T_1, f_1, r_1) \in A_4$. Since $(T, f, r) \in A_3$, we have $f_2(N_{T_2}[r_2]) \geq 1$. So $(T_2, f_2, r_2) \in B_1$ or B_2.

Case 2. $f_1(N^2_{T_1}[r_1]) = 0$. So we have $f_1 \in$ R2DF$_*(T_1^{+2})$. Then $(T_1, f_1, r_1) \in A_5$. Since $(T, f, r) \in A_3$, we obtain $f_2(N_{T_2}[r_2]) \geq 2$. Hence, $(T_2, f_2, r_2) \in B_1$. ■

Lemma 14. $A_4 = (A_4 \circ B_3) \cup (A_5 \circ B_2)$.

Proof. It is easy to check the following remarks by definitions.

(i) If $(T_1, f_1, r_1) \in A_4$ and $(T_2, f_2, r_2) \in B_3$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in A_4$.

(ii) If $(T_1, f_1, r_1) \in A_5$ and $(T_2, f_2, r_2) \in B_2$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in A_4$. Therefore, $(A_4 \circ B_3) \cup (A_5 \circ B_2) \subseteq A_4$.

On the other hand, we show $A_4 \subseteq (A_4 \circ B_3) \cup (A_5 \circ B_2)$. Let $(T, f, r) \in A_4$ and $(T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2)$. Then we have that $f \in$ R2DF$_*(T^{+1})$ and $r_1 \in C$, implying that $f(N^2_T[r_1]) = 1$. It means that $r_2 \in B$. By the definition of A_4, $f_2 \in \mathrm{R2DF}_*(T_2)$. Using the fact that $f(N^2_T[r_1]) = 1$, we deduce that $f_2(N[r_2]) < 2$. Consider the following cases.

Case 1. $f_2(N[r_2]) = 1$. It is clear that $(T_2, f_2, r_2) \in B_2$. Since $f_1(N^2_{T_1}[r_1]) = f(N^2_T[r_1]) - f_2(N[r_2]) = 0$, we obtain $(T_1, f_1, r_1) \in A_5$.

Case 2. $f_2(N[r_2]){=}0$. Then $(T_2, f_2, r_2) \in B_3$. Since $f_1(N^2_{T_1}[r_1]){=}f(N^2_T[r_1])-f_2(N[r_2]) = 1$, we have $(T_1, f_1, r_1) \in A_4$.

Consequently, we deduce that $A_4 \subseteq (A_4 \circ B_3) \cup (A_5 \circ B_2)$. ■

Lemma 15. $A_5 = A_5 \circ B_3$.

Proof. It is easy to check that $(A_5 \circ B_3) \subseteq A_5$ by the definitions. On the other hand, let $(T, f, r) \in A_5$ and $(T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2)$. Then we obtain $f \in \mathrm{R2DF}_*(T^{+2})$, $r_1 \in C$ and $f_1(N^2[r_1]) = f(N^2[r]) = 0$. It implies that $(T_1, f_1, r_1) \in A_5$ and $r_2 \in B$. Using the fact that $(T, f, r) \in A_5$, we deduce $f_2(N[r_2]) = 0$ and $f_2 \in \mathrm{R2DF}_*(T_2)$. Therefore, $(T_2, f_2, r_2) \in B_3$. Then we obtain $A_5 \subseteq (A_5 \circ B_3)$. ■

Lemma 16. $B_1 = (B_1 \circ A_1) \cup (B_1 \circ A_2) \cup (B_1 \circ A_3) \cup (B_1 \circ A_4) \cup (B_1 \circ A_5) \cup (B_2 \circ A_1) \cup (B_2 \circ A_2) \cup (B_3 \circ A_1) \cup (B_4 \circ A_1) \cup (B_4 \circ A_2) \cup (B_5 \circ A_1) \cup (B_6 \circ A_1)$.

Proof. We make some remarks.

(i) For each $1 \le i \le 5$, if $(T_1, f_1, r_1) \in B_1$ and $(T_2, f_2, r_2) \in A_i$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B_1$. It is easy to check it by the definitions of B_1 and A_i.

(ii) For each $2 \le i \le 6$, if $(T_1, f_1, r_1) \in B_i$ and $(T_2, f_2, r_2) \in A_1$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B_1$. We can easily check it by definitions too.

(iii) For each $i \in \{2, 4\}$, if $(T_1, f_1, r_1) \in B_i$ and $(T_2, f_2, r_2) \in A_2$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B_1$. Indeed, it is clear that $f \in \mathrm{R2DF}_*(T)$, $r \in B$ and $f(N[r]) = f_1(N[r_1]) + f_2(r_2) = 2$. Hence, $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B_1$.

Therefore, we need to prove $B_1 \subseteq (B_1 \circ A_1) \cup (B_1 \circ A_2) \cup (B_1 \circ A_3) \cup (B_1 \circ A_4) \cup (B_1 \circ A_5) \cup (B_2 \circ A_1) \cup (B_2 \circ A_2) \cup (B_3 \circ A_1) \cup (B_4 \circ A_1) \cup (B_4 \circ A_2) \cup (B_5 \circ A_1) \cup (B_6 \circ A_1)$. Let $(T, f, r) \in B_1$ and $(T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2)$, then we have $f \in \mathrm{R2DF}_*(T)$, $r_1 \in B$ and $f(N[r]) \ge 2$. It means that $r_2 \in C$. Consider the following cases.

Case 1. $f(r_2) = 2$. Then we have $f_2 \in \mathrm{R2DF}_*(T_2)$, impling that $(T_2, f_2, r_2) \in A_1$. If $f_1 \in \mathrm{R2DF}_*(T_1)$, we obtain $(T_1, f_1, r_1) \in B_1$, B_2 or B_3. Suppose that $f_1 \notin \mathrm{R2DF}_*(T_1)$, then $f_1 \in \mathrm{R2DF}_*(T_1^{+1})$ or $f_1 \in \mathrm{R2DF}_*(T_1^{+2})$. Hence, $(T_1, f_1, r_1) \in B_4$, B_5 or B_6.

Case 2. $f(r_2) = 1$. It is clear that $(T_2, f_2, r_2) \in A_2$. We also have $f_1(N[r_1]) = f(N[r]) - f_2(r_2) \ge 2-1 \ge 1$. If $f_1 \in \mathrm{R2DF}_*(T_1)$, we obtain $(T_1, f_1, r_1) \in B_1$ or B_2.

Suppose that $f_1 \notin \text{R2DF}_*(T_1)$, then $f_1 \in \text{R2DF}_*(T_1^{+1})$. Therefore, $(T_1, f_1, r_1) \in B_4$.

Case 3. $f(r_2) = 0$. Then we obtain $f_1(N[r_1]) = f(N[r]) - f_2(r_2) \geq 2$ and $f_1 \in \text{R2DF}_*(T_1)$, implying that $(T_1, f_1, r_1) \in B_1$. If $f_2 \in \text{R2DF}_*(T_2)$, we deduce $(T_1, f_1, r_1) \in A_3$. Suppose that $f_2 \notin \text{R2DF}_*(T_2)$, then $f_2 \in \text{R2DF}_*(T_2^{+1})$ or $f_2 \in \text{R2DF}_*(T_2^{+2})$. Therefore, $(T_2, f_2, r_2) \in A_4$ or A_5.

Hence, $B_1 \subseteq (B_1 \circ A_1) \cup (B_1 \circ A_2) \cup (B_1 \circ A_3) \cup (B_1 \circ A_4) \cup (B_1 \circ A_5) \cup (B_2 \circ A_1) \cup (B_2 \circ A_2) \cup (B_3 \circ A_1) \cup (B_4 \circ A_1) \cup (B_4 \circ A_2) \cup (B_5 \circ A_1) \cup (B_6 \circ A_1)$. ■

Lemma 17. $B_2 = (B_2 \circ A_3) \cup (B_2 \circ A_4) \cup (B_3 \circ A_2) \cup (B_5 \circ A_2)$.

Proof. We make some remarks.

(i) For each $3 \leq i \leq 4$, if $(T_1, f_1, r_1) \in B_2$ and $(T_2, f_2, r_2) \in A_i$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B_2$. It is easy to check it by the definitions.

(ii) For each $i \in \{3, 5\}$, if $(T_1, f_1, r_1) \in B_i$ and $(T_2, f_2, r_2) \in A_2$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B_2$. Indeed, if $(T_1, f_1, r_1) \in B_i$ and $(T_2, f_2, r_2) \in A_2$, we obtain that $f \in \text{R2DF}_*(T)$, $r \in B$ and $f(N[r]) = f_1(N[r_1]) + f_2(r_2) = 1$. Hence, we deduce $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B_2$. Thus, $(B_2 \circ A_3) \cup (B_2 \circ A_4) \cup (B_3 \circ A_2) \cup (B_5 \circ A_2) \subseteq B_2$.

Now we need to prove $B_2 \subseteq (B_2 \circ A_3) \cup (B_2 \circ A_4) \cup (B_3 \circ A_2) \cup (B_5 \circ A_2)$. Let $(T, f, r) \in B_2$ and $(T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2)$, then we have that $f \in \text{R2DF}_*(T)$, $r_1 \in B$ and $f(N[r]) = 1$. It implies $r_2 \in C$. Consider the following cases.

Case 1. $f(r_2) = 1$. Then we have $f_1(N[r_1]) = f(N[r]) - f(r_2) = 0$ and $f_2(r_2) = 1$, implying that $f_2 \in \text{R2DF}_*(T_2)$. So $(T_2, f_2, r_2) \in A_2$. If $f_1 \in \text{R2DF}_*(T_1)$, we obtain $(T_1, f_1, r_1) \in B_3$. Suppose that $f_1 \notin \text{R2DF}_*(T_1)$, then $f_1(r_1) = 0$ because $f \in \text{R2DF}_*(T)$. Since $f_1(N[r_1]) = 0$, we have that $(T_1, f_1, r_1) \in B_5$.

Case 2. $f(r_2) = 0$. It is clear that $f_1(N[r_1]) = f(N[r]) - f(r_2) = 1$. Since $f_1 = f|_{T_1}$ and $f \in \text{R2DF}_*(T)$, we have $f_1 \in \text{R2DF}_*(T_1)$. Hence, $(T_1, f_1, r_1) \in B_2$. If $f_2 \in \text{R2DF}_*(T_2)$, we deduce that $(T_2, f_2, r_2) \in A_3$. Suppose that $f_2 \notin \text{R2DF}_*(T_2)$, then $f_2(N^2[r_2]) = 1$. It implies $f_2 \in \text{R2DF}_*(T_2^{+1})$. Therefore, $(T_2, f_2, r_2) \in A_4$.

Hence, $B_2 \subseteq (B_2 \circ A_3) \cup (B_2 \circ A_4) \cup (B_3 \circ A_2) \cup (B_5 \circ A_2)$. ■

Lemma 18. $B_3 = B_3 \circ A_3$.

Proof. It is easy to check that $(B_3 \circ A_3) \subseteq B_3$ by the definitions. On the other hand, let $(T, f, r) \in B_3$ and $(T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2)$. Then we obtain $f_1(N[r_1]) = f(N[r]) = 0$, $r_1 \in B$ and $f(r_2) = 0$. It means that $r_2 \in C$. Since $f \in \text{R2DF}_*(T)$ and $f(r_2) = 0$, we obtain that $f_1 \in \text{R2DF}_*(T_1)$, implying that $(T_1, f_1, r_1) \in B_3$. Using the fact that $f_1(N[r_1]) = 0$ and $f(r_2) = 0$, we deduce that $f_2 \in \text{R2DF}_*(T_2)$. Therefore, $(T_2, f_2, r_2) \in A_3$. Then $B_3 \subseteq (B_3 \circ A_3)$. ■

Lemma 19. $B_4 = (B_2 \circ A_5) \cup (B_4 \circ A_3) \cup (B_4 \circ A_4) \cup (B_4 \circ A_5) \cup (B_6 \circ A_2)$.

Proof. It is easy to check the following remarks by definitions.

(i) If $(T_1, f_1, r_1) \in B_2$ and $(T_2, f_2, r_2) \in A_5$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B_4$.

(ii) For each $3 \leq i \leq 5$, if $(T_1, f_1, r_1) \in B_4$ and $(T_2, f_2, r_2) \in A_i$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B_4$.

(iii) If $(T_1, f_1, r_1) \in B_6$ and $(T_2, f_2, r_2) \in A_2$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B_4$.

Therefore, we need to prove $B_4 \subseteq (B_2 \circ A_5) \cup (B_4 \circ A_3) \cup (B_4 \circ A_4) \cup (B_4 \circ A_5) \cup (B_6 \circ A_2)$. Let $(T, f, r) \in B_4$ and $(T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2)$, then we have $f \in \text{R2DF}_*(T^{+1})$, $r_1 \in B$ and $f(N[r]) = 1$. It implies $r_2 \in C$. Consider the following cases.

Case 1. $f(r_2) = 1$. Then we have $f_1(N[r_1]) = f(N[r]) - f(r_2) = 0$ and $f_2(r_2) = 1$, implying that $f_2 \in \text{R2DF}_*(T_2)$. So $(T_2, f_2, r_2) \in A_2$ and $f_1 \notin \text{R2DF}_*(T_1)$. Since $f_1(N[r_1]) = 0$ and $(T, f, r) \in B_4$, we obtain $(T_1, f_1, r_1) \in B_6$.

Case 2. $f(r_2) = 0$. It is clear that $f_1(N[r_1]) = f(N[r]) - f(r_2) = 1$. If $f_2 \in \text{R2DF}_*(T_2)$, we deduce that $(T_2, f_2, r_2) \in A_3$, implying $(T_1, f_1, r_1) \in B_4$. Suppose that $f_2 \notin \text{R2DF}_*(T_2)$, then $f_2(N^2[r_2]) = 0$ or 1. If $f_2(N^2[r_2]) = 0$, we obtain $(T_2, f_2, r_2) \in A_5$. Then, we have $(T_1, f_1, r_1) \in B_2$ or B_4. If $f_2(N^2[r_2]) = 1$, we obtain $(T_2, f_2, r_2) \in A_4$. Then, we have $(T_1, f_1, r_1) \in B_4$.

Hence, $B_4 \subseteq (B_2 \circ A_5) \cup (B_4 \circ A_3) \cup (B_4 \circ A_4) \cup (B_4 \circ A_5) \cup (B_6 \circ A_2)$. ∎

Lemma 20. $B_5 = (B_3 \circ A_4) \cup (B_5 \circ A_3) \cup (B_5 \circ A_4)$.

Proof. It is easy to check the following remarks by definitions.

(i) If $(T_1, f_1, r_1) \in B_3$ and $(T_2, f_2, r_2) \in A_4$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B_5$.

(ii) For each $3 \leq i \leq 4$, if $(T_1, f_1, r_1) \in B_5$ and $(T_2, f_2, r_2) \in A_i$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B_5$. Thus, $(B_3 \circ A_4) \cup (B_5 \circ A_3) \cup (B_5 \circ A_4) \subseteq B_5$.

Therefore, we need to prove $B_5 \subseteq (B_3 \circ A_4) \cup (B_5 \circ A_3) \cup (B_5 \circ A_4)$. Let $(T, f, r) \in B_5$ and $(T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2)$, then we have $f \in \text{R2DF}_*(T^{+1})$, $r_1 \in B$ and $f(N[r]) = 0$. It implies $r_2 \in C$ and $f_2(r_2) = f(r_2) = 0$. Consider the following cases.

Case 1. If $f_2 \in \text{R2DF}_*(T_2)$, then we have $(T_2, f_2, r_2) \in A_3$ and $f_1 \notin \text{R2DF}_*(T_1)$. Since $f_1(N[r_1]) = 0$ and $(T, f, r) \in B_5$, we obtain $(T_1, f_1, r_1) \in B_5$.

Case 2. If $f_2 \notin \text{R2DF}_*(T_2)$, we deduce that $(T_2, f_2, r_2) \in A_4$. It is clear that $(T_1, f_1, r_1) \in B_3$ or B_5.

Hence, $B_5 \subseteq (B_3 \circ A_4) \cup (B_5 \circ A_3) \cup (B_5 \circ A_4)$. ∎

Lemma 21. $B_6 = (B_3 \circ A_5) \cup (B_5 \circ A_5) \cup (B_6 \circ A_3) \cup (B_6 \circ A_4) \cup (B_6 \circ A_5)$.

Proof. It is easy to check the following remarks by definitions.

(i) For each $i \in \{3, 5\}$, if $(T_1, f_1, r_1) \in B_i$ and $(T_2, f_2, r_2) \in A_5$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B_6$.

(ii) For each $3 \leq i \leq 5$, if $(T_1, f_1, r_1) \in B_6$ and $(T_2, f_2, r_2) \in A_i$, then $(T_1, f_1, r_1) \circ (T_2, f_2, r_2) \in B_6$.

Therefore, we need to prove $B_6 \subseteq (B_3 \circ A_5) \cup (B_5 \circ A_5) \cup (B_6 \circ A_3) \cup (B_6 \circ A_4) \cup (B_6 \circ A_5)$. Let $(T, f, r) \in B_6$ and $(T, f, r) = (T_1, f_1, r_1) \circ (T_2, f_2, r_2)$, then we have $f \in \mathrm{R2DF}_*(T^{+2})$, $r_1 \in B$ and $f(N[r]) = 0$. It implies $r_2 \in C$. Consider the following cases.

Case 1. $f_1 \in \mathrm{R2DF}_*(T_1)$. Since $f_1(N[r_1]) = f(N[r]) = 0$, we have $(T_1, f_1, r_1) \in B_3$. It implies $(T_2, f_2, r_2) \in A_5$.

Case 2. $f_1 \notin \mathrm{R2DF}_*(T_1)$. Since $f_1(N[r_1]) = f(N[r]) = 0$, then we obtain $(T_1, f_1, r_1) \in B_5$ or B_6. If $(T_1, f_1, r_1) \in B_5$, we have $f_1 \in \mathrm{R2DF}_*(T_1^{+1})$. Since $f \in \mathrm{R2DF}_*(T^{+2})$, it means that $f_2 \in \mathrm{R2DF}_*(T_2^{+2})$. Then we deduce $(T_2, f_2, r_2) \in A_5$. If $(T_1, f_1, r_1) \in B_6$, we have $f_1 \in \mathrm{R2DF}_*(T_1^{+2})$. Since $(T, f, r) \in B_6$, we deduce that $f_2(r_2) = 0$. So we obtain $(T_2, f_2, r_2) \in A_3$, A_4 or A_5.

Hence, $B_6 \subseteq (B_3 \circ A_5) \cup (B_5 \circ A_5) \cup (B_6 \circ A_3) \cup (B_6 \circ A_4) \cup (B_6 \circ A_5)$. ■

The final step is to define the initial vector. In this case, for block-cutpoint graphs, the only basis graph is a single vertex. We can use the similar method in Section 3 to initialize the vector. It is clear that if v is a C-vertex, then the initial vector is $[2, 1, \infty, \infty, 0, \infty]$; if v is a B-vertex and v represents a block of type 0, then the initial vector is $[\infty, \infty, 0, \infty, \infty, \infty]$; if v is a B-vertex and v represents a block of type 1, then the initial vector is $[\infty, 1, \infty, \infty, \infty, 0]$; if v is a B-vertex and v represents a block of type 2, then the initial vector is $[\infty, \infty, \infty, \infty, \infty, 0]$. Among them, $'\infty'$ means undefined. From the above argument, we can obtain the following theorem.

Theorem 22. *Algorithm ROMAN {2}-DOM-IN-BLOCK can output the Roman {2}-domination number of any block graphs $G = (V, E)$ in linear time $O(n)$, where $n = |V|$.*

Proof. One can prove Theorem 22 by the similar argument as in the proof of Theorem 8. ■

Now, we are ready to present the algorithm.

Algorithm 2: ROMAN {2}-DOM-IN-BLOCK

Input: A connected block graph G ($G \ncong K_n$) and its corresponding block-cutpoint graph $T = (V, E)$ with a tree ordering $v_1, v_2, \ldots, v_n$.

Output: The Roman {2}-domination number $\gamma_{\{R2\}}(G)$.

for $i := 1$ **to** n **do**
    if v_i *is a C-vertex* **then**
        initialize $h[i, 1..6]$ to $[2, 1, \infty, \infty, 0, \infty]$;
    else if v_i *is a B-vertex representing a block of type* 0 **then**
        initialize $h[i, 1..6]$ to $[\infty, \infty, 0, \infty, \infty, \infty]$;
    else if v_i *is a B-vertex representing a block of type* 1 **then**
        initialize $h[i, 1..6]$ to $[\infty, 1, \infty, \infty, \infty, 0]$;
    else
        initialize $h[i, 1..6]$ to $[\infty, \infty, \infty, \infty, \infty, 0]$;

for $j := 1$ **to** $n - 1$ **do**
    $v_k = F(v_j)$;
    if v_k *is a C-vertex* **then**
        $h[k, 1] = \min\{h[k, 1]+h[j, 1], h[k, 1]+h[j, 2], h[k, 1]+h[j, 3], h[k, 1]+ h[j, 4], h[k, 1] + h[j, 5], h[k, 1] + h[j, 6]\}$;
        $h[k, 2] = \min\{h[k, 2]+h[j, 1], h[k, 2]+h[j, 2], h[k, 2]+h[j, 3], h[k, 2]+ h[j, 4], h[k, 2] + h[j, 5]\}$;
        $h[k, 3] = \min\{h[k, 3]+h[j, 1], h[k, 3]+h[j, 2], h[k, 3]+h[j, 3], h[k, 4]+ h[j, 1], h[k, 4] + h[j, 2], h[k, 5] + h[j, 1]\}$;
        $h[k, 4] = \min\{h[k, 4] + h[j, 3], h[k, 5] + h[j, 2]\}$;
        $h[k, 5] = \min\{h[k, 5] + h[j, 3]\}$;
    else
        $S_1 = h[k, 2]$;
        $S_2 = h[k, 3]$;
        $S_3 = h[k, 5]$;
        $h[k, 1] = \min\{h[k, 1]+h[j, 1], h[k, 1]+h[j, 2], h[k, 1]+h[j, 3], h[k, 1]+ h[j, 4], h[k, 1]+h[j, 5], h[k, 2]+h[j, 1], h[k, 2]+h[j, 2], h[k, 3] + h[j, 1], h[k, 4] + h[j, 1], h[k, 4] + h[j, 2], h[k, 5] + h[j, 1], h[k, 6] + h[j, 1]\}$;
        $h[k, 2] = \min\{h[k, 2] + h[j, 3], h[k, 2] + h[j, 4], h[k, 3] + h[j, 2], h[k, 5] + h[j, 2]\}$;
        $h[k, 3] = \min\{h[k, 3] + h[j, 3]\}$;
        $h[k, 4] = \min\{S_1 + h[j, 5], h[k, 4] + h[j, 3], h[k, 4] + h[j, 4], h[k, 4]+ h[j, 5], h[k, 6] + h[j, 2]\}$;
        $h[k, 5] = \min\{S_2 + h[j, 4], h[k, 5] + h[j, 3], h[k, 5] + h[j, 4]\}$;
        $h[k, 6] = \min\{S_2+h[j, 5], S_3+h[j, 5], h[k, 6]+h[j, 3], h[k, 6]+h[j, 4], h[k, 6] + h[j, 5]\}$;

return $\gamma_{\{R2\}}(G) = \min\{h[n, 1], h[n, 2], h[n, 3]\}$;

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