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(C_3, C_4, C_5, C_7)-FREE ALMOST WELL-DOMINATED GRAPHS

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Abstract

The *domination gap* of a graph G is defined as the difference between the maximum and minimum cardinalities of a minimal dominating set in G. The term *well-dominated graphs* referring to the graphs with domination gap zero, was first introduced by Finbow *et al.* [*Well-dominated graphs: A collection of well-covered ones*, Ars Combin. 25 (1988) 5–10]. In this paper, we focus on the graphs with domination gap one which we term *almost well-dominated* graphs. While the results by Finbow *et al.* have implications for almost well-dominated graphs with girth at least 8, we extend these results to (C_3, C_4, C_5, C_7)-free almost well-dominated graphs by giving a complete structural characterization for such graphs.

Keywords: well-dominated graphs, almost well-dominated graphs, domination gap.

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1. Introduction

A *dominating set* in a graph $G = (V, E)$ is a set S such that every vertex of G is either in S or adjacent to a vertex in S. A dominating set is *minimal* if no proper subset of it is a dominating set. While the cardinality of a minimum dominating set is referred to as the *domination number* of G and denoted by $\gamma(G)$, the maximum cardinality of a minimal dominating set is called the *upper domination number* of G and denoted by $\Gamma(G)$. The *domination gap* of a graph G, denoted by $\mu_d(G)$, is defined as the difference $\Gamma(G) - \gamma(G)$.

A graph G is called *well-dominated* if $\mu_d(G) = 0$. Finbow *et al.* [3] introduced the concept of well-dominated graphs and further provided two characterization results: one for well-dominated graphs of girth at least five and the other for well-dominated bipartite graphs. Well-dominated graphs were further studied in [8].

Note that well-dominated graphs are a subclass of *well-covered* graphs, which are the graphs whose maximal independent sets have the same size. Thus, most of the research works on well-coveredness and its variants in the literature have also implications for well-dominated graphs. In this sense, Topp and Volkmann [11] provided characterizations for both well-covered and well-dominated block graphs and unicyclic graphs. Further characterization results on special subclasses of well-dominated graphs include locally well-dominated graphs and locally independent well-dominated graphs [12], 3-connected, planar, and claw-free well-dominated graphs [9], and 4-connected, 4-regular, claw-free well-dominated graphs [7]. Building upon the result of Finbow *et al.* [5] on well-covered graphs containing neither 4-cycles nor 5-cycles, Levit and Tankus [10] showed that for graphs without cycles of length 4 and 5, the family of well-dominated and well-covered graphs overlap; i.e., a graph without 4- and 5-cycles is well-dominated if and only if it is well-covered.

We say that a graph G is *almost well-dominated* (AWD) if $\mu_d(G) = 1$. With this definition, almost well-dominated graphs fall into the class of $\mathcal{D}_2$ graphs defined by Dunbar *et al.* [1]. The class $\mathcal{D}_n$ consists of graphs which have minimal dominating sets of exactly n different sizes. With this notation, $\mathcal{D}_2$ is the class of graphs having minimal dominating sets of exactly two distinct sizes. Dunbar *et al.* [1] characterized trees and split graphs in $\mathcal{D}_2$ and further gave a characterization for a subclass of bipartite graphs in $\mathcal{D}_2$ having a vertex adjacent to more than one leaf.

Similarly, Finbow *et al.* [6] denoted the graphs having exactly n distinct sizes of maximal independent sets by M_n. They investigated the graphs in the class M_2 and provided a characterization for the graphs of girth at least 8 in this class. These results have implications for almost well-dominated graphs with girth at least 8, since $AWD \subset M_2$ when restricted to girth at least 8.

Ekim *et al.* [2] dealt with a subclass of M_2 which they call *almost well-covered* graphs. Almost well-covered graphs have maximal independent sets with two distinct sizes where the difference between these two sizes is one. Ekim *et al.* [2] provided a characterization for a subclass of almost well-covered graphs with girth at least 6 and further gave a polynomial-time algorithm for the recognition of (C_3, C_4, C_5, C_7)-free almost well-covered graphs. Furthermore, they raised the characterization of almost well-covered graphs with girth at least 6 as an open problem [2].

In this paper, we study almost well-dominated graphs with restricted girth. Note that the work by Finbow *et al.* [6] implies results for almost well-dominated graphs with girth at least 8. We improve these results by providing a complete structural characterization for (C_3, C_4, C_5, C_7)-free almost well-dominated graphs. Moreover, by characterization of (C_3, C_4, C_5, C_7)-free almost well-dominated graphs, we partially answer the open question posed in [2], since almost well-dominated graphs are a subclass of almost well-covered graphs when restricted to girth at least 6.

In Section 2, after giving some graph-theoretic terms and definitions, we provide some results for the general case of almost well-dominated graphs. Then we proceed with our results for almost well-dominated graphs with restricted girth and present our characterization of (C_3, C_4, C_5, C_7)-free almost well-dominated graphs in Sections 3 and 4.

2. Preliminaries

A *graph* G is an ordered pair $(V(G), E(G))$, where $V(G)$ is the set of vertices and $E(G)$ is the set of edges each connecting a pair of vertices. Throughout this paper, G is a *simple* graph, that is, a finite, undirected, and loopless graph without multiple edges. The set of all vertices that are adjacent to a vertex v is called the *neighborhood* of v, and is denoted by $N(v)$. The *closed* neighborhood of vertex v is denoted by $N[v]$, which is the set $N(v) \cup \{v\}$. The length of a shortest cycle in G is called the *girth* of G.

By $\delta(G)$ (respectively, $\Delta(G)$), we denote the *minimum* (respectively, *maximum*) degree of G, that is, the degree of the vertex with the smallest (respectively, greatest) degree in G. While a vertex of degree zero in G is referred to as an *isolated* vertex of G, a vertex of degree one in G is a *leaf* of G and a vertex adjacent to at least one leaf is called a *stem.* Further, we denote by L_G the set of leaves in a graph G.

A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Furthermore, a subgraph of G *induced* by a set $S \subseteq V(G)$, denoted by $G[S]$, is a graph formed from the vertices of S and all edges connecting the pairs of vertices in S. We denote by P_n, C_n, and K_n a path, a cycle, and a complete graph on n vertices, respectively. We say a vertex is of *type-k* if it is adjacent to k leaves, where $k \geq 0$. Moreover, a graph G is said to be in the family $\mathcal{P}$ if every vertex of G is either a leaf or a vertex of type-1. A vertex $v \in V(G)$ is an *internal vertex* if it is not a leaf of G.

A set I of vertices in a graph G is an *independent* set if no two vertices in I are adjacent. An independent set which is not properly contained in another one is called a *maximal* independent set. The maximum size of an independent set in a graph G is called the *independence number* of G, denoted by $\alpha(G)$ and the minimum cardinality of a maximal independent set in G is denoted by $i(G)$. The following inequalities (*domination chain*) relate the aforementioned graph parameters. For any graph G, we have

$$\gamma(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G).$$

A graph is *well-covered* if all its maximal independent sets have the same cardinality, i.e., $i(G) = \alpha(G)$. It can easily be seen that every well-dominated graph is well-covered, since the equality $\gamma(G) = \Gamma(G)$ implies that $i(G) = \alpha(G)$. Furthermore, we say that a graph G is *almost well-dominated* if $\mu_d(G) = 1$.

In this section, we provide some results for the general case of almost well-dominated graphs and we then proceed with our results for almost well-dominated graphs with restricted girth in Sections 3 and 4. From now on, we restrict our attention to connected graphs due to Proposition 1.

Proposition 1. *A graph is almost well-dominated if and only if all its components are well-dominated, except one, which is almost well-dominated.*

Proof. Let G be an almost well-dominated graph and let $H_1, H_2, \ldots, H_k$ be the components of G. By the definition of $\mu_d(G)$, we have $\mu_d(G) = \sum_{n=1}^{k} \mu_d(H_n)$. Since G is almost well-dominated, then $\mu_d(G) = 1$. Thus, the domination gap is one for only one of the components and it is zero for all the other components. The converse is easy to verify. ■

Lemma 2 determines the types of vertices that can exist in a graph with domination gap k.

Lemma 2. *If $\mu_d(G) = k$ for any $k \geq 0$, then every internal vertex of G is adjacent to at most $k+1$ leaves.*

Proof. Suppose to the contrary that there exists an internal vertex x with $p \geq k+2$ leaves $l_1, l_2, \ldots, l_p$. Since the leaves of x are private neighbors of it, then there exists a minimal dominating set D including x. Consider the set $D' = D - \{x\} \cup \{l_1, l_2, \ldots, l_p\}$. Then, there exists a minimal dominating set D'' in G with $|D''| \geq |D'| = |D| + p - 1$. This implies that $\mu_d(G) \geq k+1$, contradicting the assumption $\mu_d(G) = k$. ■

Corollary 3 states an implication of Lemma 2 for almost well-dominated graphs.

Corollary 3. *Let G be an almost well-dominated graph. Then every internal vertex of G is adjacent to at most 2 leaves.*

By Corollary 3, the internal vertices of an almost well-dominated graph are of type-0, type-1, or type-2. In addition, we use the following lemma frequently in our arguments.

Lemma 4. *For every independent set I in G, $\mu_d(G - N[I]) \leq \mu_d(G)$.*

Proof. Let $H = G - N[I]$. Suppose to the contrary that $\mu_d(H) > \mu_d(G)$. Then there exist two minimal dominating sets D_1 and D_2 in H such that $|D_1| - |D_2| = \mu_d(H)$. Clearly, adding I to D_1 and D_2 results in two minimal dominating sets D_1' and D_2' in G such that $|D_1'| - |D_2'| > \mu_d(G)$, which is a contradiction. ■

An immediate result of Lemma 4 for almost well-dominated graphs is stated in the following corollary.

Corollary 5. *Let G be an almost well-dominated graph. Then for every independent set I in G, the graph $G - N[I]$ is either an almost well-dominated or a well-dominated graph.*

Our first result on almost well-dominated graphs is stated in the following lemma, which provides a basis for our characterization by restricting the number of vertices of type-2 existing in an almost well-dominated graph.

Lemma 6. *Let G be an almost well-dominated graph. Then G has at most one vertex of type-2.*

Proof. Suppose to the contrary that G has at least two vertices of type-2, say x and y with leaves $\{l_1, l_2\}$ and $\{l_3, l_4\}$, respectively. Since both of x and y have leaves (private neighbors), then there exists a minimal dominating set D_1 containing x and y. Consider the set $D = D_1 - \{x, y\} \cup \{l_1, l_2, l_3, l_4\}$. Then G has another minimal dominating set D_2 with $|D_2| \geq |D| = |D_1| + 2$, which implies that $\mu_d(G) \geq 2$, a contradiction. ■

Based on the result of Lemma 6, we continue our characterization in the following cases:

- almost well-dominated graphs containing a single vertex of type-2.
- almost well-dominated graphs containing no vertex of type-2.

3. Almost Well-Dominated Graphs Containing a Single Vertex of Type-2

Our result in this section on almost well-dominated graphs of girth at least 6 with a single vertex of type-2 is stated in Lemma 10, which follows from the results in the following two lemmas.

Lemma 7 [1]**.** *If $G \in \mathcal{D}_2$ and G has a vertex x adjacent to a set of leaves L', where $|L'| \geq 2$, then $G - (\{x\} \cup L')$ must be in $\mathcal{D}_1$.*

Lemma 8 [4]**.** *Let G be a connected well-dominated graph of girth at least 6. Then G belongs to the family $\mathcal{P}$ or G is isomorphic to K_1 or C_7.*

However, before stating the main lemma, we need to define the following graph family $\mathcal{G}_1$.

Definition 9. A graph G with girth at least 6 is in the family $\mathcal{G}_1$ if it has a single vertex of type-2 and the rest of the internal vertices, if any, are of type-1.

Lemma 10. *Let G be a connected graph of girth at least 6 with a single vertex of type-2. Then G is almost well-dominated if and only if $G \in \mathcal{G}_1$.*

Proof. Let x be a vertex of type-2 in G with two leaves, say $\{\ell_1, \ell_2\}$. We first prove that if G is almost well-dominated, then $G \in \mathcal{G}_1$. Let $G' = G - \{x, \ell_1, \ell_2\}$ and note that G' might have more than one component. By Lemma 7, we have $G' \in \mathcal{D}_1$. This means that every component of G' is well-dominated. In addition, by Lemma 8, the graphs K_1, C_7, and the family $\mathcal{P}$ are the only possible candidates for the components of G'. If there exists a component of G' isomorphic to K_1, then denote the single vertex of K_1 by y. Then the vertex x is a vertex of type-3 in G, a contradiction by Lemma 2. On the other hand, if there exists a component of G' isomorphic to a cycle $C_7 = (abcdefg)$, then due to girth at least 6, x is adjacent to exactly one vertex, say c, on C_7. Consider the independent set $I = \{a, e\}$. Then the vertex x is of type-3 in $G - N[I]$, a contradiction by Lemma 2. Now we turn our attention to the case where a component of G' belongs to the family $\mathcal{P}$. We show that x is adjacent to the components of G' through the stems of these components. Suppose to the contrary that x is adjacent to a leaf ℓ in a component $H \in \mathcal{P}$. Let s be the stem of ℓ. The stem s has at least one neighbor, say u, different from ℓ since otherwise it would not be a stem. The vertex u is not adjacent to x since otherwise $\{x, l, s, u\}$ forms a 4-cycle. Consider the independent set $I = \{u\}$. The vertex x is of type-3 in $G - N[I]$, a contradiction by Lemma 2. Hence, x is adjacent to the components of G' through the stems of these components. Thus, $G \in \mathcal{G}_1$.

In order to prove the converse, assume that $G \in \mathcal{G}_1$ and x is the only vertex of type-2. Note that from each internal vertex of type-1 and its respective leaf, only one vertex is included in any minimal dominating set D in G. Further, D includes either x and hence has cardinality $|L_G| - 1$ or D includes the leaves of x and hence has cardinality $|L_G|$. Thus, $\mu_d(G) = 1$. ∎

4. Almost Well-Dominated Graphs Containing No Vertex of Type-2

In this section we focus on almost well-dominated graphs whose internal vertices are of type-0 or type-1. Our starting point is the following proposition.

Proposition 11. *Let G be an almost well-dominated graph. If G does not contain a vertex of type-2, then it contains a vertex of type-0.*

Proof. Suppose to the contrary that there exists no vertex of type-0 in G. Then all internal vertices in G are of type-1, thus $G \in \mathcal{P}$ and hence G is well-dominated, a contradiction. ∎

Our next result restricts the number of type-0 neighbors of a type-0 vertex in (C_3, C_4, C_5, C_7)-free almost well-dominated graphs.

Lemma 12. *Let x be a vertex of type-0 in a (C_3, C_4, C_5, C_7)-free almost well-dominated graph G. Then x has at most two neighbors of type-0.*

Proof. Suppose to the contrary that x has at least three neighbors of type-0, say y, z, and w in G (see Figure 1). Note that x may also have neighbors of type-1 as shown in Figure 1. Let $N_2(x)$ and $N_3(x)$ denote the vertices at distance 2 and 3 from x, respectively. Since G is a (C_3, C_5, C_7)-free graph, both $N_2(x)$ and $N_3(x)$ are independent sets. Let $M_2(x)$ be the leaves of type-1 neighbors of x. Note that $I = N_3(x) \cup M_2(x)$ is an independent set in G. Let $H = G - N[I]$. The graph H has a vertex x with 3 leaves and hence $\mu_d(H) \geq 2$, a contradiction by Corollary 5. ∎

In the rest of the paper, a component of the subgraph induced by the vertices of type-0 is called *type-0 component*. Lemma 12 provides a tool to determine the structure of type-0 components in a (C_3, C_4, C_5, C_7)-free almost well-dominated graph.

Corollary 13. *Let G be a (C_3, C_4, C_5, C_7)-free almost well-dominated graph with no vertex of type-2. Then the graph induced by the vertices of type-0 is composed of components isomorphic to a path $P_i \in \{P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_{10}\}$ or a cycle $C_j \in \{C_6, C_8, C_9, C_{10}, C_{11}, C_{13}\}$.*

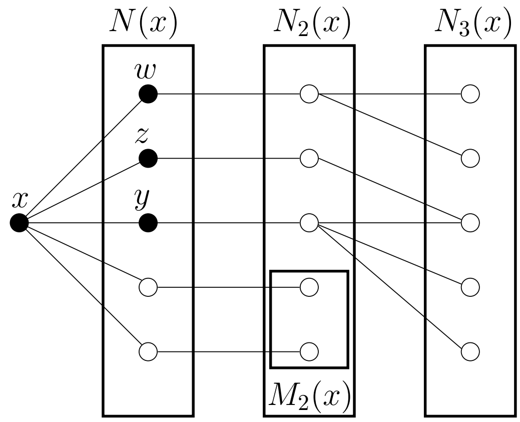

Figure 1. Type-0 vertex x with three type-0 neighbors.

Proof. Note that the graph induced by vertices of type-0 corresponds to $G - N[L_G]$ and by Lemma 12, the vertices of $G - N[L_G]$ are of degrees 0, 1 or 2. The only graph classes satisfying this degree restriction are the paths and the cycles. It follows from Lemma 4 that every component of $G - N[L_G]$ has domination gap at most 1. Note that $\gamma(P_n) = \lceil n/3 \rceil$ and $\gamma(C_n) = \lceil n/3 \rceil$, whereas $\Gamma(P_n) = \lceil n/2 \rceil$ and $\Gamma(C_n) = \lfloor n/2 \rfloor$. Thus, $P_2, P_3, P_4, P_5, P_6, P_7, P_8$, and P_{10} are the only paths having domination gap at most 1. Similarly, $C_3, C_4, C_5, C_6, C_7, C_8, C_9, C_{10}, C_{11}$, and C_{13} are the only cycles having domination gap at most 1. ■

The following lemma shows that a (C_3, C_4, C_5, C_7)-free almost well-dominated graph with no vertex of type-2 contains exactly one type-0 component.

Lemma 14. *Let G be a (C_3, C_4, C_5, C_7)-free almost well-dominated graph with no vertex of type-2. Then G has exactly one type-0 component.*

Proof. Suppose to the contrary that G has at least two type-0 components and let $H_1, H_2, \ldots, H_k$ represent the set of all type-0 components where $k \geq 2$. If $k \geq 3$, choose a minimum dominating set S_i of H_i for $3 \leq i \leq k$ and let $S = \bigcup_{i=3}^{k} S_i$. By Corollary 13, a type-0 component in a (C_3, C_4, C_5, C_7)-free almost well-dominated graph is either a path P_i, where $i \in \{1, 2, 3, 4, 5, 6, 7, 8, 10\}$ or a cycle C_j, where $j \in \{6, 8, 9, 10, 11, 13\}$.

First suppose that both H_1 and H_2 are cycles, say $H_1 \cong C_{m_1}$ and $H_2 \cong C_{m_2}$. Recall that a cycle C_n has a minimal dominating set of size $\lfloor n/2 \rfloor$. Let D_{H_1} and D_{H_2} be two minimal dominating sets of sizes $\lfloor m_1/2 \rfloor$ and $\lfloor m_2/2 \rfloor$ in H_1 and H_2, respectively. Observe that there exists a minimal dominating set D_1 in G such that $D_1 = L_G \cup D_{H_1} \cup D_{H_2} \cup S$. Then we have $|D_1| = |L_G| + \lfloor m_1/2 \rfloor + \lfloor m_2/2 \rfloor + |S|$. Note that the number of vertices of type-1 is equal to the number of leaves in G and further note that type-0 components have at least one neighbor of type-1. Let L' be a set which includes one of the vertices of type-1 adjacent to each of H_1 and H_2, and the leaves of other vertices of type-1. It is obvious that $|L'| = |L_G|$. Hence, by taking the set L', at least one vertex from each of H_1 and H_2 is dominated. Furthermore, the remaining vertices of H_1 and H_2, which induce two paths P_{m_1-1} and P_{m_2-1}, have minimal dominating sets of sizes $\lceil (m_1-1)/3 \rceil$ and $\lceil (m_2-1)/3 \rceil$, respectively. Then, there exists a minimal dominating set D_2 such that $|D_2| \leq |L_G| + \lceil (m_1-1)/3 \rceil + \lceil (m_2-1)/3 \rceil + |S|$. However, $|D_1| - |D_2| \geq \lfloor m_1/2 \rfloor - \lceil (m_1-1)/3 \rceil + \lfloor m_2/2 \rfloor - \lceil (m_2-1)/3 \rceil \geq 2$, a contradiction.

Next assume that both H_1 and H_2 are paths, say $H_1 \cong P_{m_1}$ and $H_2 \cong P_{m_2}$. Note that a path P_n has minimal dominating sets of sizes $\lceil n/2 \rceil$ and $\lceil n/3 \rceil$. Let D_{H_1} and D_{H_2} be two minimal dominating sets of sizes $\lceil m_1/2 \rceil$ and $\lceil m_2/2 \rceil$ in H_1 and H_2, respectively. Observe that the set $D_1 = L_G \cup D_{H_1} \cup D_{H_2} \cup S$ is a minimal dominating set of G. Thus, we have $|D_1| = |L_G| + \lceil m_1/2 \rceil + \lceil m_2/2 \rceil + |S|$. Note that the end vertices of a type-0 path have at least one neighbor of type-1 in G. Let L' be a set including the vertices of type-1 adjacent to the end vertices of H_1 and H_2 and the leaves of other vertices of type-1. Hence, by taking the set L', at least the end vertices of each of H_1 and H_2 are dominated. Moreover, the remaining vertices of H_1 and H_2, which induce two paths P_{m_1-2} and P_{m_2-2}, have minimal dominating sets of sizes $\lceil (m_1-2)/2 \rceil$ and $\lceil (m_2-2)/2 \rceil$, respectively. Thus, there exists a minimal dominating set D_2 such that

$$\begin{aligned} |D_2| &\leq |L_G| + \lceil (m_1-2)/2 \rceil + \lceil (m_2-2)/2 \rceil + |S| \\ &= |L_G| + \lceil m_1/2 \rceil - 1 + \lceil m_2/2 \rceil - 1 + |S|. \end{aligned}$$

It follows that $|D_1| - |D_2| \geq 2$, a contradiction.

In the last case, we suppose that one of the components, say H_1, is a cycle C_{m_1}, and the other, namely H_2, is a path P_{m_2}. Let D_{H_1} and D_{H_2} be two minimal dominating sets of sizes $\lfloor m_1/2 \rfloor$ and $\lceil m_2/2 \rceil$ in H_1 and H_2, respectively. Similarly, the set $D_1 = L_G \cup D_{H_1} \cup D_{H_2} \cup S$ is a minimal dominating set of G. Thus, we have $|D_1| = |L_G| + \lfloor m_1/2 \rfloor + \lceil m_2/2 \rceil + |S|$. Notice that H_1 has at least one neighbor of type-1 and the end vertices of H_2 both have neighbors of type-1. Let L' be a set including the vertices of type-1 adjacent to the type-0 components and the leaves of other vertices of type-1. Hence, by taking the set L', at least one vertex from H_1 and two end vertices of H_2 are dominated. Therefore, the remaining vertices of H_1, which induce a path P_{m_1-1} and the remaining vertices of H_2, which induce a path P_{m_2-2} have minimal dominating sets of sizes $\lceil (m_1-1)/3 \rceil$ and $\lceil (m_2-2)/2 \rceil$, respectively. Hence, there exists a minimal dominating set D_2 such that

$$\begin{aligned} |D_2| &\leq |L_G| + \lceil (m_1-1)/3 \rceil + \lceil (m_2-2)/2 \rceil + |S| \\ &= |L_G| + \lceil (m_1-1)/3 \rceil + \lceil m_2/2 \rceil - 1 + |S|. \end{aligned}$$

It follows that $|D_1| - |D_2| = \lfloor m_1/2 \rfloor - \lceil (m_1 - 1)/3 \rceil + 1 \geq 2$, a contradiction. ■

From here onwards, we denote the type-0 component of G by G_0. Recall that L_G denotes the set of leaves in a graph G. We will use the following proposition frequently in our proofs.

Proposition 15. *Let G be a graph with no vertex of type-k for $k \geq 2$. Then, $\Gamma(G) = |L_G| + \Gamma(G_0)$.*

Proof. Let G be a graph with no vertex of type-k for $k \geq 2$. Note that the set of leaves of G together with a maximum minimal dominating set of G_0 is a minimal dominating set of size $|L_G| + \Gamma(G_0)$ in G. Furthermore, we show that there is no minimal dominating set of size at least $|L_G| + \Gamma(G_0) + 1$ in G. First notice that any minimal dominating set of G contains exactly one vertex from each stem-leaf pair since otherwise it is not minimal. Now consider a dominating set D of size at least $|L_G| + \Gamma(G_0) + 1$ in G. Then D contains either at least $\Gamma(G_0) + 1$ vertices from G_0 or at least $|L_G| + 1$ vertices from the stem-leaf pairs. Both cases imply that D is not minimal. Thus, $\Gamma(G) = |L_G| + \Gamma(G_0)$. ■

In what follows, we focus on the cases where G_0 is isomorphic to one of the paths or cycles mentioned in Corollary 13. Using the previous results and lemmas, we show that some of these cases yield families of (C_3, C_4, C_5, C_7)-free almost well-dominated graphs.

4.1. Type-0 component is a path

In this section, we analyze almost well-dominated graphs with a type-0 component isomorphic to a path P_n. Recall that a path P_n has $\gamma(P_n) = \lceil n/3 \rceil$ and $\Gamma(P_n) = \lceil n/2 \rceil$. First let $G_0 \cong P_1$. We define the graph family $\mathcal{G}_2$ and then state the result for this case in Lemma 17.

Definition 16. A (C_3, C_4, C_5, C_7)-free graph G is in the family $\mathcal{G}_2$, if it has a single vertex of type-0 with at least two neighbors of type-1 and the rest of the internal vertices, if any exist, are of type-1.

Lemma 17. *Let G be a (C_3, C_4, C_5, C_7)-free graph without a vertex of type-2. Then G is almost well-dominated with $G_0 \cong P_1$ if and only if $G \in \mathcal{G}_2$.*

Proof. Let G be a (C_3, C_4, C_5, C_7)-free graph without a vertex of type-2. If G is almost well-dominated with $G_0 \cong P_1$, then $G \in \mathcal{G}_2$ by definition of $\mathcal{G}_2$.

To prove the converse, we assume that $G \in \mathcal{G}_2$ and let v be the vertex of type-0 in G. By Proposition 15, we have $\Gamma(G) = |L_G| + 1$. Note further that every minimal dominating set D includes exactly one vertex from each stem-leaf pair; thus, $|D| \geq |L_G|$. If any stem adjacent to v is included in a minimal dominating set D_1, then $v \notin D_1$ and thus $|D_1| = |L_G|$. On the other hand, if none of the stems adjacent to v are included in a minimal dominating set D_2, then $v \in D_2$, and thus $|D_2| = |L_G| + 1$. Hence, $\mu_d(G) = 1$. ■

Next suppose that $G_0 \cong P_2$. In this case we obtain a graph family $\mathcal{G}_3$ which is defined in Definition 18.

Definition 18. A (C_3, C_4, C_5, C_7)-free graph G is in the family $\mathcal{G}_3$, if it has one type-0 component $H \cong P_2$ where the end vertices of H have at least one neighbor of type-1 in G and the rest of the internal vertices, if any, are of type-1.

Lemma 19. *Let G be a (C_3, C_4, C_5, C_7)-free graph without a vertex of type-2. Then G is almost well-dominated with $G_0 \cong P_2$ if and only if $G \in \mathcal{G}_3$.*

Proof. Let G be a (C_3, C_4, C_5, C_7)-free graph without a vertex of type-2. If G is almost well-dominated with $G_0 \cong P_2$, then each end vertex of P_2 has at least one neighbor of type-1 in G and the rest of the internal vertices (if any) are of type-1. Hence, $G \in \mathcal{G}_3$.

To prove the converse, let $G \in \mathcal{G}_3$. Note that every minimal dominating set includes exactly one vertex from each stem-leaf pair; thus, each minimal dominating set is of size at least $|L_G|$. Furthermore, by Proposition 15, we have $\Gamma(G) = |L_G| + 1$. Therefore, it remains to show that G has two minimal dominating sets of sizes $|L_G|$ and $|L_G| + 1$. If both stems adjacent to the end vertices of P_2 are included in a minimal dominating set, then no vertex from P_2 can be added to this minimal dominating set; hence, such a minimal dominating has size $|L_G|$. However, if none of the stems adjacent to the P_2 are included in a minimal dominating set, one vertex from P_2 can be added to this minimal dominating set, which has size $|L_G| + 1$. Thus, G is an almost well-dominated graph since all minimal dominating sets are of size either $|L_G|$ or $|L_G| + 1$. ■

In the case of $G_0 \cong P_3$, we define the graph family $\mathcal{G}_4$ in Definition 20 and state the result for this case in Lemma 21.

Definition 20. A (C_3, C_4, C_5, C_7)-free graph G is in the family $\mathcal{G}_4$ if it has one type-0 component $H \cong P_3$, where the end vertices of H have at least one neighbor of type-1 in G, the middle vertex in H has no neighbors of type-1 in G, and the rest of the internal vertices, if any, are of type-1.

Lemma 21. *Let G be a (C_3, C_4, C_5, C_7)-free graph without a vertex of type-2. Then G is almost well-dominated with $G_0 \cong P_3$ if and only if $G \in \mathcal{G}_4$.*

Proof. Let G be a (C_3, C_4, C_5, C_7)-free graph without a vertex of type-2. Suppose that G is almost well-dominated with $G_0 \cong P_3$ and let $P_3 = [abc]$. Since a and c are of type-0, they have at least one neighbor of type-1. Further, we claim that the middle vertex of P_3, namely b, does not have a neighbor of type-1. Suppose to the contrary that b has at least one neighbor of type-1. Then, the set of leaves L_G together with $\{a, c\}$ form a minimal dominating set D_1 of size $|L_G| + 2$. On the other hand, consider a minimal dominating set D_2 which includes the type-1 neighbors of a, b and c. Such a minimal dominating set includes no vertices from $\{a, b, c\}$ and is of size $|L_G|$. Hence $\mu_d(G) \geq 2$, a contradiction. Thus, c has no neighbor of type-1 and hence $G \in \mathcal{G}_4$.

To prove the converse, suppose that $G \in \mathcal{G}_4$. By Proposition 15, we have $\Gamma(G) = |L_G| + 2$. Moreover, note that every minimal dominating set in a graph $G \in \mathcal{G}_4$ includes $|L_G|$ vertices from stem-leaf pairs and either one (the vertex b) or two vertices (a and c) from P_3. Thus, all minimal dominating sets are of size either $|L_G| + 1$ or $|L_G| + 2$ and hence G is an almost well-dominated graph. ■

We proceed with the case $G_0 \cong P_4$. This case yields another family of almost well-dominated graphs $\mathcal{G}_5$ defined in Definition 22.

Definition 22. A (C_3, C_4, C_5, C_7)-free graph G is in the family $\mathcal{G}_5$ if it has one type-0 component $H \cong P_4 = [abcd]$ where the end vertices of H, namely a and d have at least one neighbor of type-1 in G, at least one of the middle vertices of H, say b has no neighbors of type-1 in G, and the rest of the internal vertices, if any, are of type-1.

Lemma 23. *Let G be a (C_3, C_4, C_5, C_7)-free graph without a vertex of type-2. Then G is almost well-dominated with $G_0 \cong P_4$ if and only if $G \in \mathcal{G}_5$.*

Proof. Let G be a (C_3, C_4, C_5, C_7)-free graph without a vertex of type-2. Suppose that G is almost well-dominated with $G_0 \cong P_4$ and let $P_4 = [abcd]$. Since the end vertices a and d are of type-0, they have at least one neighbor of type-1 in G. Furthermore, we show that since G is almost well-dominated, at least one of the middle vertices, namely b or c, does not have a neighbor of type-1 in G. Suppose to the contrary that both of b and c have neighbors of type-1 in G. Then the set of leaves L_G together with two vertices from P_4, say $\{a, c\}$, form a minimal dominating set D_1 of size $|L_G| + 2$ in G. On the other hand, consider a minimal dominating set D_2 which includes the type-1 neighbors of a, b, c and d. While such a minimal dominating set includes no vertices from P_4, it includes exactly one vertex from each stem-leaf pair and has size $|L_G|$. Thus, $\mu_d(G) \geq 2$, a contradiction. Therefore, at least one of the middle vertices of P_4 has no type-1 neighbors in G. Hence, $G \in \mathcal{G}_5$.

For the converse, assume that $G \in \mathcal{G}_5$. By Proposition 15, we have $\Gamma(G) = |L_G| + 2$. Moreover, notice that every minimal dominating set in a graph $G \in \mathcal{G}_5$ includes $|L_G|$ vertices from stem-leaf pairs and either one or two vertices from P_4. Thus, all minimal dominating sets are of size either $|L_G| + 1$ or $|L_G| + 2$ and hence G is almost well-dominated. ■

The last case which we analyze in this section is the case of $G_0 \cong P_6$. However, we first deal with the cases where $G_0 \in \{P_5, P_7, P_8, P_{10}\}$ and show that in these cases G is not almost well-dominated.

Lemma 24. *Let G be a (C_3, C_4, C_5, C_7)-free graph without a vertex of type-2 and suppose that the graph induced by the vertices of type-0 in G is isomorphic to a path P_m for $m \in \{5, 7, 8, 10\}$. Then G is not almost well-dominated.*

Proof. Let G be a (C_3, C_4, C_5, C_7)-free graph without a vertex of type-2 and let H be the graph induced by the vertices of type-0 in G. Suppose that $H \cong P_m$ where $m \in \{5, 7, 8, 10\}$. Note that a path on m vertices has two minimal dominating sets of sizes $\lceil m/2 \rceil$ and $\lceil m/3 \rceil$. Consider a minimal dominating set of cardinality $\lceil m/2 \rceil$ in H, say D_H. Then, the set of leaves L_G together with D_H form a minimal dominating set D_1 of size $|L_G| + \lceil m/2 \rceil$ in G. On the other hand, let S be the set of stems in G. Note that $|S| = |L_G|$ and S definitely dominates the end vertices of H. Let further H' be the graph induced by the internal vertices of H. Since $H' \cong P_{m-2}$, a minimal dominating set $D_{H'}$ in H' together with S form a minimal dominating set D_2 of size at most $|L_G| + \lceil (m-2)/3 \rceil$ in G. For $m \in \{5, 7, 8, 10\}$, we get $|D_1| - |D_2| \geq 2$, thus G is not almost well-dominated. ∎

The case of $G_0 \cong P_6$ leads to a family of almost well-dominated graphs $\mathcal{G}_6$ defined in Definition 25.

Definition 25. A (C_3, C_4, C_5, C_7)-free graph G is in the family $\mathcal{G}_6$ if it has one type-0 component $H \cong P_6 = [abcdef]$ where the end vertices of H have at least one neighbor of type-1 in G, the vertices adjacent to the end vertices of H, namely $\{b, e\}$, have no neighbors of type-1 in G, and the rest of the internal vertices, if any, are of type-1.

Lemma 26. *Let G be a (C_3, C_4, C_5, C_7)-free graph without a vertex of type-2. Then G is almost well-dominated with $G_0 \cong P_6$ if and only if $G \in \mathcal{G}_6$.*

Proof. Let G be a (C_3, C_4, C_5, C_7)-free graph without a vertex of type-2. Suppose that G is almost well-dominated with $G_0 \cong P_6$ and let $P_6 = [abcdef]$. As the end vertices a and f are of type-0, they have at least one neighbor of type-1 in G. Furthermore, it is easy to see that G has two minimal dominating sets $D_1 = L_G \cup \{a, c, e\}$ and $D_2 = L_G \cup \{b, e\}$ of sizes $|L_G| + 3$ and $|L_G| + 2$, respectively. Thus, in order for G to be almost well-dominated, the minimal dominating sets of size smaller than $|L_G| + 2$ must be avoided. We show that the vertices b and e have no neighbors of type-1 in G. Suppose for a contradiction that at least one of b and e, say b, has neighbors of type-1 in G. Then, consider a minimal dominating set D_3 which includes the vertex d from P_6 and the type-1 neighbors of a, b, and f. Such a minimal dominating set includes exactly one vertex from each stem-leaf pair and the vertex d. Hence, $|D_3| = |L_G| + 1$. Thus, $\mu_d(G) \geq 2$, a contradiction. Therefore, none of b and e have neighbors of type-1 in G. Thus, $G \in \mathcal{G}_6$.

To prove the converse, suppose that $G \in \mathcal{G}_6$. By Proposition 15, we have $\Gamma(G) = |L_G| + 3$. Furthermore, note that every minimal dominating set in a graph $G \in \mathcal{G}_6$ includes $|L_G|$ vertices from stem-leaf pairs and either two or three vertices from P_6. Thus, all minimal dominating sets of G have size either $|L_G|+2$ or $|L_G| + 3$. Hence, G is almost well-dominated. ∎

4.2. Type-0 component is a cycle

In this section we investigate the cases where the type-0 component is isomorphic to a cycle C_n, where $n \in \{6, 8, 9, 10, 11, 13\}$. Recall that a cycle C_n has $\gamma(C_n) = \lceil n/3 \rceil$ and $\Gamma(C_n) = \lfloor n/2 \rfloor$. Let us first assume that $G_0 \cong C_6$. We will define the following graph family in order to state our result in Lemma 28.

Definition 27. A (C_3, C_4, C_5, C_7)-free graph G is in the family $\mathcal{G}_7$ if it has one type-0 component $H \cong C_6$ where no three consecutive vertices on H have neighbors of type-1 in G and the rest of the internal vertices, if any, are of type-1.

Lemma 28. *Let G be a (C_3, C_4, C_5, C_7)-free graph without a vertex of type-2. Then G is almost well-dominated with $G_0 \cong C_6$ if and only if $G \in \mathcal{G}_7$.*

Proof. Let G be a (C_3, C_4, C_5, C_7)-free graph without a vertex of type-2. Suppose that G is almost well-dominated with $G_0 \cong C_6$. Note that C_6 has two minimal dominating sets of sizes 2 and 3; thus, together with the set L_G, the graph G has minimal dominating sets of sizes $|L_G| + 2$ and $|L_G| + 3$. By Proposition 15, we have that $\Gamma(G) = |L_G| + 3$. Now it remains to ensure that the cases which lead to minimal dominating sets with size at most $|L_G| + 1$ are avoided. These cases are as follows.

- If all vertices of C_6 are adjacent to stems, then the stems constitute a minimal dominating set of size $|L_G|$ in G.
- If the vertices of C_6 which are not adjacent to stems induce a path P_i where $i \in \{1, 2, 3\}$, then one vertex from P_i together with the stems form a minimal dominating set of size $|L_G| + 1$ in G.

In order to avoid the above cases, no three consecutive vertices on C_6 must have neighbors of type-1; therefore $G \in \mathcal{G}_7$.

To prove the converse suppose that $G \in \mathcal{G}_7$. By Proposition 15, we have $\Gamma(G) = |L_G| + 3$. Furthermore, by the definition of $\mathcal{G}_7$, no three consecutive vertices on C_6 have type-1 neighbors, which implies that all the minimal dominating sets of G include at least two vertices from C_6 and hence of size at least $|L_G| + 2$. Hence, G is almost well-dominated. ∎

Next we assume $G_0 \cong C_8$. In Definition 29, we define an almost well-dominated graph family $\mathcal{G}_8$ which has a type-0 component isomorphic to C_8.

Definition 29. A (C_3, C_4, C_5, C_7)-free graph G is in the family $\mathcal{G}_8$ if it has one type-0 component $H \cong C_8 = (abcdefgh)$ where neither two consecutive vertices nor two vertices at distance 4 (say for example a and e) on H have type-1 neighbors in G, and the rest of the vertices, if any, are of type-1.

Lemma 30. *Let G be a (C_3, C_4, C_5, C_7)-free graph without a vertex of type-2. Then G is almost well-dominated with $G_0 \cong C_8$ if and only if $G \in \mathcal{G}_8$.*

Proof. Let G be a (C_3, C_4, C_5, C_7)-free graph without a vertex of type-2. Suppose that G is almost well-dominated with $G_0 \cong C_8$ and let $C_8 = (abcdefgh)$. Note that C_8 has two minimal dominating sets of sizes 3 and 4; thus, together with the set L_G, the graph G has minimal dominating sets of sizes $|L_G| + 3$ and $|L_G| + 4$. By Proposition 15, we have $\Gamma(G) = |L_G| + 4$. Furthermore, since G is almost well-dominated, the cases leading to minimal dominating sets of size at most $|L_G|+2$ must be avoided. The cases which require that at most two vertices from C_8 being included in a minimal dominating set are as follows.

- If the vertices of C_8 which are not adjacent to type-1 neighbors induce a single path P_m where $m \leq 6$, then $\lceil m/3 \rceil$ vertices from P_m together with the stems constitute a minimal dominating set D of size $|L_G| + \lceil m/3 \rceil$ in G. For $m \leq 6$, we have that $\lceil m/3 \rceil \leq 2$. Hence, $|D| \leq |L_G| + 2$.
- If two vertices of C_8 with distance 4, say a and e, have neighbors of type-1, then the stems together with two vertices from C_8, namely c and g, form a dominating set of size $|L_G| + 2$, which in turn includes a minimal dominating set of size at most $|L_G| + 2$ in G.

In order to avoid the above cases, neither two consecutive vertices nor two vertices at distance 4 on C_8 have type-1 neighbors in G. Therefore, $G \in \mathcal{G}_8$.

To prove the converse suppose that $G \in \mathcal{G}_8$. By Proposition 15, $\Gamma(G) = |L_G| + 4$. Then, by definition of $\mathcal{G}_8$, all the minimal dominating sets of G include at least three vertices from C_8 and thus, have size at least $|L_G| + 3$. Therefore, G is almost well-dominated. ■

We proceed with the case where $G_0 \cong C_9$.

Definition 31. A (C_3, C_4, C_5, C_7)-free graph G is in the family $\mathcal{G}_9$, if it has one type-0 component $H \cong C_9 = (abcdefghi)$ with the following properties.

- No three consecutive vertices on H have type-1 neighbors in G.
- No two consecutive vertices on H, say $\{a, b\}$, together with a vertex at distance 4 from both a and b on H, say f, have type-1 neighbors in G.

Lemma 32. *Let G be a (C_3, C_4, C_5, C_7)-free graph without a vertex of type-2. Then G is almost well-dominated with $G_0 \cong C_9$ if and only if $G \in \mathcal{G}_9$.*

Proof. Let G be a (C_3, C_4, C_5, C_7)-free graph without a vertex of type-2. Suppose that G is almost well-dominated with $G_0 \cong C_9$. Note that C_9 has two minimal dominating sets of sizes 3 and 4; thus, together with the set L_G, the graph G has minimal dominating sets of sizes $|L_G| + 3$ and $|L_G| + 4$. By Proposition 15, we have that $\Gamma(G) = |L_G| + 4$. Then, it suffices to guarantee that the cases which lead to minimal dominating sets of size at most $|L_G|+2$ are prevented; since otherwise, the domination gap becomes at least two. The cases which require that at most two vertices from C_9 are included in a minimal dominating set are as follows.

- If the vertices of C_9 which do not have neighbors of type-1 in G induce a single path P_m for $m \leq 6$, then the stems together with $\lceil m/3 \rceil$ vertices from P_m form a minimal dominating set D of size $|L_G| + \lceil m/3 \rceil$. Since $m \leq 6$, we have that $\lceil m/3 \rceil \leq 2$. Then $|D| \leq |L_G| + 2$.
- If the vertices of C_9 which are not adjacent to neighbors of type-1 in G induce two disjoint paths P_i and P_j on C_9 for $i \leq 3$ and $j \leq 3$, then the stems together with $\lceil i/3 \rceil$ vertices from P_i and $\lceil j/3 \rceil$ vertices from P_j constitute a minimal dominating set D of size $|L_G| + \lceil i/3 \rceil + \lceil j/3 \rceil$ in G. However, $|D| \leq |L_G| + 2$ since we have that $\lceil i/3 \rceil \leq 1$ and $\lceil j/3 \rceil \leq 1$ for $i \leq 3$ and $j \leq 3$.

In order to avoid the first case, no three consecutive vertices on C_9 must have type-1 neighbors in G. Furthermore, to prevent the second case, no two consecutive vertices together with a vertex at distance 4 from these consecutive vertices on C_9 must have type-1 neighbors in G. Hence, $G \in \mathcal{G}_9$.

To prove the converse, suppose that $G \in \mathcal{G}_9$. It follows from Proposition 15 that $\Gamma(G) = |L_G| + 4$. Furthermore, by definition of $\mathcal{G}_9$, any minimal dominating set of G includes at least three vertices from C_9 and hence has size at least $|L_G| + 3$. Therefore, G is almost well-dominated. ■

In the case of $G_0 \cong C_m$ where $m \in \{10, 13\}$, we show that there exists a unique almost well-dominated graph for each value of m.

Lemma 33. *Let G be a (C_3, C_4, C_5, C_7)-free graph without a vertex of type-2. Then G is almost well-dominated with $G_0 \cong C_m$ for $m \in \{10, 13\}$ if and only if $G \cong C_m$ for $m \in \{10, 13\}$.*

Proof. Let G be a (C_3, C_4, C_5, C_7)-free graph without a vertex of type-2. Suppose that G is almost well-dominated with $G_0 \cong C_m$ for $m \in \{10, 13\}$. Suppose to the contrary that C_m has at least one neighbor of type-1 in G, say u. Let l be the leaf neighbor of u in G. Note that C_m has two minimal dominating sets of sizes $\lfloor m/2 \rfloor$ and $\lceil m/3 \rceil$; thus, together with the set L_G, G has two minimal dominating sets: D_1 of size $|L_G| + \lfloor m/2 \rfloor$ and D_2 of size $|L_G| + \lceil m/3 \rceil$. Note that the vertex u has at least one neighbor, say v, on C_m. Observe that the set $L_G - \{l\} \cup \{u\}$, which is of size $|L_G|$, dominates at least the vertex v from C_m. Hence, the vertices of C_m different from v, which induce a path P_{m-1}, has a minimal dominating set of size $\lceil (m-1)/3 \rceil$. Thus, the set $L_G - \{l\} \cup \{u\}$ together with $\lceil (m-1)/3 \rceil$ vertices from P_{m-1} form a dominating set D of size $|L_G| + \lceil (m-1)/3 \rceil$, which implies a minimal dominating set D_3 of size at most $|L_G| + \lceil (m-1)/3 \rceil$. However, $|D_1| - |D_3| \geq 2$ for $m \in \{10, 13\}$, a contradiction.

The proof for the converse is straightforward since it is easy to verify that C_{10} and C_{13} are almost well-dominated graphs. ■

The last case we settle in this section is the case where $G_0 \cong C_{11}$. We obtain a family of almost well-dominated graphs $\mathcal{G}_{10}$ defined in Definition 34.

Definition 34. A (C_3, C_4, C_5, C_7)-free graph G is in the family $\mathcal{G}_{10}$ if it has a type-0 component $H \cong C_{11} = (abcdefghijk)$ with the following properties.

- No two consecutive vertices on H have type-1 neighbors in G.
- No two vertices at distance 4 on H, say a and e, have type-1 neighbors in G.

Lemma 35. *Let G be a (C_3, C_4, C_5, C_7)-free graph without a vertex of type-2. Then G is almost well-dominated with $G_0 \cong C_{11}$ if and only if $G \in \mathcal{G}_{10}$.*

Proof. Let G be a (C_3, C_4, C_5, C_7)-free graph without a vertex of type-2. Suppose that G is almost well-dominated with $G_0 \cong C_{11} = (abcdefghijk)$. Note that C_{11} has two minimal dominating sets of sizes 4 and 5; thus, together with the set of leaves L_G, the graph G has minimal dominating sets of sizes $|L_G| + 4$ and $|L_G| + 5$. Notice that $\Gamma(G) = |L_G| + 5$ by Proposition 15. Therefore, the cases leading to a minimal dominating set of size at most $|L_G| + 3$ must be prevented since otherwise, the domination gap becomes at least two. The cases which require that at most three vertices from C_{11} be included in a minimal dominating set are as follows.

- If the vertices of C_{11} which do not have neighbors of type-1 in G induce a single path P_m for $m \leq 9$, then the stems together with $\lceil m/3 \rceil$ vertices from P_m form a minimal dominating set D of size $|L_G| + \lceil m/3 \rceil$. Since $m \leq 9$, we have that $\lceil m/3 \rceil \leq 3$. Thus, $|D| \leq |L_G| + 3$.
- If the vertices of C_{11} which do not have neighbors of type-1 in G induce two disjoint paths P_3 and P_6, say $[abc]$ and $[efghij]$, respectively, then the set L_G together with one vertex from P_3, namely b, and two vertices from P_6, namely f and i, form a dominating set D of size $|L_G| + 3$, which implies a minimal dominating set of size at most $|L_G| + 3$ in G.

In order to avoid the first case, no two consecutive vertices on C_{11} must have type-1 neighbors in G. Furthermore, to prevent the second case, no two vertices at distance 4 on C_{11} must have type-1 neighbors in G. Hence, $G \in \mathcal{G}_{10}$.

To prove the converse suppose that $G \in \mathcal{G}_{10}$. It follows from Proposition 15 that $\Gamma(G) = |L_G| + 5$. Moreover, by the definition of $\mathcal{G}_{10}$, all minimal dominating sets in G include at least four vertices from C_{11} and thus, have size at least $|L_G| + 4$. Hence, G is almost well-dominated. ■

Our main result for (C_3, C_4, C_5, C_7)-free almost well-dominated graphs is stated in the following theorem.

Theorem 36. *Let G be a (C_3, C_4, C_5, C_7)-free graph. Then, G is an almost well-dominated graph if and only if one of the following holds.*

- *G has a single vertex of type-2 and $G \in \mathcal{G}_1$.*
- *G has no vertex of type-2 and $G \in \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4 \cup \mathcal{G}_5 \cup \mathcal{G}_6 \cup \mathcal{G}_7 \cup \mathcal{G}_8 \cup \mathcal{G}_9 \cup \mathcal{G}_{10} \cup \{C_{10}, C_{13}\}$.*

Proof. It is first followed by Lemma 6 that a (C_3, C_4, C_5, C_7)-free graph G has at most one vertex of type-2. Then we proceed the proof in two cases: G has a single vertex of type-2 and G has no vertex of type-2. While the first case follows from Lemma 3, the latter follows from Lemmas 17, 19, 21, 23, 24, 26, 28, 30, 32, 33, and 35. ■

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