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BOUNDS ON THE DOUBLE ITALIAN DOMINATION NUMBER OF A GRAPH

Farzaneh Azvin and Nader Jafari Rad

Department of Mathematics, Shahed University, Tehran, Iran

e-mail: n.jafarirad@gmail.com

Abstract

For a graph G, a Roman $\{3\}$-dominating function is a function $f : V \longrightarrow \{0, 1, 2, 3\}$ having the property that for every vertex $u \in V$, if $f(u) \in \{0, 1\}$, then $f(N[u]) \geq 3$. The weight of a Roman $\{3\}$-dominating function is the sum $w(f) = f(V) = \sum_{v \in V} f(v)$, and the minimum weight of a Roman $\{3\}$-dominating function is the *Roman* $\{3\}$*-domination number*, denoted by $\gamma_{\{R3\}}(G)$. In this paper, we present a sharp lower bound for the double Italian domination number of a graph, and improve previous bounds given in [D.A. Mojdeh and L. Volkmann, *Roman* $\{3\}$*-domination* (*double Italian domination*), Discrete Appl. Math. 283 (2022) 555–564]. We also present a probabilistic upper bound for a generalized version of double Italian domination number of a graph, and show that the given bound is asymptotically best possible.

Keywords: Italian domination, double Italian domination, probabilistic methods.

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1. Introduction

For a (simple) graph $G = (V, E)$ with vertex set $V = V(G)$ and edge set $E = E(G)$, we denote by $|V(G)| = n(G) = n$ the *order* of G. The *open neighborhood* of a vertex v is $N(v) = \{u \in V : uv \in E\}$ and the *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood of S is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighborhood of S is $N[S] = N(S) \cup S$. The *degree* of a vertex v is $\deg(v) = |N(v)|$. The maximum and minimum degree among the vertices of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. For a set $S \subseteq V$ in a graph G and a vertex $v \in V$, we say that S *dominates* v if $v \in S$ or

v is adjacent to some vertex of S. A set S is called a *dominating set* in G if S dominates every vertex of G. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set in G. For other definitions and notations not given here we refer to [6].

Cockayne *et al.* [5] introduced the concept of Roman domination in graphs, although this notion was inspired by the work of ReVelle *et al.* in [11], and Stewart in [12]. Let $f : V(G) \longrightarrow \{0, 1, 2\}$ be a function having the property that for every vertex $v \in V$ with $f(v) = 0$, there exists a neighbor $u \in N(v)$ with $f(u) = 2$. Such a function is called a *Roman dominating function* or just an RDF. The weight of an RDF f is the sum $f(V) = \sum_{v \in V} f(v)$. The minimum weight of an RDF on G is called the *Roman domination number* of G, and is denoted by $\gamma_R(G)$. Several varieties of Roman domination are already studied, and the reader can consult [3, 4].

A generalization of Roman domination called *Italian domination* (or Roman $\{2\}$-domination) was introduced by Chellali *et al.* in [2], Klostermeyer and MacGillivray [8], and Henning and Klostermeyer [7]. An *Italian dominating function (IDF)* on a graph $G = (V, E)$ is a function $f : V \longrightarrow \{0, 1, 2\}$ satisfying the property that for every vertex $v \in V$, with $f(v) = 0$, $\sum_{u \in N(v)} f(u) \geq 2$. The weight of an IDF f is the value $w(f) = f(V) = \sum_{u \in V} f(u)$. The minimum weight of an IDF on a graph G is called the *Italian domination number* of G, denoted by $\gamma_I(G)$. This same concept was called *Roman* $\{2\}$*-domination* and what we called $\gamma_I(G)$ is called $\gamma_{\{R2\}}(G)$. A $\gamma_{\{R2\}}(G)$-function f can be represented by a triple $f = (V_0, V_1, V_2)$ (or $f = (V_0^f, V_1^f, V_2^f)$ to refer to f), where $V_i = \{v \in V(G) : f(v) = i\}$ for $i = 0, 1, 2$.

Beeler *et al.* [1] introduced the concept of double Roman domination in graphs. A function $f : V \longrightarrow \{0, 1, 2, 3\}$ is a *double Roman dominating function* (or just DRDF) on a graph G if the following conditions hold, where V_i denote the set of vertices assigned i under f, for $i = 0, 1, 2, 3$: (1) If $f(v) = 0$, then v must have at least two neighbors in V_2 or one neighbor in V_3; (2) If $f(v) = 1$, then v must have at least one neighbor in $V_2 \cup V_3$. The weight of a DRDF f is the value $w(f) = f(V) = \sum_{v \in V} f(v)$. The *double Roman domination number*, $\gamma_{dR}(G)$, is the minimum weight of a DRDF on G, and a DRDF of G with weight $\gamma_{dR}(G)$ is called a γ_{dR}-function of G.

Recently, Mojdeh and Volkmann [9] considered an extension of Roman $\{2\}$-domination as follows. For a graph G, a *Roman* $\{3\}$*-dominating function* is a function $f : V \longrightarrow \{0, 1, 2, 3\}$ having the property that for every vertex $u \in V$, if $f(u) \in \{0, 1\}$, then $f(N[u]) \geq 3$. The weight of a Roman $\{3\}$-dominating function is the sum $w(f) = f(V) = \sum_{v \in V} f(v)$, and the minimum weight of a Roman $\{3\}$-dominating function is the *Roman* $\{3\}$*-domination number*, denoted by $\gamma_{\{R3\}}(G)$. For a Roman $\{3\}$-dominating function f, one can denote $f = (V_0, V_1, V_2, V_3)$, where $V_i = \{v \in V : f(v) = i\}$, for $i = 0, 1, 2, 3$. This concept was further studied in [10]. Among other results, Mojdeh *et al.* presented the following lower bound in [9].

Theorem 1 (Mojdeh and Volkmann [9]). *If G is a connected graph of order n and maximum degree Δ, then $\gamma_{\{R3\}}(G) \geq \min\left\{\frac{3n}{\Delta+2}, \frac{2n+\Delta}{\Delta+1}\right\}$.*

In this paper we present upper and lower bounds for the Roman {3}-domination number of a graph. In Section 2, we present a sharp lower bound for the Roman {3}-domination number of a graph and improve the bound given in Theorem 1. In Section 3, we present a probabilistic upper bound for a generalized version of the Roman {3}-domination number, namely, the Roman $\{k\}$-domination number for every $k \geq 3$, of a graph and show that the given bound is asymptotically best possible.

In this paper, for an event F we denote by $Pr(F)$ the probability that F occurs. We also denote by $\mathbb{E}(X)$ the expectation of X if X is a random variable.

2. Lower Bound

In this section we present a new sharp lower bound for the Roman {3}-domination number of a graph. We begin with the following observation.

Observation 2. *For every connected graph G of order n and maximum degree Δ, $\gamma_{\{R3\}}(G) \leq 2(n-\Delta)+1$.*

Proof. Let v be a vertex of maximum degree. Let f be a function defined on $V(G)$ by $f(v) = 3$, $f(x) = 0$ if $x \in N(v)$ and $f(x) = 2$ otherwise. Then f is a R3DF for G, and so $\gamma_{\{R3\}}(G) \leq 2(n-\Delta-1)+3 = 2(n-\Delta)+1$, as desired. ■

Lemma 3. *If G is a connected graph of maximum degree $\Delta(G) = \Delta \geq 1$ and $f = (V_0, V_1, V_2, V_3)$ is a $\gamma_{\{R3\}}(G)$-function, then $3|V_0| \leq (\Delta-2)|V_1| + 2\Delta|V_2| + 3\Delta|V_3|$.*

Proof. Let G be a connected graph of maximum degree $\Delta(G) = \Delta \geq 1$ and $f = (V_0, V_1, V_2, V_3)$ be a $\gamma_{\{R3\}}(G)$-function. If $\Delta = 1$, then $G = K_2$, and since $\gamma_{\{R3\}}(K_2) = 3$, we obtain that either $|V_0| = |V_3| = 1$ and $|V_1| = |V_2| = 0$ or $|V_0| = |V_3| = 0$ and $|V_1| = |V_2| = 1$. Thus the inequality holds.

Hence we assume that $\Delta \geq 2$. We partition V_0 into four sets V_0^3, V_0^{12}, V_0^1 and V_0^2, and V_1 into three sets V_1^1, V_1^2 and V_1^3 as follows. Let

$$\begin{aligned}
V_0^3 &= \{v \in V_0 : N(v) \cap V_3 \neq \emptyset\},\\
V_0^{12} &= \left\{v \in V_0 \setminus V_0^3 : N(v) \cap V_1 \neq \emptyset, N(v) \cap V_2 \neq \emptyset\right\},\\
V_0^1 &= \left\{v \in V_0 \setminus \left(V_0^3 \cup V_0^{12}\right) : N(v) \subseteq V_0 \cup V_1\right\},\\
V_0^2 &= \left\{v \in V_0 \setminus \left(V_0^3 \cup V_0^{12}\right) : N(v) \subseteq V_0 \cup V_2\right\}.
\end{aligned}$$

$$\begin{aligned} V_1^1 &= \{x \in V_1 : N(x) \cap (V_2 \cup V_3) = \emptyset\}, \\ V_1^2 &= \{x \in V_1 : N(x) \cap V_2 \neq \emptyset\}, \\ V_1^3 &= V_1 \setminus \left(V_1^1 \cup V_1^2\right). \end{aligned}$$

For $i = 1, 2, 3$, let $|V_1^i| = m_i$. We first present an upper bound for $|V_0^3|$ in terms of $|V_3|$ and m_3. Each vertex in V_3 with no neighbor in V_1 dominates at most Δ vertices of V_0^3, and every vertex in V_3 with at least one neighbor in V_1 dominates at most $\Delta - 1$ vertices of V_0^3. Thus,

$$\left|V_0^3\right| \leq \Delta(|V_3| - m_3) + (\Delta - 1)m_3 = \Delta|V_3| - m_3. \tag{1}$$

Let $|V_0^{12}| = x$. We next present an upper bound for $|V_0^2|$ in terms of $|V_2|$, x and m_2. Clearly, every vertex of $V_1^2 \cup V_0^{12}$ has a neighbor in V_2. Since $|V_1^2| = m_2$ and $|V_0^{12}| = x$, there are at most $\Delta|V_2| - x - m_2$ edges which have an end-point in V_2. Since any vertex of V_0^2 is adjacent to at least two vertices of V_2, we obtain that

$$\left|V_0^2\right| \leq \frac{\Delta|V_2| - x - m_2}{2}. \tag{2}$$

We next present an upper bound for $\left|V_0^1\right|$ in terms of $|V_1|$, x, m_1, m_2 and m_3. Note that every vertex of V_1^1 is adjacent to at least two vertices of V_1, every vertex of V_1^2 is adjacent to at least one vertex of V_2 and every vertex of V_1^3 is adjacent to at least one vertex of V_3. Also every vertex of V_0^{12} is adjacent to a vertex in V_1. Thus, there are at most $\Delta|V_1| - 2m_1 - m_2 - m_3 - x$ edges which have an end-point in V_1. Since any vertex of V_0^1 is adjacent to at least three vertices of V_1, we obtain that

$$\left|V_0^1\right| \leq \frac{\Delta|V_1| - 2m_1 - m_2 - m_3 - x}{3}. \tag{3}$$

Since $|V_0| = \left|V_0^1\right| + \left|V_0^2\right| + \left|V_0^{12}\right| + \left|V_0^3\right|$, from (1), (2) and (3) we obtain that

$$3|V_0| \leq \Delta|V_1| - 2m_1 - m_2 - m_3 - x + \frac{3\Delta|V_2|}{2} - \frac{3x}{2} - \frac{3m_2}{2} + 3x + 3\Delta|V_3| - 3m_3.$$

Since $|V_1| = m_1 + m_2 + m_3$, we obtain that

$$3|V_0| \leq (\Delta - 2)|V_1| + 2\Delta|V_2| + 3\Delta|V_3| - \frac{m_2}{2} - 2m_3 + \frac{x}{2} - \frac{\Delta|V_2|}{2}.$$

It is evident that $x \leq \Delta|V_2|$. Thus $3|V_0| \leq (\Delta - 2)|V_1| + 2\Delta|V_2| + 3\Delta|V_3|$, as desired. ■

Corollary 4. *If G is a connected graph of order n with maximum degree $\Delta(G) = \Delta \geq 1$ and $f = (V_0, V_1, V_2, V_3)$ is a $\gamma_{\{R3\}}(G)$-function, then $\gamma_{\{R3\}}(G) \geq \frac{3n - |V_2|}{\Delta + 1}$.*

Proof. Let $f = (V_0, V_1, V_2, V_3)$ be a $\gamma_{\{R3\}}(G)$-function for G. Then

$$\begin{aligned}(\Delta+1)\gamma_{\{R3\}}(G) &= (\Delta+1)|V_1| + 2(\Delta+1)|V_2| + 3(\Delta+1)|V_3| \\ &= (\Delta-2)|V_1| + 2\Delta|V_2| + 3\Delta|V_3| + 3|V_1| + 2|V_2| + 3|V_3| \\ &\geq 3|V_0| + 3|V_1| + 2|V_2| + 3|V_3| \qquad \text{(by Lemma 3)} \\ &= 3n - |V_2|.\end{aligned}$$

Thus the result follows. ■

Now we present the main result of this section.

Theorem 5. *If G is a connected graph of order $n > 1$ and maximum degree $\Delta \geq 1$, then*

$$\gamma_{\{R3\}}(G) \geq \left\lfloor \max\left\{\frac{3n}{\Delta+2}, \frac{2n+\Delta}{\Delta+1}\right\}\right\rfloor + 1.$$

This bound is sharp.

Proof. Let G be a connected graph of order n and maximum degree Δ and $f = (V_0, V_1, V_2, V_3)$ be a $\gamma_{\{R3\}}(G)$-function for G. If $|V_2| = 0$, then from Corollary 4, we obtain that $\gamma_{\{R3\}}(G) \geq \frac{3n}{\Delta+1} > \max\left\{\frac{3n}{\Delta+2}, \frac{2n+\Delta}{\Delta+1}\right\}$, and so $\gamma_{\{R3\}}(G) \geq \left\lfloor \max\left\{\frac{3n}{\Delta+2}, \frac{2n+\Delta}{\Delta+1}\right\}\right\rfloor + 1$.

Thus assume that $|V_2| \neq 0$. By Corollary 4, $\gamma_{\{R3\}}(G) \geq \frac{3n-|V_2|}{\Delta+1}$. Using Observation 2, we obtain $|V_2| \leq \frac{\gamma_{\{R3\}}(G)}{2} \leq n - \Delta + \frac{1}{2}$ which implies that $|V_2| \leq n - \Delta$.

We first show that $\gamma_{\{R3\}}(G) > \frac{2n+\Delta}{\Delta+1}$. If $|V_2| < n - \Delta$, then we obtain that $\gamma_{\{R3\}}(G) \geq \frac{3n-|V_2|}{\Delta+1} > \frac{2n+\Delta}{\Delta+1}$, as desired. Thus assume that $|V_2| = n - \Delta$. Suppose that $\gamma_{\{R3\}}(G) = \frac{3n-|V_2|}{\Delta+1}$. Let $|V_i| = v_i$ for $i = 0, 1, 2, 3$. Then from $\gamma_{\{R3\}}(G) = \frac{3n-|V_2|}{\Delta+1}$ we find that $(\Delta+1)(v_1 + 2v_2 + 3v_3) = 3v_0 + 3v_1 + 2v_2 + 3v_3$, since $n = v_0 + v_1 + v_2 + v_3$. Since $\Delta = n - v_2 = v_0 + v_1 + v_3$, we obtain by a simple calculation that $v_1 + 2v_2 + 3v_3 = \frac{3v_0+2v_1}{n-v_2} = \frac{3v_0+2v_1}{v_0+v_1+v_3}$ and this implies that $2v_2 = 3-(v_1+3v_3)-\frac{v_1+3v_3}{v_0+v_1+v_3}$. If $v_3 = 0$ and $v_1 = 0$, then $2v_2 = 3$, a contradiction. If $v_3 = 0$ and $v_1 \neq 0$, then $2v_2 = 3 - v_1 - \frac{v_1}{v_0+v_1} < 2$, a contradiction. Thus $v_3 \neq 0$. If $v_1 = 0$, then $2v_2 = 3 - 3v_3 - \frac{3v_3}{v_0+v_3} < 0$, a contradiction. Thus $v_1 \neq 0$. Then $2v_2 = 3 - (v_1 + 3v_3) - \frac{v_1+3v_3}{v_0+v_1+v_3} < 0$, a contradiction. We conclude that $\gamma_{\{R3\}}(G) \neq \frac{3n-|V_2|}{\Delta+1}$, and so $\gamma_{\{R3\}}(G) > \frac{3n-|V_2|}{\Delta+1} \geq \frac{2n+\Delta}{\Delta+1}$, as desired.

We next show that $\gamma_{\{R3\}}(G) > \frac{3n}{\Delta+2}$. If $|V_2| \geq \frac{3n}{\Delta+2}$, then $\gamma_{\{R3\}}(G) \geq 2|V_2| > \frac{3n}{\Delta+2}$. Thus assume that $|V_2| < \frac{3n}{\Delta+2}$. Then a simple calculation shows that $\frac{3n-|V_2|}{\Delta+1} > \frac{3n}{\Delta+2}$, as desired.

Hence, $\gamma_{\{R3\}}(G) > \max\left\{\frac{2n+\Delta}{\Delta+1}, \frac{3n}{\Delta+2}\right\}$. This completes the proof of lower bound.

To see the sharpness, consider a complete graph of order at least two. ■

We end this section by remarking that Lemma 3 holds for each R3DF. Moreover, it holds if $\Delta = 0$.

3. Upper Bound

In this section we present an upper bound for a generalization of the Roman $\{3\}$-domination number namely *Roman $\{k\}$-domination number* for every integer $k \geq 3$ that is defined as follows. For a graph G and an integer $k \geq 3$, a *Roman $\{k\}$-dominating function* is a function $f : V \longrightarrow \{0, 1, \ldots, k\}$ having the property that for every vertex $u \in V$, if $f(u) < \lceil \frac{k}{2} \rceil$, then $f(N[u]) \geq k$. The weight of a Roman $\{k\}$-dominating function is the sum $w(f) = f(V) = \sum_{v \in V} f(v)$, and the minimum weight of a Roman $\{k\}$-dominating function is the *Roman $\{k\}$-domination number*, denoted by $\gamma_{\{Rk\}}(G)$. For a Roman $\{k\}$-dominating function f, we denote $f = (V_0, V_1, \ldots, V_k)$, where $V_i = \{v \in V : f(v) = i\}$, for $i = 0, 1, \ldots, k$.

Theorem 6. *If G is a graph of order n with minimum degree $\delta(G) = \delta \geq 1$, then*

$$\gamma_{\{Rk\}}(G) \leq \frac{k\left(\ln \lceil \frac{k}{2} \rceil + \ln(1+\delta) - \ln k + 1\right)}{1+\delta} n.$$

Proof. Let G be a graph of order n with minimum degree $\delta(G) = \delta \geq 1$. Let $S \subseteq V(G)$ be a set obtained by choosing each vertex $v \in V(G)$, independently, with probability $p \in [0,1]$, and let $T = V(G) \setminus N[S]$. We form sets V_i, $i = 0, 1, \ldots, k$ as follows. Let $V_0 = V(G) \setminus (S \cup T)$, $V_{\lceil \frac{k}{2} \rceil} = T$, $V_k = S$ and $V_i = \emptyset$ for $i = 1, 2, \ldots, k-1, i \neq \lceil \frac{k}{2} \rceil$. Then the function $f = (V_0, V_1, \ldots, V_k)$ is a Roman $\{k\}$-dominating function for G. We compute the expected value of $w(f)$. Note that

$$\mathbb{E}(w(f)) = \mathbb{E}(k|S| + \left\lceil \frac{k}{2} \right\rceil |T|) = k\mathbb{E}(|S|) + \left\lceil \frac{k}{2} \right\rceil \mathbb{E}(|T|).$$

Clearly, $\mathbb{E}(|S|) = np$. If $v \in T$, then $v \notin S$ and $v \notin N(S)$. Thus, $Pr(v \notin N[S]) = (1-p)^{1+\deg(v)} \leq (1-p)^{1+\delta}$. Using the fact that $1 - p \leq e^{-p}$ for $p \geq 0$, we find that $Pr(v \in T) \leq e^{-p(1+\delta)}$, and so $\mathbb{E}(|T|) \leq ne^{-p(1+\delta)}$. Therefore,

$$\mathbb{E}(w(f)) = \mathbb{E}\left(k|S| + \left\lceil \frac{k}{2} \right\rceil |T|\right) \leq knp + \left\lceil \frac{k}{2} \right\rceil ne^{-p(1+\delta)}. \tag{4}$$

Taking derivative of the function $g(p) = kp + \left\lceil \frac{k}{2} \right\rceil e^{-p(1+\delta)}$ and solving the equation $g'(p) = 0$, we obtain that $g(p)$ is minimized at $p = \frac{\ln \frac{\lceil \frac{k}{2} \rceil (1+\delta)}{k}}{1+\delta}$. Then by putting these values in (4) we obtain

$$\mathbb{E}(w(f)) \leq nk \left(\frac{\ln \frac{\lceil \frac{k}{2} \rceil (1+\delta)}{k} + 1}{1+\delta} \right) =: \alpha.$$

Since the average of $w(f)$ is not more than α, there is a Roman $\{k\}$-dominating function with weight at most α, i.e., $\gamma_{\{Rk\}}(G) \leq \alpha$, as desired. ■

We now prove that the upper bound of Theorem 6 is asymptotically best possible.

Theorem 7. *When n is large, there exists a graph G of order n and minimum degree δ such that*

$$\gamma_{\{Rk\}}(G) \geq \frac{k\left(\ln \left\lceil \frac{k}{2} \right\rceil + \ln(1+\delta) - \ln k + 1\right)}{1+\delta} n(1 + o(1)).$$

Proof. Let H a complete graph with $\lfloor \delta \ln \delta \rfloor$ vertices and let $V(H) = V$. We add a set of new vertices $V' = \{v_1, v_2, \ldots, v_\delta\}$ and join each of them to δ vertices of $V(H)$ which are chosen randomly. Let G be the resulted graph. Therefore G has $n = \lfloor \delta \ln \delta \rfloor + \delta$ vertices. We show that

$$\gamma_{\{Rk\}}(G) \geq \frac{k \ln \delta}{\delta} n(1 + o_\delta(1)) = \frac{k \ln \delta}{\delta} (\delta \ln \delta + \delta)(1 + o_\delta(1)) = k \ln^2 \delta (1 + o_\delta(1)).$$

Let $f = (V_0, V_1, \ldots, V_k)$ be a $\gamma_{\{Rk\}}$-function for G. If $|V_k| \geq \ln^2 \delta - \ln \delta \, \ln \ln^4 \delta$, then

$$\gamma_{\{Rk\}}(G) \geq k|V_k| \geq k \ln^2 \delta - k \ln \delta \, \ln \ln^4 \delta = k \ln^2 \delta (1 + o_\delta(1)),$$

as desired. Thus assume for the next that $|V_k| < \ln^2 \delta - \ln \delta \, \ln \ln^4 \delta$.

We compute the probability that V_k dominates an element of V'. Note that we can assume $V_k \subseteq V$. For a vertex $v_i \in V'$, we have

$Pr[V_k$ does not dominate $v_i]$

$$= \frac{\binom{|V| - |V_k|}{\delta}}{\binom{|V|}{\delta}} \geq \left(\frac{|V| - |V_k| - \delta}{|V| - \delta} \right)^\delta = \left(1 - \frac{|V_k|}{|V| - \delta} \right)^\delta.$$

Using the fact that $1 - x \geq e^{-x}(1 - x^2)$ for $x \leq 1$, we find that

$$\begin{aligned} Pr[V_k \text{ does not dominate } v_i] &\geq e^{-\frac{|V_k|}{|V|-\delta}\delta} \left(1 - \left(\frac{|V_k|}{|V| - \delta}\right)^2\right)^{\delta} \\ &\geq e^{-\frac{\ln^2\delta - \ln\delta \, \ln\ln^4\delta}{\delta\ln\delta - \delta}\delta} (1 + o_\delta(1)) \\ &\geq e^{-\frac{\ln\delta - \ln\ln^4\delta}{1-\frac{1}{\ln\delta}}} (1 + o_\delta(1)) \\ &\geq e^{\ln\left(\frac{\ln^4\delta}{\delta}\right)(1+o_\delta(1))} (1 + o_\delta(1)) \\ &\geq \left(\frac{\ln^4\delta}{\delta}\right)^{(1+o_\delta(1))} (1 + o_\delta(1)) \geq \frac{\ln^3\delta}{\delta}. \end{aligned}$$

Thus $Pr[V_k \text{ dominates } v_i] \leq 1 - \frac{\ln^3\delta}{\delta}$. Now the expected value of the random variable $|N(V_k) \cap V'|$ is bounded above as follows

$$\mathbb{E}\left(|N(V_k) \cap V'|\right) = \sum_{i=1}^{\delta} Pr[V_k \text{ dominates } v_i] \leq \delta\left(1 - \frac{\ln^3\delta}{\delta}\right) = \delta - \ln^3\delta.$$

Consequently, $|V' \setminus N(V_k)| \geq \ln^3\delta$. Since $V_k \subseteq V$, we conclude that there exists a graph G for which

$$\gamma_{\{Rk\}}(G) \geq |V' \setminus N(V_k)| \geq \ln^3\delta > k\ln^2\delta(1 + o_\delta(1)),$$

as desired. ■

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