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ON M_f-EDGE COLORINGS OF GRAPHS

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Abstract

An edge coloring φ of a graph G is called an M_f-edge coloring if $|\varphi(v)| \leq f(v)$ for every vertex v of G, where $\varphi(v)$ is the set of colors of edges incident with v and f is a function which assigns a positive integer $f(v)$ to each vertex v. Let $\mathcal{K}_f(G)$ denote the maximum number of colors used in an M_f-edge coloring of G. In this paper we establish some bounds on $\mathcal{K}_f(G)$, present some graphs achieving the bounds and determine exact values of $\mathcal{K}_f(G)$ for some special classes of graphs.

Keywords: edge coloring, anti-Ramsey number, dominating set.

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1. Introduction

We consider finite undirected graphs without loops and multiple edges. If G is a graph, then $V(G)$ and $E(G)$ stand for the vertex set and edge set of G, respectively. The subgraph of a graph G induced by $U \subseteq V(G)$ is denoted by $G[U]$. Similarly, if $A \subseteq E(G)$, then $G[A]$ denotes the subgraph of G induced by A (i.e., the subgraph with the edge set A and the vertex set consisting of all vertices incident with an edge in A). The set of vertices of G adjacent to a vertex $v \in V(G)$ is denoted by $N_G(v)$. The cardinality of this set, denoted $\deg_G(v)$, is called the degree of v. As usual $\Delta(G)$ and $\delta(G)$ stand for the maximum and minimum degree among vertices of G. The set of vertices of degree d in G is denoted by $V_d(G)$.

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An edge coloring of a graph G is an assignment of colors to the edges of G, one color to each edge. So any mapping φ from $E(G)$ onto a non-empty set is an edge coloring of G. The set of colors used in an edge coloring φ of G is denoted by $\varphi(G)$, i.e., $\varphi(G) = \{\varphi(e) : e \in E(G)\}$. For any vertex $v \in V(G)$, let $\varphi(v)$ denote the set of colors of edges incident with v, i.e., $\varphi(v) = \{\varphi(vu) : u \in N_G(v)\}$. Let f be a function which assigns a positive integer $f(v)$ to each vertex $v \in V(G)$. An edge coloring φ of G is an M_f*-edge coloring* if at most $f(v)$ colors appear at any vertex v of G, i.e., $|\varphi(v)| \leq f(v)$ for every vertex $v \in V(G)$. The maximum number of colors used in an M_f-edge coloring of G is denoted by $\mathcal{K}_f(G)$. If $f(v) = i$ for all $v \in V(G)$, then an M_f-edge coloring is called an M_i*-edge coloring* and the maximum number of colors used in an M_i-edge coloring is denoted by $\mathcal{K}_i(G)$.

The M_f-edge coloring is a natural generalization of the M_i-edge coloring. The concept of M_i-edge colorings was introduced by Czap [4]. In [3] authors establish a tight bound on $\mathcal{K}_2(G)$ depending on the size of a maximum matching in G. In [4] and [5], the exact values of $\mathcal{K}_2(G)$ for subcubic graphs and complete graphs are determined. In [7] it is determined $\mathcal{K}_2(G)$ for cacti, trees, graph joins and complete multipartite graphs. In [10] there are established some bounds on $\mathcal{K}_2(G)$ and presented graphs achieving the bounds. Exact values of $\mathcal{K}_2(G)$ for dense graphs are also determined. $\mathcal{K}_3(G)$ and $\mathcal{K}_4(G)$ for complete graphs are determined in [6]. A vertex variant of the M_2-edge coloring was studied in [2].

However before, Feng *et al.* [8] introduced a *maximum edge q-coloring problem* which arises from wireless mesh networks. It is really the problem of finding an M_q-edge coloring of a given graph G which uses $\mathcal{K}_q(G)$ colors (for an integer q, $q \geq 2$). There are studied mainly algorithmic aspects of the maximum edge q-coloring problem. In [8] there is provided a 2-approximation algorithm for $q = 2$ and a $\left(1 + \frac{4q-2}{3q^2-5q+2}\right)$-approximation for $q > 2$. In [1] there is proved that the maximum edge q-coloring problem is NP-Hard. Also, for graphs with perfect matching there is presented a $\frac{5}{3}$-approximation algorithm in case $q = 2$. A related problem is studied in [12].

The *anti-Ramsey number* has been extensively studied in the area of extremal graph theory (see [9] for a survey). For given graphs G and H the anti-Ramsey number $\mathrm{ar}(G, H)$ is defined to be the maximum number k such that there exists an assignment of k colors to the edges of G in which every copy of H in G has at least two edges with the same color. A coloring of G is an M_q-edge coloring if and only if each subgraph $K_{1,q+1}$ (a star with $q + 1$ edges) of G has two edges with the same color. Therefore $\mathcal{K}_q(G)$ is equal to $\mathrm{ar}(G, K_{1,q+1})$. Thereby, in [11] there is determined $\mathcal{K}_q(K_{n,n})$ exactly and $\mathcal{K}_q(K_n)$ within 1, for all positive integers n and q. Similarly, an upper bound on the value of $\mathcal{K}_q(G)$ if $\delta(G) \geq q + 5$, or if G is K_3-free and $\delta(G) \geq q + 2$, is presented in [13]. Some applications of this bound (e.g., exact values of $\mathcal{K}_q(G)$ for hypercubes) are also produced.

In this paper we establish some bounds of $\mathcal{K}_f(G)$ depending on dominating sets of G. We also determine exact values of $\mathcal{K}_f(G)$ for some particular classes of graphs, especially for trees, forests, some cactuses, and dense graphs with a dominating vertex. Accordingly, we extend some known results, proved in [11] and [13], on $\mathcal{K}_q(G)$ (as anti-Ramsey number) for complete graphs and complete multipartite graphs.

2. Auxiliary Results

It is easy to see that $|\varphi(v)| \leq \deg_G(v)$ for any edge coloring φ of a graph G and each vertex $v \in V(G)$. Therefore, throughout the paper we suppose that the function f satisfies

$$1 \leq f(v) \leq \deg_G(v) \quad \text{for every } v \in V(G). \tag{1}$$

The following two claims are evident.

Observation 1. *Let f be a function from the vertex set of a graph G to positive integers. Assume that G has k connected components. Let G_j, $j \in \{1, \ldots, k\}$, be a component of the graph G and let f_j be a restriction of f to $V(G_j)$. Then*

$$\mathcal{K}_f(G) = \sum_{j=1}^{k} \mathcal{K}_{f_j}(G_j).$$

Given a graph G, let $e = uv$ be an edge of G such that $\deg_G(v) \geq 2$. By $S(G; e, v)$ we denote the graph with the vertex set $V(G) \cup \{v'\}$ and the edge set $(E(G) \setminus \{e\}) \cup \{uv'\}$.

Observation 2. *Let f be a function from the vertex set of a graph G to integers satisfying* (1)*. Let v be a vertex of G such that $f(v) = \deg_G(v) \geq 2$. For an edge e incident with v let h be a function from the vertex set of $S(G; e, v)$ to integers given by*

$$h(u) = \begin{cases} f(u) & \text{if } u \notin \{v, v'\}, \\ \deg_{S(G;e,v)}(u) & \text{if } u \in \{v, v'\}. \end{cases}$$

Then

$$\mathcal{K}_f(G) = \mathcal{K}_h\big(S(G; e, v)\big).$$

Let φ be an M_f-edge coloring of G. For a set $U \subseteq V(G)$, let $\varphi(U)$ denote the set of colors of edges incident with vertices of U in G. Thus, $\varphi(U) = \bigcup_{v \in U} \varphi(v)$.

Lemma 1. *Let φ be an M_f-edge coloring of a graph G and let U be a non-empty subset of $V(G)$. Then the following statements hold.*

(i) $|\varphi(U)| \leq c + \sum_{u\in U}\big(f(u)-1\big)$, *where c denotes the number of connected components of $G[U]$.*

(ii) *If $G[U]$ is a 2-connected graph and $|\varphi(U)| = 1 + \sum_{u\in U}\big(f(u)-1\big)$, then $|\{\varphi(e) : e \in E(G[U])\}| = 1$.*

Proof. (i) First suppose that $G[U]$ is a connected graph. Denote the vertices of U by $u_1, u_2, \ldots, u_k$ in such a way that the set $X_i = \{u_1, u_2, \ldots, u_i\}$ induces a connected subgraph of G for every $i \in \{1, 2, \ldots, k\}$. As $G[X_i]$ is connected for $i \geq 2$, there is j $(1 \leq j < i)$ such that u_iu_j is an edge of G. Therefore, $\varphi(u_iu_j) \in \varphi(X_{i-1}) \cap \varphi(u_i)$ and

$$\begin{aligned}|\varphi(X_i)| = |\varphi(X_{i-1}) \cup \varphi(u_i)| &= |\varphi(X_{i-1})| + |\varphi(u_i)| - |\varphi(X_{i-1}) \cap \varphi(u_i)|\\ &\leq |\varphi(X_{i-1})| + f(u_i) - 1.\end{aligned}$$

Clearly, $|\varphi(X_1)| = |\varphi(u_1)| \leq f(u_1) = 1 + \sum_{u\in X_1}\big(f(u)-1\big)$. Thus, by induction we get

$$|\varphi(X_i)| \leq |\varphi(X_{i-1})| + \big(f(u_i)-1\big) \leq 1 + \sum_{u\in X_i}\big(f(u)-1\big)$$

and consequently $|\varphi(U)| = |\varphi(X_k)| \leq 1 + \sum_{u\in U}\big(f(u)-1\big)$.

If $G[U]$ is a disconnected graph, then the set U can be partitioned into disjoint subsets $U_1, U_2, \ldots, U_c$ in such a way that $G[U_i]$ is a connected component of $G[U]$ for every $i \in \{1, 2, \ldots, c\}$. Therefore,

$$\begin{aligned}|\varphi(U)| = \left|\varphi\left(\bigcup_{i=1}^{c} U_i\right)\right| \leq \sum_{i=1}^{c} |\varphi(U_i)| &\leq \sum_{i=1}^{c}\left(1 + \sum_{u\in U_i}\big(f(u)-1\big)\right)\\ &= c + \sum_{u\in U}\big(f(u)-1\big).\end{aligned}$$

(ii) Now suppose that $G[U]$ is 2-connected and $|\{\varphi(e) : e \in E(G[U])\}| > 1$. Then there are edges uw and vw in $E(G[U])$ such that $\varphi(uw) \neq \varphi(vw)$. Therefore, $|\varphi(w) \cap \big(\varphi(u) \cup \varphi(v)\big)| \geq 2$ and consequently $|\varphi(w) \cap \varphi(U \setminus \{w\})| \geq 2$. As $G[U]$ is 2-connected, $G[U \setminus \{w\}]$ is connected and by (i)

$$|\varphi(U \setminus \{w\})| \leq 1 + \sum_{u\in U\setminus\{w\}}\big(f(u)-1\big).$$

Hence $|\varphi(U)| \leq |\varphi(U \setminus \{w\})| + f(w) - 2 \leq \sum_{u\in U}\big(f(u)-1\big)$, which completes the proof. ∎

A subgraph H of a graph G is called an *f-subgraph* of G if $\deg_H(v) < f(v)$ for every $v \in V(H)$. The maximum number of edges in an f-subgraph of G is denoted by $\alpha_f(G)$ and the maximum number of edges in an f-subgraph of $G[U]$ ($U \subset V(G)$) is denoted by $\alpha_f(U)$ (i.e., $\alpha_f(G) = \alpha_f(V(G))$). If $f(v) = i$ for all $v \in V(G)$, then $\alpha_f(G)$ and $\alpha_f(U)$ is denoted by $\alpha_i(G)$ and $\alpha_i(U)$, respectively.

Lemma 2. *Let H be an f-subgraph of a graph G. Then there is an* M_f*-edge coloring of G such that $|\varphi(G)| = c + |E(H)|$, where c denotes the number of connected components of $G[E(G) \setminus E(H)]$,*

Proof. Denote by $e_1, e_2, \ldots, e_h$ edges of H and by $C_1, C_2, \ldots, C_c$ components of $G[E(G) \setminus E(H)]$ ($c = 0$ when $E(H) = E(G)$). Consider a mapping φ from $E(G)$ onto $\{1, 2, \ldots, h + c\}$ given by

$$\varphi(e) = \begin{cases} j & \text{if } e \in E(H) \text{ and } e = e_j, \\ h + j & \text{if } e \notin E(H) \text{ and } e \in C_j. \end{cases}$$

Clearly, $|\varphi(v)| \leq \deg_H(v) + 1 \leq f(v)$, for any vertex $v \in V(G)$. Therefore, φ is a desired M_f-edge coloring of G. ■

Lemma 3. *Let G be a connected graph of order at least 2. Let $c(v)$ denote the number of components of $G - v$ and $d(v) = \min\{c(v), f(v)\}$ for every $v \in V(G)$. Then there is an* M_f*-edge coloring φ of G such that*

$$|\varphi(G)| = 1 + \sum_{v \in V(G)} (d(v) - 1) \text{ and } |\varphi(v)| = d(v) \text{ for every } v \in V(G).$$

Proof. Denote vertices of $U = \{u \in V(G) : d(u) > 1\}$ by $u_1, u_2, \ldots, u_k$. Put $U_0 = \emptyset$, $s_0 = 0$ and $U_i = U_{i-1} \cup \{u_i\}$, $s_i = s_{i-1} + d(u_i) - 1$, for $i \in \{1, 2, \ldots, k\}$. Evidently, $s_i = \sum_{v \in U_i}(d(v) - 1)$. For all $i \in \{0, 1, \ldots, k\}$, define the M_f-edge coloring φ_i of G recursively in the following way.

Let φ_0 be a mapping from $E(G)$ to $\{0\}$. As $\varphi_0(e) = 0$, for every edge $e \in E(G)$, $|\varphi_0(G)| = 1 = 1 + s_0$ and $|\varphi_0(v)| = 1$ for each $v \in V(G)$.

Now suppose that a mapping φ_i from $E(G)$ onto $\{0, 1, \ldots, s_i\}$ is an M_f-edge coloring of G such that $|\varphi_i(v)| = d(v)$ for $v \in U_i$ and $|\varphi_i(v)| = 1$ for $v \in V(G) \setminus U_i$. As $u_{i+1} \notin U_i$, $|\varphi_i(u_{i+1})| = 1$. Since $d(u_{i+1}) > 1$, the graph $G - u_{i+1}$ is disconnected with $c(u_{i+1})$ components. As $c(u_{i+1}) \geq d(u_{i+1})$, we can choose components $C_1, C_2, \ldots, C_t$ (where $t = d(u_{i+1}) - 1$) of $G - u_{i+1}$. For each $j \in \{1, 2, \ldots, t\}$, let H_j be a subgraph of G induced by $V(C_j) \cup \{u_{i+1}\}$. Consider a mapping φ_{i+1} from $E(G)$ onto $\{0, 1, \ldots, s_{i+1}\}$ given by

$$\varphi_{i+1}(e) = \begin{cases} s_i + j & \text{if } \varphi_i(e) \in \varphi_i(u_{i+1}) \text{ and } e \in E(H_j), \\ \varphi_i(e) & \text{otherwise.} \end{cases}$$

Evidently, $|\varphi_{i+1}(v)| = |\varphi_i(v)|$ for $v \in V(G) \setminus \{u_{i+1}\}$, and $|\varphi_{i+1}(u_{i+1})| = 1 + t = d(u_{i+1})$. Therefore, φ_{i+1} is an M_f-edge coloring of G such that $|\varphi_{i+1}(G)| =$

$1 + s_i + t = 1 + s_{i+1}$. Moreover, $|\varphi_{i+1}(v)| = d(v)$ for $v \in U_{i+1}$ and $|\varphi_{i+1}(v)| = 1$ for $v \in V(G) \setminus U_{i+1}$.

Thus, there is an M_f-edge coloring φ ($\varphi = \varphi_k$) of G such that $|\varphi(G)| = 1+s_k$, $|\varphi(v)| = d(v)$ for $v \in U_k = U$, and $|\varphi(v)| = 1$ for $v \in V(G) \setminus U$. As $d(v) = 1$ for each $v \in V(G) \setminus U$, φ is a desired coloring. ■

3. Main Results

A set $D \subseteq V(G)$ is called *dominating* in G, if for each $v \in V(G) \setminus D$ there exists a vertex $u \in D$ adjacent to v.

Theorem 1. *Let D be a dominating set of a graph G. If c denotes the number of connected components of $G[D]$, then*

$$\mathcal{K}_f(G) \le c + \sum_{u \in D} \big(f(u) - 1\big) + \alpha_f\big(V(G) \setminus D\big).$$

Proof. Let φ be an M_f-edge coloring of G which uses $\mathcal{K}_f(G)$ colors, i.e., $|\varphi(G)| = \mathcal{K}_f(G)$. Suppose that A is a subset of $E(G)$ containing exactly one edge of each color belonging to $\varphi(G) \setminus \varphi(D)$. Let H be a subgraph of G induced by A. Evidently, the graph H is an f-subgraph of G and $V(H) \subseteq V(G) \setminus D$. Therefore, $|A| = |E(H)| \le \alpha_f\big(V(G) \setminus D\big)$. Thus,

$$\mathcal{K}_f(G) = |\varphi(D)| + |A| \le |\varphi(D)| + \alpha_f(V(G) \setminus D).$$

According to Lemma 1, $|\varphi(D)| \le c + \sum_{u \in D} \big(f(u) - 1\big)$ and the desired inequality follows. ■

The following result present some graphs achieving the bound established in Theorem 1.

Theorem 2. *Let D be a dominating set of a connected graph G satisfying*

(i) $|D| \ge 2$;

(ii) *$G[D]$ is a connected subgraph of G;*

(iii) *if $u \in D$ and $c(u)$ is the number of connected components of $G[D] - u$, then there is at least $f(u) - c(u)$ vertices in $V(G) \setminus D$ adjacent to u;*

(iv) *$f(v) = \deg_G(v)$ for all $v \in V(G) \setminus D$.*

Then

$$\mathcal{K}_f(G) = 1 + \big|E\big(G[V(G) \setminus D]\big)\big| + \sum_{u \in D} \big(f(u) - 1\big).$$

Proof. For each vertex $v \in V(G) \setminus D$ there is a vertex in D adjacent to v. Thus, $\deg_{G[V(G)\setminus D]}(v) < \deg_G(v) = f(v)$. Therefore, $G[V(G) \setminus D]$ is an f-subgraph of G and $\alpha_f\big(V(G) \setminus D\big) = \big|E\big(G[V(G) \setminus D]\big)\big|$. According to (ii), $G[D]$ is a connected subgraph of G, and by Theorem 1 we have

$$\begin{aligned}\mathcal{K}_f(G) &\le 1 + \sum_{u \in D} \big(f(u) - 1\big) + \alpha_f\big(V(G) \setminus D\big)\\ &= 1 + \sum_{u \in D} \big(f(u) - 1\big) + \big|E\big(G[V(G) \setminus D]\big)\big|.\end{aligned}$$

On the other hand, according to (i) and (ii), $G[D]$ is a connected graph of order at least 2. For every vertex $u \in D$, set $A(u) = \{uv \in E(G) : v \in V(G) \setminus D\}$, $d(u) = \min\{c(u), f(u)\}$, and $t(u) = f(u) - d(u)$. By (iii), $|A(u)| \ge t(u)$. Thus, there is a set $A^*(u)$ such that $A^*(u) \subseteq A(u)$ and $|A^*(u)| = t(u)$. Clearly,

$$\left|E\big(G[V(G) \setminus D]\big) \cup \bigcup_{u \in D} A^*(u)\right| = \big|E\big(G[V(G) \setminus D]\big)\big| + \sum_{u \in D} t(u).$$

Therefore, there is a bijection ζ from $E\big(G[V(G) \setminus D]\big) \cup \bigcup_{u \in D} A^*(u)$ onto a set B, where $|B| = \big|E\big(G[V(G) \setminus D]\big)\big| + \sum_{u \in D} t(u)$.

According to Lemma 3, there is an M_f-edge coloring φ of $G[D]$ such that $|\varphi(G[D])| = 1 + \sum_{u \in D} \big(d(u) - 1\big)$ and $|\varphi(u)| = d(u)$ for each $u \in D$. Moreover, we can assume that $\varphi(G[D])$ and B are disjoint sets. Now suppose that ξ is any mapping from D to $\varphi(G[D])$ satisfying $\xi(u) \in \varphi(u)$ for each $u \in D$. Consider the edge coloring ψ of G defined in the following way

$$\psi(e) = \begin{cases} \varphi(e) & \text{if } e \in E(G[D]),\\ \zeta(e) & \text{if } e \in A^*(u),\\ \xi(u) & \text{if } e \in A(u) \setminus A^*(u),\\ \zeta(e) & \text{if } e \in E\big(G[V(G) \setminus D]\big).\end{cases}$$

We have $|\psi(u)| = |\varphi(u)| + |A^*(u)| = d(u) + t(u) = f(u)$, for any vertex $u \in D$, and $|\psi(v)| \le \deg_G(v) = f(v)$, for any vertex $v \in V(G) \setminus D$. So, ψ is an M_f-edge coloring of G which uses $\big|\varphi\big(G[D]\big)\big| + |B|$ colors. Hence

$$\begin{aligned}\big|\psi(G)\big| &= 1 + \sum_{u \in D} \big(d(u) - 1\big) + \big|E\big(G[V(G) \setminus D]\big)\big| + \sum_{u \in D} t(u)\\ &= 1 + \sum_{u \in D} \big(d(u) - 1 + t(u)\big) + \big|E\big(G[V(G) \setminus D]\big)\big|\\ &= 1 + \sum_{u \in D} \big(f(u) - 1\big) + \big|E\big(G[V(G) \setminus D]\big)\big|,\end{aligned}$$

i.e., $\mathcal{K}_f(G) \ge 1 + \sum_{u \in D} \big(f(u) - 1\big) + \big|E\big(G[V(G) \setminus D]\big)\big|$. ■

Recall that a connected graph in which every edge belongs to at most one cycle is called a *cactus*.

Corollary 3. *Let G be a cactus of order at least 2. For every vertex $u \in V(G)$, let $\nu(u)$ denote the number of cycles of G containing u. If $f(u)+\nu(u) \leq \deg_G(u)$, for all $u \in V(G)$, then*

$$\mathcal{K}_f(G) = 1 + \sum_{u \in V(G)} \big(f(u) - 1\big).$$

Proof. Evidently, $D = V(G)$ is a dominating set of G. As G is a cactus, $c(u)$, the number of connected components of $G-u$, is equal to $\deg_G(u)-\nu(u)$ for every vertex $u \in V(G)$. Then, $f(u) - c(u) = f(u) + \nu(u) - \deg_G(u) \leq 0$. Therefore, the conditions of Theorem 2 are satisfied. Moreover, $\big|E\big(G[V(G) \setminus D]\big)\big| = 0$. According to Theorem 2, the result follows. ■

Corollary 4. *Let T be a tree of order at least 2. Let f be a function from $V(T)$ to positive integers satisfying* (1)*. Then*

$$\mathcal{K}_f(T) = 1 + \sum_{u \in V(T)} \big(f(u) - 1\big) = |E(T)| - \sum_{u \in V(T)} \big(\deg_T(u) - f(u)\big).$$

Especially, if q is a positive integer, then

$$\mathcal{K}_q(T) = 1 + \big(q-1\big)\big|V(T)\big| - \sum_{j=1}^{q-1} \big(q-j\big)\big|V_j(T)\big|.$$

Proof. Each tree is a cactus without cycles. Therefore, by Corollary 3,

$$\begin{aligned}
\mathcal{K}_f(T) &= 1 + \sum_{u \in V(T)} \big(f(u) - 1\big) = 1 + \sum_{u \in V(T)} f(u) - \big|V(T)\big| \\
&= \sum_{u \in V(T)} f(u) - \big|E(T)\big| = \big|E(T)\big| + \sum_{u \in V(T)} f(u) - 2\big|E(T)\big| \\
&= \big|E(T)\big| + \sum_{u \in V(T)} f(u) - \sum_{u \in V(T)} \deg_T(u) \\
&= \big|E(T)\big| - \sum_{u \in V(T)} \big(\deg_T(u) - f(u)\big).
\end{aligned}$$

Now consider a function t from $V(T)$ to positive integers given by

$$t(u) = \min\{\deg_T(u), q\}.$$

Then

$$\sum_{u\in V(T)} t(u) = \sum_{j=1}^{\Delta(T)} \left(\sum_{\substack{u\in V(T)\\ \deg_T(u)=j}} t(u) \right) = \sum_{j=1}^{q-1} j\big|V_j(T)\big| + \sum_{j=q}^{\Delta(T)} q\big|V_j(T)\big|$$
$$= \sum_{j=1}^{\Delta(T)} q\big|V_j(T)\big| - \sum_{j=1}^{q-1}(q-j)\big|V_j(T)\big| = q\big|V(T)\big| - \sum_{j=1}^{q-1}(q-j)\big|V_j(T)\big|.$$

Evidently, $\mathcal{K}_q(T) = \mathcal{K}_t(T)$. Thus

$$\mathcal{K}_q(T) = 1 + \sum_{u\in V(T)} \big(t(u)-1\big) = 1 + \sum_{u\in V(T)} t(u) - \big|V(T)\big|$$
$$= 1 + q\big|V(T)\big| - \sum_{j=1}^{q-1}(q-j)\big|V_j(T)\big| - \big|V(T)\big|$$
$$= 1 + (q-1)\big|V(T)\big| - \sum_{j=1}^{q-1}(q-j)\big|V_j(T)\big|,$$

which completes the proof. ■

Corollary 5. *Let F be a forest whose every component is of order at least 2. Let f be a function from $V(F)$ to positive integers satisfying (1). Then*

$$\mathcal{K}_f(F) = |E(F)| - \sum_{u\in V(F)} \big(\deg_F(u) - f(u)\big).$$

Proof. Let T_j, $j \in \{1,\dots,k\}$, be a component of F and let f_j be a restriction of f to $V(T_j)$. Every component of F is a tree, thus, by Observation 1 and Corollary 4, we have

$$\mathcal{K}_f(F) = \sum_{j=1}^{k} \mathcal{K}_{f_j}(T_j) = \sum_{j=1}^{k} \left(|E(T_j)| - \sum_{u\in V(T_j)} \big(\deg_{T_j}(u) - f(u)\big) \right)$$
$$= |E(F)| - \sum_{u\in V(F)} \big(\deg_F(u) - f(u)\big),$$

which completes the proof. ■

Corollary 6. *Let f be a function from the vertex set of a graph G to positive integers satisfying (1). If every cycle of G contains a vertex v such that $f(v) = \deg_G(v)$, then*

$$\mathcal{K}_f(G) = |E(G)| - \sum_{u\in V(G)} \big(\deg_G(u) - f(u)\big).$$

Proof. Suppose that G is a counterexample with the minimum number of cycles. According to Corollary 5, G contains a cycle C. Then there is a vertex v of C such that $f(v) = \deg_G(v)$. Let e be an edge of the cycle C incident with v. Consider a graph $H = S(G; e, v)$ and a function h from $V(H)$ to positive integers defined by

$$h(u) = \begin{cases} f(u) & \text{if } u \in V(H) \setminus \{v, v'\}, \\ \deg_H(u) & \text{if } u \in \{v, v'\}. \end{cases}$$

Clearly, every cycle of H is also a cycle in G and it contains a vertex w such that $h(w) = \deg_H(w)$. Moreover, H has less cycles than G and so it is not a counterexample. Then, $\mathcal{K}_h(H) = |E(H)| - \sum_{u\in V(H)} \big(\deg_H(u) - h(u)\big)$. By Observation 2, $\mathcal{K}_f(G) = \mathcal{K}_h(H)$. Therefore,

$$\begin{aligned} \mathcal{K}_f(G) = \mathcal{K}_h(H) &= |E(H)| - \sum_{u\in V(H)} \big(\deg_H(u) - h(u)\big) \\ &= |E(G)| - \sum_{u\in V(G)} \big(\deg_G(u) - f(u)\big), \end{aligned}$$

a contradiction to the choice of G. ■

The following result present other graphs achieving the bound established in Theorem 1.

Theorem 3. *Let D be a dominating set of a graph G such that $|D| \geq 2$ and $G[D]$ is a connected subgraph of G. Let I be a set of isolated vertices in $G[V(G) \setminus D]$. If there is a spanning subgraph B of G satisfying*

(i) *every edge of B is incident with a vertex in I,*

(ii) $\deg_B(u) = f(u) - 1$ *if $u \in D$,*

(iii) $\deg_B(u) < f(u)$ *if $u \in I$ and* $\deg_G(u) > f(u)$,

then

$$\mathcal{K}_f(G) = 1 + \sum_{u\in D} \big(f(u) - 1\big) + \alpha_f\big(V(G) \setminus D\big).$$

Proof. Set $k = \sum_{u\in D} \big(f(u) - 1\big)$ and $\alpha = \alpha_f\big(V(G) \setminus D\big)$. According to (i), every edge of B connects a vertex from I with one from D. Moreover, by (ii), $|E(B)| = k$. Let H be an f-subgraph of $G[V(G) \setminus D]$ having α edges. Clearly, no edge of H is incident with a vertex in I.

Denote by $e_1, e_2, \ldots, e_k$ edges of B and by $a_1, a_2, \ldots, a_\alpha$ edges of H. Consider the mapping ψ from $E(G)$ onto $\{1, 2, \ldots, 1 + k + \alpha\}$ given by

$$\psi(e) = \begin{cases} j & \text{if } e \in E(B) \text{ and } e = e_j, \\ k + j & \text{if } e \in E(H) \text{ and } e = a_j, \\ 1 + k + \alpha & \text{if } e \notin E(B) \cup E(H). \end{cases}$$

According to (ii), $|\psi(u)| = f(u)$, for any vertex $u \in D$. By (iii), $|\psi(u)| \leq f(u)$, for any vertex $u \in I$. Similarly, $|\psi(u)| \leq f(u)$, for any vertex $u \in V(G) \setminus (D \cup I)$, because H is an f-subgraph. Therefore, ψ is an M_f-edge coloring of G. Consequently, $\mathcal{K}_f(G) \geq |\psi(G)| = 1 + k + \alpha$. The opposite inequality follows from Theorem 1. ■

Recall that the *join* of two graphs G and H is obtained from vertex-disjoint copies of G and H by adding all edges between $V(G)$ and $V(H)$.

Corollary 7. *Let q, n and m be integers such that $q \geq 2$, $n \geq 2$, $m \geq q-1$ when $n \leq q$, and $m \geq n$ when $n > q$. Let G_1 and G_2 be disjoint graphs such that $|V(G_1)| = n$, G_2 contains m isolated vertices, and let G be the join of G_1 and G_2. Then*

$$\mathcal{K}_q(G) = 1 + n(q-1) + \alpha_q\big(V(G_2)\big).$$

Proof. Clearly, $V(G_1)$ is a dominating set of G. Let I be the set of isolated vertices in G_2. Then G contains the complete bipartite subgraph with parts $V(G_1)$ and I (i.e., the subgraph isomorphic to $K_{n,m}$). The graph $K_{n,m}$ contains either a subgraph isomorphic to $K_{n,q-1}$ (if $n \leq q$), or a $(q-1)$-regular subgraph of order $2n$ (if $n > q$). Thus, there is a spanning subgraph B of G satisfying conditions (i)–(iii) from Theorem 3, and the assertion follows. ■

A *complete k-partite graph* is a graph whose vertices can be partitioned into $k \geq 2$ disjoint classes $V_1, \ldots, V_k$ such that two vertices are adjacent whenever they belong to distinct classes. If $|V_i| = n_i$, $i = 1, \ldots, k$, then the complete k-partite graph is denoted by $K_{n_1,\ldots,n_k}$.

In [13] there are stated some results on $\mathcal{K}_q(G)$ for complete multipartite graphs with parts of size at least $q-1$. In the following assertion we consider complete multipartite graphs that can contain parts of size less than $q-1$, so we extend the result from [13]. The complete k-partite graph $K_{n_1,\ldots,n_{k-1},n_k}$ is the join of $K_{n_1,\ldots,n_{k-1}}$ and the totally disconnected graph of order n_k. Thus, according to Corollary 7, we immediately have the following statement.

Corollary 8. *Let q, k, $n_1, \ldots, n_k$ and p be integers such that $q \geq 2$, $k \geq 3$, $1 \leq n_1 \leq \cdots \leq n_k$, $p = \sum_{j=1}^{k-1} n_j$, $n_k \geq q-1$ when $p \leq q$, and $n_k \geq p$ when $p > q$. Then*

$$\mathcal{K}_q\big(K_{n_1,\ldots,n_k}\big) = 1 + p(q-1).$$

The *corona* $G \odot H$ of graphs G and H is obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining by an edge the i'th vertex of G to every vertex in the i'th copy of H.

According to Theorem 3, we immediately have the following assertion.

Corollary 9. *Let q be a positive integer. Let G be a connected graph of order at least two and let H be a graph containing at least $q-1$ isolated vertices. Then*

$$\mathcal{K}_q(G \odot H) = 1 + |V(G)|\big(q - 1 + \alpha_q(H)\big).$$

A vertex of a graph G is called a *dominating vertex* if it is adjacent to every other vertex of G.

Theorem 4. *Let w be a dominating vertex of a graph G. Let f be a function from $V(G)$ to positive integers such that $\deg_G(u) \geq f(u) + \big\lfloor \big(|V(G)| + f(w) - 3\big)/2 \big\rfloor$, for every vertex u of G. Then*

$$\mathcal{K}_f(G) = 1 + \alpha_f(G).$$

Proof. Suppose that φ is an M_f-edge coloring of G which uses $\mathcal{K}_f(G)$ colors (i.e., $|\varphi(G)| = \mathcal{K}_f(G)$). Denote colors of $\varphi(w)$ by $c_1, \ldots, c_k$ ($k = |\varphi(w)|$) and set $U_j = \big\{u \in V(G) \setminus \{w\} : \varphi(wu) = c_j\big\}$ for each $j \in \{1, \ldots, k\}$.

Let A be a subset of $E(G)$ containing exactly one edge of each color belonging to $\varphi(G) \setminus \varphi(w)$. Let H be a subgraph of G such that $V(H) = V(G) \setminus \{w\}$ and $E(H) = A$. Evidently, the graph H is an f-subgraph of G. Set

$$X = \big\{v \in V(H) : \deg_H(v) = f(v) - 1\big\} \text{ and}$$
$$Y = \big\{v \in V(H) : \deg_H(v) < f(v) - 1\big\}.$$

First suppose that $|Y| \leq k - 2$. As $U_1, U_2, \ldots, U_k$ are pairwise disjoint, at most $|Y|$ sets of them contain a vertex of Y. Then there are at least two sets, without loss of generality U_1 and U_2, such that $U_1 \cap Y = \emptyset = U_2 \cap Y$. Moreover, we can assume that $|U_1| \leq |U_2|$. Thus, $|U_1| \leq \lfloor |X|/2 \rfloor = \big\lfloor (|V(G)| - 1 - |Y|)/2 \big\rfloor$. Let u^* be a vertex of U_1. As

$$\begin{aligned} |\{w\}| + |U_1 \setminus \{u^*\}| + |Y| &\leq 1 + \left(\left\lfloor \frac{|V(G)| - 1 - |Y|}{2} \right\rfloor - 1 \right) + |Y| \\ &= \left\lfloor \frac{|V(G)| + |Y| - 1}{2} \right\rfloor \leq \left\lfloor \frac{|V(G)| + k - 3}{2} \right\rfloor \leq \left\lfloor \frac{|V(G)| + f(w) - 3}{2} \right\rfloor, \end{aligned}$$

there are at least $f(u^*)$ vertices of $X \setminus U_1$ that are adjacent to u^* in G. Since $\deg_H(u^*) = f(u^*) - 1$, there is a vertex $v^* \in X \setminus U_1$ such that $u^*v^* \in E(G)$ and $u^*v^* \notin E(H)$. As $v^* \in X \setminus U_1$, there is i, $2 \leq i \leq k$, such that $v^* \in U_i$. Since $\deg_H(v^*) = f(v^*) - 1$, for each color $c \in \varphi(v^*) \setminus \{c_i\}$, there is a vertex $x \in N_H(v^*)$ such that $\varphi(v^*x) = c$. Similarly, for each color $c \in \varphi(u^*) \setminus \{c_1\}$, there is a vertex $x \in N_H(u^*)$ such that $\varphi(u^*x) = c$. Therefore, $\big(\varphi(u^*) \setminus \{c_1\}\big) \cap \big(\varphi(v^*) \setminus \{c_i\}\big) = \emptyset$, because the vertices u^* and v^* are not adjacent in H. As the colors c_1 and c_i are distinct, $\varphi(u^*) \cap \varphi(v^*) = \emptyset$. Consequently, $\varphi(u^*v^*) \in \varphi(u^*) \cap \varphi(v^*) = \emptyset$, a contradiction. So, this case is impossible.

Then $|Y| \geq k-1$ and there are vertices $y_1, \ldots, y_{k-1}$ belonging to Y. Set $A^* = A \cup \{wy_j : 1 \leq j \leq k-1\}$ and consider a subgraph F of G induced by A^*. Clearly, F is an f-subgraph of G and so $|A^*| \leq \alpha_f(G)$. Hence

$$\mathcal{K}_f(G) = |\varphi(G)| = |\varphi(w)| + |A| = 1 + (k-1) + |A| = 1 + |A^*| \leq 1 + \alpha_f(G).$$

The opposite inequality follows from Lemma 2. ■

Corollary 10. *Let q be a positive integer. Let G be a graph such that*

$$\Delta(G) = |V(G)| - 1 \quad \textit{and} \quad \delta(G) \geq \lfloor (|V(G)| + 3q - 3)/2 \rfloor.$$

Then

$$\mathcal{K}_q(G) = 1 + \left\lfloor \frac{(q-1)|V(G)|}{2} \right\rfloor.$$

Proof. The case when $q = 1$ is evident, so next we consider $q \geq 2$.

As $\delta(G) \geq \lfloor (|V(G)| + 3q - 3)/2 \rfloor \geq (3q-4)/2 + |V(G)|/2$, there are pairwise edge-disjoint Hamilton cycles $C_1, C_2, \ldots, C_k$, where $k = \lceil (q-1)/2 \rceil$, in G (because of Dirac's theorem). Suppose that A is a subset of $E(C_1)$ such that it consists of either $\lfloor |V(G)|/2 \rfloor$ independent edges, when q is even, or all edges of C_1, when q is odd. Set $A^* = A \cup \bigcup_{j=2}^{k} E(C_j)$. It is easy to see that the subgraph of G induced by A^* is a q-subgraph with the maximum number of edges, i.e., $\alpha_q(G) = |A^*| = \lfloor (q-1)|V(G)|/2 \rfloor$. Therefore, according to Theorem 4, we have the assertion. ■

In [11] there is determined $\mathcal{K}_q(K_n)$ within 1, for $n \geq q+2$. Note that, by Corollary 10, $\mathcal{K}_q(K_n) = 1 + \lfloor (q-1)n/2 \rfloor$, for $n \geq 3q-1$, which is an extension of the result from [11].

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