# MINIMAL GRAPHS WITH DISJOINT DOMINATING AND PAIRED-DOMINATING SETS 

Michael A. Henning<br>Department of Pure and Applied Mathematics<br>University of Johannesburg<br>Auckland Park 2006, South Africa<br>e-mail: mahenning@uj.ac.za

AND
Jerzy Topp
The State University of Applied Sciences in Elblgg, Poland
e-mail: j.topp@inf.ug.edu.pl


#### Abstract

A subset $D \subseteq V_{G}$ is a dominating set of $G$ if every vertex in $V_{G}-D$ has a neighbor in $D$, while $D$ is a paired-dominating set of $G$ if $D$ is a dominating set and the subgraph induced by $D$ contains a perfect matching. A graph $G$ is a $D P D P$-graph if it has a pair $(D, P)$ of disjoint sets of vertices of $G$ such that $D$ is a dominating set and $P$ is a paired-dominating set of $G$. The study of the $D P D P$-graphs was initiated by Southey and Henning [Cent. Eur. J. Math. 8 (2010) 459-467; J. Comb. Optim. 22 (2011) 217-234]. In this paper, we provide conditions which ensure that a graph is a $D P D P$-graph. In particular, we characterize the minimal $D P D P$-graphs.


Keywords: domination, paired-domination.
2010 Mathematics Subject Classification: 05C69, 05C85.

## 1. Introduction

Let $G=\left(V_{G}, E_{G}\right)$ be a graph with vertex set $V(G)=V_{G}$ and edge set $E(G)=E_{G}$, where we allow multiple edges and loops. A set of vertices $D \subseteq V_{G}$ is a dominating set of $G$ if every vertex in $V_{G} \backslash D$ has a neighbor in $D$, while $D$ is 2-dominating set of $G$ if every vertex in $V_{G} \backslash D$ has at least two neighbors in $D$. A set $D \subseteq V_{G}$ is
a total dominating set of $G$ if every vertex has a neighbor in $D$. A set $D \subseteq V_{G}$ is a paired-dominating set of $G$ if $D$ is a dominating set and the subgraph induced by $D$ contains a perfect matching.

Ore [23] was the first to observe that a graph with no isolated vertex contains two disjoint dominating sets. Consequently, the vertex set of a graph without isolated vertices can be partitioned into two dominating sets. Various graph theoretic properties and parameters of graphs having disjoint dominating sets are studied in $[1,8-10,14,20,21]$. Characterizations of graphs with disjoint dominating and total dominating sets are given in $[11-13,16,17,19,25]$, while in $[2,4-6,18]$ graphs which have the property that their vertex set can be partitioned into two disjoint total dominating sets are studied. Conditions which guarantee the existence of a dominating set whose complement contains a 2-dominating set, a paired-dominating set or an independent dominating set are presented in $[7,12$, $15,19,20,22,26]$.

In this paper we restrict our attention to conditions which ensure a partition of vertex set of a graph into a dominating set and a paired-dominating set. The study of graphs having a dominating set whose complement is a paireddominating set was initiated by Southey and Henning [24, 26]. They define a $D P$-pair in a graph $G$ to be a pair $(D, P)$ of disjoint sets of vertices of $G$ such that $V(G)=D \cup P$ where $D$ is a dominating set and $P$ is a paired-dominating set of $G$. A graph that has a $D P$-pair is called a DPDP-graph (standing, as in [24,26], for "dominating, paired dominating, partitionable graph"). It is easy to observe that a complete graph $K_{n}$ is a $D P D P$-graph if $n \geq 3$ (and $K_{3}$ is the smallest $D P D P$-graph), a path $P_{n}$ is a $D P D P$-graph if and only if $n \in \mathbb{N} \backslash\{1,2,3,5,6,9\}$, while a cycle $C_{n}$ is a $D P D P$-graph if $n \geq 3$ and $n \neq 5$. It was also proved in [24] that every cubic graph is a $D P D P$-graph. In [26] the $D P D P$-graphs (and, in particular, the $D P D P$-trees) were characterized as the graphs which can be constructed from a labeled $P_{4}$ by applying eight (four, respectively) operations.

For notation and graph theory terminology we in general follow [3]. Specifically, for a vertex $v$ of a graph $G=\left(V_{G}, E_{G}\right)$, its neighborhood, denoted by $N_{G}(v)$, is the set of all vertices adjacent to $v$, and the cardinality of $N_{G}(v)$, denoted by $d_{G}(v)$, is called the degree of $v$. The closed neighborhood of $v$, denoted by $N_{G}[v]$, is the set $N_{G}(v) \cup\{v\}$. In general, for a subset $X \subseteq V_{G}$ of vertices, the neighborhood of $X$, denoted by $N_{G}(X)$, is defined to be $\bigcup_{v \in X} N_{G}(v)$, and the closed neighborhood of $X$, denoted by $N_{G}[X]$, is the set $N_{G}(X) \cup X$. The minimum degree of a vertex in $G$ is denoted by $\delta(G)$. A vertex of degree one is called a leaf, and the only neighbor of a leaf is called its support vertex (or simply, its support). If a support vertex has at least two leaves as neighbors, we call it a strong support, otherwise it is a weak support. The set of leaves, the set of weak supports, the set of strong supports, and the set of all supports of $G$ is denoted by $L_{G}, S_{G}^{\prime}, S_{G}^{\prime \prime}$, and $S_{G}$, respectively. If $v$ is a vertex of $G$, then by
$E_{G}(v)$ and $\mathcal{L}_{G}(v)$ we denote the set of edges and the set of loops incident with $v$ in $G$, respectively.

We denote the path, cycle, and complete graph on $n$ vertices by $P_{n}, C_{n}$, and $K_{n}$, respectively. The complete bipartite graph with one partite set of size $n$ and the other of size $m$ is denoted by $K_{n, m}$. A star is the tree $K_{1, k}$ for some $k \geq 1$. For $r, s \geq 1$, a double star $S(r, s)$ is the tree with exactly two vertices that are not leaves, one of which has $r$ leaf neighbors and the other $s$ leaf neighbors. We define a pendant edge of a graph to be an edge incident with a vertex of degree 1. We use the standard notation $[k]=\{1, \ldots, k\}$.

## 2. 2-Subdivision Graphs of a Graph

Let $H=\left(V_{H}, E_{H}\right)$ be a graph with no isolated vertices and with possible multiedges and multi-loops. By $\varphi_{H}$ we denote a function from $E_{H}$ to $2^{V_{H}}$ that associates with each $e \in E_{H}$ the set $\varphi_{H}(e)$ of vertices incident with $e$. Let $X_{2}$ be a set of 2-element subsets of an arbitrary set (disjoint with $V_{H} \cup E_{H}$ ), and let $\xi: E_{H} \rightarrow X_{2}$ be a function such that $\xi(e) \cap \xi(f)=\emptyset$ if $e$ and $f$ are distinct elements of $E_{H}$. If $e \in E_{H}$ and $\varphi_{H}(e)=\{u, v\}\left(\varphi_{H}(e)=\{v\}\right.$, respectively), then we write $\xi(e)=\left\{u_{e}, v_{e}\right\}\left(\xi(e)=\left\{v_{e}^{1}, v_{e}^{2}\right\}\right.$, respectively). If $\alpha: L_{H} \rightarrow \mathbb{N}$ is a function, then let $\Phi_{\alpha}: L_{H} \rightarrow L_{H} \times \mathbb{N}$ be a function such that $\Phi_{\alpha}(v)=\{(v, i): i \in[\alpha(v)]\}$ for $v \in L_{H}$.

Now we say that a graph $S_{2}(H)=\left(V_{S_{2}(H)}, E_{S_{2}(H)}\right)$ is the 2-subdivision graph of $H$ (with respect to the functions $\xi: E_{H} \rightarrow X_{2}$ and $\alpha: L_{H} \rightarrow \mathbb{N}$ ), if $V_{S_{2}(H)}=$ $V_{S_{2}(H)}^{o} \cup V_{S_{2}(H)}^{n}$, where

$$
V_{S_{2}(H)}^{o}=\left(V_{H} \backslash L_{H}\right) \cup \bigcup_{v \in L_{H}} \Phi_{\alpha}(v) \quad \text { and } \quad V_{S_{2}(H)}^{n}=\bigcup_{e \in E_{H}} \xi(e)
$$

and

$$
\begin{aligned}
E_{S_{2}(H)} & =\bigcup_{e \in E_{H}}\{x y: \xi(e)=\{x, y\}\} \cup \bigcup_{v \in L_{H}}\left\{v_{e}(v, i): e \in E_{H}(v), i \in[\alpha(v)]\right\} \\
& \cup \bigcup_{v \in V_{H} \backslash L_{H}}\left(\left\{v v_{e}: e \in E_{H}(v)\right\} \cup\left\{v v_{e}^{1}, v v_{e}^{2}: e \in \mathcal{L}_{H}(v)\right\}\right)
\end{aligned}
$$

## 3. Main Result

In this paper, our aim is to characterize $D P D P$-graphs. The following result provides a characterization of minimal $D P D P$-graphs, where a good subgraph is defined in Section 5.

Theorem 3.1. If $G$ is a connected graph of order at least three, then the following statements are equivalent.
(1) $G$ is a minimal DPDP-graph.
(2) $G=S_{2}(H)$ for some connected graph $H$, and either $\left(V_{S_{2}(H)}^{o}, V_{S_{2}(H)}^{n}\right)$ is the unique DP-pair in $G$ or $G$ is a cycle of length 3,6 or 9 .
(3) $G=S_{2}(H)$ for some connected graph $H$ that has neither an isolated vertex nor a good subgraph.
(4) $G=S_{2}(H)$ for some connected graph $H$ and no proper spanning subgraph of $G$ without isolated vertices is a 2-subdivision graph.

## 4. Properties of 2-Subdivision Graphs

We remark that 2-subdivision graphs are defined only for graphs without isolated vertices and, intuitively, $S_{2}(H)$ is the graph obtained from $H$ by inserting two new vertices into each edge and each loop of $H$, and then replacing each pendant edge $v_{e} v$ by pendant edges $v_{e}(v, 1), \ldots, v_{e}(v, \alpha(v))$. In particular, it follows from this definition that every tree of diameter three (i.e., every double star) is a 2 subdivision graph of $K_{2}$. Moreover, a path $P_{n}$ (of order $n$ ) is a 2-subdivision graph (of a path) if and only if $n=3 k+1$ for every positive integer $k$ and here $\alpha$ assign to each leaf the value 1. Figure 1 shows a graph $H$ and a possible 2-subdivision graph $S_{2}(H)$ of $H$ where $\alpha: L_{H} \rightarrow\{3\}$.


Figure 1. A 2-subdivision graph $S_{2}(H)$ of a graph $H$.

Observation 4.1. Let $H$ be a graph with no isolated vertex, and let $G=S_{2}(H)$ be the 2-subdivision graph of $H$ (with respect to functions $\xi: E_{H} \rightarrow X_{2}$ and $\alpha: L_{H} \rightarrow \mathbb{N}$ ). Then the following statements hold.
(1) $d_{G}(v)=d_{H}(v)$ if $v \in V_{H} \backslash L_{H}$, and $d_{G}((v, i))=1$ if $v \in L_{H}$ and $i \in[\alpha(v)]$.
(2) $d_{G}(x)=2$ if $x \in V_{S_{2}(H)}^{n} \backslash S_{G}$, and $d_{G}\left(v_{e}\right)=1+\alpha(v)$ if $v \in L_{H}$ and $e \in E_{H}(v)$.
(3) If $x, y \in V_{G} \backslash V_{S_{2}(H)}^{n}$ are distinct, and belong to the same component of $G$, then either $d_{G}(x, y) \equiv 0(\bmod 3)$ or $x, y \in L_{G}$ and $d_{G}(x, y)=2$.
(4) If $x \in S_{G}$, then $\left|N_{G}(x) \cap V_{S_{2}(H)}^{n}\right|=1$ and $N_{G}(x) \backslash V_{S_{2}(H)}^{n} \subseteq L_{G}$.
(5) If $x \in V_{G}$, then the following hold.
(a) If $d_{G}(x)>2$, then either $x \in V_{H}$ or $x \in S_{G}$ and $\left|N_{G}(x) \backslash L_{G}\right|=1$.
(b) If $x \in V_{S_{2}(H)}^{n}$, then either $d_{G}(x)=2$ or $d_{G}(x)>2$ and $x \in S_{G}$.
(c) If $x \in V_{S_{2}(H)}^{n}$ and $d_{G}(x)>2$, then $x \in S_{G}$.
(6) Let $G^{\prime}$ be a 2-subdivision graph which is a spanning subgraph of $G$. If $F$ is a component of $G^{\prime}$, then $F$ has exactly one of the following properties.
(a) $F$ is an induced subgraph of $G$ if no leaf of $F$ is in $V_{S_{2}(H)}^{n}$.
(b) $F$ is a 2-subdivision graph of a path $P_{k+1}(k \geq 1)$ and $F$ has at most one strong support if at least one leaf of $F$ is in $V_{S_{2}(H)}^{n}$. In addition, exactly one of these support vertices is in $V_{H}$. Moreover, if $F$ has a strong support, then this strong support vertex is in $V_{H}$.

Proof. The statements (1)-(5) are immediate consequences of the definition of the 2 -subdivision graph. To prove (6), let $G^{\prime}$ be a spanning subgraph of $G$ that is a 2-subdivision graph and let $F$ be a component of $G^{\prime}$. Since $G^{\prime}$ is a 2 -subdivision graph, so too is the graph $F$, i.e., $F=S_{2}\left(H^{\prime}\right)$ for some connected graph $H^{\prime}$ (and some functions $\xi^{\prime}: E_{H^{\prime}} \rightarrow X_{2}$ and $\left.\alpha^{\prime}: L_{H^{\prime}} \rightarrow \mathbb{N}\right)$.

Case 1. $L_{F} \cap V_{S_{2}(H)}^{n}=\emptyset$. Since $F$ is a 2-subdivision graph, the sets $V_{F} \cap V_{H}$ and $V_{F} \cap V_{S_{2}(H)}^{n}$ are nonempty. Assume first that $v \in V_{F} \cap V_{H}$ and $e$ is a loop at $v$ in $H$. We claim that the vertices $v_{e}^{1}$ and $v_{e}^{2}$, and the edges $v v_{e}^{1}, v_{e}^{1} v_{e}^{2}, v v_{e}^{2}$ belong to $F$. If $v_{e}^{1}$ or $v_{e}^{2}$ were not in $F$, then $G^{\prime}$ (which is a spanning subgraph of $G$ ) would have a component of order one or two, which is impossible in a 2 subdivision graph. Now, since neither $v_{e}^{1}$ nor $v_{e}^{2}$ is a leaf in $F$, both $v_{e}^{1}$ and $v_{e}^{2}$ are of degree 2 in $F$ and this proves that the edges $v v_{e}^{1}, v_{e}^{1} v_{e}^{2}, v v_{e}^{2}$ belong to $F$. We can similarly show that if $u, v \in V_{F} \cap V_{H}$ and $e$ is an edge joining $u$ to $v$ in $H$, then the vertices $u_{e}$ and $v_{e}$, and the edges $v v_{e}, v_{e} u_{e}, u_{e} u$ belong to $F$. From this it follows that $F$ is a 2-subdivision graph of the induced subgraph $H\left[V_{F} \cap V_{H}\right]$ and, therefore, $F$ is an induced subgraph of $G$.

Case 2. $L_{F} \cap V_{S_{2}(H)}^{n} \neq \emptyset$. Let $x_{0}$ be a leaf of $F$ which belongs to $V_{S_{2}(H)}^{n}$ in $G$. Since $G^{\prime}$ is a 2-subdivision graph, the vertex $x_{0}$ does not belong to $N_{G}\left[S_{G}\right]$. In addition, if $\Delta(F) \leq 2$, then $F$ is a path, and, since $F$ is a 2 -subdivision graph, we note that $F=P_{3 k+1}=S_{2}\left(P_{k+1}\right)$ (for some positive integer $k$ ), as desired. Thus assume that $\Delta(F) \geq 3$. Let $x$ be a vertex of degree at least 3 in $F$. It follows from (5) applied to the graph $F=S_{2}\left(H^{\prime}\right)$ that either $x \in V_{H^{\prime}}$ or $x \in S_{F}$ and $\left|N_{F}(x) \backslash L_{F}\right|=1$. However, such a vertex $x$ cannot be in $V_{H^{\prime}}=\{y \in$
$\left.V_{F}: d_{F}\left(x_{0}, y\right) \equiv 0(\bmod 3)\right\}$, as every vertex belonging to $V_{H^{\prime}} \backslash L_{H^{\prime}} \subseteq V_{S_{2}(H)}^{n}$ is of degree 2 in $G$ and in $F$, while vertices in $F$ corresponding to elements of $L_{H^{\prime}}$ are of degree 1 . This proves that $x \in S_{F}$ and $\left|N_{F}(x) \backslash L_{F}\right|=1$, that is, every vertex of degree at least 3 in $F$ is a strong support vertex and it has only one neighbor which is not a leaf. From this it follows that $F$ is a 2 -subdivision graph $S_{2}\left(P_{k+1}\right)$ with at least one strong support vertex (for some positive integer $k$ ).

It remains to show that $F$ cannot have two strong support vertices. Suppose, for the sake of contradiction, that $s_{1}$ and $s_{2}$ are distinct strong support vertices in $F$. Let $\ell_{1}$ and $\ell_{2}$ be leaves in $F$ adjacent to $s_{1}$ and $s_{2}$, respectively. Since $s_{1}$ and $s_{2}$ are vertices of degree at least three in $G=S_{2}(H)$, it follows from (5) that each of them belongs to $V_{H}$ or $S_{G}$. There are three cases to consider. If $s_{1}, s_{2} \in V_{H}$, then it follows from (3) that $d_{G}\left(s_{1}, s_{2}\right) \equiv 0(\bmod 3)$, implying that $d_{F}\left(\ell_{1}, \ell_{2}\right) \equiv 2$ $(\bmod 3)$ and $d_{F}\left(\ell_{1}, \ell_{2}\right) \neq 2$, contradicting (3) in $F$. Hence renaming $s_{1}$ and $s_{2}$ if necessary, we may assume that $s_{2} \in S_{G}$. If $s_{1} \in V_{H}$, then it follows from (3) that $d_{G}\left(s_{1}, \ell_{2}\right) \equiv 0(\bmod 3)$, implying that $d_{F}\left(\ell_{1}, \ell_{2}\right) \equiv 1(\bmod 3)$, contradicting (3) in $F$. Hence, $s_{1} \in S_{G}$. Thus, no leaf of $F$ belongs to $V_{S_{2}(H)}^{n}$ in $G$, contradicting our choice of $F$. This completes the proof of the statement (6).

We next present the following elementary property of a $D P D P$-graph.
Observation 4.2. If $(D, P)$ is a $D P$-pair in a graph $G$, then every leaf of $G$ belongs to $D$, while every support of $G$ is in $P$, that is, $L_{G} \subseteq D$ and $S_{G} \subseteq P$.

A connected graph $G$ is said to be a minimal $D P D P$-graph, if $G$ is a $D P D P$ graph and no proper spanning subgraph of $G$ is a $D P D P$-graph.

We remark that a complete graph $K_{n}$ is a minimal $D P D P$-graph only if $n=3$. We observe that a path $P_{n}$ is a minimal $D P D P$-graph if and only if $n \in\{4,7,10,13\}$, while a cycle $C_{n}$ is a minimal $D P D P$-graph if and only if $n \in\{3,6,9\}$. From the definition of a minimal $D P D P$-graph we immediately have the following important (and intuitively easy) observation.

Observation 4.3. Every spanning supergraph of a DPDP-graph is a DPDPgraph, and, trivially, every DPDP-graph is a spanning supergraph of some minimal DPDP-graph.

We show next that the 2 -subdivision graph of an isolate-free graph is a $D P D P$ graph.

Proposition 4.4. If a graph $H$ has no isolated vertex, then its 2 -subdivision graph $S_{2}(H)$ is a DPDP-graph.

Proof. Let $S_{2}(H)$ be the subdivision graph of $H$ (with respect to functions $\xi: E_{H} \rightarrow X_{2}$ and $\left.\alpha: L_{H} \rightarrow \mathbb{N}\right)$. We shall prove that $(D, P)$ is a $D P$-pair in
$S_{2}(H)$, where

$$
D=V_{S_{2}(H)}^{o}=\left(V_{H} \backslash L_{H}\right) \cup \bigcup_{v \in L_{H}} \Phi_{\alpha}(v)
$$

and $P=V_{S_{2}(H)}^{n}=V_{S_{2}(H)} \backslash D$. If $x \in P$, then $x \in \xi(e)$ for some $e \in E_{H}$, and $x$ is adjacent in $S_{2}(H)$ to a vertex incident with $e$ in $H$. This proves that $D$ is a dominating set of $S_{2}(H)$. Assume now that $y \in D$. If $y \in V_{H} \backslash L_{H}$, then, since $H$ has no isolated vertex, there is an edge $f$ incident with $y$ in $H$, and therefore $y$ is adjacent to $y_{f} \in P$ (or to $y_{f}^{1} \in P$ and $y_{f}^{2} \in P$ if $f$ is a loop) in $S_{2}(H)$. If $y \in \Phi_{\alpha}(v)$ for some $v \in L_{H}$, then $y$ is adjacent to $v_{e}$ in $S_{2}(H)$, where $e$ is the only pendant edge incident with $v$ in $H$. Consequently, $P$ is a dominating set of $S_{2}(H)$. In addition, since the two vertices of $\xi(e)$ are adjacent in $S_{2}(H)$ for every $e \in E_{H}$, the set $P=\bigcup_{e \in E_{H}} \xi(e)$ is a paired-dominating set of $S_{2}(H)$. This proves that $S_{2}(H)$ is a $D P D P$-graph.

Since every graph is homeomorphic to its 2-subdivision graph, it follows from Proposition 4.4 that every graph without isolated vertices is homeomorphic to a $D P D P$-graph. Consequently, the structure of $D P D P$-graphs becomes more complex.

The next theorem presents general properties of $D P$-pairs in a minimal $D P D P$-graph.

Theorem 4.5. If $G$ is a minimal $D P D P$-graph and $(D, P)$ is a $D P$-pair in $G$, then the following four statements hold.
(1) $D$ is a maximal independent set in $G$.
(2) The induced graph $G[P]$ consists of independent edges, that is, $\delta(G[P])=$ $\Delta(G[P])=1$.
(3) If $x \in P$, then $\left|N_{G}(x) \backslash P\right|=1$ or $N_{G}(x) \backslash P$ is a nonempty subset of $L_{G}$.
(4) $G$ is a 2-subdivision graph of some graph $H$.

Proof. (1) If $D$ is not an independent set, then $D$ contains two vertices, say $x$ and $y$, that are adjacent. In this case, $(D, P)$ would be a $D P$-pair in $G-x y$, contradicting the minimality of $G$. Hence, the set $D$ is both an independent and dominating set of $G$, implying that $D$ is a maximal independent set in $G$.
(2) Since $P$ is a paired-dominating set of $G$, by definition, $G[P]$ has a perfect matching, say $M$. If $x y$ is an edge of $G[P]$ which is not in $M$, then $(D, P)$ would be a $D P$-pair in $G-x y$, violating the minimality of $G$. Hence, the edges of $M$ are the only edges of $G[P]$.
(3) Assume that $x \in P$. It follows from (2) that $x$ has exactly one neighbor in $P$, say $x^{\prime}$. Thus since $(D, P)$ is a $D P$-pair in $G$, we note that $N_{G}(x) \backslash\left\{x^{\prime}\right\}$ is a nonempty subset of the dominating set $D$ of $G$. If every neighbor of $x$ in $D$ is a leaf, then $N_{G}(x) \backslash P$ is a nonempty subset of $L_{G}$. Hence we may assume that
$x$ contains a neighbor $y$ in $D$ that is not a leaf, for otherwise the desired result follows. If $x$ contains a neighbor in $D$ different from $y$, then $x$ is dominated by a vertex belonging to $D \backslash\{y\}$ and $y$ is dominated by some vertex in $P \backslash\{x\}$, implying that $(D, P)$ is a $D P$-pair in $G-x y$, contradicting the minimality of $G$. Hence in this case, the vertex $y$ is the only neighbor of $x$ in $D$, and so $N_{G}(x)=\left\{x^{\prime}, y\right\}$ and $\left|N_{G}(x) \backslash P\right|=1$.
(4) Let $G$ be a minimal $D P D P$-graph, and let $(D, P)$ be a $D P$-pair in $G$. For a support vertex $s$, the set of leaves adjacent to $s$ is denoted by $L_{G}(s)$, i.e., $L_{G}(s)=N_{G}(s) \cap L_{G}$. Let $G^{*}$ denote the graph resulting from $G$ by replacing the vertices of $L_{G}(s)$ by a new vertex $v_{s}$ and joining $v_{s}$ to $s$, for every $s \in S_{G}$, i.e., $G^{*}=\left(V_{G^{*}}, E_{G^{*}}\right)$, where $V_{G^{*}}=\left(V_{G} \backslash L_{G}\right) \cup\left\{v_{s}: s \in S_{G}\right\}$ and $E_{G^{*}}=E_{G-L_{G}} \cup$ $\left\{s v_{s}: s \in S_{G}\right\}$. By (2) and (3) above, we note that every vertex of $P$ has degree 2 in $G^{*}$. Further, every vertex of $P$ has exactly one neighbor in $P$.

We define a graph $H=\left(V_{H}, E_{H}, \varphi_{H}\right)$ as follows. Let $V_{H}=V_{G^{*}} \backslash P$. For every edge $v_{1} v_{2}$ in $G^{*}$ that joins two vertices of $P$ we do the following. If $v_{1}$ and $v_{2}$ have a common neighbor, say $v$, in $G^{*}$, then in $H$ we add a loop in $H$ at the vertex $v$. If $v_{1}$ and $v_{2}$ do not have a common neighbor in $G^{*}$, then we add the edge $u_{1} u_{2}$ to $H$ where $u_{1}$ is the neighbor of $v_{1}$ different from $v_{2}$ and where $u_{2}$ is the neighbor of $v_{2}$ different from $v_{1}$ (and so $u_{1} v_{1} v_{2} u_{2}$ is a path in $G^{*}$ ). We let $\varphi_{H}: E_{H} \rightarrow 2^{V_{H}}$ be the function such that $\varphi_{H}(m)=N_{G^{*}}(m) \backslash m$ if $m \in E_{H}$. Now let $\xi: E_{H} \rightarrow 2^{V_{H}}$ and $\alpha: L_{H} \rightarrow \mathbb{N}$ be functions such that $\xi(e)=e$ if $e \in E_{H}$, and $\alpha\left(v_{s}\right)=\left|L_{G}(s)\right|$ if $v_{s} \in L_{H}$. With these definitions, the graph $G$ is isomorphic to the 2-subdivision graph $S_{2}(H)$ of $H$ (with respect to functions $\xi: E_{H} \rightarrow 2^{V_{H}}$ and $\alpha: L_{H} \rightarrow \mathbb{N}$ ). That means that we can restore the graph $G$ by applying the operation $S_{2}$ to the graph $H$.

By Theorem 4.5 (4) every minimal $D P D P$-graph is a 2 -subdivision graph of some graph. The converse, however, is not true in general. For example, if $H$ is the underlying graph of any of the graphs in Figure 2, then its 2-subdivision graph $S_{2}(H)$ is a $D P D P$-graph, but it is not a minimal $D P D P$-graph. The following result, which is a special case of Theorem 6.2 proven later in the paper, will be useful to establish which 2-subdivision graphs are not minimal $D P D P$-graphs.

Proposition 4.6. Let $x$ and $y$ be adjacent vertices of degree 2 in a graph $H$ without isolated vertices, and let $x^{\prime}$ and $y^{\prime}$ be the vertices such that $N_{H}(x) \backslash$ $\{y\}=\left\{x^{\prime}\right\}$ and $N_{H}(y) \backslash\{x\}=\left\{y^{\prime}\right\}$, respectively. If the sets $N_{H}\left(x^{\prime}\right) \backslash\{x, y\}$ and $N_{H}\left(y^{\prime}\right) \backslash\{x, y\}$ are both nonempty, then the 2-subdivision graph $S_{2}(H)$ is a DPDP-graph but not a minimal DPDP-graph.

Proof. It follows from Proposition 4.4 that $S_{2}(H)$ is a $D P D P$-graph and $(D, P)$ is a $D P$-pair in $S_{2}(H)$, where $D=V_{S_{2}(H)}^{o}$ and $P=V_{S_{2}(H)}^{n}$. The pair ( $D^{\prime}, P^{\prime}$ ), where $D^{\prime}=(D \backslash\{x, y\}) \cup\left\{x_{x y}, y_{x y}, x_{x x^{\prime}}^{\prime}, y_{y y^{\prime}}^{\prime}\right\}$ and $P^{\prime}=\left(P \backslash\left\{x_{x x^{\prime}}^{\prime}, y_{y y^{\prime}}^{\prime}\right\}\right) \cup\{x, y\}$
is a $D P$-pair in the proper spanning subgraph $S_{2}(H) \backslash\left\{x_{x y} y_{x y}, x^{\prime} x_{x x^{\prime}}^{\prime}, y^{\prime} y_{y y^{\prime}}^{\prime}\right\}$ of $S_{2}(H)$. Thus, $S_{2}(H)$ is a $D P D P$-graph but not a minimal $D P D P$-graph.

As a consequence of Proposition 4.6, we can readily determine the minimal $D P D P$-paths and minimal $D P D P$-cycles.
Corollary 4.7. The following holds.
(a) If $P_{n}$ is a path of order $n$, then $S_{2}\left(P_{n}\right)$ is a DPDP-graph for every $n \geq 2$, and $S_{2}\left(P_{n}\right)$ is a minimal DPDP-graph if and only if $n \in\{2,3,4,5\}$.
(b) If $C_{m}$ is a cycle of size $m$, then $S_{2}\left(C_{m}\right)$ is a DPDP-graph for every positive integer $m$, and $S_{2}\left(C_{m}\right)$ is a minimal DPDP-graph if and only if $m \in[3]$.

## 5. Good Subgraphs of a Graph

In this section, we define a good subgraph of a graph. Let $Q$ be a subgraph without isolated vertices of a graph $H$, and let $E_{Q}^{-}$denote the set of edges belonging to $E_{H} \backslash E_{Q}$ that are incident with a vertex of $Q$. Let $E$ be a set such that $E_{Q}^{-} \subseteq E \subseteq E_{H} \backslash E_{Q}$, and let $A_{E}$ is a set of arcs obtained by assigning an orientation for each edge in $E$. Then by $H\left(A_{E}\right)$ we denote the partially oriented graph obtained from $H$ by replacing the edges in $E$ by the arcs belonging to $A_{E}$. If $e \in E$, then by $e_{A}$ we denote the only arc in $A_{H}$ that corresponds to $e$. By $H_{0}$ we denote the subgraph of $H\left(A_{E}\right)$ induced by the vertices that are not the initial vertex of an arc belonging to $A_{E}$, i.e., by the set $\left\{v \in V_{H}: d_{H\left(A_{E}\right)}^{+}(v)=0\right\}$.

We say that $Q$ is a good subgraph of $H$ if there exist a set of edges $E$ (where $\left.E_{Q}^{-} \subseteq E \subseteq E_{H} \backslash E_{Q}\right)$ and a set of arcs $A_{E}$ such that in the resulting graph $H\left(A_{E}\right)$, which we simply denote by $H$ for notational convenience, the arcs in $A_{E}$ form a family $\mathcal{P}=\left\{P_{x}: x \in V_{Q}\right\}$ of oriented paths indexed by the vertices of $Q$ and such that the following holds.
(1) Every vertex of $Q$ is an initial vertex of exactly one path belonging to $\mathcal{P}$. For each vertex $v \in Q$, we denote the (unique) path belonging to $\mathcal{P}$ that begins at $v$ by $P_{v}$. Thus, if $v \in V_{Q}$, then $d_{H}^{+}(v)=1$ and $d_{H}^{-}(v)=d_{H}(v)-d_{Q}(v)-1$.
(2) If $x$ is an inner vertex of a path $P_{v} \in \mathcal{P}$, then $d_{H}^{+}(x)=1$ and $d_{H}^{-}(x)=$ $d_{H}(x)-1$.
(3) If $x$ is a end vertex of a path $P_{v} \in \mathcal{P}$, then $d_{H}^{-}(x)<d_{H}(x)$.

Examples of good subgraphs in small graphs are presented in Figure 2. For clarity, the edges of a good subgraph $Q$ are drawn in bold, the arcs belonging to oriented paths are thin (and their orientations are represented by arrows), and all other edges, if any, belong to the subgraph $H_{0}$, are thin and without arrows.

We remark that not every graph has a good subgraph (see also Observation 7.3 and Corollary 7.2). On the other hand, if $Q$ is a graph with no isolated
vertex, and $H$ is the graph obtained from $Q$ by attaching one pendant edge to each vertex of $Q$ and then subdividing this edge, then $Q$ is a good subgraph in $H$, implying that every graph without isolated vertices can be a good subgraph of some graph.



Figure 2. Examples of good subgraphs (drawn in bold) in small graphs.
From the definition of a good subgraph we immediately have the following observation.

Observation 5.1. Neither a leaf nor a support vertex of a graph $H$ belongs to a good subgraph in $H$.

## 6. Structural Characterization of $D P D P$-Graphs

In this section, we present a proof of our main result, namely Theorem 3.1, which provides a characterization of minimal $D P D P$-graphs. We proceed further with the following result.

Theorem 6.1. If $G$ is a connected graph of order at least 3 , then $G$ is a minimal DPDP-graph if and only if $G=S_{2}(H)$ for some connected graph $H$, and either $\left(V_{S_{2}(H)}^{o}, V_{S_{2}(H)}^{n}\right)$ is the only DP-pair in $S_{2}(H)$ or $S_{2}(H)$ is a cycle of length 3,6 or 9 .

Proof. If $G=S_{2}(H)$ is a cycle of length 3,6 or 9 , then $G$ is clearly a minimal $D P D P$-graph, as claimed. Thus assume that $\left(V_{S_{2}(H)}^{o}, V_{S_{2}(H)}^{n}\right)$ is the only $D P$ pair in $G=S_{2}(H)$. Certainly, $G$ is a $D P D P$-graph, and we shall prove that $G$ is a minimal $D P D P$-graph. Suppose, to the contrary, that $G$ is not a minimal $D P D P$ graph. Then some proper spanning subgraph $G^{\prime}$ of $G$ is a $D P D P$-graph. Let ( $D^{\prime}, P^{\prime}$ ) be a $D P$-pair in $G^{\prime}$ and, consequently, in $G$ (by Observation 4.3). Thus $\left(V_{S_{2}(H)}^{o}, V_{S_{2}(H)}^{n}\right)$ and $\left(D^{\prime}, P^{\prime}\right)$ are $D P$-pairs in $G$, and $\left(V_{S_{2}(H)}^{o}, V_{S_{2}(H)}^{n}\right) \neq\left(D^{\prime}, P^{\prime}\right)$, noting that $\left(V_{S_{2}(H)}^{o}, V_{S_{2}(H)}^{n}\right)$ is a $D P$-pair in no proper spanning subgraph of $G=S_{2}(H)$. This contradicts the uniqueness of a $D P$-pair in $G$ and proves that $G$ is a minimal $D P D P$-graph.

Suppose next that $G$ is a minimal $D P D P$-graph. By Theorem $4.5, G$ is a $2-$ subdivision graph of some connected graph $H$, i.e., $G=S_{2}(H)$, and the pair $(D, P)=\left(V_{S_{2}(H)}^{o}, V_{S_{2}(H)}^{n}\right)$ is a $D P$-pair in $S_{2}(H)$. It remains to prove that either $\left(V_{S_{2}(H)}^{o}, V_{S_{2}(H)}^{n}\right)$ is the only $D P$-pair in $S_{2}(H)$ or $S_{2}(H)$ is a cycle of length 3,6 or 9 . We consider three cases depending on $\Delta(H)$.

Case 1. $\Delta(H)=1$. In this case, $H=P_{2}$, and its 2-subdivision graph $S_{2}\left(P_{2}\right)$ (which is a double star $S(r, s)$ for some positive integers $r$ and $s$ ) has the desired property.

Case 2. $\Delta(H)=2$. In this case, $H$ is a cycle $C_{m}$ where $m \geq 1$ or a path $P_{n}$ where $n \geq 3$. Now, since $S_{2}(H)$ is a minimal $D P D P$-graph, Corollary 4.7, implies that $H=C_{m}$ and $m \in[3]$, or $H=P_{n}$ and $n \in\{3,4,5\}$. In each of these six cases $S_{2}(H)$ has the desired property.

Case 3. $\Delta(H) \geq 3$. In this case, we claim that $(D, P)=\left(V_{S_{2}(H)}^{o}, V_{S_{2}(H)}^{n}\right)$ is the only $D P$-pair in $S_{2}(H)$. Suppose to the contrary that $\left(D^{\prime}, P^{\prime}\right)$ is another $D P$-pair in $G$. Then, since $D$ and $D^{\prime}$ are maximal independent sets in $G$ (by Theorem 4.5) and $D \neq D^{\prime}$, each of the sets $D \backslash D^{\prime}$ and $D^{\prime} \backslash D$ is a nonempty subset of $P^{\prime}$ and $P$, respectively. Let $v$ be a vertex of maximum degree among all vertices in $D \backslash D^{\prime} \subseteq P^{\prime}$. Since $v \in P^{\prime}$, it follows from Theorem 4.5 that $d_{H}(v) \geq 2$. We deal with the two cases when $d_{H}(v)=2$ and $d_{H}(v) \geq 3$ in turn.

Case 3.1. $d_{H}(v) \geq 3$. We distinguish three subcases.
Subcase 3.1.1. There are only loops at $v$ in $H$. Since $d_{H}(v) \geq 3$, there are at least two loops at $v$, say $e$ and $f$. Renaming loops if necessary, we may assume that $v_{e}^{1}$ is the (unique) neighbor of $v$ belonging to $P^{\prime}$. We note that $v_{e}^{2} \in D^{\prime}$ and that all other neighbors of $v$ in $G$, including $v_{f}^{1}$ and $v_{f}^{2}$, belong to $D^{\prime}$. Therefore, $\left(D^{\prime}, P^{\prime}\right)$ is also a $D P$-pair in the proper subgraph $G-v v_{e}^{2}$ of $G$, contradicting the minimality of $G$.

Subcase 3.1.2. There is exactly one loop at $v$ in $H$. Let $e$ be the loop at $v$ in $H$ and let $f$ be an edge of $H$ incident with $v$. If $v_{e}^{1}\left(v_{e}^{2}\right.$, respectively) is the (unique) neighbor of $v$ belonging to $P^{\prime}$, then as in Subcase 3.1.1 we infer that ( $D^{\prime}, P^{\prime}$ ) is a $D P$-pair in the subgraph $G-v v_{e}^{2}\left(G-v v_{e}^{1}\right.$, respectively) of $G$. If $v_{f}$ is the (unique) neighbor of $v$ belonging to $P^{\prime}$, then $\left(D^{\prime}, P^{\prime}\right)$ is a $D P$-pair in the subgraph $G-v_{e}^{1} v_{e}^{2}$ of $G$. In both cases we get a contradiction to the minimality of $G$.

Subcase 3.1.3. There is no loop at $v$ in $H$. In this case, there are three distinct edges, say $e, f$, and $g$, incident with $v$ joining $v$ to $u$, $w$, and $z$, respectively. Assume first that $u, w$, and $z$ are distinct and, without loss of generality, $v_{e}$ is the (unique) neighbor of $v$ which belongs to $P^{\prime}$. Then, since $G$ is a minimal $D P D P$-graph and $\left(D^{\prime}, P^{\prime}\right)$ is a $D P$-pair in $G$, Theorem 4.5 implies that the vertices $u_{e}, v_{f}, v_{g}$ belong to $D^{\prime}$, while $u, w, w_{f}, z$, and $z_{g}$ belong to $P^{\prime}$. This implies that ( $D^{\prime}, P^{\prime}$ ) is a $D P$-pair in $G-v v_{g}$, contradicting the minimality of $G$. We derive similar contradictions if $u, w$, and $z$ are not distinct, and one of the vertices $v_{e}, v_{f}$, $v_{g}$ is the (unique) neighbor of $v$ that belongs to $P^{\prime}$. We omit the proofs of these cases which are analogous to the previous case when $u, w$, and $z$ are distinct.

Case 3.2. $\quad d_{H}(v)=2$. By our choice of the vertex $v$, this implies that $d_{H}(x)=2$ for every $x \in D \backslash D^{\prime}$. Since $\Delta(H) \geq 3$, we note that $H$ is not a cycle, implying that there is no loop at $v$. Let $e$ and $f$ be the two edges incident with $v$. Renaming the edges $e$ and $f$ if necessary, we may assume that $v_{e}$ is the (unique) neighbor of $v$ in $P^{\prime}$.

Suppose that $e$ and $f$ are parallel edges. Let $u$ be the second common vertex of $e$ and $f$. In this case, we note that $d_{H}(u) \geq 3$ as $H$ is not a cycle. Since $G$ is a minimal $D P D P$-graph and $\left(D^{\prime}, P^{\prime}\right)$ is a $D P$-pair in $G$, Theorem 4.5 implies that the vertices $u_{e}$ and $v_{f}$ belong to $D^{\prime}$, while $u$ and $u_{f}$ belong to $P^{\prime}$. In particular, $u \in P^{\prime}, d_{H}(u) \geq 3$, and $u_{e}$ is a neighbor of $u$ not in $P^{\prime}$ of degree 2 . This contradicts Theorem 4.5 which states that every neighbor of $u$ not in $P^{\prime}$ is a leaf of $G$. Hence, the edges $e$ and $f$ are not parallel edges. Thus, $e$ and $f$ join $v$ to distinct vertices $u$ and $w$, respectively.

Recall that by our earlier assumption, $v_{e}$ is the (unique) neighbor of $v$ in $P^{\prime}$. Theorem 4.5 implies that the vertices $u_{e}$ and $v_{f}$ belong to $D^{\prime}$, while $u$, $w$ and $v_{f}$ belong to $P^{\prime}$. If $d_{H}(u) \geq 3$, then noting that $u_{e}$ is a neighbor of $u$ not in $P^{\prime}$ of degree 2 , we contradict Theorem 4.5. Hence, $d_{H}(u)=2$. Analogously, $d_{H}(w)=2$. Let $u^{\prime}$ and $w^{\prime}$ be the neighbor of $u$ and $w$, respectively, different from $v$ in $H$, and so $N_{H}(u) \backslash\{v\}=\left\{u^{\prime}\right\}$ and $N_{H}(w) \backslash\{v\}=\left\{w^{\prime}\right\}$. Since $H \neq C_{3}$, we note that $w^{\prime} \neq u$ (and $u^{\prime} \neq w$ ). We remark that possibly, $u^{\prime}=w^{\prime}$. Since $\Delta(H) \geq 3$, at least one of the vertices $u^{\prime}$ and $w^{\prime}$ is not a leaf in $H$. By symmetry, we may assume that $u^{\prime}$ is not a leaf in $H$, and so $d_{H}\left(u^{\prime}\right) \geq 2$. Proposition 4.6 with $x=v, y=u, x^{\prime}=w$, and $y^{\prime}=u^{\prime}$ implies that $G$ is not a minimal $D P D P$-graph, the final contradiction which completes the proof of Theorem 6.1.

We next provide a characterization of minimal $D P D P$-graphs in terms of good subgraphs. In the next theorem we prove that minimal $D P D P$-graphs are precisely 2 -subdivision graphs of graphs that have neither an isolated vertex nor a good subgraph.

Theorem 6.2. A graph $G$ is a minimal DPDP-graph if and only if $G=S_{2}(H)$, where $H$ is a graph that has neither an isolated vertex nor a good subgraph.

Proof. Assume first that $G$ is a minimal $D P D P$-graph, and let $(D, P)$ be a $D P$ pair in $G$. It follows from Theorem 4.5 that $G=S_{2}(H)$ for some graph $H$. Since no $D P D P$-graph has an isolated vertex, neither $S_{2}(H)$ nor $H$ has an isolated vertex. We now claim that $H$ has no good subgraph. Suppose, to the contrary, that $Q$ is a good subgraph in $H$. By definition, there exist a set of edges $E$ (where $E_{Q}^{-} \subseteq E \subseteq E_{H} \backslash E_{Q}$ ) and an orientation $A_{E}$ of $E$ such that in the partially oriented graph $H\left(A_{E}\right)$ there exists a family of oriented paths $\mathcal{P}=\left\{P_{x}: x \in V_{Q}\right\}$ satisfying the properties (1)-(3) stated in the definition of a good subgraph.

We adopt the following notation: If $e$ is an edge belonging to $E, \varphi_{H}(e)=$ $\{v, u\}, \xi(e)=\left\{v_{e}, u_{e}\right\}$, and $e_{A}=(v, u)$, then $v, v_{e}, u_{e}, u$ is the 4-path corresponding to $e$ in $S_{2}(H)$, and we write $p_{1}(e)=v, p_{2}(e)=v_{e}, p_{3}(e)=u_{e}$, and $p_{4}(e)=u$. If $e$ is a loop belonging to $E, \varphi_{H}(e)=\{v\}, \xi(e)=\left\{v_{e}^{1}, v_{e}^{2}\right\}$, then $v, v_{e}^{1}, v_{e}^{2}, v$ is the 3 -cycle corresponding to $e$ in $S_{2}(H)$, and we write $p_{1}(e)=v$, $p_{2}(e)=v_{e}^{1}, p_{3}(e)=v_{e}^{2}$, and $p_{4}(e)=v$. Finally, we denote by $e\left(P_{x}\right)$ the edge in $E$ corresponding to the last arc (or loop) in the oriented path $P_{x} \in \mathcal{P}$.

Let us consider now the spanning subgraph $G^{\prime}$ of $G=S_{2}(H)$ in which

$$
E_{G^{\prime}}=E_{S_{2}(H)} \backslash\left(\bigcup_{e \in E_{Q}}\{x y: \xi(e)=\{x, y\}\} \cup\left\{p_{3}\left(e\left(P_{x}\right)\right) p_{4}\left(e\left(P_{x}\right)\right): P_{x} \in \mathcal{P}\right\}\right)
$$

More intuitively, $G^{\prime}$ is the graph obtained from $S_{2}(H)$ by removing the middle edge from the 4 -path corresponding to each edge of $Q$, and the third edge from the 4 -path corresponding to the last arc in every path $P_{x} \in \mathcal{P}$. A graph $H$, its 2 subdivision graph $S_{2}(H)$, and the subgraph $G^{\prime}$ of $S_{2}(H)$ corresponding to a good subgraph $Q$ in $H$ (drawn in bold) and a family of oriented paths $\mathcal{P}=\left\{P_{x}: x \in\right.$ $\left.V_{Q}\right\}$ are shown in Figure 3. Formally, $H, S_{2}(H)$, and $G^{\prime}$ are the underlying graphs of the graphs in Figure 3.

We note that the sets

$$
D^{\prime}=V_{H_{0}} \cup\left\{p_{3}\left(e_{A}\right): e_{A} \in A_{E}\right\} \cup \bigcup_{e \in E_{Q}} \xi(e)
$$

and

$$
P^{\prime}=\bigcup_{e_{A} \in A_{E}}\left\{p_{1}\left(e_{A}\right), p_{2}\left(e_{A}\right)\right\} \cup \bigcup_{e \in E_{H_{0}}} \xi(e)
$$

form a partition of the vertex set of $G^{\prime}$. We now claim that $\left(D^{\prime}, P^{\prime}\right)$ is a $D P$-pair in $G^{\prime}$. If $V_{H_{0}} \neq \emptyset$, then it follows from the construction of $G^{\prime}$ that $G^{\prime}\left[V_{H_{0}} \cup\right.$ $\left.\bigcup_{e \in E_{H_{0}}} \xi(e)\right]=S_{2}\left(H_{0}\right)$ and therefore, as it follows from the proof of Proposition 4.4, the pair $\left(V_{H_{0}}, \bigcup_{e \in E_{H_{0}}} \xi(e)\right)$ is a $D P$-pair in $S_{2}\left(H_{0}\right)$. Thus, it remains to prove that the sets $D^{\prime \prime}=D^{\prime} \backslash V_{H_{0}}$ and $P^{\prime \prime}=P^{\prime} \backslash \bigcup_{e \in E_{H_{0}}} \xi(e)$ form a $D P$-pair in $G^{\prime \prime}=G^{\prime}-S_{2}\left(H_{0}\right)$.

We show firstly that $D^{\prime \prime}$ is a dominating set of $G^{\prime \prime}$. Let $x$ be an arbitrary vertex in $V_{G^{\prime \prime}} \backslash D^{\prime \prime}=P^{\prime \prime}$. Then either $x=p_{2}\left(e_{A}\right)$ or $x=p_{1}\left(e_{A}\right)$ for some $e_{A} \in A_{E}$. In the first case $x$ is adjacent to $p_{3}\left(e_{A}\right) \in D^{\prime \prime}$. Thus assume that $x=p_{1}\left(e_{A}\right)$ and $e_{A} \in A_{E}$. If $x=p_{1}\left(e_{A}\right) \in V_{Q}$, then there exists an edge $f$ in $Q$ incident with $x$, and therefore $x$ is adjacent to $v_{f} \in \xi(f) \subseteq D^{\prime \prime}$. Finally assume that $x=p_{1}\left(e_{A}\right) \notin V_{Q}$. Now $e_{A}$ belongs to some oriented path $P_{v} \in \mathcal{P}$. Since $x=p_{1}\left(e_{A}\right) \notin V_{Q}$, there exists an arc $f_{A}$ on $P_{v}$ such that $p_{4}\left(f_{A}\right)=x=p_{1}\left(e_{A}\right)$, and therefore $x$ is adjacent to $p_{3}\left(f_{A}\right) \in D^{\prime \prime}$. This proves that $D^{\prime \prime}$ is a dominating set of $G^{\prime \prime}$.

We show next that $P^{\prime \prime}$ is a dominating set of $G^{\prime \prime}$. Let $y$ be an arbitrary vertex in $V_{G^{\prime \prime}} \backslash P^{\prime \prime}=D^{\prime \prime}$. If $y=p_{3}\left(e_{A}\right)$ for some $e_{A} \in A_{E}$, then $y$ is adjacent to $p_{2}\left(e_{A}\right) \in P^{\prime \prime}$. Finally assume that $y \in \xi(e)$ for some $e \in E_{Q}$. Without loss of generality, we may assume that $\varphi_{H}(e)=\{u, v\}, \xi(e)=\left\{v_{e}, u_{e}\right\}$, and $y=v_{e}$. Thus, $y$ is adjacent to $p_{1}\left(f_{A}\right) \in P^{\prime \prime}$ where $f_{A}$ is the first arc in the unique path $P_{v} \in \mathcal{P}$ starting at $v$. This implies that $P^{\prime \prime}$ is a dominating set of $G^{\prime \prime}$. In addition, $P^{\prime \prime}$ is a paired-dominating set of $G^{\prime \prime}$, as the edges $p_{1}\left(e_{A}\right) p_{2}\left(e_{A}\right)$, where $e_{A} \in A_{E}$, form a perfect matching in the subgraph induced by $P^{\prime \prime}$. This proves that $\left(D^{\prime \prime}, P^{\prime \prime}\right)$ is a $D P$-pair in $G^{\prime \prime}$, and implies that $\left(D^{\prime}, P^{\prime}\right)$ is a $D P$-pair in a proper spanning subgraph $G^{\prime}$ of $G$, contradicting the minimality of $G$.

Assume now that $H$ is a graph that has neither an isolated vertex nor a good subgraph. By Proposition 4.4, the 2-subdivision graph $G=S_{2}(H)$ of $H$ is a $D P D P$-graph. We claim that $G$ is a minimal $D P D P$-graph. Suppose, to the contrary, that $G$ is not a minimal $D P D P$-graph. Thus some proper spanning subgraph $G^{\prime}$ of $G$ is a minimal $D P D P$-graph, and it follows from Theorem 4.5 that $G^{\prime}$ is a 2 -subdivision graph of some graph $H^{\prime}$, i.e., $G^{\prime}=S_{2}\left(H^{\prime}\right)$.

Since $G^{\prime}$ is a proper spanning subgraph of $G$, the set $E_{G} \backslash E_{G^{\prime}}$ (of the edges removed from $G$ ) is nonempty and it is the union of disjoint subsets $E_{n n}^{\prime}=$ $\left(E_{G} \backslash E_{G^{\prime}}\right) \cap E_{n n}$ and $E_{n o}^{\prime}=\left(E_{G} \backslash E_{G^{\prime}}\right) \backslash E_{n n}$, where $E_{n n}$ is the set of edges of $G$ each of which joins two vertices in $\bigcup_{e \in E_{H}} \xi(e)$. It follows from the definition of the 2-subdivision graph that if $x y \in E_{G} \backslash E_{G^{\prime}}$, then both $x$ and $y$ are leaves in $G^{\prime}$ if $x y \in E_{n n}^{\prime}$ and at least one of the vertices $x$ and $y$ is a leaf in $G^{\prime}$ if $x y \in E_{n o}^{\prime}$, and $\{x, y\} \cap N_{G}\left[L_{G}\right]=\emptyset$ (since $G^{\prime}$ is a $D P D P$-graph). This implies that $G^{\prime}$ has two types of components: those which have at least one leaf belonging to the set $V_{S_{2}(H)}^{n}$, and those in which no leaf belongs to $V_{S_{2}(H)}^{n}$. From this and from Observation 4.1 (6) (and Corollary 4.7) it follows that if $F$ is a component of $G^{\prime}$,
then $F=S_{2}\left(P_{k+1}\right)$ for some $k \in[4]$ and $F$ has at most one strong support vertex if $L_{F} \cap V_{S_{2}(H)}^{n} \neq \emptyset$ or $F$ is an induced subgraph of $G$ if $L_{F} \cap V_{S_{2}(H)}^{n}=\emptyset$.

Let $F_{1}, \ldots, F_{\ell}$ be that components of $G^{\prime}$ for which $L_{F_{i}} \cap V_{S_{2}(H)}^{n} \neq \emptyset$ where $i \in[\ell]$. From this and from the fact that $F_{i}=S_{2}\left(P_{k_{i}+1}\right)$ is of diameter $3 k_{i}+1$ it follows that exactly one support vertex of $F_{i}$ is a vertex of $H$, say $\left\{v^{i}\right\}=S_{F_{i}} \cap V_{H}$ for $i \in[\ell]$. Let $\bar{v}^{i}$ be the (unique) leaf farthest from $v^{i}$ in $F_{i}$, and let $\widetilde{v}^{i}$ be the only vertex in $N_{G}\left(\bar{v}^{i}\right) \backslash N_{G^{\prime}}\left(\bar{v}^{i}\right) \subseteq V_{H}$. Let $\bar{P}_{i}$ be the $v^{i}-\bar{v}^{i}$ path in $F_{i}$, and let $\widetilde{P}_{i}$ be the $v^{i}-\widetilde{v}^{i}$ path obtained from $\bar{P}_{i}$ by adding $\widetilde{v}^{i}$ and the edge $\bar{v}^{i} \widetilde{v}^{i}$. Since $v^{i}, \widetilde{v}^{i} \in V_{G^{\prime}}$ and $d_{G^{\prime}}\left(v^{i}, \widetilde{v}^{i}\right)=3 k_{i}-1$ for some $k_{i} \in[4]$, we may assume that $\bar{P}_{i}$ is the path $v^{i}=x^{0}, x^{1}, \ldots, x^{3 k_{i}-1}=\bar{v}^{i}$ and $\widetilde{P}_{i}$ is the path $v^{i}=x^{0}, x^{1}, \ldots, x^{3 k_{i}-1}=$ $\bar{v}^{i}, x^{3 k_{i}}=\widetilde{v}^{i}$, where $x^{0}, x^{3}, \ldots, x^{3 k_{i}} \in V_{H}$, while $x^{3 j+1}=x_{e}^{3 j}$ and $x^{3 j+2}=x_{e}^{3 j+3}$, where $e$ is an edge joining $x^{3 j}$ and $x^{3 j+3}$ in $H$ for $j \in\{0\} \cup\left[k_{i}-1\right]$ (or $x^{3 j+1}=$ $x_{e}^{3 j 1}$ and $x^{3 j+2}=x_{e}^{3 j 2}$ if $e$ is a loop at $x^{3 j}$ and $j=k_{i}-1$ ). Now let $P_{i}$ be the oriented path $\left(x^{0}, a\left(x^{0}, x^{3}\right), x^{3}, \ldots, x^{3 k_{i}-3}, a\left(x^{3 k_{i}-3}, x^{3 k_{i}}\right), x^{3 k_{i}}\right)$ in $H$, where $a\left(x^{3 j}, x^{3 j+3}\right)$ is the arc which goes from $x^{3 j}$ to $x^{3 j+3}$ and which corresponds to the path $\left(x^{3 j}, x^{3 j+1}, x^{3 j+2}, x^{3 j+3}\right)$ in the path $\widetilde{P}_{i}$ for $j \in\{0\} \cup\left[k_{i}-1\right]$.

Let $Q=\left(V_{Q}, E_{Q}\right)$ be the subgraph of $H$, where $V_{Q}$ consists of those vertices of $H$ which are support vertices in $F_{1}, \ldots, F_{\ell}$, that is, $V_{Q}=\left\{v^{1}, v^{2}, \ldots, v^{\ell}\right\}$, and $E_{Q}$ consists of those edges (and loops) of $H$ whose middle edges were removed in the process of forming $G^{\prime}$ from $G$, i.e., $E_{Q}=\left\{e \in E_{H}: \xi(e)=\{x, y\}\right.$ and $\left.x y \in E_{n n}^{\prime}\right\}$ (see Figure 3, where $Q$ (defined by $G^{\prime}$ ) is the bold subgraph of the underlying graph of $H$ ). All that remains to prove is that $Q$ is a good subgraph in $H$.


Figure 3. Graphs $H, S_{2}(H)$, and a minimal spanning $D P D P$-subgraph $G^{\prime}$ of $S_{2}(H)$.
Since the paths $\widetilde{P}_{1}, \ldots, \widetilde{P}_{\ell}$ are edge-disjoint in $G^{\prime}$, it follows from the definition of $P_{1}, \ldots, P_{\ell}$ that $\mathcal{P}=\left\{P_{1}, \ldots, P_{\ell}\right\}$ is a family of arc-disjoint (not necessarily vertex-disjoint) oriented paths (in $H$ ) indexed by the vertices of $Q$. In addition, $P_{i}$ is the only path belonging to $\mathcal{P}$ and growing out from the vertex $v^{i} \in V_{Q}$, implying that $d_{H}^{+}\left(v^{i}\right)=1$ and $d_{H}^{-}\left(v^{i}\right)=d_{H}\left(v^{i}\right)-d_{Q}\left(v^{i}\right)-1$ for $i \in[\ell]$. From the same fact it follows that if the paths $P_{i}, P_{j} \in \mathcal{P}$, where $i \neq j$, are not vertexdisjoint, then the end vertex of (at least) one of them is the only vertex belonging
to the second one. Consequently, if $x$ is a non-end vertex of a path $P_{i} \in \mathcal{P}$, then $d_{H}^{+}(x)=1$ (and $\left.d_{H}^{-}(x)=d_{H}(x)-1\right)$. Finally assume that $y$ is an end vertex of a path $P_{i} \in \mathcal{P}$. If $d_{H}^{-}(y) \geq d_{H}(y)$, then $y$ would be an isolated vertex in a $D P D P$ graph $G^{\prime}$, which is impossible. Therefore, $d_{H}^{-}(y)<d_{H}(y)$. This proves that $Q$ is a good subgraph in $H$ and this completes the proof of Theorem 6.2.

We are now in a position to present a proof of our main result, namely Theorem 3.1. Recall its statement.

Theorem 3.1. If $G$ is a connected graph of order at least three, then the following statements are equivalent.
(1) $G$ is a minimal DPDP-graph.
(2) $G=S_{2}(H)$ for some connected graph $H$, and either $\left(V_{S_{2}(H)}^{o}, V_{S_{2}(H)}^{n}\right)$ is the unique $D P$-pair in $G$ or $G$ is a cycle of length 3,6 or 9 .
(3) $G=S_{2}(H)$ for some connected graph $H$ that has neither an isolated vertex nor a good subgraph.
(4) $G=S_{2}(H)$ for some connected graph $H$ and no proper spanning subgraph of $G$ without isolated vertices is a 2-subdivision graph.

Proof. The statements (1), (2), and (3) are equivalent by Theorems 6.1 and 6.2. We shall prove that (1) and (4) are equivalent.

Assume that $G$ is a minimal $D P D P$-graph. By Theorem 4.5, $G=S_{2}(H)$ for some connected graph $H$. In addition, since $G$ is a minimal $D P D P$-graph, no proper spanning subgraph of $G$ is a $D P D P$-graph. Thus no proper spanning subgraph of $G$ having no isolated vertex is a 2 -subdivision graph, as, by Proposition 4.4 , every 2 -subdivision graph of a graph with no isolated vertex is a $D P D P$-graph. This proves the implication $(1) \Rightarrow(4)$.

If $G=S_{2}(H)$ for some connected graph $H$, then $G$ is a $D P D P$-graph (by Proposition 4.4). Assume that no proper spanning subgraph of $G$ without isolated vertices is a 2-subdivision graph. We claim that $G$ is a minimal $D P D P$ graph. Suppose, to the contrary, that $G$ is not a minimal $D P D P$-graph. Then, since $G$ is a $D P D P$-graph, some proper spanning subgraph $G^{\prime}$ of $G$ is a minimal $D P D P$-graph. Consequently, $G^{\prime}$ has no isolated vertex (as no $D P D P$-graph has an isolated vertex). In addition, from the minimality of $G^{\prime}$ and from Theorem 4.5 it follows that $G^{\prime}$ is a 2 -subdivision graph. But this contradicts the statement (4) and proves the implication $(4) \Rightarrow(1)$.

The corona $F \circ K_{1}$ of a graph $F$ is the graph obtained from $F$ by adding a pendant edge to each vertex of $F$. A corona graph is a graph obtained from a graph $F$ by attaching any number of pendant edges to each vertex of $F$. In particular, the corona $F \circ K_{1}$ of a graph $F$ is a corona graph.

Corollary 6.3. If $H$ is a corona graph, then its 2-subdivision graph $S_{2}(H)$ is a minimal DPDP-graph. In particular, $S_{2}\left(F \circ K_{1}\right)$ is a minimal DPDP-graph for every graph $F$.

Proof. Since every vertex of a corona graph is a leaf or a support vertex, it follows from Observation 5.1 that $H$ has no good subgraph, and, therefore, $S_{2}(H)$ is a minimal $D P D P$-graph, by Theorem 6.2.

Corollary 6.4. If $H$ is a connected graph, then $S_{2}\left(S_{2}(H)\right)$ is a minimal DPDPgraph if and only if $H$ has either exactly one edge or exactly one loop.
Proof. If $E_{H}=\emptyset$, then $H$ consists of an isolated vertex, and $S_{2}\left(S_{2}(H)\right)=$ $S_{2}(H)=H$ is not a $D P D P$-graph. If $\left|E_{H}\right|=1$, then $H=P_{2}$ (or $H=C_{1}$, respectively), and $S_{2}\left(S_{2}(H)\right)=P_{10}\left(\right.$ or $S_{2}\left(S_{2}(H)\right)=C_{9}$, respectively) is a minimal $D P D P$-graph. Assume now that $\left|E_{H}\right| \geq 2$. Thus, $V_{H} \backslash L_{H} \neq \emptyset$. If $v \in V_{H} \backslash L_{H}$, then $\left|E_{H}(v)\right| \geq 2$ and we consider two cases. Assume first that there is a loop $e$ in $E_{H}(v)$. In this case the vertices $v_{e}^{1}, v_{e}^{2}$, and the edge $v_{e}^{1} v_{e}^{2}$ form a good subgraph in $S_{2}(H)$. Consequently, by Theorem $6.2, S_{2}\left(S_{2}(H)\right)$ is not a minimal DPDPgraph. Assume now that $E_{H}(v)=\left\{e_{1}, \ldots, e_{k}\right\}$ where $k \geq 2$, and no loop belongs to $E_{H}(v)$. Then the vertices $v, v_{e_{1}}, \ldots, v_{e_{k-1}}$, and the edges $v v_{e_{1}}, v v_{e_{2}}, \ldots, v v_{e_{k-1}}$ form a good subgraph in $S_{2}(H)$. From this and from Theorem 6.2 it again follows that $S_{2}\left(S_{2}(H)\right)$ is not a minimal $D P D P$-graph.

## 7. DPDP-Trees

In this section we study the $D P D P$-trees, minimal $D P D P$-trees, and good subgraphs in trees. We begin with the following characterization of $D P D P$-trees.

Proposition 7.1. A tree $T$ is a DPDP-tree if and only if $T$ is a spanning supergraph of a 2-subdivision graph of a forest without isolated vertices and good subgraphs.
Proof. If $H$ is a forest without isolated vertices, then the forest $S_{2}(H)$ is a $D P D P$ graph (by Proposition 4.4) and every spanning supergraph of $S_{2}(H)$ is a $D P D P$ graph. In particular, any tree which is a spanning supergraph of $S_{2}(H)$ is a $D P D P$-tree.

Assume now that a tree $T$ is a $D P D P$-graph. Let $R$ be a spanning minimal $D P D P$-subgraph of $T$. Then $R$ is a forest and it follows from Theorems 4.5 (4) and 6.2 that $R=S_{2}(F)$ for some forest $F$ (without isolated vertices and good subgraphs) and therefore $T$ is a spanning supergraph of $S_{2}(F)$.

We are interested in recognizing the structure of trees having a good subgraph. The following result shows that if a tree has a good forest, then it also has a good subtree.

Proposition 7.2. A tree has a good subgraph if and only if it has a good subtree.
Proof. Assume that a forest $Q$ is a good subgraph in a tree $H$. Let $Q_{1}, \ldots, Q_{k}$ ( $k \geq 2$ ) be the components of $Q$. It suffices to prove that one of the components $Q_{1}, \ldots, Q_{k}$ is a good subgraph in $H$. Let $\mathcal{P}=\left\{P_{v}: v \in V_{Q}\right\}$ be a family of oriented paths indexed by the vertices of $Q$ and having the properties (1)-(3) stated in the definition of a good subgraph (for some subset $E$, where $E_{Q}^{-} \subseteq E \subseteq E_{H} \backslash E_{Q}$, and some orientation $A_{E}$ of the edges in $E$ ). Let $\mathcal{P}_{i}$ denote the family $\left\{P_{v}: v \in V_{Q_{i}}\right\}$ where $i \in[k]$. From the properties of $\mathcal{P}$ and from the fact that $H$ is a tree it follows that $\mathcal{P}_{i}$ is a family of vertex-disjoint paths, each vertex of $Q_{i}$ is the initial vertex of exactly one path belonging to $\mathcal{P}_{i}$, and no path $P_{v} \in \mathcal{P}_{i}$ terminates at a vertex of $Q_{i}$ or at a leaf of $H$. (Although, this time a path belonging to $\mathcal{P}_{i}$ can terminate at a vertex belonging to $Q_{j}$ or to a path in $\mathcal{P}_{j}, j \neq i$.) However, from the same facts it follows that there exists a subtree $Q_{i_{0}} \in\left\{Q_{1}, \ldots, Q_{k}\right\}$ such that no path $P_{v} \in \bigcup_{j \neq i_{0}} \mathcal{P}_{j}$ terminates at $Q_{i_{0}}$. Now $Q_{i_{0}}$ is a good subtree in $H$ as the family $\mathcal{P}_{i_{0}}$ has the properties (1)-(3) stated in the definition of a good subgraph (for the partially ordered graph $H\left[A_{E_{i_{0}}}\right]$, where $A_{E_{i_{0}}}$ is the set of arcs belonging to $A_{E}$ and covered by the paths of $\mathcal{P}_{i_{0}}$, see $Q_{2}$ or $Q_{5}$ in Figure 4).


Figure 4. A good forest in a tree.
We observe that every tree can be a good subtree in a tree. The following result describes the place of a good subtree in a tree and connections between this good subtree and the rest of the tree.

Proposition 7.3. A tree $Q$ is a good subgraph in a tree $H$ if and only if no leaf of $H$ is a neighbor of $Q$ and the subgraph of $H$ induced by the set $N_{H}\left[V_{Q}\right]$ is a corona graph, that is, if and only if $N_{H}\left[V_{Q}\right] \cap L_{H}=\emptyset$ and $H\left[N_{H}\left[V_{Q}\right]\right]=Q \circ K_{1}$.

Proof. Let $Q$ be a good subgraph of $H$ and let $\mathcal{P}=\left\{P_{v}: v \in V_{Q}\right\}$ be a family of oriented paths indexed by the vertices of $Q$ and having the properties (1)-(3)
stated in the definition of a good subgraph (for some subset $E$, where $E_{Q}^{-} \subseteq E \subseteq$ $E_{H} \backslash E_{Q}$, and some orientation $A_{E}$ of the edges in $E$ ). From these properties and from the fact that $H$ is a tree it follows that $\mathcal{P}$ is a family of vertex-disjoint paths, each vertex of $Q$ is the initial vertex of exactly one path belonging to $\mathcal{P}$, and no path $P_{v} \in \mathcal{P}$ terminates at a vertex of $Q$ or at a leaf of $H$. This proves that $N_{H}\left[V_{Q}\right] \cap L_{H}=\emptyset$. (The same follows directly from Observation 5.1.) In addition, every vertex $v$ of $Q$ is adjacent to exactly one vertex in $V_{H} \backslash V_{Q}$, say $s_{v}$, which is the terminal vertex of the first arc in $P_{v}$. Since $H$ is a tree, the set $\left\{s_{v}: v \in V_{Q}\right\}$ is independent and, consequently, the subgraph of $H$ induced by $V_{Q} \cup\left\{s_{v}: v \in V_{Q}\right\}\left(=V_{H}\left[V_{Q}\right]\right)$ is a corona graph isomorphic to $Q \circ K_{1}$.

Now assume that $Q$ is a subtree of $H$ such that $N_{H}\left[V_{Q}\right] \cap L_{H}=\emptyset$ and $H\left[N_{H}\left[V_{Q}\right]\right]=Q \circ K_{1}$. For a vertex $v$ of $Q$, let $v_{\ell}$ denote the only vertex in $N_{H}(v) \backslash V_{Q}$. Since the edge set $E=\left\{v v_{\ell}: v \in V_{Q}\right\}$, the arc set $A_{E}=\left\{\left(v, v_{\ell}\right): v \in\right.$ $\left.V_{Q}\right\}$, and the family of oriented paths $\mathcal{P}=A_{E}$ have properties (1)-(3) of the definition of a good subgraph, we note that $Q$ is a good subgraph in $H$.

Corollary 7.4. If $H$ is a tree of order at least two, then $S_{2}(H)$ is a DPDP-tree. In addition, the DPDP-tree $S_{2}(H)$ is not a minimal DPDP-tree if and only if there is a tree $Q$ in $H-\left(L_{H} \cup S_{H}\right)$ such that $Q \circ K_{1}$ is a subtree in $H-L_{H}$ and $d_{H}(x)=d_{Q}(x)+1$ for each vertex $x$ of $Q$.

## 8. Open Problems

We close this paper with the following list of open problems that we have yet to settle.
(a) How difficult is it to recognize graphs having good subgraphs?
(b) How difficult is it to recognize whether a given graph is a good subgraph in a graph?
(c) How difficult is it to recognize whether a given tree has good subtree?
(d) Provide an algorithm for the problem of determining a good subgraph of a graph.
(e) Since every graph without isolated vertices is homeomorphic to a $D P D P$ graph, it would be interesting to find the smallest number of subdivisions of edges of a graph in order to obtain a $D P D P$-graph.

## References

[1] V. Anusuya and R. Kala, A note on disjoint dominating sets in graphs, Int. J. Contemp. Math. Sci. 7 (2012) 2099-2110.
[2] I. Broere, M. Dorfling, W. Goddard, J.H. Hattingh, M.A. Henning and E. Ungerer, Augmenting trees to have two disjoint total dominating sets, Bull. Inst. Combin. Appl. 42 (2004) 12-18.
[3] G. Chartrand, L. Lesniak and P. Zhang, Graphs and Digraphs (CRC Press, Boca Raton, 2016)
[4] P. Delgado, W.J. Desormeaux and T.W. Haynes, Partitioning the vertices of a graph into two total dominating sets, Quaest. Math. 39 (2016) 863-873. doi:10.2989/16073606.2016.1188862
[5] W.J. Desormeaux, T.W. Haynes and M.A. Henning, Partitioning the vertices of a cubic graph into two total dominating sets, Discrete Appl. Math. 223 (2017) 52-63.
doi:10.1016/j.dam.2017.01.032
[6] M. Dorfling, W. Goddard, J.H. Hattingh and M.A. Henning, Augmenting a graph of minimum degree 2 to have two disjoint total dominating sets, Discrete Math. $\mathbf{3 0 0}$ (2005) 82-90. doi:10.1016/j.disc.2005.06.020
[7] T.W. Haynes and M.A. Henning, Trees with two disjoint minimum independent dominating sets, Discrete Math. 304 (2005) 69-78.
doi:10.1016/j.disc.2005.09.012
[8] S.M. Hedetniemi, S.T. Hedetniemi, R.C. Laskar, L. Markus and P.J. Slater, Disjoint dominating sets in graphs, in: Proc. ICDM 2006, Ramanujan Math. Soc. Lect. Notes Ser. 7 (2008) 87-100.
[9] P. Heggernes and J.A. Telle, Partitioning graphs into generalized dominating sets, Nordic J. Comput. 5 (1988) 128-142.
[10] M.A. Henning, C. Löwenstein and D. Rautenbach, Remarks about disjoint dominating sets, Discrete Math. 309 (2009) 6451-6458. doi:10.1016/j.disc.2009.06.017
[11] M.A. Henning, C. Löwenstein and D. Rautenbach, Partitioning a graph into a dominating set, a total dominating set, and something else, Discuss. Math. Graph Theory 30 (2010) 563-574.
doi:10.7151/dmgt. 1514
[12] M.A. Henning, C. Löwenstein and D. Rautenbach, An independent dominating set in the complement of a minimum dominating set of a tree, Appl. Math. Lett. 23 (2010) 79-81. doi:10.1016/j.aml.2009.08.008
[13] M.A. Henning, C. Löwenstein, D. Rautenbach and J. Southey, Disjoint dominating and total dominating sets in graphs, Discrete Appl. Math. 158 (2010) 1615-1623. doi:10.1016/j.dam.2010.06.004
[14] M.A. Henning and A.J. Marcon, Semitotal domination in graphs: Partition and algorithmic results, Util. Math. 106 (2018) 165-184.
[15] M.A. Henning and D.F. Rall, On graphs with disjoint dominating and 2-dominating sets, Discuss. Math. Graph Theory 33 (2013) 139-146. doi:10.7151/dmgt. 1652
[16] M.A. Henning and J. Southey, A note on graphs with disjoint dominating and total dominating sets, Ars Combin. 89 (2008) 159-162.
[17] M.A. Henning and J. Southey, A characterization of graphs with disjoint dominating and total dominating sets, Quaest. Math. 32 (2009) 119-129. doi:10.2989/QM.2009.32.1.10.712
[18] M.A. Henning and A. Yeo, Total Domination in Graphs (Springer Monographs in Mathematics, Springer, 2013). doi:10.1007/978-1-4614-6525-6
[19] E.M. Kiunisala and F.P. Jamil, On pairs of disjoint dominating sets in a graph, Int. J. Math. Anal. 10 (2016) 623-637. doi:10.12988/ijma.2016.6343
[20] V.R. Kulli and S.C. Sigarkanti, Inverse domination in graphs, Nat. Acad. Sci. Lett. 14 (1991) 473-475.
[21] C. Löwenstein and D. Rautenbach, Pairs of disjoint dominating sets and the minimum degree of graphs, Graphs Combin. 26 (2010) 407-424.
doi:10.1007/s00373-010-0918-9
[22] M. Miotk, J. Topp and P. Żyliński, Disjoint dominating and 2-dominating sets in graphs, Discrete Optim. 35 (2020) 100553. doi:10.1016/j.disopt.2019.100553
[23] O. Ore, Theory of Graphs (Providence, RI, 1962).
[24] J. Southey and M.A. Henning, Graphs with disjoint dominating and paireddominating sets, Cent. Eur. J. Math. 8 (2010) 459-467. doi:10.2478/s11533-010-0033-4
[25] J. Southey and M.A. Henning, Dominating and total dominating partitions in cubic graphs, Cent. Eur. J. Math. 9 (2011) 699-708. doi:10.2478/s11533-011-0014-2
[26] J. Southey and M.A. Henning, A characterization of graphs with disjoint dominating and paired-dominating sets, J. Comb. Optim. 22 (2011) 217-234. doi:10.1007/s10878-009-9274-1

