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MINIMAL GRAPHS WITH DISJOINT DOMINATING AND PAIRED-DOMINATING SETS

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Abstract

A subset $D \subseteq V_G$ is a dominating set of G if every vertex in $V_G - D$ has a neighbor in D, while D is a paired-dominating set of G if D is a dominating set and the subgraph induced by D contains a perfect matching. A graph G is a *DPDP*-graph if it has a pair (D, P) of disjoint sets of vertices of G such that D is a dominating set and P is a paired-dominating set of G. The study of the *DPDP*-graphs was initiated by Southey and Henning [Cent. Eur. J. Math. 8 (2010) 459–467; J. Comb. Optim. 22 (2011) 217–234]. In this paper, we provide conditions which ensure that a graph is a *DPDP*-graph. In particular, we characterize the minimal *DPDP*-graphs.

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1. Introduction

Let $G = (V_G, E_G)$ be a graph with vertex set $V(G) = V_G$ and edge set $E(G) = E_G$, where we allow multiple edges and loops. A set of vertices $D \subseteq V_G$ is a *dominating set* of G if every vertex in $V_G \setminus D$ has a neighbor in D, while D is 2-*dominating set* of G if every vertex in $V_G \setminus D$ has at least two neighbors in D. A set $D \subseteq V_G$ is a *total dominating set* of G if every vertex has a neighbor in D. A set $D \subseteq V_G$ is a *paired-dominating set* of G if D is a dominating set and the subgraph induced by D contains a perfect matching.

Ore [23] was the first to observe that a graph with no isolated vertex contains two disjoint dominating sets. Consequently, the vertex set of a graph without isolated vertices can be partitioned into two dominating sets. Various graph theoretic properties and parameters of graphs having disjoint dominating sets are studied in [1,8–10,14,20,21]. Characterizations of graphs with disjoint dominating and total dominating sets are given in [11–13, 16, 17, 19, 25], while in [2, 4–6, 18] graphs which have the property that their vertex set can be partitioned into two disjoint total dominating sets are studied. Conditions which guarantee the existence of a dominating set whose complement contains a 2-dominating set, a paired-dominating set or an independent dominating set are presented in [7, 12, 15, 19, 20, 22, 26].

In this paper we restrict our attention to conditions which ensure a partition of vertex set of a graph into a dominating set and a paired-dominating set. The study of graphs having a dominating set whose complement is a paired-dominating set was initiated by Southey and Henning [24, 26]. They define a *DP-pair* in a graph G to be a pair (D, P) of disjoint sets of vertices of G such that $V(G) = D \cup P$ where D is a dominating set and P is a paired-dominating set of G. A graph that has a *DP*-pair is called a *DPDP-graph* (standing, as in [24,26], for "dominating, paired dominating, partitionable graph"). It is easy to observe that a complete graph K_n is a *DPDP*-graph if $n \geq 3$ (and K_3 is the smallest *DPDP*-graph), a path P_n is a *DPDP*-graph if and only if $n \in \mathbb{N} \setminus \{1, 2, 3, 5, 6, 9\}$, while a cycle C_n is a *DPDP*-graph if $n \geq 3$ and $n \neq 5$. It was also proved in [24] that every cubic graph is a *DPDP*-graph. In [26] the *DPDP*-graphs (and, in particular, the *DPDP*-trees) were characterized as the graphs which can be constructed from a labeled P_4 by applying eight (four, respectively) operations.

For notation and graph theory terminology we in general follow [3]. Specifically, for a vertex v of a graph $G = (V_G, E_G)$, its *neighborhood*, denoted by $N_G(v)$, is the set of all vertices adjacent to v, and the cardinality of $N_G(v)$, denoted by $d_G(v)$, is called the *degree* of v. The *closed neighborhood* of v, denoted by $N_G[v]$, is the set $N_G(v) \cup \{v\}$. In general, for a subset $X \subseteq V_G$ of vertices, the *neighborhood* of X, denoted by $N_G(X)$, is defined to be $\bigcup_{v \in X} N_G(v)$, and the *closed* neighborhood of X, denoted by $N_G[X]$, is the set $N_G(X) \cup X$. The minimum degree of a vertex in G is denoted by $\delta(G)$. A vertex of degree one is called a *leaf*, and the only neighbor of a leaf is called its *support vertex* (or simply, its *support*). If a support vertex has at least two leaves as neighbors, we call it a *strong* support, otherwise it is a *weak* support. The set of leaves, the set of weak supports, the set of strong supports, and the set of all supports of G is denoted by L_G, S'_G, S''_G, and S_G, respectively. If v is a vertex of G, then by

$E_G(v)$ and $\mathcal{L}_G(v)$ we denote the set of edges and the set of loops incident with v in G, respectively.

We denote the *path*, *cycle*, and *complete graph* on n vertices by P_n, C_n, and K_n, respectively. The *complete bipartite graph* with one partite set of size n and the other of size m is denoted by $K_{n,m}$. A *star* is the tree $K_{1,k}$ for some $k \geq 1$. For $r, s \geq 1$, a *double star* $S(r, s)$ is the tree with exactly two vertices that are not leaves, one of which has r leaf neighbors and the other s leaf neighbors. We define a *pendant edge* of a graph to be an edge incident with a vertex of degree 1. We use the standard notation $[k] = \{1, \ldots, k\}$.

2. 2-Subdivision Graphs of a Graph

Let $H = (V_H, E_H)$ be a graph with no isolated vertices and with possible multi-edges and multi-loops. By φ_H we denote a function from E_H to 2^{V_H} that associates with each $e \in E_H$ the set $\varphi_H(e)$ of vertices incident with e. Let X_2 be a set of 2-element subsets of an arbitrary set (disjoint with $V_H \cup E_H$), and let $\xi\colon E_H \to X_2$ be a function such that $\xi(e) \cap \xi(f) = \emptyset$ if e and f are distinct elements of E_H. If $e \in E_H$ and $\varphi_H(e) = \{u, v\}$ ($\varphi_H(e) = \{v\}$, respectively), then we write $\xi(e) = \{u_e, v_e\}$ ($\xi(e) = \{v_e^1, v_e^2\}$, respectively). If $\alpha\colon L_H \to \mathbb{N}$ is a function, then let $\Phi_\alpha\colon L_H \to L_H \times \mathbb{N}$ be a function such that $\Phi_\alpha(v) = \{(v, i)\colon i \in [\alpha(v)]\}$ for $v \in L_H$.

Now we say that a graph $S_2(H) = (V_{S_2(H)}, E_{S_2(H)})$ is the 2-*subdivision graph* of H (with respect to the functions $\xi\colon E_H \to X_2$ and $\alpha\colon L_H \to \mathbb{N}$), if $V_{S_2(H)} = V^o_{S_2(H)} \cup V^n_{S_2(H)}$, where

$$V^o_{S_2(H)} = (V_H \setminus L_H) \cup \bigcup_{v \in L_H} \Phi_\alpha(v) \quad \text{and} \quad V^n_{S_2(H)} = \bigcup_{e \in E_H} \xi(e),$$

and

$$\begin{aligned} E_{S_2(H)} = & \bigcup_{e \in E_H} \{xy\colon \xi(e) = \{x, y\}\} \cup \bigcup_{v \in L_H} \{v_e(v, i)\colon e \in E_H(v),\ i \in [\alpha(v)]\} \\ & \cup \bigcup_{v \in V_H \setminus L_H} \left(\{vv_e\colon e \in E_H(v)\} \cup \{vv_e^1, vv_e^2\colon e \in \mathcal{L}_H(v)\}\right). \end{aligned}$$

3. Main Result

In this paper, our aim is to characterize *DPDP*-graphs. The following result provides a characterization of minimal *DPDP*-graphs, where a good subgraph is defined in Section 5.

Theorem 3.1. *If G is a connected graph of order at least three, then the following statements are equivalent.*

(1) *G is a minimal DPDP-graph.*

(2) *$G = S_2(H)$ for some connected graph H, and either $\left(V^o_{S_2(H)}, V^n_{S_2(H)}\right)$ is the unique DP-pair in G or G is a cycle of length $3, 6$ or 9.*

(3) *$G = S_2(H)$ for some connected graph H that has neither an isolated vertex nor a good subgraph.*

(4) *$G = S_2(H)$ for some connected graph H and no proper spanning subgraph of G without isolated vertices is a 2-subdivision graph.*

4. Properties of 2-Subdivision Graphs

We remark that 2-subdivision graphs are defined only for graphs without isolated vertices and, intuitively, $S_2(H)$ is the graph obtained from H by inserting two new vertices into each edge and each loop of H, and then replacing each pendant edge $v_e v$ by pendant edges $v_e(v, 1), \ldots, v_e(v, \alpha(v))$. In particular, it follows from this definition that every tree of diameter three (i.e., every double star) is a 2-subdivision graph of K_2. Moreover, a path P_n (of order n) is a 2-subdivision graph (of a path) if and only if $n = 3k + 1$ for every positive integer k and here α assign to each leaf the value 1. Figure 1 shows a graph H and a possible 2-subdivision graph $S_2(H)$ of H where $\alpha\colon L_H \to \{3\}$.

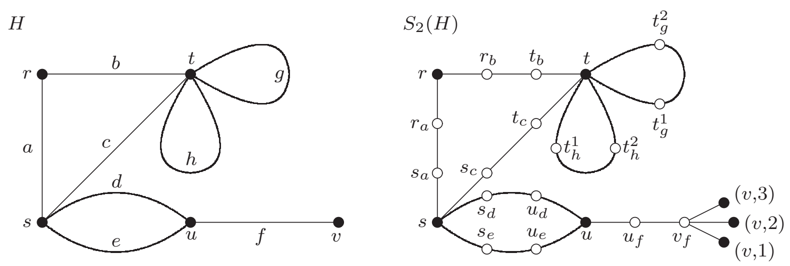

Figure 1. A 2-subdivision graph $S_2(H)$ of a graph H.

Observation 4.1. *Let H be a graph with no isolated vertex, and let $G = S_2(H)$ be the 2-subdivision graph of H (with respect to functions $\xi\colon E_H \to X_2$ and $\alpha\colon L_H \to \mathbb{N}$). Then the following statements hold.*

(1) *$d_G(v) = d_H(v)$ if $v \in V_H \setminus L_H$, and $d_G((v, i)) = 1$ if $v \in L_H$ and $i \in [\alpha(v)]$.*

(2) *$d_G(x) = 2$ if $x \in V^n_{S_2(H)} \setminus S_G$, and $d_G(v_e) = 1 + \alpha(v)$ if $v \in L_H$ and $e \in E_H(v)$.*

(3) *If* $x, y \in V_G \setminus V^n_{S_2(H)}$ *are distinct, and belong to the same component of* G*, then either* $d_G(x,y) \equiv 0 \pmod 3$ *or* $x, y \in L_G$ *and* $d_G(x,y) = 2$*.*

(4) *If* $x \in S_G$*, then* $\left|N_G(x) \cap V^n_{S_2(H)}\right| = 1$ *and* $N_G(x) \setminus V^n_{S_2(H)} \subseteq L_G$*.*

(5) *If* $x \in V_G$*, then the following hold.*

  (a) *If* $d_G(x) > 2$*, then either* $x \in V_H$ *or* $x \in S_G$ *and* $|N_G(x) \setminus L_G| = 1$*.*

  (b) *If* $x \in V^n_{S_2(H)}$*, then either* $d_G(x) = 2$ *or* $d_G(x) > 2$ *and* $x \in S_G$*.*

  (c) *If* $x \in V^n_{S_2(H)}$ *and* $d_G(x) > 2$*, then* $x \in S_G$*.*

(6) *Let* G' *be a* 2*-subdivision graph which is a spanning subgraph of* G*. If* F *is a component of* G'*, then* F *has exactly one of the following properties.*

  (a) F *is an induced subgraph of* G *if no leaf of* F *is in* $V^n_{S_2(H)}$*.*

  (b) F *is a* 2*-subdivision graph of a path* P_{k+1} $(k \geq 1)$ *and* F *has at most one strong support if at least one leaf of* F *is in* $V^n_{S_2(H)}$*. In addition, exactly one of these support vertices is in* V_H*. Moreover, if* F *has a strong support, then this strong support vertex is in* V_H*.*

Proof. The statements (1)–(5) are immediate consequences of the definition of the 2-subdivision graph. To prove (6), let G' be a spanning subgraph of G that is a 2-subdivision graph and let F be a component of G'. Since G' is a 2-subdivision graph, so too is the graph F, i.e., $F = S_2(H')$ for some connected graph H' (and some functions $\xi' \colon E_{H'} \to X_2$ and $\alpha' \colon L_{H'} \to \mathbb{N}$).

Case 1. $L_F \cap V^n_{S_2(H)} = \emptyset$. Since F is a 2-subdivision graph, the sets $V_F \cap V_H$ and $V_F \cap V^n_{S_2(H)}$ are nonempty. Assume first that $v \in V_F \cap V_H$ and e is a loop at v in H. We claim that the vertices v^1_e and v^2_e, and the edges vv^1_e, $v^1_e v^2_e$, vv^2_e belong to F. If v^1_e or v^2_e were not in F, then G' (which is a spanning subgraph of G) would have a component of order one or two, which is impossible in a 2-subdivision graph. Now, since neither v^1_e nor v^2_e is a leaf in F, both v^1_e and v^2_e are of degree 2 in F and this proves that the edges vv^1_e, $v^1_e v^2_e$, vv^2_e belong to F. We can similarly show that if $u, v \in V_F \cap V_H$ and e is an edge joining u to v in H, then the vertices u_e and v_e, and the edges vv_e, $v_e u_e$, $u_e u$ belong to F. From this it follows that F is a 2-subdivision graph of the induced subgraph $H[V_F \cap V_H]$ and, therefore, F is an induced subgraph of G.

Case 2. $L_F \cap V^n_{S_2(H)} \neq \emptyset$. Let x_0 be a leaf of F which belongs to $V^n_{S_2(H)}$ in G. Since G' is a 2-subdivision graph, the vertex x_0 does not belong to $N_G[S_G]$. In addition, if $\Delta(F) \leq 2$, then F is a path, and, since F is a 2-subdivision graph, we note that $F = P_{3k+1} = S_2(P_{k+1})$ (for some positive integer k), as desired. Thus assume that $\Delta(F) \geq 3$. Let x be a vertex of degree at least 3 in F. It follows from (5) applied to the graph $F = S_2(H')$ that either $x \in V_{H'}$ or $x \in S_F$ and $|N_F(x) \setminus L_F| = 1$. However, such a vertex x cannot be in $V_{H'} = \{y \in$

$V_F\colon d_F(x_0,y)\equiv 0\,(\mathrm{mod}\ 3)\}$, as every vertex belonging to $V_{H'}\setminus L_{H'}\subseteq V^n_{S_2(H)}$ is of degree 2 in G and in F, while vertices in F corresponding to elements of $L_{H'}$ are of degree 1. This proves that $x\in S_F$ and $|N_F(x)\setminus L_F|=1$, that is, every vertex of degree at least 3 in F is a strong support vertex and it has only one neighbor which is not a leaf. From this it follows that F is a 2-subdivision graph $S_2(P_{k+1})$ with at least one strong support vertex (for some positive integer k).

It remains to show that F cannot have two strong support vertices. Suppose, for the sake of contradiction, that s_1 and s_2 are distinct strong support vertices in F. Let ℓ_1 and ℓ_2 be leaves in F adjacent to s_1 and s_2, respectively. Since s_1 and s_2 are vertices of degree at least three in $G=S_2(H)$, it follows from (5) that each of them belongs to V_H or S_G. There are three cases to consider. If $s_1,s_2\in V_H$, then it follows from (3) that $d_G(s_1,s_2)\equiv 0\,(\mathrm{mod}\ 3)$, implying that $d_F(\ell_1,\ell_2)\equiv 2$ $(\mathrm{mod}\ 3)$ and $d_F(\ell_1,\ell_2)\neq 2$, contradicting (3) in F. Hence renaming s_1 and s_2 if necessary, we may assume that $s_2\in S_G$. If $s_1\in V_H$, then it follows from (3) that $d_G(s_1,\ell_2)\equiv 0\,(\mathrm{mod}\ 3)$, implying that $d_F(\ell_1,\ell_2)\equiv 1\,(\mathrm{mod}\ 3)$, contradicting (3) in F. Hence, $s_1\in S_G$. Thus, no leaf of F belongs to $V^n_{S_2(H)}$ in G, contradicting our choice of F. This completes the proof of the statement (6). ■

We next present the following elementary property of a *DPDP*-graph.

Observation 4.2. *If (D,P) is a DP-pair in a graph G, then every leaf of G belongs to D, while every support of G is in P, that is, $L_G\subseteq D$ and $S_G\subseteq P$.*

A connected graph G is said to be a *minimal DPDP-graph*, if G is a *DPDP*-graph and no proper spanning subgraph of G is a *DPDP*-graph.

We remark that a complete graph K_n is a minimal *DPDP*-graph only if $n=3$. We observe that a path P_n is a minimal *DPDP*-graph if and only if $n\in\{4,7,10,13\}$, while a cycle C_n is a minimal *DPDP*-graph if and only if $n\in\{3,6,9\}$. From the definition of a minimal *DPDP*-graph we immediately have the following important (and intuitively easy) observation.

Observation 4.3. *Every spanning supergraph of a DPDP-graph is a DPDP-graph, and, trivially, every DPDP-graph is a spanning supergraph of some minimal DPDP-graph.*

We show next that the 2-subdivision graph of an isolate-free graph is a *DPDP*-graph.

Proposition 4.4. *If a graph H has no isolated vertex, then its 2-subdivision graph $S_2(H)$ is a DPDP-graph.*

Proof. Let $S_2(H)$ be the subdivision graph of H (with respect to functions $\xi\colon E_H\to X_2$ and $\alpha\colon L_H\to\mathbb{N}$). We shall prove that (D,P) is a *DP*-pair in

$S_2(H)$, where

$$D = V^o_{S_2(H)} = (V_H \setminus L_H) \cup \bigcup_{v \in L_H} \Phi_\alpha(v)$$

and $P = V^n_{S_2(H)} = V_{S_2(H)} \setminus D$. If $x \in P$, then $x \in \xi(e)$ for some $e \in E_H$, and x is adjacent in $S_2(H)$ to a vertex incident with e in H. This proves that D is a dominating set of $S_2(H)$. Assume now that $y \in D$. If $y \in V_H \setminus L_H$, then, since H has no isolated vertex, there is an edge f incident with y in H, and therefore y is adjacent to $y_f \in P$ (or to $y^1_f \in P$ and $y^2_f \in P$ if f is a loop) in $S_2(H)$. If $y \in \Phi_\alpha(v)$ for some $v \in L_H$, then y is adjacent to v_e in $S_2(H)$, where e is the only pendant edge incident with v in H. Consequently, P is a dominating set of $S_2(H)$. In addition, since the two vertices of $\xi(e)$ are adjacent in $S_2(H)$ for every $e \in E_H$, the set $P = \bigcup_{e \in E_H} \xi(e)$ is a paired-dominating set of $S_2(H)$. This proves that $S_2(H)$ is a *DPDP*-graph. ■

Since every graph is homeomorphic to its 2-subdivision graph, it follows from Proposition 4.4 that every graph without isolated vertices is homeomorphic to a *DPDP*-graph. Consequently, the structure of *DPDP*-graphs becomes more complex.

The next theorem presents general properties of *DP*-pairs in a minimal *DPDP*-graph.

Theorem 4.5. *If G is a minimal DPDP-graph and (D, P) is a DP-pair in G, then the following four statements hold.*

(1) *D is a maximal independent set in G.*

(2) *The induced graph $G[P]$ consists of independent edges, that is, $\delta(G[P]) = \Delta(G[P]) = 1$.*

(3) *If $x \in P$, then $|N_G(x) \setminus P| = 1$ or $N_G(x) \setminus P$ is a nonempty subset of L_G.*

(4) *G is a 2-subdivision graph of some graph H.*

Proof. (1) If D is not an independent set, then D contains two vertices, say x and y, that are adjacent. In this case, (D, P) would be a *DP*-pair in $G - xy$, contradicting the minimality of G. Hence, the set D is both an independent and dominating set of G, implying that D is a maximal independent set in G.

(2) Since P is a paired-dominating set of G, by definition, $G[P]$ has a perfect matching, say M. If xy is an edge of $G[P]$ which is not in M, then (D, P) would be a *DP*-pair in $G - xy$, violating the minimality of G. Hence, the edges of M are the only edges of $G[P]$.

(3) Assume that $x \in P$. It follows from (2) that x has exactly one neighbor in P, say x'. Thus since (D, P) is a *DP*-pair in G, we note that $N_G(x) \setminus \{x'\}$ is a nonempty subset of the dominating set D of G. If every neighbor of x in D is a leaf, then $N_G(x) \setminus P$ is a nonempty subset of L_G. Hence we may assume that

x contains a neighbor y in D that is not a leaf, for otherwise the desired result follows. If x contains a neighbor in D different from y, then x is dominated by a vertex belonging to $D \setminus \{y\}$ and y is dominated by some vertex in $P \setminus \{x\}$, implying that (D, P) is a *DP*-pair in $G - xy$, contradicting the minimality of G. Hence in this case, the vertex y is the only neighbor of x in D, and so $N_G(x) = \{x', y\}$ and $|N_G(x) \setminus P| = 1$.

(4) Let G be a minimal *DPDP*-graph, and let (D, P) be a *DP*-pair in G. For a support vertex s, the set of leaves adjacent to s is denoted by $L_G(s)$, i.e., $L_G(s) = N_G(s) \cap L_G$. Let G^* denote the graph resulting from G by replacing the vertices of $L_G(s)$ by a new vertex v_s and joining v_s to s, for every $s \in S_G$, i.e., $G^* = (V_{G^*}, E_{G^*})$, where $V_{G^*} = (V_G \setminus L_G) \cup \{v_s \colon s \in S_G\}$ and $E_{G^*} = E_{G-L_G} \cup \{sv_s \colon s \in S_G\}$. By (2) and (3) above, we note that every vertex of P has degree 2 in G^*. Further, every vertex of P has exactly one neighbor in P.

We define a graph $H = (V_H, E_H, \varphi_H)$ as follows. Let $V_H = V_{G^*} \setminus P$. For every edge v_1v_2 in G^* that joins two vertices of P we do the following. If v_1 and v_2 have a common neighbor, say v, in G^*, then in H we add a loop in H at the vertex v. If v_1 and v_2 do not have a common neighbor in G^*, then we add the edge u_1u_2 to H where u_1 is the neighbor of v_1 different from v_2 and where u_2 is the neighbor of v_2 different from v_1 (and so $u_1v_1v_2u_2$ is a path in G^*). We let $\varphi_H \colon E_H \to 2^{V_H}$ be the function such that $\varphi_H(m) = N_{G^*}(m) \setminus m$ if $m \in E_H$. Now let $\xi \colon E_H \to 2^{V_H}$ and $\alpha \colon L_H \to \mathbb{N}$ be functions such that $\xi(e) = e$ if $e \in E_H$, and $\alpha(v_s) = |L_G(s)|$ if $v_s \in L_H$. With these definitions, the graph G is isomorphic to the 2-subdivision graph $S_2(H)$ of H (with respect to functions $\xi \colon E_H \to 2^{V_H}$ and $\alpha \colon L_H \to \mathbb{N}$). That means that we can restore the graph G by applying the operation S_2 to the graph H. ■

By Theorem 4.5 (4) every minimal *DPDP*-graph is a 2-subdivision graph of some graph. The converse, however, is not true in general. For example, if H is the underlying graph of any of the graphs in Figure 2, then its 2-subdivision graph $S_2(H)$ is a *DPDP*-graph, but it is not a minimal *DPDP*-graph. The following result, which is a special case of Theorem 6.2 proven later in the paper, will be useful to establish which 2-subdivision graphs are not minimal *DPDP*-graphs.

Proposition 4.6. *Let x and y be adjacent vertices of degree 2 in a graph H without isolated vertices, and let x' and y' be the vertices such that $N_H(x) \setminus \{y\} = \{x'\}$ and $N_H(y) \setminus \{x\} = \{y'\}$, respectively. If the sets $N_H(x') \setminus \{x, y\}$ and $N_H(y') \setminus \{x, y\}$ are both nonempty, then the 2-subdivision graph $S_2(H)$ is a DPDP-graph but not a minimal DPDP-graph.*

Proof. It follows from Proposition 4.4 that $S_2(H)$ is a *DPDP*-graph and (D, P) is a *DP*-pair in $S_2(H)$, where $D = V^o_{S_2(H)}$ and $P = V^n_{S_2(H)}$. The pair (D', P'), where $D' = (D \setminus \{x, y\}) \cup \{x_{xy}, y_{xy}, x'_{xx'}, y'_{yy'}\}$ and $P' = \big(P \setminus \{x'_{xx'}, y'_{yy'}\}\big) \cup \{x, y\}$

is a *DP*-pair in the proper spanning subgraph $S_2(H) \setminus \{x_{xy}y_{xy}, x'x'_{xx'}, y'y'_{yy'}\}$ of $S_2(H)$. Thus, $S_2(H)$ is a *DPDP*-graph but not a minimal *DPDP*-graph. ■

As a consequence of Proposition 4.6, we can readily determine the minimal *DPDP*-paths and minimal *DPDP*-cycles.

Corollary 4.7. *The following holds.*

(a) *If P_n is a path of order n, then $S_2(P_n)$ is a DPDP-graph for every $n \geq 2$, and $S_2(P_n)$ is a minimal DPDP-graph if and only if $n \in \{2, 3, 4, 5\}$.*

(b) *If C_m is a cycle of size m, then $S_2(C_m)$ is a DPDP-graph for every positive integer m, and $S_2(C_m)$ is a minimal DPDP-graph if and only if $m \in [3]$.*

5. Good Subgraphs of a Graph

In this section, we define a good subgraph of a graph. Let Q be a subgraph without isolated vertices of a graph H, and let E_Q^- denote the set of edges belonging to $E_H \setminus E_Q$ that are incident with a vertex of Q. Let E be a set such that $E_Q^- \subseteq E \subseteq E_H \setminus E_Q$, and let A_E is a set of arcs obtained by assigning an orientation for each edge in E. Then by $H(A_E)$ we denote the partially oriented graph obtained from H by replacing the edges in E by the arcs belonging to A_E. If $e \in E$, then by e_A we denote the only arc in A_H that corresponds to e. By H_0 we denote the subgraph of $H(A_E)$ induced by the vertices that are not the initial vertex of an arc belonging to A_E, i.e., by the set $\left\{v \in V_H \colon d^+_{H(A_E)}(v) = 0\right\}$.

We say that Q is a *good subgraph of* H if there exist a set of edges E (where $E_Q^- \subseteq E \subseteq E_H \setminus E_Q$) and a set of arcs A_E such that in the resulting graph $H(A_E)$, which we simply denote by H for notational convenience, the arcs in A_E form a family $\mathcal{P} = \{P_x \colon x \in V_Q\}$ of oriented paths indexed by the vertices of Q and such that the following holds.

(1) Every vertex of Q is an initial vertex of exactly one path belonging to $\mathcal{P}$. For each vertex $v \in Q$, we denote the (unique) path belonging to $\mathcal{P}$ that begins at v by P_v. Thus, if $v \in V_Q$, then $d^+_H(v) = 1$ and $d^-_H(v) = d_H(v) - d_Q(v) - 1$.

(2) If x is an inner vertex of a path $P_v \in \mathcal{P}$, then $d^+_H(x) = 1$ and $d^-_H(x) = d_H(x) - 1$.

(3) If x is a end vertex of a path $P_v \in \mathcal{P}$, then $d^-_H(x) < d_H(x)$.

Examples of good subgraphs in small graphs are presented in Figure 2. For clarity, the edges of a good subgraph Q are drawn in bold, the arcs belonging to oriented paths are thin (and their orientations are represented by arrows), and all other edges, if any, belong to the subgraph H_0, are thin and without arrows.

We remark that not every graph has a good subgraph (see also Observation 7.3 and Corollary 7.2). On the other hand, if Q is a graph with no isolated vertex, and H is the graph obtained from Q by attaching one pendant edge to each vertex of Q and then subdividing this edge, then Q is a good subgraph in H, implying that every graph without isolated vertices can be a good subgraph of some graph.

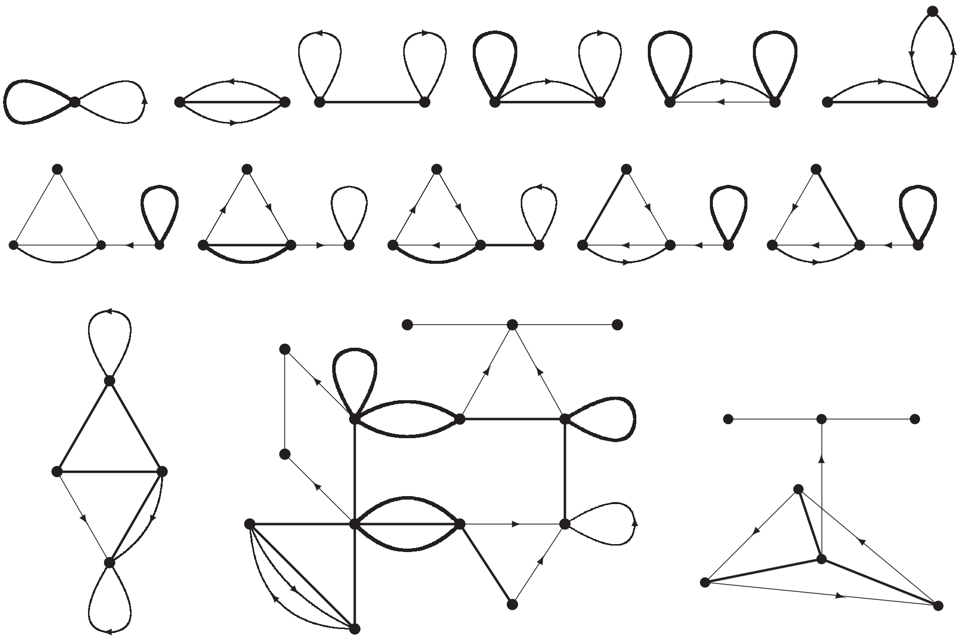

Figure 2. Examples of good subgraphs (drawn in bold) in small graphs.

From the definition of a good subgraph we immediately have the following observation.

Observation 5.1. *Neither a leaf nor a support vertex of a graph H belongs to a good subgraph in H.*

6. Structural Characterization of *DPDP*-graphs

In this section, we present a proof of our main result, namely Theorem 3.1, which provides a characterization of minimal *DPDP*-graphs. We proceed further with the following result.

Theorem 6.1. *If G is a connected graph of order at least 3, then G is a minimal DPDP-graph if and only if $G = S_2(H)$ for some connected graph H, and either $\left(V^o_{S_2(H)}, V^n_{S_2(H)}\right)$ is the only DP-pair in $S_2(H)$ or $S_2(H)$ is a cycle of length 3, 6 or 9.*

Proof. If $G = S_2(H)$ is a cycle of length 3, 6 or 9, then G is clearly a minimal *DPDP*-graph, as claimed. Thus assume that $\left(V^o_{S_2(H)}, V^n_{S_2(H)}\right)$ is the only *DP*-pair in $G = S_2(H)$. Certainly, G is a *DPDP*-graph, and we shall prove that G is a minimal *DPDP*-graph. Suppose, to the contrary, that G is not a minimal *DPDP*-graph. Then some proper spanning subgraph G' of G is a *DPDP*-graph. Let (D', P') be a *DP*-pair in G' and, consequently, in G (by Observation 4.3). Thus $\left(V^o_{S_2(H)}, V^n_{S_2(H)}\right)$ and (D', P') are *DP*-pairs in G, and $\left(V^o_{S_2(H)}, V^n_{S_2(H)}\right) \neq (D', P')$, noting that $\left(V^o_{S_2(H)}, V^n_{S_2(H)}\right)$ is a *DP*-pair in no proper spanning subgraph of $G = S_2(H)$. This contradicts the uniqueness of a *DP*-pair in G and proves that G is a minimal *DPDP*-graph.

Suppose next that G is a minimal *DPDP*-graph. By Theorem 4.5, G is a 2-subdivision graph of some connected graph H, i.e., $G = S_2(H)$, and the pair $(D, P) = \left(V^o_{S_2(H)}, V^n_{S_2(H)}\right)$ is a *DP*-pair in $S_2(H)$. It remains to prove that either $\left(V^o_{S_2(H)}, V^n_{S_2(H)}\right)$ is the only *DP*-pair in $S_2(H)$ or $S_2(H)$ is a cycle of length 3, 6 or 9. We consider three cases depending on $\Delta(H)$.

Case 1. $\Delta(H) = 1$. In this case, $H = P_2$, and its 2-subdivision graph $S_2(P_2)$ (which is a double star $S(r, s)$ for some positive integers r and s) has the desired property.

Case 2. $\Delta(H) = 2$. In this case, H is a cycle C_m where $m \geq 1$ or a path P_n where $n \geq 3$. Now, since $S_2(H)$ is a minimal *DPDP*-graph, Corollary 4.7, implies that $H = C_m$ and $m \in [3]$, or $H = P_n$ and $n \in \{3, 4, 5\}$. In each of these six cases $S_2(H)$ has the desired property.

Case 3. $\Delta(H) \geq 3$. In this case, we claim that $(D, P) = \left(V^o_{S_2(H)}, V^n_{S_2(H)}\right)$ is the only *DP*-pair in $S_2(H)$. Suppose to the contrary that (D', P') is another *DP*-pair in G. Then, since D and D' are maximal independent sets in G (by Theorem 4.5) and $D \neq D'$, each of the sets $D \setminus D'$ and $D' \setminus D$ is a nonempty subset of P' and P, respectively. Let v be a vertex of maximum degree among all vertices in $D \setminus D' \subseteq P'$. Since $v \in P'$, it follows from Theorem 4.5 that $d_H(v) \geq 2$. We deal with the two cases when $d_H(v) = 2$ and $d_H(v) \geq 3$ in turn.

Case 3.1. $d_H(v) \geq 3$. We distinguish three subcases.

Subcase 3.1.1. There are only loops at v in H. Since $d_H(v) \geq 3$, there are at least two loops at v, say e and f. Renaming loops if necessary, we may assume that v^1_e is the (unique) neighbor of v belonging to P'. We note that $v^2_e \in D'$ and that all other neighbors of v in G, including v^1_f and v^2_f, belong to D'. Therefore, (D', P') is also a *DP*-pair in the proper subgraph $G - vv^2_e$ of G, contradicting the minimality of G.

Subcase 3.1.2. There is exactly one loop at v in H. Let e be the loop at v in H and let f be an edge of H incident with v. If v_e^1 (v_e^2, respectively) is the (unique) neighbor of v belonging to P', then as in Subcase 3.1.1 we infer that (D', P') is a *DP*-pair in the subgraph $G - vv_e^2$ ($G - vv_e^1$, respectively) of G. If v_f is the (unique) neighbor of v belonging to P', then (D', P') is a *DP*-pair in the subgraph $G - v_e^1 v_e^2$ of G. In both cases we get a contradiction to the minimality of G.

Subcase 3.1.3. There is no loop at v in H. In this case, there are three distinct edges, say e, f, and g, incident with v joining v to u, w, and z, respectively. Assume first that u, w, and z are distinct and, without loss of generality, v_e is the (unique) neighbor of v which belongs to P'. Then, since G is a minimal *DPDP*-graph and (D', P') is a *DP*-pair in G, Theorem 4.5 implies that the vertices u_e, v_f, v_g belong to D', while u, w, w_f, z, and z_g belong to P'. This implies that (D', P') is a *DP*-pair in $G - vv_g$, contradicting the minimality of G. We derive similar contradictions if u, w, and z are not distinct, and one of the vertices v_e, v_f, v_g is the (unique) neighbor of v that belongs to P'. We omit the proofs of these cases which are analogous to the previous case when u, w, and z are distinct.

Case 3.2. $d_H(v) = 2$. By our choice of the vertex v, this implies that $d_H(x) = 2$ for every $x \in D \setminus D'$. Since $\Delta(H) \geq 3$, we note that H is not a cycle, implying that there is no loop at v. Let e and f be the two edges incident with v. Renaming the edges e and f if necessary, we may assume that v_e is the (unique) neighbor of v in P'.

Suppose that e and f are parallel edges. Let u be the second common vertex of e and f. In this case, we note that $d_H(u) \geq 3$ as H is not a cycle. Since G is a minimal *DPDP*-graph and (D', P') is a *DP*-pair in G, Theorem 4.5 implies that the vertices u_e and v_f belong to D', while u and u_f belong to P'. In particular, $u \in P'$, $d_H(u) \geq 3$, and u_e is a neighbor of u not in P' of degree 2. This contradicts Theorem 4.5 which states that every neighbor of u not in P' is a leaf of G. Hence, the edges e and f are not parallel edges. Thus, e and f join v to distinct vertices u and w, respectively.

Recall that by our earlier assumption, v_e is the (unique) neighbor of v in P'. Theorem 4.5 implies that the vertices u_e and v_f belong to D', while u, w and v_f belong to P'. If $d_H(u) \geq 3$, then noting that u_e is a neighbor of u not in P' of degree 2, we contradict Theorem 4.5. Hence, $d_H(u) = 2$. Analogously, $d_H(w) = 2$. Let u' and w' be the neighbor of u and w, respectively, different from v in H, and so $N_H(u) \setminus \{v\} = \{u'\}$ and $N_H(w) \setminus \{v\} = \{w'\}$. Since $H \neq C_3$, we note that $w' \neq u$ (and $u' \neq w$). We remark that possibly, $u' = w'$. Since $\Delta(H) \geq 3$, at least one of the vertices u' and w' is not a leaf in H. By symmetry, we may assume that u' is not a leaf in H, and so $d_H(u') \geq 2$. Proposition 4.6 with $x = v$, $y = u$, $x' = w$, and $y' = u'$ implies that G is not a minimal *DPDP*-graph, the final contradiction which completes the proof of Theorem 6.1. ■

We next provide a characterization of minimal *DPDP*-graphs in terms of good subgraphs. In the next theorem we prove that minimal *DPDP*-graphs are precisely 2-subdivision graphs of graphs that have neither an isolated vertex nor a good subgraph.

Theorem 6.2. *A graph G is a minimal DPDP-graph if and only if $G = S_2(H)$, where H is a graph that has neither an isolated vertex nor a good subgraph.*

Proof. Assume first that G is a minimal *DPDP*-graph, and let (D, P) be a *DP*-pair in G. It follows from Theorem 4.5 that $G = S_2(H)$ for some graph H. Since no *DPDP*-graph has an isolated vertex, neither $S_2(H)$ nor H has an isolated vertex. We now claim that H has no good subgraph. Suppose, to the contrary, that Q is a good subgraph in H. By definition, there exist a set of edges E (where $E_Q^- \subseteq E \subseteq E_H \setminus E_Q$) and an orientation A_E of E such that in the partially oriented graph $H(A_E)$ there exists a family of oriented paths $\mathcal{P} = \{P_x \colon x \in V_Q\}$ satisfying the properties (1)–(3) stated in the definition of a good subgraph.

We adopt the following notation: If e is an edge belonging to E, $\varphi_H(e) = \{v, u\}$, $\xi(e) = \{v_e, u_e\}$, and $e_A = (v, u)$, then v, v_e, u_e, u is the 4-path corresponding to e in $S_2(H)$, and we write $p_1(e) = v$, $p_2(e) = v_e$, $p_3(e) = u_e$, and $p_4(e) = u$. If e is a loop belonging to E, $\varphi_H(e) = \{v\}$, $\xi(e) = \{v_e^1, v_e^2\}$, then v, v_e^1, v_e^2, v is the 3-cycle corresponding to e in $S_2(H)$, and we write $p_1(e) = v$, $p_2(e) = v_e^1$, $p_3(e) = v_e^2$, and $p_4(e) = v$. Finally, we denote by $e(P_x)$ the edge in E corresponding to the last arc (or loop) in the oriented path $P_x \in \mathcal{P}$.

Let us consider now the spanning subgraph G' of $G = S_2(H)$ in which

$$E_{G'} = E_{S_2(H)} \setminus \left(\bigcup_{e \in E_Q} \{xy \colon \xi(e) = \{x, y\}\} \cup \{p_3\big(e(P_x)\big)p_4\big(e(P_x)\big) \colon P_x \in \mathcal{P}\} \right).$$

More intuitively, G' is the graph obtained from $S_2(H)$ by removing the middle edge from the 4-path corresponding to each edge of Q, and the third edge from the 4-path corresponding to the last arc in every path $P_x \in \mathcal{P}$. A graph H, its 2-subdivision graph $S_2(H)$, and the subgraph G' of $S_2(H)$ corresponding to a good subgraph Q in H (drawn in bold) and a family of oriented paths $\mathcal{P} = \{P_x \colon x \in V_Q\}$ are shown in Figure 3. Formally, H, $S_2(H)$, and G' are the underlying graphs of the graphs in Figure 3.

We note that the sets

$$D' = V_{H_0} \cup \{p_3(e_A) \colon e_A \in A_E\} \cup \bigcup_{e \in E_Q} \xi(e)$$

and

$$P' = \bigcup_{e_A \in A_E} \{p_1(e_A), p_2(e_A)\} \cup \bigcup_{e \in E_{H_0}} \xi(e)$$

form a partition of the vertex set of G'. We now claim that (D', P') is a *DP*-pair in G'. If $V_{H_0} \neq \emptyset$, then it follows from the construction of G' that $G'[V_{H_0} \cup \bigcup_{e\in E_{H_0}} \xi(e)] = S_2(H_0)$ and therefore, as it follows from the proof of Proposition 4.4, the pair $(V_{H_0}, \bigcup_{e\in E_{H_0}} \xi(e))$ is a *DP*-pair in $S_2(H_0)$. Thus, it remains to prove that the sets $D'' = D' \setminus V_{H_0}$ and $P'' = P' \setminus \bigcup_{e\in E_{H_0}} \xi(e)$ form a *DP*-pair in $G'' = G' - S_2(H_0)$.

We show firstly that D'' is a dominating set of G''. Let x be an arbitrary vertex in $V_{G''} \setminus D'' = P''$. Then either $x = p_2(e_A)$ or $x = p_1(e_A)$ for some $e_A \in A_E$. In the first case x is adjacent to $p_3(e_A) \in D''$. Thus assume that $x = p_1(e_A)$ and $e_A \in A_E$. If $x = p_1(e_A) \in V_Q$, then there exists an edge f in Q incident with x, and therefore x is adjacent to $v_f \in \xi(f) \subseteq D''$. Finally assume that $x = p_1(e_A) \notin V_Q$. Now e_A belongs to some oriented path $P_v \in \mathcal{P}$. Since $x = p_1(e_A) \notin V_Q$, there exists an arc f_A on P_v such that $p_4(f_A) = x = p_1(e_A)$, and therefore x is adjacent to $p_3(f_A) \in D''$. This proves that D'' is a dominating set of G''.

We show next that P'' is a dominating set of G''. Let y be an arbitrary vertex in $V_{G''} \setminus P'' = D''$. If $y = p_3(e_A)$ for some $e_A \in A_E$, then y is adjacent to $p_2(e_A) \in P''$. Finally assume that $y \in \xi(e)$ for some $e \in E_Q$. Without loss of generality, we may assume that $\varphi_H(e) = \{u, v\}$, $\xi(e) = \{v_e, u_e\}$, and $y = v_e$. Thus, y is adjacent to $p_1(f_A) \in P''$ where f_A is the first arc in the unique path $P_v \in \mathcal{P}$ starting at v. This implies that P'' is a dominating set of G''. In addition, P'' is a paired-dominating set of G'', as the edges $p_1(e_A)p_2(e_A)$, where $e_A \in A_E$, form a perfect matching in the subgraph induced by P''. This proves that (D'', P'') is a *DP*-pair in G'', and implies that (D', P') is a *DP*-pair in a proper spanning subgraph G' of G, contradicting the minimality of G.

Assume now that H is a graph that has neither an isolated vertex nor a good subgraph. By Proposition 4.4, the 2-subdivision graph $G = S_2(H)$ of H is a *DPDP*-graph. We claim that G is a minimal *DPDP*-graph. Suppose, to the contrary, that G is not a minimal *DPDP*-graph. Thus some proper spanning subgraph G' of G is a minimal *DPDP*-graph, and it follows from Theorem 4.5 that G' is a 2-subdivision graph of some graph H', i.e., $G' = S_2(H')$.

Since G' is a proper spanning subgraph of G, the set $E_G \setminus E_{G'}$ (of the edges removed from G) is nonempty and it is the union of disjoint subsets $E'_{nn} = (E_G \setminus E_{G'}) \cap E_{nn}$ and $E'_{no} = (E_G \setminus E_{G'}) \setminus E_{nn}$, where E_{nn} is the set of edges of G each of which joins two vertices in $\bigcup_{e\in E_H} \xi(e)$. It follows from the definition of the 2-subdivision graph that if $xy \in E_G \setminus E_{G'}$, then both x and y are leaves in G' if $xy \in E'_{nn}$ and at least one of the vertices x and y is a leaf in G' if $xy \in E'_{no}$, and $\{x, y\} \cap N_G[L_G] = \emptyset$ (since G' is a *DPDP*-graph). This implies that G' has two types of components: those which have at least one leaf belonging to the set $V^n_{S_2(H)}$, and those in which no leaf belongs to $V^n_{S_2(H)}$. From this and from Observation 4.1 (6) (and Corollary 4.7) it follows that if F is a component of G', then $F = S_2(P_{k+1})$ for some $k \in [4]$ and F has at most one strong support vertex if $L_F \cap V^n_{S_2(H)} \neq \emptyset$ or F is an induced subgraph of G if $L_F \cap V^n_{S_2(H)} = \emptyset$.

Let $F_1, \ldots, F_\ell$ be that components of G' for which $L_{F_i} \cap V^n_{S_2(H)} \neq \emptyset$ where $i \in [\ell]$. From this and from the fact that $F_i = S_2(P_{k_i+1})$ is of diameter $3k_i + 1$ it follows that exactly one support vertex of F_i is a vertex of H, say $\{v^i\} = S_{F_i} \cap V_H$ for $i \in [\ell]$. Let $\overline{v}^i$ be the (unique) leaf farthest from v^i in F_i, and let $\widetilde{v}^i$ be the only vertex in $N_G(\overline{v}^i) \setminus N_{G'}(\overline{v}^i) \subseteq V_H$. Let $\overline{P}_i$ be the $v^i - \overline{v}^i$ path in F_i, and let $\widetilde{P}_i$ be the $v^i - \widetilde{v}^i$ path obtained from $\overline{P}_i$ by adding $\widetilde{v}^i$ and the edge $\overline{v}^i\widetilde{v}^i$. Since $v^i, \widetilde{v}^i \in V_{G'}$ and $d_{G'}(v^i, \widetilde{v}^i) = 3k_i - 1$ for some $k_i \in [4]$, we may assume that $\overline{P}_i$ is the path $v^i = x^0, x^1, \ldots, x^{3k_i-1} = \overline{v}^i$ and $\widetilde{P}_i$ is the path $v^i = x^0, x^1, \ldots, x^{3k_i-1} = \overline{v}^i, x^{3k_i} = \widetilde{v}^i$, where $x^0, x^3, \ldots, x^{3k_i} \in V_H$, while $x^{3j+1} = x_e^{3j}$ and $x^{3j+2} = x_e^{3j+3}$, where e is an edge joining x^{3j} and x^{3j+3} in H for $j \in \{0\} \cup [k_i - 1]$ (or $x^{3j+1} = x_e^{3j\,1}$ and $x^{3j+2} = x_e^{3j\,2}$ if e is a loop at x^{3j} and $j = k_i - 1$). Now let P_i be the oriented path $\big(x^0, a\big(x^0, x^3\big), x^3, \ldots, x^{3k_i-3}, a\big(x^{3k_i-3}, x^{3k_i}\big), x^{3k_i}\big)$ in H, where $a(x^{3j}, x^{3j+3})$ is the arc which goes from x^{3j} to x^{3j+3} and which corresponds to the path $\big(x^{3j}, x^{3j+1}, x^{3j+2}, x^{3j+3}\big)$ in the path $\widetilde{P}_i$ for $j \in \{0\} \cup [k_i - 1]$.

Let $Q = (V_Q, E_Q)$ be the subgraph of H, where V_Q consists of those vertices of H which are support vertices in $F_1, \ldots, F_\ell$, that is, $V_Q = \{v^1, v^2, \ldots, v^\ell\}$, and E_Q consists of those edges (and loops) of H whose middle edges were removed in the process of forming G' from G, i.e., $E_Q = \{e \in E_H \colon \xi(e) = \{x, y\} \text{ and } xy \in E'_{nn}\}$ (see Figure 3, where Q (defined by G') is the bold subgraph of the underlying graph of H). All that remains to prove is that Q is a good subgraph in H.

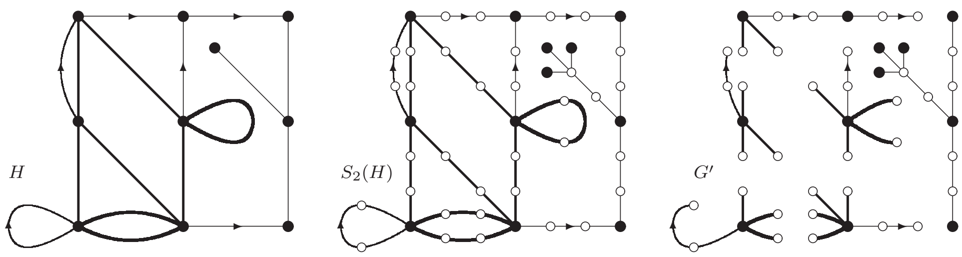

Figure 3. Graphs H, $S_2(H)$, and a minimal spanning *DPDP*-subgraph G' of $S_2(H)$.

Since the paths $\widetilde{P}_1, \ldots, \widetilde{P}_\ell$ are edge-disjoint in G', it follows from the definition of $P_1, \ldots, P_\ell$ that $\mathcal{P} = \{P_1, \ldots, P_\ell\}$ is a family of arc-disjoint (not necessarily vertex-disjoint) oriented paths (in H) indexed by the vertices of Q. In addition, P_i is the only path belonging to $\mathcal{P}$ and growing out from the vertex $v^i \in V_Q$, implying that $d_H^+(v^i) = 1$ and $d_H^-(v^i) = d_H(v^i) - d_Q(v^i) - 1$ for $i \in [\ell]$. From the same fact it follows that if the paths $P_i, P_j \in \mathcal{P}$, where $i \neq j$, are not vertex-disjoint, then the end vertex of (at least) one of them is the only vertex belonging to the second one. Consequently, if x is a non-end vertex of a path $P_i \in \mathcal{P}$, then $d^+_H(x) = 1$ (and $d^-_H(x) = d_H(x) - 1$). Finally assume that y is an end vertex of a path $P_i \in \mathcal{P}$. If $d^-_H(y) \geq d_H(y)$, then y would be an isolated vertex in a *DPDP*-graph G', which is impossible. Therefore, $d^-_H(y) < d_H(y)$. This proves that Q is a good subgraph in H and this completes the proof of Theorem 6.2. ∎

We are now in a position to present a proof of our main result, namely Theorem 3.1. Recall its statement.

Theorem 3.1. *If G is a connected graph of order at least three, then the following statements are equivalent.*

(1) *G is a minimal DPDP-graph.*

(2) *$G = S_2(H)$ for some connected graph H, and either $\left(V^o_{S_2(H)}, V^n_{S_2(H)}\right)$ is the unique DP-pair in G or G is a cycle of length $3, 6$ or 9.*

(3) *$G = S_2(H)$ for some connected graph H that has neither an isolated vertex nor a good subgraph.*

(4) *$G = S_2(H)$ for some connected graph H and no proper spanning subgraph of G without isolated vertices is a 2-subdivision graph.*

Proof. The statements (1), (2), and (3) are equivalent by Theorems 6.1 and 6.2. We shall prove that (1) and (4) are equivalent.

Assume that G is a minimal *DPDP*-graph. By Theorem 4.5, $G = S_2(H)$ for some connected graph H. In addition, since G is a minimal *DPDP*-graph, no proper spanning subgraph of G is a *DPDP*-graph. Thus no proper spanning subgraph of G having no isolated vertex is a 2-subdivision graph, as, by Proposition 4.4, every 2-subdivision graph of a graph with no isolated vertex is a *DPDP*-graph. This proves the implication (1) $\Rightarrow$ (4).

If $G = S_2(H)$ for some connected graph H, then G is a *DPDP*-graph (by Proposition 4.4). Assume that no proper spanning subgraph of G without isolated vertices is a 2-subdivision graph. We claim that G is a minimal *DPDP*-graph. Suppose, to the contrary, that G is not a minimal *DPDP*-graph. Then, since G is a *DPDP*-graph, some proper spanning subgraph G' of G is a minimal *DPDP*-graph. Consequently, G' has no isolated vertex (as no *DPDP*-graph has an isolated vertex). In addition, from the minimality of G' and from Theorem 4.5 it follows that G' is a 2-subdivision graph. But this contradicts the statement (4) and proves the implication (4) $\Rightarrow$ (1). ∎

The *corona* $F \circ K_1$ of a graph F is the graph obtained from F by adding a pendant edge to each vertex of F. A *corona graph* is a graph obtained from a graph F by attaching any number of pendant edges to each vertex of F. In particular, the corona $F \circ K_1$ of a graph F is a corona graph.

Corollary 6.3. *If H is a corona graph, then its 2-subdivision graph $S_2(H)$ is a minimal DPDP-graph. In particular, $S_2(F \circ K_1)$ is a minimal DPDP-graph for every graph F.*

Proof. Since every vertex of a corona graph is a leaf or a support vertex, it follows from Observation 5.1 that H has no good subgraph, and, therefore, $S_2(H)$ is a minimal *DPDP*-graph, by Theorem 6.2. ■

Corollary 6.4. *If H is a connected graph, then $S_2(S_2(H))$ is a minimal DPDP-graph if and only if H has either exactly one edge or exactly one loop.*

Proof. If $E_H = \emptyset$, then H consists of an isolated vertex, and $S_2(S_2(H)) = S_2(H) = H$ is not a *DPDP*-graph. If $|E_H| = 1$, then $H = P_2$ (or $H = C_1$, respectively), and $S_2(S_2(H)) = P_{10}$ (or $S_2(S_2(H)) = C_9$, respectively) is a minimal *DPDP*-graph. Assume now that $|E_H| \geq 2$. Thus, $V_H \setminus L_H \neq \emptyset$. If $v \in V_H \setminus L_H$, then $|E_H(v)| \geq 2$ and we consider two cases. Assume first that there is a loop e in $E_H(v)$. In this case the vertices v_e^1, v_e^2, and the edge $v_e^1 v_e^2$ form a good subgraph in $S_2(H)$. Consequently, by Theorem 6.2, $S_2(S_2(H))$ is not a minimal *DPDP*-graph. Assume now that $E_H(v) = \{e_1, \ldots, e_k\}$ where $k \geq 2$, and no loop belongs to $E_H(v)$. Then the vertices $v, v_{e_1}, \ldots, v_{e_{k-1}}$, and the edges $vv_{e_1}, vv_{e_2}, \ldots, vv_{e_{k-1}}$ form a good subgraph in $S_2(H)$. From this and from Theorem 6.2 it again follows that $S_2(S_2(H))$ is not a minimal *DPDP*-graph. ■

7. *DPDP*-Trees

In this section we study the *DPDP*-trees, minimal *DPDP*-trees, and good subgraphs in trees. We begin with the following characterization of *DPDP*-trees.

Proposition 7.1. *A tree T is a DPDP-tree if and only if T is a spanning supergraph of a 2-subdivision graph of a forest without isolated vertices and good subgraphs.*

Proof. If H is a forest without isolated vertices, then the forest $S_2(H)$ is a *DPDP*-graph (by Proposition 4.4) and every spanning supergraph of $S_2(H)$ is a *DPDP*-graph. In particular, any tree which is a spanning supergraph of $S_2(H)$ is a *DPDP*-tree.

Assume now that a tree T is a *DPDP*-graph. Let R be a spanning minimal *DPDP*-subgraph of T. Then R is a forest and it follows from Theorems 4.5 (4) and 6.2 that $R = S_2(F)$ for some forest F (without isolated vertices and good subgraphs) and therefore T is a spanning supergraph of $S_2(F)$. ■

We are interested in recognizing the structure of trees having a good subgraph. The following result shows that if a tree has a good forest, then it also has a good subtree.

Proposition 7.2. *A tree has a good subgraph if and only if it has a good subtree.*

Proof. Assume that a forest Q is a good subgraph in a tree H. Let $Q_1, \ldots, Q_k$ ($k \geq 2$) be the components of Q. It suffices to prove that one of the components $Q_1, \ldots, Q_k$ is a good subgraph in H. Let $\mathcal{P} = \{P_v : v \in V_Q\}$ be a family of oriented paths indexed by the vertices of Q and having the properties (1)–(3) stated in the definition of a good subgraph (for some subset E, where $E_Q^- \subseteq E \subseteq E_H \setminus E_Q$, and some orientation A_E of the edges in E). Let $\mathcal{P}_i$ denote the family $\{P_v : v \in V_{Q_i}\}$ where $i \in [k]$. From the properties of $\mathcal{P}$ and from the fact that H is a tree it follows that $\mathcal{P}_i$ is a family of vertex-disjoint paths, each vertex of Q_i is the initial vertex of exactly one path belonging to $\mathcal{P}_i$, and no path $P_v \in \mathcal{P}_i$ terminates at a vertex of Q_i or at a leaf of H. (Although, this time a path belonging to $\mathcal{P}_i$ can terminate at a vertex belonging to Q_j or to a path in $\mathcal{P}_j$, $j \neq i$.) However, from the same facts it follows that there exists a subtree $Q_{i_0} \in \{Q_1, \ldots, Q_k\}$ such that no path $P_v \in \bigcup_{j \neq i_0} \mathcal{P}_j$ terminates at Q_{i_0}. Now Q_{i_0} is a good subtree in H as the family $\mathcal{P}_{i_0}$ has the properties (1)–(3) stated in the definition of a good subgraph (for the partially ordered graph $H[A_{E_{i_0}}]$, where $A_{E_{i_0}}$ is the set of arcs belonging to A_E and covered by the paths of $\mathcal{P}_{i_0}$, see Q_2 or Q_5 in Figure 4). ∎

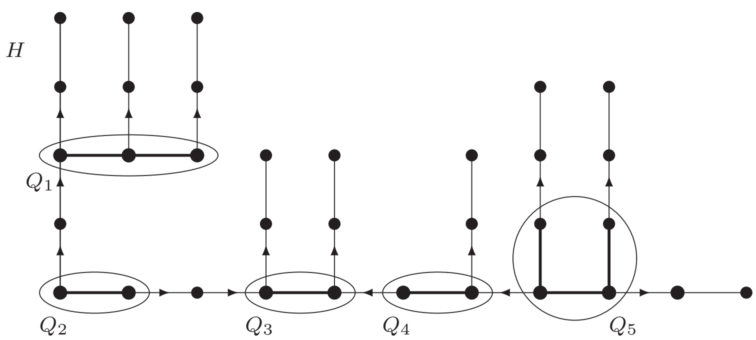

Figure 4. A good forest in a tree.

We observe that every tree can be a good subtree in a tree. The following result describes the place of a good subtree in a tree and connections between this good subtree and the rest of the tree.

Proposition 7.3. *A tree Q is a good subgraph in a tree H if and only if no leaf of H is a neighbor of Q and the subgraph of H induced by the set $N_H[V_Q]$ is a corona graph, that is, if and only if $N_H[V_Q] \cap L_H = \emptyset$ and $H[N_H[V_Q]] = Q \circ K_1$.*

Proof. Let Q be a good subgraph of H and let $\mathcal{P} = \{P_v : v \in V_Q\}$ be a family of oriented paths indexed by the vertices of Q and having the properties (1)–(3)

stated in the definition of a good subgraph (for some subset E, where $E_Q^- \subseteq E \subseteq E_H \setminus E_Q$, and some orientation A_E of the edges in E). From these properties and from the fact that H is a tree it follows that $\mathcal{P}$ is a family of vertex-disjoint paths, each vertex of Q is the initial vertex of exactly one path belonging to $\mathcal{P}$, and no path $P_v \in \mathcal{P}$ terminates at a vertex of Q or at a leaf of H. This proves that $N_H[V_Q] \cap L_H = \emptyset$. (The same follows directly from Observation 5.1.) In addition, every vertex v of Q is adjacent to exactly one vertex in $V_H \setminus V_Q$, say s_v, which is the terminal vertex of the first arc in P_v. Since H is a tree, the set $\{s_v : v \in V_Q\}$ is independent and, consequently, the subgraph of H induced by $V_Q \cup \{s_v : v \in V_Q\}$ $(= V_H[V_Q])$ is a corona graph isomorphic to $Q \circ K_1$.

Now assume that Q is a subtree of H such that $N_H[V_Q] \cap L_H = \emptyset$ and $H[N_H[V_Q]] = Q \circ K_1$. For a vertex v of Q, let v_ℓ denote the only vertex in $N_H(v) \setminus V_Q$. Since the edge set $E = \{vv_\ell : v \in V_Q\}$, the arc set $A_E = \{(v, v_\ell) : v \in V_Q\}$, and the family of oriented paths $\mathcal{P} = A_E$ have properties (1)–(3) of the definition of a good subgraph, we note that Q is a good subgraph in H. ■

Corollary 7.4. *If H is a tree of order at least two, then $S_2(H)$ is a DPDP-tree. In addition, the DPDP-tree $S_2(H)$ is not a minimal DPDP-tree if and only if there is a tree Q in $H - (L_H \cup S_H)$ such that $Q \circ K_1$ is a subtree in $H - L_H$ and $d_H(x) = d_Q(x) + 1$ for each vertex x of Q.*

8. Open Problems

We close this paper with the following list of open problems that we have yet to settle.

(a) How difficult is it to recognize graphs having good subgraphs?

(b) How difficult is it to recognize whether a given graph is a good subgraph in a graph?

(c) How difficult is it to recognize whether a given tree has good subtree?

(d) Provide an algorithm for the problem of determining a good subgraph of a graph.

(e) Since every graph without isolated vertices is homeomorphic to a *DPDP*-graph, it would be interesting to find the smallest number of subdivisions of edges of a graph in order to obtain a *DPDP*-graph.

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