Discussiones Mathematicae
Graph Theory 41 (2021) 809–826
doi:10.7151/dmgt.2326

PROPER RAINBOW CONNECTION NUMBER OF GRAPHS

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Abstract

A path in an edge-coloured graph is called a *rainbow path* if its edges receive pairwise distinct colours. An edge-coloured graph is said to be *rainbow connected* if any two distinct vertices of the graph are connected by a rainbow path. The minimum k for which there exists such an edge-colouring is the rainbow connection number $rc(G)$ of G. Recently, Bau *et al.* [*Rainbow connectivity in some Cayley graphs*, Australas. J. Combin. 71 (2018) 381–393] introduced this concept with the additional requirement that the edge-colouring must be proper. The *proper rainbow connection number* of G, denoted by $prc(G)$, is the minimum number of colours needed in order to make it properly rainbow connected. Obviously, $prc(G) \geq \max\{rc(G), \chi'(G)\}$.

In this paper we first prove an improved upper bound $prc(G) \leq n$ for every connected graph G of order $n \geq 3$. Next we show that the difference $prc(G) - \max\{rc(G), \chi'(G)\}$ can be arbitrarily large. Finally, we present several sufficient conditions for graph classes satisfying $prc(G) = \chi'(G)$.

Keywords: edge-colouring, rainbow connection number, proper rainbow connection number.

2010 Mathematics Subject Classification: 05C15, 05C40, 05C07.

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[1]Part of this research was done while the first author was a postdoctoral fellow at the Institute of Mathematics, Vietnam Academy of Science and Technology and he wishes to express gratitude for the very good conditions and the pleasant environments of work there. The first author is also thankful to Prof. Ngo Dac Tan, Prof. Phan Thi Ha Duong, and other people for their help at the time. This research is supported by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 08/2020/STS01.

1. Introduction

We use [20] for terminology and notation not defined here and consider only simple, finite and undirected graphs. Let G be a graph. We denote by $V(G), E(G)$, $n, m, \Delta(G)$, $diam(G)$ the vertex set, the edge set, number of vertices, number of edges, maximum degree, and diameter of G, respectively. Let K_n, C_n, P_n be a complete graph, a cycle and a path on n vertices, respectively. By $N_G(u)$ we denote the set of neighbours of a vertex $u \in V(G)$ and by $d(u)$ its degree in G. Let us denote by $d(u, v)$ and $d(uPv)$ the distance between two vertices u, v and the length of a u, v-path P, respectively. For each integer $n \geq 4$, the wheel is defined as $W_n = C_n + K_1$, the join of C_n and K_1. For simplifying notation, let $[k]$ be the set $\{1, 2, \ldots, k\}$ for some positive integer k.

Let $c : E(G) \to [k]$ be an edge-colouring of G. If adjacent edges of G receive different colours by c, then c is a *proper colouring*. The smallest number of colours needed in a proper colouring of G, denoted by $\chi'(G)$, is called the *chromatic index* of G. Vizing *et al.* [19] proved that for any graph G, $\chi'(G)$ is either its maximum degree $\Delta(G)$ or $\Delta(G) + 1$. If $\chi'(G) = \Delta(G)$, then G is in *class* 1. Otherwise, G is in *class* 2.

A path P in an edge-coloured graph G is called a *rainbow path* if its edges have different colours. An edge-coloured graph G is *rainbow connected* if every two vertices are connected by at least one rainbow path in G. For a connected graph G, the *rainbow connection number* of G, denoted by $rc(G)$, is defined as the smallest number of colours required to make it rainbow connected. The concept of rainbow connection was first introduced by Chartrand *et al.* [4] and well-studied since then. Readers who are interested in this topic are referred to [15, 17].

As an extension of proper colouring and motivated by rainbow connections of graphs, Bau *et al.* [1] introduced the concept of proper rainbow connections in connected graphs. Let G be a nontrivial connected graph. The proper edge-coloured graph G is said to be *properly rainbow connected* if any two vertices $u, v \in V(G)$ are connected by a rainbow path. The *proper rainbow connection number* $prc(G)$ of a connected graph G is the smallest number of colours needed to colour G properly rainbow connected.

By the definition above, if an edge-coloured graph G is properly rainbow connected, then G is properly coloured and rainbow connected. Hence, the lower bound of the proper rainbow connection number was obtained by the following proposition.

Proposition 1 (Bau *et al.* [1]). *Let G be a connected graph. Then*

$$diam(G) \leq rc(G) \leq prc(G)$$

and

$$\chi'(G) \leq prc(G).$$

On the other hand, if every edge of G receives a distinct colour from $[m]$, where m is the number edges of G, then G is properly rainbow connected. By using Proposition 1 and Vizing's Theorem in [19], the proper rainbow connection number of an arbitrary connected graph is bounded as follows.

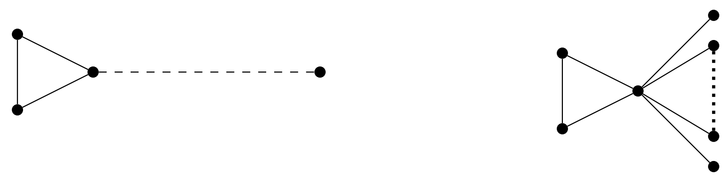
Corollary 2. *Let G be a connected graph of size m. Then*

$$\max\{rc(G), \chi'(G)\} \leq prc(G) \leq m.$$

The authors in [7] determined some graphs with large proper rainbow connection number. First of all, they characterized all graphs whose proper connection numbers equal their size.

Theorem 3 (Jiang *et al.* [7])**.** *Let G be a connected graph of size m. Then $prc(G) = m$ if and only if G is a tree or K_3.*

After that, they also classified connected graphs whose proper rainbow connection numbers are close to the maximum possible value. Let $\mathcal{H}'$ and $\mathcal{H}''$ be two graph classes as shown in Figure 1, where the order of $H' \in \mathcal{H}'$ is at least 4 and the order of $H'' \in \mathcal{H}''$ is at least 5, respectively.



H' $\qquad$ H'

Figure 1. The graphs $H' \in \mathcal{H}'$ and $H'' \in \mathcal{H}''$.

Theorem 4 (Jiang *et al.* [7])**.** *If G is a connected graph of size m, then $prc(G) = m - 1$ if and only if $G \in \mathcal{H}'$ or $G \in \mathcal{H}''$.*

Next, the proper rainbow connection numbers of special graphs were considered by Bau *et al.* [1] and Jiang *et al.* [7].

Theorem 5 (Bau *et al.* [1])**.** *For each integer $n \geq 2$,*

$$prc(K_n) = \chi'(K_n) = \begin{cases} n-1, & \text{if } n \text{ is even,} \\ n, & \text{if } n \text{ is odd.} \end{cases}$$

Corollary 6 (Jiang *et al.* [7]). *For each interger $n \geq 4$, $prc(C_n) = \left\lceil \frac{n}{2} \right\rceil$.*

Chartrand *et al.* [4] determined the rainbow connection numbers of complete graphs.

Proposition 7 (Chartrand *et al.* [4]). *Let $n \geq 2$ be an integer. Then $rc(K_n) = 1$.*

By Theorem 5 and Proposition 7, it can be readily seen that the difference $prc(G) - rc(G)$ can be arbitrarily large.

The *Cartesian product* of two graphs G and H written $G \square H$ is the graph with vertex set $V(G) \times V(H)$ specified by putting (u_1, u_2) adjacent to (v_1, v_2) if and only if (1) $u_1 = u_2$ and $v_1v_2 \in E(H)$, or (2) $v_1 = v_2$ and $u_1u_2 \in E(G)$. Bau *et al.* [1] and Jiang *et al.* [7] determined proper connection numbers of Cartesian products by the following results.

Proposition 8 (Bau *et al.* [1]). *Let $n, p_1, \ldots, p_n > 1$ be integers and $G = K_{p_1} \square \cdots \square K_{p_n}$. Then*

$$\sum_{i=1}^{n}(p_i - 1) \leq prc(G) \leq \sum_{i=1}^{n} \chi'\left(K_{p_i}\right).$$

Theorem 9 (Jiang *et al.* [7]). *Suppose that $n \geq 1$, and $p_1, \ldots, p_n > 1$ are integers. If $G = K_{p_1} \square \cdots \square K_{p_n}$, then $prc(G) = \chi'(G)$.*

2. Upper Bounds

In this section we will show improved upper bounds for the proper rainbow connection number of graphs.

The concept of rainbow connection was first introduced by Chartrand *et al.* [4]. Moreover, they gave the relation between rainbow connection number of a connected graph and rainbow connection number of its spanning tree as follows.

Proposition 10 (Chartrand *et al.* [4]). *Let G be a connected graph and T be a spanning tree of G. Then $rc(G) \leq rc(T)$.*

By Proposition 10, it can be readily seen that if G has n vertices, then $rc(G) \leq n - 1$. Moreover, by Theorem 5 and Corollary 6, proper connection numbers of complete graphs and cycles do not exceed their number of vertices. These facts are our motivation to improve the upper bound for the proper rainbow connection number as follows.

Theorem 11. *Let G be a nontrivial, connected graph of order n and maximum degree $\Delta(G)$. Then*

$$\max\{\Delta(G), diam(G)\} \leq prc(G) \leq \chi'(G) + (n - 1 - \triangle(G)) = \begin{cases} n, & \text{if } G \text{ is in class } 2, \\ n-1, & \text{if } G \text{ is in class } 1. \end{cases}$$

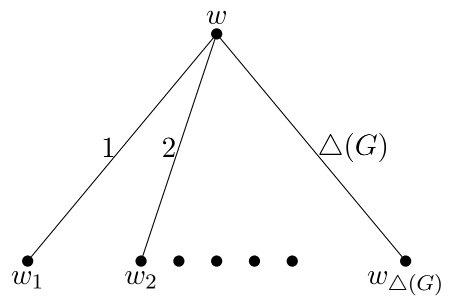


Figure 2. Tree $K_{1,\triangle(G)}$.

Proof. Since $\Delta(G) \leq \chi'(G)$ by Vizing's Theorem in [19] and $diam(G) \leq rc(G)$ by Chartrand *et al.* [4], the lower bound is easily obtained.

Next, we consider the upper bound. Since G has maximum degree $\triangle(G)$, there exists a vertex $w \in V(G)$ such that $d_G(w) = \triangle(G)$. Let $N_G(w) = \{w_1, w_2, \ldots, w_{\triangle(G)}\}$ be the neighbour set of $w \in V(G)$. We construct a tree $T \cong K_{1,\triangle(G)}$, which consists of the vertex w as a root of the tree $K_{1,\triangle(G)}$ and all the vertices in the set $N_G(w)$. Let c be a proper edge-colouring of G with $\chi'(G)$ colours $\{1, 2, \ldots, \chi'(G)\}$. We may assume that the edges of T have colours $1, 2, \ldots, \Delta(G)$. Since G is connected, we can extend the tree T to a spanning tree T' of G by properly adding $n - 1 - \Delta(G)$ edges. Now we recolour these $n - 1 - \Delta(G)$ edges by using $n - 1 - \Delta(G)$ new colours. This leads to an proper edge-colouring c' of G, since every new colour is used exactly once. Moreover, the tree T' is rainbow coloured, which shows that G is properly rainbow-connected.

By using Vizing's Theorem in [19], $prc(G) \leq n$, if G is in class 2 or $prc(G) \leq n - 1$, if G is in class 1. We obtain the result. ∎

Now, we improve the upper bound by requiring a structural condition.

Proposition 12. *Let G be a connected graph with maximum degree $\Delta(G) \leq n-2$ and $w \in V(G)$ such that $d(w) = \Delta(G)$. If there is a Hamiltonian cycle in $G - N[w]$, then $prc(G) \leq \frac{n+\Delta(G)}{2} + 1$.*

Proof. Suppose that G has a vertex w with $d(w) = \Delta(G)$ and there is a Hamiltonian cycle, say C, in $G - N[w]$. Let us colour all the edges of $E(G) \setminus E(C)$ by $c : E(G) \setminus E(C) \to [\chi'(G)]$ in order to make it a proper colouring. Next, we continue to colour all remaining edges of C with $\left\lceil \frac{|V(C)|}{2} \right\rceil$ new colours. It can be readily seen that G is properly rainbow connected using $\chi'(G) + \left\lceil \frac{|V(C)|}{2} \right\rceil$ colours. Since $\Delta(G) \leq n - 2$, it can be readily seen that

$$prc(G) \leq \Delta(G) + 1 + \left\lceil \frac{n - 1 - \Delta(G)}{2} \right\rceil \leq \frac{n + \Delta(G)}{2} + 1.$$

The result is obtained. ∎

3. Estimating the Difference $prc(G) - rc(G)$

Next observe that

$$0 \leq prc(G) - rc(G) \leq n - 1$$

for all connected graphs G, where the upper bound is attained for K_n if n is odd. We now extend this observation as follows.

Proposition 13. *Let G be a connected graph of order $n \geq 3$ with clique number $\omega(G)$. If $\frac{n+1}{2} \leq \omega(G) \leq n - 1$, then*

$$prc(G) - rc(G) \geq 2\omega(G) - n - 1.$$

Proof. First observe that $rc(G) \leq n + 1 - \omega(G)$. To see this, take a clique of size $\omega(G)$ and colour all edges between its vertices by one colour. Next we add $n - \omega(G)$ edges to obtain a spanning subgraph of G. We colour each edge by a new colour and can colour all remaining edges arbitrarily. Then this colouring makes G rainbow connected. Since G is connected and $\omega(G) \leq n - 1$, we deduce that $\Delta(G) \geq \omega(G)$. Hence by Theorem 11 we obtain $prc(G) \geq \omega(G)$. Now the inequality follows. ■

Note that for $G \cong K_n$ and n even it holds $prc(K_n) - rc(K_n) = 2\omega(K_n) - n - 2$.

Next we analyse the values of $prc(G)$ and $rc(G)$ for graphs with respect to their minimum degree.

Theorem 14. *Let G be a connected graph of order $n \geq 3$ and minimum degree $\delta = \delta(G)$. If*

1. $\delta \geq 3$, *then* $rc(G) \leq \frac{3n}{4}$ ([18]),
2. $\delta \geq 4$, *then* $rc(G) \leq \frac{3n}{\delta+1} + 3$ ([3]).

Now observe that $\delta \geq \frac{3n}{\delta+1} + 3$ if $\delta \geq 1 + \sqrt{3n+4}$. With $prc(G) \geq \chi'(G) \geq \Delta(G) \geq \delta(G)$ we thus obtain

Theorem 15. *Let G be a connected graph of order n and with $\delta(G) \geq 1 + \sqrt{3n+4}$. Then*

$$prc(G) - rc(G) \geq \delta - \left(\frac{3n}{\delta + 1} + 3\right).$$

Further observe that

$$0 \leq prc(G) - \chi'(G) \leq (n - 1) - 2 = n - 3$$

for all connected graphs with $n \geq 3$, where the upper bound is attained for the path P_n.

Since $prc(G) \geq \max\{rc(G), \chi'(G)\}$ by Corollary 2, it is natural to ask whether the difference $prc(G) - \max\{rc(G), \chi'(G)\}$ is unbounded as well. In our next theorem we show that the difference $prc(G) - \max\{rc(G), \chi'(G)\}$ can be arbitrarily large.

Theorem 16. *Let k, t be two integers, where $k \geq t \geq 1$. There always exists a connected graph $G_{k,t}$ with $\Delta(G) = 2t^2 + 1$ and $diam(G) = 2t^2 + 1 + k$ such that $prc(G) \geq \max\{rc(G), \chi'(G)\} + t$.*

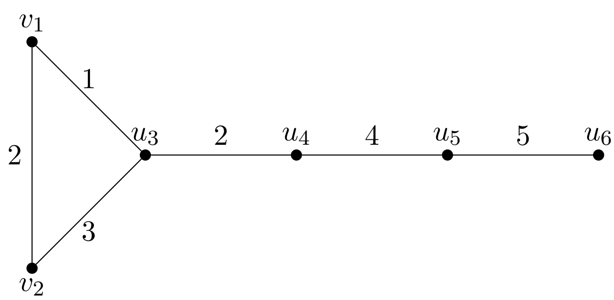


Figure 3. Graph $G_{1,1}$ with $rc(G) = 4$ and $prc(G) = rc(G) + 1 = 5$.

Proof. Firstly, if $k = t = 1$, then we take a connected graph with $\Delta(G) = 3$ and $diam(G) = 4$ as shown in Figure 3. Clearly, $rc(G) = 4, \chi'(G) = 3$ and $prc(G) = 5 = rc(G) + t$.

Now, we consider $t \geq 2$. Let W_{2t^2} be a wheel consisting of a cycle $C = v_1 \cdots v_{2t^2} v_1$ and a center vertex v. Let G be a connected graph constructed from W_{2t^2} and a path $P = u_{2t^2+1} \cdots u_{4t^2+1+k}$ of order $2t^2 + 1 + k$ by identifying v and u_{2t^2+1} as shown in Figure 4. It can be readily seen that $\Delta(G) = 2t^2 + 1$ and $diam(G) = 2t^2 + 1 + k$. Hence, $\chi'(G) \leq \Delta(G) + 1 = 2t^2 + 2$ and thus $\max\{rc(G), \chi'(G)\} = rc(G) \geq 2t^2 + 1 + k$. Let us define a colouring c with $2t^2 + 1 + k$ colours to colour all the edges of G as follows.

$$c(e) = \begin{cases} 1 & \text{if } e = vv_i, i \in \left[2t^2\right], \\ i + 1 - 2t^2 & \text{if } e = u_i u_{i+1}, i \in \left[2t^2 + 1, 4t^2 + k\right], \\ i & \text{if } e = v_i v_{i+1}, i \in \left[t^2\right], \\ i - t^2 & \text{if } e = v_i v_{i+1}, i \in \left[t^2 + 1, 2t^2 - 1\right], \\ t^2 & \text{if } e = v_{2t^2} v_1. \end{cases}$$

It can be readily seen that G is rainbow connected with $2t^2 + 1 + k$ colours. Thus, $rc(G) \leq 2t^2 + 1 + k$. So we deduce that $rc(G) = 2t^2 + 1 + k$.

Next, we show that $prc(G) \geq 2t^2 + 1 + k + t$.

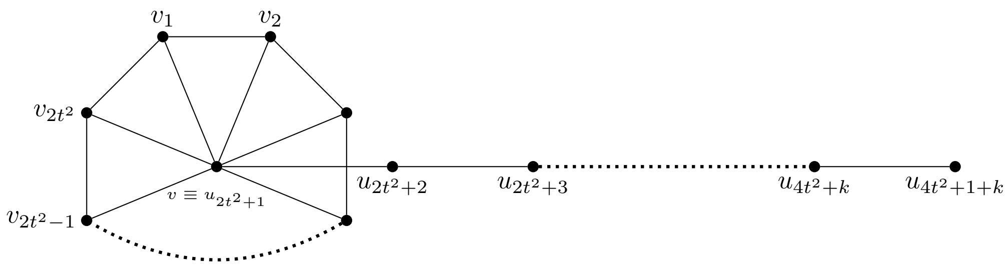


Figure 4. The graph $G_{k,t}$ for $t \geq 2$.

Suppose that $prc(G) \leq 2t^2+k+t$. Then there is a colouring c with $2t^2+k+t$ colours which makes G properly rainbow connected. Since P is the only path from u_{2t^2+1} to u_{4t^2+1+k}, P is a rainbow path. Hence we may assume that its $2t^2+k$ edges are coloured with the colours $1, 2, \ldots, 2t^2+k$. Since G is properly rainbow connected, all the edges that are incident to v receive distinct colours. Moreover, every rainbow path from u_{4t^2+1+k} to a vertex $v_i, 1 \leq i \leq 2t^2$, uses exactly one edge vv_j, whose colour is distinct from $1, 2, \ldots, 2t^2+k$. Hence we may assume that p of these edges, where $1 \leq p \leq t$, have a colour from the set $2t^2+k+1, \ldots, 2t^2+k+t$. Suppose first that $p = 1$. We may assume that vv_1 has the only colour from the set $2t^2+k+1, \ldots, 2t^2+k+t$. Then any rainbow path from v_{t^2+1} to u_{4t^2+1+k} has $t^2+1+(2t^2+k) > 2t^2+k+t$ edges, a contradiction. Next suppose that $p \geq 2$. So there are p integers $1 \leq i_1 < i_2 < \cdots < i_p \leq 2t^2$ such that the edges $vv_{i_1}, \ldots, vv_{i_p}$ have these p colours. So there is a partition of the cycle C into p paths, each connecting v_{i_j} with $v_{i_{j+1}}$ along the cycle of length $|i_{j+1} - i_j|$ (modulo $2t^2$). Hence the longest of these p paths has length at least $\frac{2t^2}{p} \geq \frac{2t^2}{t} = 2t$. We may assume that the path from v_{i_1} to v_{i_2} has length at least $2t$ and that $i_1 = 1$. Then any rainbow path from v_{t+1} to u_{4t^2+1+k} has at least $t+1+2t^2+k$ edges, a contradiction. ■

4. Graph Classes with $prc(G) = rc(G)$

Proposition 17. *Let C_n by a cycle of order $n \geq 4$. Then*

$$prc(C_n) = rc(C_n) = \left\lceil \frac{n}{2} \right\rceil .$$

Next observe that any colouring, which makes a tree rainbow connected, is a proper colouring. So we deduce that

Proposition 18. *Let T by a tree of order $n \geq 2$. Then*

$$prc(T) = rc(T) = n - 1.$$

Starting with a given tree T we can generate a large variety of classes of graphs satisfying $prc(G) = rc(G)$. For example we can attach to each leaf of T a finite number of cycles each of length at least 4.

Another class of graphs is described in [7]. Here $g(G)$ denotes the girth of G.

Proposition 19. *Let G be a connected graph with $rc(G) < g(G) - 2$. Then*

$$prc(G) = rc(G).$$

5. GRAPH CLASSES WITH $prc(G) = \chi'(G)$

The proper rainbow connection numbers of complete graphs K_n are determined in Theorem 5. Now we consider the proper rainbow connection number of connected graphs whose diameter is 2.

Proposition 20. *Let G be a connected graph of order $n \geq 3$. If $diam(G) = 2$, then $prc(G) = \chi'(G)$.*

Proof. Let $c : E(G) \to [\chi'(G)]$ be a proper edge-colouring of G. Now we show that for every pair ofvertices $u, v \in V(G)$, there is at least one rainbow path. If $uv \in E(G)$, then uv is the rainbow path between the two vertices u, v. On the other hand, if $uv \notin E(G)$, there is at least one vertex, say w, such that $w \in N_G(u)$ and $w \in N_G(v)$, since $diam(G) = 2$. Clearly, $c(uw) \neq c(wv)$, since G is proper. Hence, uwv is the rainbow path connecting two vertices u, v. We conclude that G is properly rainbow connected. Thus, $prc(G) \leq \chi'(G)$.

With the aid of Proposition 1, we are now able to obtain that $prc(G) = \chi'(G)$. ■

By using Proposition 20, we determine proper rainbow connection numbers of some graphs whose diameter equals 2.

First, we determine the proper rainbow connection number for wheels.

Proposition 21. *For each integer $n \geq 4$, $prc(W_n) = n$.*

Proof. Suppose that a wheel W_n of order $n + 1$ consists of a cycle $C_n = v_1 v_2 \cdots v_n v_1$ and a single vertex w joined to all vertices of the cycle C_n. Now observe that $\chi'(W_n) = n$ for $n \geq 4$.

Clearly, $diam(W_n) = 2$. By using Proposition 20, $prc(W_n) = \chi'(n) = n$. We obtain the result. ■

Next we determine the proper rainbow connection number for the complete bipartite graph $K_{s,t}$.

Theorem 22. *Let s, t be two integers. If $K_{s,t}$ is a complete bipartite graph, then $prc(K_{s,t}) = \max\{s, t\}$.*

Let us mention the following result which is very useful to prove our Theorem 22.

Theorem 23 (König *et al.* [10])**.** *If G is bipartite, then G is in class* 1*.*

Now we are able to prove Thereom 22.

Proof. With the aid of Theorem 23, it can be readily seen that $\chi'(K_{s,t}) = \Delta(K_{s,t}) = \max\{s, t\}$. On the other hand, $diam(K_{s,t}) = 2$. By using Proposition 20, $prc(K_{s,t}) = \chi'(K_{s,t})$.

We conclude that $prc(K_{s,t}) = \max\{s, t\}$. ■

We know that the chromatic index $\chi'(G)$ depends on the property of G being overfull or not overfull. G is called *overfull* if the number of vertices n is odd, and the number of edges m is greater than $\frac{1}{2} \triangle(G)(n-1)$. Let us mention the result of Hoffman *et al.* [6] on the chromatic index of complete multipartite graph.

Lemma 24 (Hoffman *et al.* [6])**.** *Let G be a complete multipartite graph. Then $\chi'(G) = \triangle(G)$ if G is not overfull. Otherwise, $\chi'(G) = \triangle(G) + 1$.*

Now the, proper connection number of a complete multipartite graph is determined as follows.

Proposition 25. *Let G be a complete multipartite graph. If G is overfull, then $prc(G) = \triangle(G) + 1$. Otherwise, $prc(G) = \triangle(G)$.*

Proof. Suppose that G is a complete multipartite graph. It can be readily seen that $diam(G) = 2$. By using Proposition 20, $prc(G) = \chi'(G)$.

Now, applying Lemma 24, $\chi'(G) = \triangle(G)+1$ if G is overfull or $\chi'(G) = \triangle(G)$ if G is not overfull. Hence, the result is obtained. ■

In [4], Chartrand *et al.* showed that $rc(G) = 1$ if and only if G is complete. After that, Caro *et al.* [2] investigated graphs with small rainbow connection numbers and they gave a sufficient condition that guarantees $rc(G) = 2$.

Theorem 26 (Caro *et al.* [2])**.** *Let G be a nontrivial, connected graph of minimum degree $\delta(G)$. If $\delta(G) \geq \frac{n}{2} + \log_2 n$, then $rc(G) = 2$*

Next, we show that dense graphs have large proper rainbow connection number.

Proposition 27. *Let G be a proper edge-coloured graph of order n and minimum degree $\delta(G)$. If $\delta(G) \geq \frac{n-1}{2}$, then $prc(G) = \chi'(G)$.*

Proof. We show that $diam(G) \leq 2$. Let $u, v \in V(G)$ be two non adjacent vertices. Then $d(u) + d(v) = |N(u) \cup N(V| + |N(u) \cap N(v)| \geq 2 \cdot \frac{n-1}{2} = n - 1$. Since $|N(u) \cup N(v)| \leq n - 2$, we conclude $|N(u) \cap N(v)| \geq 1$. Hence there is a proper coloured path uwv for a vertex $w \in N(u) \cap N(v)$. This shows that $diam(G) \leq 2$. Now $prc(G) = \chi'(G)$ follows by Proposition 20. ■

This proof immediately leads to the following extension.

Proposition 28. *Let G be a proper edge-coloured graph of order $n \geq 3$. If $d(u) + d(v) \geq n - 1$ for every pair of non adjacent vertices $u, v \in V(G)$, then $prc(G) = \chi'(G)$.*

Proposition 29. *Let G be a proper edge-coloured graph of order $n \geq 9$ and minimum degree $\delta(G)$. If $\delta(G) \geq \frac{n-2}{2}$, then $prc(G) = \chi'(G)$.*

Proof. Let $u, w \in V(G)$ be any two vertices. If $d(u, w) \leq 2$, then u and w are connected by a rainbow path of length at most two. Hence we may assume that $d(u, w) \geq 3$. Then $N[u] \cup N[w] = V(G)$ implying $\delta(G) = \frac{n-2}{2}$. Since G is connected we conclude that $d(u, w) = 3$. We may assume $4 \leq d(u) \leq d(w)$, since $n \geq 9$. Let $U = N(u) = \{u_1, u_2, \ldots, u_{d(u)}\}$ and $W = N(w) = \{w_1, w_2, \ldots, w_{d(w)}\}$. Suppose $u_iw_j, u_iw_k \in E(G)$. Then at least one of the two paths uu_iw_jw and uu_iw_kw is a rainbow uw-path. By symmetry we conclude that $E(U, W)$ is an induced matching. Suppose $u_1w_1 \in E(G)$, but uu_1w_1w is no rainbow uw-path. We may assume that $c(uu_1) = c(w_1w) = 1, c(u_1w_1) = 2$. Since $N(u_1) \cap W = \{w_1\}$, we conclude that $|N(u_1) \cap (U \setminus \{u_1\})| \geq \delta(G) - 2 \geq \frac{n-6}{2} \geq 2$. We may assume that $u_1u_2, u_1u_3 \in E(G)$. Now $uu_2u_1w_1w$ or $uu_3u_1w_1w$ is a rainbow uw-path. Hence G is rainbow connected. Now $prc(G) = \chi'(G)$ follows by Proposition 20. ■

Sharpness. For $n = 8$ consider the following graph F_8 with vertices $V(F_8) = \{u, u_1, u_2, u_3, w, w_1, w_2, w_3\}$ and edges $E(F_8) = \{uu_1, uu_2, uu_3, u_1u_2, u_2u_3, u_1w_1, u_3w_3, w_1w_2, w_2w_3, ww_1, ww_2, ww_3\}$. Then $\chi'(F_8) = 3$ and we may assume that $c(uu_i) = i$ for $1 \leq i \leq 3$. Then the colours of five further edges are uniquely determined as follows: $c(u_1u_2) = 3, c(u_2u_3) = 1, c(u_1w_1) = c(u_3w_3) = c(ww_2) = 2$. For the four remaining edges we obtain (1) $c(w_2w_3) = c(ww_1) = 3$ and $c(w_1w_2) = c(ww_3) = 1$ or (2) $c(w_2w_3) = c(ww_1) = 1$ and $c(w_1w_2) = c(ww_3) = 3$.

This is a proper edge-colouring of F_8, but F_8 is not rainbow connected. If we recolour in (1) the edges u_2u_3 and w_2w_3 by colour 4, then F_8 becomes proper rainbow connected. Hence $prc(F_8) = 4$. If we switch in (1) the colours of u_1u_2, u_2u_3 and w_1w_2, w_2w_3, then F_8 becomes rainbow connected showing that $rc(F_8) = 3 = diam(F_8)$.

This proof immediately leads to the following extension.

Proposition 30. *Let G be a proper edge-coloured graph of order $n \geq 9$. If $d(u) + d(v) \geq n - 2$ for every pair of non adjacent vertices $u, v \in V(G)$, then $prc(G) = \chi'(G)$.*

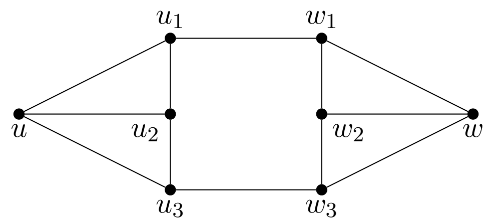


Figure 5. Graph F_8 with $\chi'(F_8) = 3$ but $prc(G) = 4$.

Proposition 31. *Let G be a proper edge-coloured graph of order $n \geq 9$ and minimum degree $\delta(G)$. If $\delta(G) \geq \frac{n+k}{3}$ for an integer $k \geq 3$, then $prc(G) = \chi'(G)$.*

Proof. Let $u, w \in V(G)$ be any two vertices. If $d(u, w) \leq 2$, then u and w are connected by a rainbow path of length at most two. Hence we may assume that $d(u, w) \geq 3$.

Case 1. $d(u, w) = 3$. Let $R = V(G) \setminus (N[u] \cup N[w])$. Then $|R| \leq n - 2(\delta + 1)$. Following arguments in the previous proof we conclude that $E(U, W)$ is an induced matching. Then $\delta \leq d(u_1) \leq |R| + 4 \leq n - 2\delta + 2$ implying $\delta \leq \frac{n}{3}$, a contradiction.

Case 2. $d(u, w) = 4$. Let uu_1xw_1w be a uw-path of length 4, and let $R = V(G) \setminus (N[u] \cup N[w])$. Thus $x \in R$ and $|R| \leq n - 2(\delta + 1)$. If $|N(x) \cap (U \cup W)| \geq 5$, then there is always a rainbow uw-path uu_ixw_jw for two vertices u_i and w_j. Hence we may assume that $|N(x) \cap (U \cup W)| \leq 4$ implying $\delta - 4 \leq d(x) - 4 \leq |R| - 1 \leq n - 2\delta - 3$. This gives $\delta \leq \frac{n+1}{3}$, a contradiction.

Case 3. $d(u, w) = 5$. Let $uu_1x_1x_2w_1w$ be a uw-path of length 5, and let $R = V(G) \setminus (N[u] \cup N[w])$. Thus $x_1, x_2 \in R$ and $|R| \leq n - 2(\delta + 1)$. Note that $N(x_1) \cap W = N(x_2) \cap U = \emptyset$. If $|N(x_1) \cap U| \leq 5$, then $\delta - 5 \leq d(x) - 5 \leq |R| - 1 \leq n - 2\delta - 3$. This gives $\delta \leq \frac{n+2}{3}$, a contradiction. Hence we may assume that $|N(x_1) \cap U \geq 6$. Now there is always a rainbow uw-path $uu_ix_1x_2w_1w$ for a vertex u_i.

Case 4. $d(u, w) = t \geq 6$. Let $uu_1x_1x_2 \cdots x_{t-3}w_1w$ be a uw-path of length t. Then $N[x_2] \cap N[u] = N[x_2] \cap N[w] = \emptyset$ implying $3(\delta + 1) \leq n$, a contradiction. ■

6. Lower Bound

In this section, we consider the lower bound of proper rainbow connection in dense graphs and some conditions on size of graphs.

6.1. Dense graphs

Dense graphs tend to have a small rainbow connection number. However, dense graphs have a large proper rainbow connection number. This follows immediately from its average degree.

Proposition 32. *Let G be a nontrivial, connected graph of order $n \geq 2$ and average degree $\overline{d}(G) = \frac{2|E(G)|}{n}$. Then $prc(G) \geq \lceil \overline{d}(G) \rceil$.*

Proof. It can be readily seen that $prc(G) \geq \triangle(G)$ since $prc(G) \geq \chi'(G)$ by Proposition 1 and $\chi'(G) \geq \triangle(G)$ by Vizing's Theorem. On the other hand, $\triangle(G) \geq \lceil \overline{d}(G) \rceil$. Hence, $prc(G) \geq \lceil \overline{d}(G) \rceil$.

We obtain the result. ∎

Proposition 33. *Let G be a connected graph of order n and size $|E(G)| \geq k\frac{n}{2}$. Then $prc(G) \geq k$.*

Proof. By the handshaking lemma we obtain $\frac{2|E(G)|}{n} = \overline{d}(G) \geq k$. Now the result follows by Proposition 32. ∎

6.2. Size of graphs

The problem of rainbow connection depending on size of graphs are studied by Kemnitz and Schiermeyer in [9] as follows. For every integer k with $1 \leq k \leq n-1$, compute and minimize the function $f(n,k)$ with the following property. If $|E(G)| \geq f(n,k)$, then $rc(G) \leq k$, where

$$f(n,k) \geq \binom{n-k+1}{2} + (k-1).$$

It has been shown in [9, 12, 8] that equality holds for $k = 1, 2, 3, 4, n-6, n-5, n-4, n-3, n-2, n-1$. Now, we obtain the following result.

Theorem 34. *Let G be a connected proper edge-coloured graph of order $n \geq 3$. If*

$$|E(G)| \geq \binom{n-2}{2} + 2,$$

then $prc(G) = \chi'(G)$ or $G \in \{P_4, Z_2, G_{6.3}\}$.

Proof. We perform an induction on the order n of G. For $n = 3$ we obtain $|E(G)| \geq 2$ and verify that $prc(G) = \chi'(G)$. Now let G be a connected graph of order $n \geq 4$ and size $|E(G)| \geq \binom{n-2}{2} + 2$. Then $\Delta(G) \geq \lceil \overline{d}(G) \rceil \geq \lceil n - 5 + \frac{10}{n} \rceil \geq n-4$.

Let $w \in V(G)$ be a vertex with $d(w) = \Delta(G)$, and let $N(w) = W = \{w_1, w_2, \ldots, w_{\Delta(G)}\}$ be its neighours. We distinguish four cases.

1. If $\Delta(G) = n-1$, then $G[\{ww_i|1 \le i \le n-1\}]$ induces a spanning subgraph H of G, which is rainbow-connected. Hence G is rainbow-connected.

2. If $\Delta(G) = n-2$, then let $V(G) \setminus N[w] = \{u\}$. First observe that $N(u) \subset N(w)$. So we may assume that $uw_i \in E(G)$ for $1 \le i \le d(u)$. If $d(u) \ge 2$, then $G[\{ww_i|1 \le i \le n-2\} \cup \{uw_1, uw_2\}]$ induces a spanning subgraph H of G, which is rainbow-connected. Hence G is rainbow-connected. If $d(u) = 1$, let $c(w_1u) = 1, c(ww_1) = 2$. If $c(ww_i) \ne 1$ for $2 \le i \le n-2$, then G is rainbow-connected. Hence we may assume that $c(ww_2) = 1$. Then we are sure that all pairs of vertices $x, y \in V(G)$ are rainbow-connected except for the pair (u, w_2), if $d(u, w_2) \ge 3$. Hence we may assume that $w_1w_2 \notin E(G)$. Suppose there is a vertex $w_i \in N(w_1) \cap N(w_2)$ for some $3 \le i \le n-2$. We may assume that $c(w_1w_i) = 3$. Then $c(w_2w_i) \ne 1, 3$ and so $uw_1w_iw_2$ is a rainbow path. Suppose there is no such vertex w_i. Then $|E(\overline{G})| \ge 1 + (n-3) + 1 + (n-4) = 2n-5$, which implies that $|E(G)| = \binom{n-2}{2} + 2$. Thus we have $d(u) = 1$ and $d(w_1) + d(w_2) = 3 + (n-4) = n-1$. Therefore, $G - \{w_1, w_2, u\} \cong K_{n-3}$. We may further assume that $w_2w_i \in E(G)$ for $3 \le i \le d(w_2) + 1$.

Suppose first that $d(w_2) \ge 3$. If $c(w_2w_3) = 2$, then $uw_1ww_4w_2$ is a rainbow uw_2-path. If $c(w_2w_3) \ne 2$, then $uw_1ww_3w_2$ is a rainbow uw_2-path. Suppose next that $d(w_2) = 2$, which implies $n \ge 5$. If $c(w_2w_3) \ne 2$, then $uw_1ww_3w_2$ is a rainbow uw_2-path. Hence we may assume that $c(w_2w_3) = 2$. Then $uw_1w_4w_3w_2$ is a rainbow uw_2-path for $n \ge 6$. If $n = 5$, then $G \cong Z_2$. Note that $\chi'(Z_2) = rc(Z_2) = 3$, but $prc(Z_2) = 4$. Hence Z_2 is an exceptional graph.

Finally suppose that $d(w_2) = 1$. Then G consists of a complete graph of order $n-2$ induced by $V(G) \setminus \{u, w_2\}$ and two pendant edges attached at w and w_1. If $n \ge 4$ is odd, then $\chi'(G) = n-2 = \Delta(G)$. Observe that the K_{n-2} is coloured with $n-2$ colours and that w and w_1 have distinct palettes of colours for their incident edges. Hence the pendant edges ww_2 and w_1u have distinct colours. This shows that G is rainbow connected. If $n \ge 4$ is even, then $G \cong P_4$ for $n = 4$ implying $prc(G) = rc(G) = 3$. Therefore, P_4 is an exceptional graph, since $\chi'(P_4) = 2$.

If $n \ge 6$, take an optimal edge colouring of the K_{n-2} using $n-3$ colours. Now switch the colour from ww_3 to the edge ww_2, and colour the edges ww_3 and w_1u with a new colour. Now observe that this colouring makes G properly rainbow connected.

3. If $\Delta(G) = n-3$, then let $V(G) \setminus N[w] = U = \{u_1, u_2\}$. We first distinguish two cases.

Case 1. $u_1u_2 \in E(G)$. We first show that $G - w_i$ is connected for $1 \le i \le n-3$. Suppose that $G - w_i$ is disconnected for some i with $1 \le i \le n-3$. Then $|E(G)| \le |E(G[N[w] - w_i])| + |E(G[\{u_1, u_2\}])| + d(w_i) \le \binom{n-3}{2} + 1 + (n-3) < \binom{n-2}{2} + 2$, a contradiction. Now $G - w_1$ and $G - w_2$ both have size at least $\binom{n-2}{2} + 2 - (n-3) = \binom{(n-1)-2}{2} + 2$. So by induction both $G - w_1$ and $G - w_2$ are

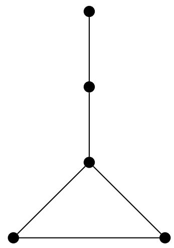


Figure 6. Graph Z_2.

rainbow connected or an exceptional graph. Furthermore w_1 and w_2 are rainbow connected, since $1 \leq d(w_1, w_2) \leq 2$. This shows that G is rainbow connected or an exceptional graph.

Case 2. $u_1u_2 \notin E(G)$. Let $d(u_1) \geq d(u_2) \geq 1$. Then $d(w) + \sum_{i=1}^{n-3} d(w_i) \leq (n-3) + (n-3)(n-3) - d(u_1) - d(u_2)$ implying $|E(G)| \leq \binom{n-2}{2} + \frac{d(u_1)+d(u_2)}{2} < \binom{n-2}{2} + 2$ for $d(u_1) + d(u_2) \leq 3$, a contradiction. Hence we may assume that $d(u_1) + d(u_2) \geq 4$. Now if $d(u_1) \geq 3$, $d(u_2) = 1$ or $d(u_1) \geq 2$, $d(u_2) \geq 2$, then there are always two vertices w_i, w_j such that $G - w_i$ and $G - w_j$ are both connected. This shows that G is rainbow connected or an exceptional graph.

This discussion shows, that in both cases G is either properly rainbow connected or an exceptional graph. So suppose that $m(G-w_i) = \binom{n-3}{2}+2$ and $G-w_i$ is not properly rainbow connected for some $1 \leq i \leq d(w)$. We may choose the labeling of the vertices of G such that $G - w_1$ is not properly rainbow connected.

If $n = 5$, then $G - w_1 \cong P_4$. Taking into account that $d(w_1) = 2 = \Delta(G)$ we conclude that $G \cong C_5$. Now we have $prc(C_5) = 3 = rc(C_5) = \chi'(C_5)$.

If $n = 6$, then $G - w_1 \cong Z_2$. Now up to isomorphism the following three graphs $G_{6.1}, G_{6.2}$ and $G_{6.3}$ are possible.

(a) $G_{6.1} : c(ww_2) = c(w_1w_3) = c(u_1u_2) = 1, c(ww_1) = c(w_2w_3) = 2, c(ww_3) = c(w_1u_1) = c(w_2u_2) = 3$. This shows that $prc(G_{6.1}) = rc(G_{6.1}) = \chi'(G_{6.1}) = 3$.

(b) $G_{6.2} : c(ww_2) = c(w_1w_3) = c(u_1u_2) = 1, c(ww_1) = c(w_2w_3) = 2, c(ww_3) = c(w_1u_1) = 3, c(w_2u_1) = 4$. Observe that $\chi'(G_{6.2}) = 4$. This shows that $prc(G_{6.2}) = \chi'(G_{6.2}) = 4$, whereas $rc(G_{6.2}) = 3$.

(c) For $G_{6.3}$ we can show that $rc(G_{6.3}) = \chi'(G_{6.3}) = 3$, whereas $prc(G_{6.3}) = 4$. Up to a permutation of the colours $G_{6.3}$ has to be coloured as follows: $G_{6.3} : c(ww_1) = c(w_2w_3) = c(u_1u_2) = 1, c(ww_2) = c(w_1u_2) = 2, c(ww_3) = c(w_2u_2) = c(w_1u_1) = 3$. Thus $G_{6.3}$ is an exceptional graph.

If $n = 7$, then $G - w_1 \cong G_{6.3}$. Observe that $d(w_1) = 4$. Now we can always find two vertices $x, y \in V(G_{6.3})$ such that $d(x, y) \leq 2$, $G - x$ and $G - y$ are connected, and $\Delta(G - x) = \Delta(G - y) = 4$. Hence, G is no exceptional graph.

Now by induction it follows that there are no exceptional graphs G with size $\binom{n-2}{2}+2$ for all $n \geq 7$.

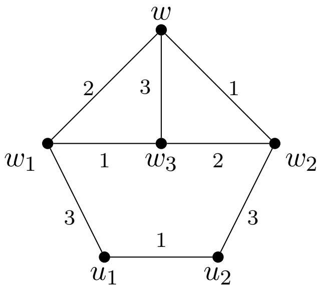


Figure 7. Graph $G_{6.1}$.

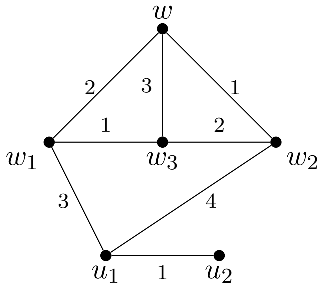


Figure 8. Graph $G_{6.2}$.

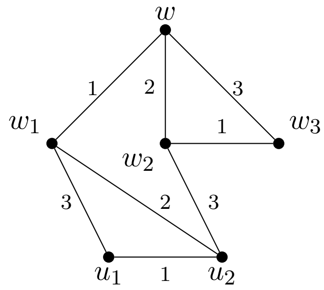


Figure 9. Graph $G_{6.3}$.

4. If $\Delta(G) = n-4$, then $n \geq 7$. Let $V(G) \setminus N[w] = U = \{u_1, u_2, u_3\}$. We distinguish three cases.

Case 1. U is connected. We first show that $G - w_i$ is connected for $1 \leq i \leq n-4$. Suppose that $G - w_i$ is disconnected for some i with $1 \leq i \leq n-4$. Then $|E(G)| \leq |E(G[N[w]-w_i])| + |E(G[\{u_1,u_2,u_3\}])| + d(w_i) \leq \binom{n-4}{2}+3+(n-4) = \binom{n-3}{2}+3 < \binom{n-2}{2}+2$, a contradiction. Hence we may assume that $|N(U) \cap W| \geq 2$.

Fact 1. Then there are always two vertices w_i, w_j such that $G - w_i$ and $G - w_j$ are both connected. Now $G - w_i$ and $G - w_j$ both have size at least $\binom{n-2}{2}+3-(n-4) = \binom{(n-1)-2}{2}+4$. So by induction both $G - w_i$ and $G - w_j$ are rainbow connected. Furthermore w_1 and w_2 are rainbow connected, since $1 \leq d(w_1, w_2) \leq 2$. This shows that G is rainbow connected.

Case 2. $|E(U)| = 1$. We may assume that $E(U) = u_1u_2$. If $|E(U,W)| \leq 3$, then $\sum_{v \in V(G)} d(v) \leq (n-3)(n-4)+2+3$ implying $m(G) \leq \binom{n-3}{2}+2 < \binom{n-2}{2}+2$, a contradiction. Hence we may assume that $|E(U,W)| \geq 4$. Now considering the two components $\{u_1, u_2\}$ and u_3 of U as two vertices we can follow the previous Case 2 for $\Delta(G) = n-3$.

Case 3. $E(U) = \emptyset$. Let $d(u_1) \geq d(u_2) \geq d(u_3) \geq 1$. If $d(u_3) \geq 2$, then there are always two vertices w_i, w_j such that $G - w_i$ and $G - w_j$ are connected. Moreover, $d(w_i, w_j) \leq 2$ and we apply Fact 1. So we may assume that $d(u_3) = 1$. Let $u_3w_1 \in E(G)$. If $|E(\{u_1,u_2\}, W - w_1| \geq 4$, then there are two vertices $w_i, w_j, 2 \leq i < j \leq j$ such that $G - w_i$ and $G - w_j$ are connected and we apply Fact 1. Hence we may assume that $|E(U,W)| \leq 3+3 = 6$. This implies $|E(G)| \leq \binom{n-3}{2}+6 < \binom{n-2}{2}+2$, a contradiction. ■

Sharpness. For even $n \geq 6$ take a complete graph of order $n-2$ and label its vertices $v_1, v_2, \ldots, v_{n-2}$. Now we add two vertices v_{n-1}, v_n, add the edges

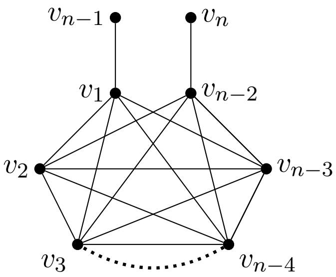


Figure 10. Graph F_n, where n is even.

$v_1v_{n-1}, v_{n-2}v_n$, and delete the edge v_1v_{n-2}. Let F_n denote the resulting graph. Then $|E(F_n)| = \binom{n-2}{2} + 1, \Delta(F_n) = \chi'(F_n) = n-3, rc(F_n) = diam(F_n) = 4$, but $prc(G) = n-2$, if n is even. This can be seen as follows. In any edge colouring of F_n using $n-3$ colours, $F_n - \{v_{n-2}, v_{n-1}, v_n\}$ uses all $n-3$ colours, each on $\frac{n-4}{2}$ edges. Applying the same argument on $F_n - \{v_1, v_{n-1}, v_n\}$, we deduce that v_1 and v_{n-2} have the same palette of $n-4$ colours for their incident edges in $F_n - \{v_{n-1}, v_n\}$. Suppose these colours are $1, 2, \ldots, n-4$. Then both edges v_1v_{n-1} and $v_{n-2}v_n$ obtain colour $n-3$. But then there is no rainbow $v_{n-1}v_n$-path.

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Received 2 December 2019
Revised 22 April 2020
Accepted 22 April 2020