DISTANCE-LOCAL RAINBOW CONNECTION NUMBER

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Abstract

Under an edge coloring (not necessarily proper), a rainbow path is a path whose edge colors are all distinct. The d-local rainbow connection number $lrc_d(G)$ (respectively, d-local strong rainbow connection number $lsrc_d(G)$) is the smallest number of colors needed to color the edges of G such that any two vertices with distance at most d can be connected by a rainbow path (respectively, rainbow geodesic). This generalizes rainbow connection numbers, which are the special case $d = diam(G)$. We discuss some bounds and exact values. Moreover, we also characterize all triples of positive integers d, a, b such that there is a connected graph G with $lrc_d(G) = a$ and $lsrc_d(G) = b$.

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1. Introduction

In 2008, Chartrand *et al.* introduced rainbow connection numbers [2]. We recall the basic definitions. We call any map $c : E(G) \to \{1, \ldots, k\}$ an edge coloring. This map is not necessarily proper so that adjacent edges may have the same color. A rainbow path is a path whose edge colors are all distinct. A rainbow coloring is an edge coloring such that any two vertices can be connected by a rainbow path. A strong rainbow coloring is an edge coloring such that any two vertices can be connected by a rainbow geodesic. These colorings always exist. In fact, an all-distinct coloring (i.e., an edge coloring where all edges receive distinct colors) is a (strong) rainbow coloring, since under such a coloring every

path becomes a rainbow path. We can often be more efficient. The rainbow connection number $rc(G)$ is the smallest number of colors in a rainbow coloring of G. The strong rainbow connection number $src(G)$ is the smallest number of colors in a strong rainbow coloring of G. The reader is referred to the book [9] and the dynamic survey [10] for a comprehensive overview of these topics.

A rainbow connection can be applied to the secure transfer of classified information (cf. [7]). In a network of government agencies, they wish to have a procedure that allows the sharing of information between appropriate parties but secure enough against intruders. They may assign some number of passwords or firewalls between agencies such that there is always a secure information transfer paths between any two agencies, possibly through intermediaries, and no passwords in that path are repeated. Then it is natural to ask the minimum number of passwords or firewalls needed.

There are several generalizations of rainbow connection. For instance, rainbow k-connectivity [3], k-rainbow index [4], the vertex version [8], total version [12], directed version [6], and rainbow connection for hypergraphs [1].

We propose yet another generalization, to "localize" some properties of the rainbow connection. In a rainbow coloring, every pair of vertices is connected by a rainbow path. Now, we consider only those pairs with distance up to d, a given positive integer. We define a d-local rainbow coloring as an edge coloring such that any two vertices with distance at most d can be connected by a rainbow path, and we define d-local rainbow connection number $lrc_d(G)$ as the smallest number of colors in such a coloring. Similarly, we define d-local strong rainbow coloring and d-local strong rainbow connection number $lsrc_d(G)$ by replacing the word "path" with "geodesic". Referring to the secure transfer of important information, we need not have different passwords for every pair of agencies, but it may be enough to have different passwords for distance d. The problem then can be represented by d-local rainbow coloring.

In the next sections, we discuss some bounds and exact values. We also discuss the equality cases of these bounds. Finally, we consider the problem of realizing positive integers d, a, b as $lrc_d(G) = a$ and $lsrc_d(G) = b$ for some connected graph G.

2. Some Bounds and Exact Values

In [2], it was observed that

$$(1) \qquad\qquad diam(G) \leq rc(G) \leq src(G) \leq |E(G)|.$$

This elementary but very useful fact can be generalized. We recall that $\chi(G)$ denotes the vertex chromatic number of a graph G, the smallest number of colors

needed to color the vertices of G such that adjacent vertices receive distinct colors. The line graph $L(G)$ is a graph with vertex set $V(L(G)) = E(G)$ such that $e_1, e_2 \in E(G)$ are adjacent in $L(G)$ if and only if they share a common endpoint. For a positive integer k, the k'th power G^k is a graph with vertex set $V(G^k) = V(G)$ and edge set $E(G^k) = \{xy : 1 \leq d_G(x, y) \leq k\}$.

Lemma 1. *If G is any graph and d does not exceed the maximum diameter of a connected component of G, then*

$$(2) \qquad d \leq lrc_d(G) \leq lsrc_d(G) \leq \chi(L(G)^{d-1}) \leq |E(G)|.$$

Proof. For the leftmost bound, take any $x, y \in V(G)$ with distance d. Any rainbow path between them must use at least d colors, so $lrc_d(G) \geq d$.

Now let c be a proper vertex coloring on $L(G)^{d-1}$. Any two vertices with distance at most $d-1$ in $L(G)$ have distinct colors, so any path in G of length at most d is rainbow and c is a d-local strong rainbow coloring on G. This proves $lsrc_d(G) \leq \chi(L(G)^{d-1})$. For the rightmost bound, it is clear that $\chi(L(G)^{d-1}) \leq |V(L(G))| = |E(G)|$. ∎

There is a simple case where the $\chi(L(G)^{d-1})$ bound is exact. Recall that the girth $g(G)$ is the smallest size of a cycle in G, or $g(G) = \infty$ if G has no cycle.

Lemma 2. *If $d < g(G)/2$, then $lsrc_d(G) = \chi(L(G)^{d-1})$.*

Proof. Let c be a d-local strong rainbow coloring on G. Since $g(G) > 2d$, any $x, y \in V(G)$ with distance d are connected by a unique geodesic. Therefore, every path in G of length at most d is rainbow. So any two edges of G with distance at most $d-1$ have distinct colors, and c is a proper vertex coloring on $L(G)^{d-1}$. ∎

Corollary 3. *If $g(G) \geq 5$, then $lsrc_2(G) = \chi(L(G)) = \chi'(G)$, where $\chi'(G)$ is the edge chromatic number of G.*

For the Petersen graph P (with diameter 2) it is known that $rc(P) = 3$ and $src(P) = 4$ (see [2]). We reprove this, and more generally we compute the 2-local rc and src of generalized Petersen graph $P(n, k)$. We recall that $P(n, k)$ denotes a graph with $2n$ vertices $u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}$ whose edges are $u_i u_{i+1}$, $u_i v_i$, and $v_i v_{i+k}$ for each $i \in \{0, 1, \ldots, n-1\}$, the indices considered modulo n. The usual Petersen graph is $P = P(5, 2)$. Note that $P(n, k) \cong P(n, n - k)$. We assume k is chosen such that there is no $l < k$ with $P(n, k) \cong P(n, l)$. First, an observation.

Lemma 4. *For any graph G, $lrc_2(G) = 2$ if and only if $lsrc_2(G) = 2$.*

Proof. Any path of length two between non-adjacent vertices is a geodesic, so any 2-local rainbow coloring with two colors is also a 2-local strong rainbow. ∎

Theorem 5. *Let $G = P(n,k)$ be a generalized Petersen graph, with $n \geq 3$ and $k \geq 1$ such that there is no $l < k$ with $P(n,l) \cong G$.*

1. *If $(n,k) = (5,2)$, then $lrc_2(G) = 3$ and $lsrc_2(G) = 4$.*

2. *If $(n,k) \neq (5,2)$, then*

$$\text{(3)} \qquad lrc_2(G) = lsrc_2(G) = \begin{cases} 2, & \text{if } n = 3, \text{ or } n \geq 4 \text{ is even and } k = 1, \\ 3, & \text{otherwise.} \end{cases}$$

Proof. We will need the edge chromatic number, which is well-known,

$$\text{(4)} \qquad\qquad \chi'(P(n,k)) = \begin{cases} 4, & \text{if } (n,k) = (5,2), \\ 3, & \text{otherwise.} \end{cases}$$

First, assume $(n,k) = (5,2)$. Then $g(G) = 5$, so $lsrc_2(G) = \chi'(G) = 4$. The following figure shows a 2-local rainbow coloring on G, so $lrc_2(G) \leq 3$. If $lrc_2(G) = 2$, we would also have $lsrc_2(G) = 2$, a contradiction. So $lrc_2(G) = 3$.
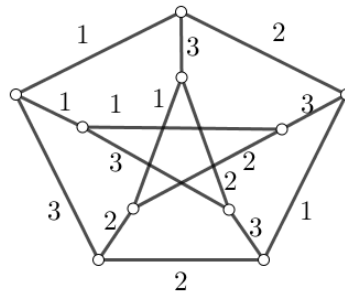

Figure 1. A 2-local rainbow coloring on the Petersen graph.

For the rest, assume $(n,k) \neq (5,2)$. Then $lrc_2(G) \leq lsrc_2(G) \leq \chi'(G) = 3$. By the previous lemma, $lrc_2(G) = lsrc_2(G)$. If $n = 3$, or $n \geq 4$ is even and $k = 1$, we construct a 2-local (strong) rainbow coloring on G as follows.
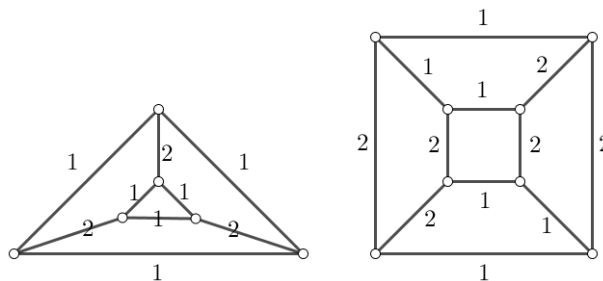

Figure 2. 2-local rainbow colorings with two colors.

Suppose $n \geq 4$ is odd and $k = 1$. We will show $lsrc_2(G) = 3$. The graph is a prism with two rims (C_n's). Suppose there is a 2-local rainbow coloring c on G with two colors. Since there are only two colors, any two vertices with distance 2 in the inner or outer rim must be connected by a rainbow path within that rim. So c restricts to a 2-local rainbow coloring on C_n and the edge colors alternate $1, 2, 1, 2, \ldots$. But n is odd, so the first and last edges have the same color and it is impossible to connect their endpoints with a rainbow path.

Now suppose $n \geq 4$ and $k \geq 2$. We will show $lsrc_2(G) = 3$. Suppose there is a 2-local strong rainbow coloring c on G with two colors. Suppose $u_0 - u_1 - u_2$ is a rainbow geodesic between u_0 and u_2, with $c(u_0 u_1) = 1$ and $c(u_1 u_2) = 2$. Since there are only two colors, we may assume $c(u_1 v_1) = 1$. Since $d_G(u_0, v_1) = 2$, there must be a rainbow geodesic between them. But $k \geq 2$, so the only geodesic between them is $u_0 - u_1 - v_1$ which is not rainbow. ∎

As functions of d, the d-local rc and src are non-decreasing.

Lemma 6. *For any graph G and any positive integer d,*

$$(5) \qquad\qquad lrc_d(G) \leq lrc_{d+1}(G),$$
$$(6) \qquad\qquad lsrc_d(G) \leq lsrc_{d+1}(G).$$

Proof. Any $(d+1)$-local (strong) rainbow coloring is automatically d-local. ∎

The following example shows that this bound can be sharp.

Theorem 7. *Let G be the vertex amalgamation of n copies of C_n, where $n \geq 3$. That is, G is obtained from n disjoint C_n's by choosing one vertex in each cycle and identifying those vertices as a single vertex. Then, for each $d \geq 2$,*

$$(7) \qquad\qquad lrc_d(G) = n.$$

Proof. Put the colors $1, 2, \ldots, n$ in each cycle. It is easy to see that this is a rainbow coloring, so $rc(G) \leq n$. Now, because

$$(8) \qquad lrc_2(G) \leq lrc_3(G) \leq lrc_4(G) \leq \cdots \leq rc(G) = n$$

it remains to show $lrc_2(G) \geq n$. Suppose there is a 2-local rainbow coloring c on G with $n - 1$ colors. Let the i'th cycle be $u_{i,1} - u_{i,2} - \cdots - u_{i,n} - u_{i,1}$, with $v = u_{1,1} = u_{2,1} = \cdots = u_{n,1}$ being the common vertex, where $1 \leq i \leq n$. Consider $u_{i,2}$ and $u_{j,2}$ with $i \neq j$. The shortest path between them is $u_{i,2} - v - u_{j,2}$. Any other path has length at least n. So $c(vu_{i,2}) \neq c(vu_{j,2})$ for all $i \neq j$, and at least n colors are needed. ∎

2.1. Trees

For a tree G, it is known that $rc(G) = src(G) = |E(G)|$ (see [2]). We will compute the local rc and src. From Lemma 2 we already know $lsrc_d(G) = \chi(L(G)^{d-1})$, but we still need to compute the right hand side.

Theorem 8. *Let G be a tree and $d \leq diam(G)$. Then*

$$(9) \quad lrc_d(G) = lsrc_d(G) = \max\{|E(S)| : S \text{ a subtree of } G, diam(S) = d\}.$$

Proof. Let $f(G)$ be the right hand side, that is, the maximum number of edges in a subtree of G with diameter exactly d. Because $d \leq diam(G)$, any subtree with diameter less than d can always be enlarged to a subtree with diameter exactly d. So $f(G)$ is also the maximum number of edges in a subtree of G with diameter at most d.

Every path in a tree is a geodesic, so $lrc_d(G) = lsrc_d(G)$. If c is a d-local rainbow coloring on G and S is a subtree with $diam(S) = d$, then the restriction of c to S is a rainbow coloring. So $lrc_d(G) \geq f(G)$.

We prove $lrc_d(G) \leq f(G)$ inductively. In G, choose a vertex x with maximum eccentricity, and a path $L : x - x_1 - x_2 - \cdots - x_p$ with $p = diam(G)$. We define the d-step open neighborhood of x as

$$(10) \quad N_G^d(x) = \{y \in V(G) : 1 \leq d(x,y) \leq d\}.$$

Let S be the subtree induced by the d-step *closed* neighborhood of x, which is defined as $N_G^d[x] = \{x\} \cup N_G^d(x)$.

Claim. $diam(S) = d$.

Proof. Suppose there are $y_1, y_2 \in N_G^d(x)$ with $d_G(y_1, y_2) > d$. Let z be where y_1 and y_2 meet, and let x_i be where z meets the path L.
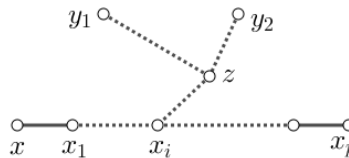

Figure 3. Branches connecting y_1 and y_2.

Since $d_G(y_1, y_2) > d$, we may assume $d_G(y_1, z) > d/2$ so $i = d(x, x_i) \leq d(x, z) < d/2$. Then $d_G(y_1, x_p) > d_G(z, x_p) + d/2 \geq p - i + d/2 > p$, impossible. $\quad \square$

Continuing the proof of the theorem, let $S' = S \backslash x$ and $G' = G \backslash x$ be the graphs obtained by deleting x from S and G, respectively. If x has another neighbor $w \neq x_1$, then by the uniqueness of paths in a tree, the only path from w to x_p is $w - x - x_1 - \cdots - x_p$ so $d_G(w, x_p) = 1 + p$, a contradiction. Therefore, x only has one neighbor in G, namely x_1. As a result, S' and G' are connected graphs.

Moreover, $diam(S') \leq diam(S) = d$, $diam(G') \leq diam(G)$, and $f(G') \leq f(G)$. By induction, $lrc_d(G') \leq f(G')$ so G' has a d-local rainbow coloring with $f(G')$ colors. Consider x_1, the (only) vertex adjacent to x. If $f(G') \leq f(G) - 1$, then we can put a new color on $x_1 x$. If $f(G') = f(G)$, then $|E(S')| = |E(S)| - 1 \leq f(G) - 1 = f(G') - 1$ so S' does not use up all $f(G')$ colors, and we can put one unused color on $x_1 x$. ∎

For small d, we can be more explicit.

Corollary 9. *If G is a tree, then*

1. $lrc_2(G) = lsrc_2(G) = \Delta(G)$,
2. $lrc_3(G) = lsrc_3(G) = \max\{\deg(x) + \deg(y) - 1 : xy \in E(G)\}$.

Trees are the only graphs with $src(G) = |E(G)|$ (see [2]). A similar fact also holds for local rc and src.

Corollary 10. *If G is connected and $d \leq diam(G)$, then the following are equivalent.*

1. $lrc_d(G) = |E(G)|$.
2. $lsrc_d(G) = |E(G)|$.
3. G *is a tree and* $d = diam(G)$.

Proof. The nontrivial implication is 2⟹3. Suppose $lsrc_d(G) = |E(G)|$. Since $lsrc_d(G) \leq src(G)$, this implies $src(G) = |E(G)|$ so G is a tree. By the previous result, $lsrc_d(G) = |E(S)|$ with $diam(S) = d$, so $|E(S)| = |E(G)|$ and $S = G$. ∎

2.2. Cycles

For the cycle, we know $rc(C_n) = src(C_n) = \lceil n/2 \rceil$ (see [2]). Here, we compute the local rc and src.

Theorem 11. *If $n \geq 3$ and $d \leq n/2$, then*

$$(11) \qquad lrc_d(C_n) = lsrc_d(C_n) = \left\lceil \frac{n}{\lfloor n/d \rfloor} \right\rceil .$$

Proof. Let $n = dq + r$ with $0 \leq r \leq d - 1$ and $q = \lfloor n/d \rfloor$. Note that $q \geq 2$ since $n \geq 2d$. Let $b = \lceil n/q \rceil$. Then

$$(12) \qquad b = \left\lceil \frac{n}{q} \right\rceil = \left\lceil \frac{dq + r}{q} \right\rceil = d + \left\lceil \frac{r}{q} \right\rceil.$$

First, we prove $lsrc_d(C_n) \leq b$. If $r = 0$, then $n = dq$ and we can put the colors $1, 2, \ldots, d$ consecutively on the edges of C_n. This is clearly a d-local strong rainbow coloring. Now suppose $r > 0$, so that $b = d + \lceil r/q \rceil \geq d + 1$. Let $t = (b - d)q - r$. Then $t = \left\lceil \frac{r}{q} \right\rceil q - r \geq \frac{r}{q} q - r = 0$ and $t < \left(1 + \frac{r}{q} \right) q - r = q + r - r = q$ so $q > t \geq 0$. Note that

$$(13) \qquad n = qd + r = qb - t = (q - t)b + t(b - 1).$$

On the edges of C_n, put the colors $1, 2, \ldots, b$ consecutively for $q - t$ times, and then put the colors $1, 2, \ldots, b - 1$ consecutively for t times. Since $b - 1 \geq d$, this is a d-local strong rainbow coloring on C_n, and we have proved that $lsrc_d(C_n) \leq b$.

Now we prove $lsrc_d(C_n) \geq b$. Suppose otherwise, so C_n has a d-local strong rainbow coloring c with $b - 1$ colors. Let e be an edge of C_n with color $c(e) = i$. Let L_e and R_e be the set of the first $d - 1$ edges from e in the left and right direction, respectively. Then $L_e \cap R_e = \emptyset$ because $n > 2d - 1$. Since the coloring is d-local strong rainbow, the color i cannot occur again on $L_e \cup R_e$. Therefore, the color i can only occur again on the remaining $n - 2(d - 1) - 1 = n - 2d + 1$ edges. From each further occurence of the color i, the next $d - 1$ edges are free of the color i, and so on. In total, the number of occurence of color i is at most $\frac{n-2d+1}{d} + 1 = \frac{n+1}{d} - 1$. Note that

$$(14) \qquad \frac{n+1}{d} - 1 = \frac{dq + r + 1}{d} - 1 = q + \frac{r+1}{d} - 1 \leq q + 1 - 1 = q.$$

Since there are n edges, and each edge must be colored with one of the $b - 1$ colors, we have $n \leq (b - 1)q$. This contradicts $n/q > b - 1$.

It remains to prove $lrc_d(C_n) \geq b$. Suppose otherwise, so C_n has a d-local rainbow coloring c with $b - 1$ colors. Let $x, y \in V(C_n)$ with $d(x, y) \leq d$. There are two paths between them: one of length d, and one of length $n - d$. Note that

$$(15) \qquad n - d \geq \frac{2(n - d)}{q} = \frac{n + n - 2d}{q} \geq \frac{n}{q} > b - 1,$$

so the shorter path is rainbow. Therefore, c is actually a d-local strong rainbow coloring, contradicting $lsrc_d(C_n) = b$. ∎

As noted in the proof, another way to express this result is

$$(16) \qquad lrc_d(C_n) = lsrc_d(C_n) = d + \left\lceil \frac{r}{q} \right\rceil,$$

where $n = dq + r$ with q, r integers such that $0 \leq r \leq d - 1$ and $q \geq 2$.

From here, we may conclude

$$(17) \qquad d \leq lrc_d(C_n) = lsrc_d(C_n) \leq d + \left\lceil \frac{d-1}{2} \right\rceil.$$

For small d we have the following results.

Corollary 12.

1. *For $n \geq 4$, $lrc_2(C_n) = lsrc_2(C_n) = \begin{cases} 2, & \text{if } n \text{ is even,} \\ 3, & \text{if } n \text{ is odd.} \end{cases}$*

2. *For $n \geq 6$, $lrc_3(C_n) = lsrc_3(C_n) = \begin{cases} 3, & \text{if } 3 \mid n, \\ 4, & \text{otherwise.} \end{cases}$*

3. *For $n \geq 8$, $lrc_4(C_n) = lsrc_4(C_n) = \begin{cases} 4, & \text{if } 4 \mid n, \\ 5, & \text{if } 4 \nmid n \text{ and } n \neq 11, \\ 6, & \text{if } n = 11. \end{cases}$*

3. GRAPHS WITH $lrc_d(G) = d$

The bound $rc(G) \geq diam(G)$ is useful because it often becomes equality. However, this makes it difficult to completely characterize the equality cases. Surprisingly, for the local version $lrc_d(G) \geq d$, it is possible to characterize equality, at least when d is small compared to the girth. First, an observation.

Lemma 13. *If $g(G) > D + lrc_D(G)$ for some D, then $lrc_d(G) = lsrc_d(G)$ for all $d \leq D$.*

Proof. Note $g(G) > D + lrc_D(G) \geq d + lrc_d(G)$. Let c be a d-local rainbow coloring on G with $k = lrc_d(G)$ colors. We show that this coloring is actually strong. Suppose otherwise. Then some x, y with $d_G(x, y) \leq d$ are not connected by a rainbow geodesic. Let L_1 be any geodesic in G between x and y. Let L_2 be a rainbow path in G between x and y. Let z be the first vertex after x that the rainbow path L_2 intersects the non-rainbow geodesic L_1 (possibly $z = y$). The concatenated paths $x L_1 z$ and $x L_2 z$ combine to form a cycle of length at least $g(G)$. Since $x L_1 z$ has at most d edges, the rainbow path $x L_2 z$ has at least $g(G) - d > k$ edges. But there are only k colors, a contradiction. ∎

As a side note, we get the following.

Corollary 14. *Let G be a connected graph which is not a tree, and suppose $g(G) > rc(G) + diam(G)$. Then $rc(G) = src(G) = diam(G) = \frac{g(G)-1}{2}$ and $lrc_d(G) = lsrc_d(G)$ for each $d \leq diam(G)$.*

Proof. Let $g = g(G)$. It is well known that $g \leq 2\,diam(G) + 1$, so $diam(G) \geq \lfloor g/2 \rfloor$ and $g > rc(G) + diam(G) \geq 2\lfloor \frac{g}{2} \rfloor$. Then g is odd, $rc(G) + diam(G) = g - 1$, and $g = 2\,diam(G) + 1$. We are done by Lemma 13. ∎

Now we prove the main result.

Theorem 15. *If G is connected and $d < g(G)/2$, then the following are equivalent.*

1. *$lrc_d(G) = d$.*
2. *$lsrc_d(G) = d$.*
3. *G is either a path P_n with $n \geq d + 1$ or a cycle C_{dq} with $q \geq 3$.*

Proof. The non-trivial implication is 1$\Longrightarrow$3. Suppose $lrc_d(G) = d$. Then $g(G) > 2d = d + lrc_d(G)$, so Lemma 13 gives $d = lsrc_d(G) = \chi(L(G)^{d-1})$. If G is a tree, Theorem 8 can be used to show $G = P_n$. We assume G has a cycle.

For any graph H we have $\chi(H) \geq \omega(H)$, where $\omega(H)$ is the clique number of H, i.e., the largest number of pairwise adjacent vertices in H. We have $\omega(L(G)^{d-1}) \leq \chi(L(G)^{d-1}) = d$, so there are at most d vertices in $L(G)$ with pairwise distance not exceeding $d - 1$.

Suppose $d = 2$. Then $2 = \chi(L(G)) \geq \Delta(G)$, and G is a cycle. It must be even by Corollary 12.

Finally, suppose $d \geq 3$ and consider a shortest cycle C_g in G, where $g = g(G)$. This cycle has no chord. We claim that $G = C_g$. Suppose otherwise. Then there is a vertex v outside of this cycle that is adjacent to some vertex x in the cycle. Consider a path $x_1 - \cdots - x_d - x_{d+1}$ in the cycle with $x_2 = x$. Let S be the set of edges in this path together with the edge xv. Then any two edges in S have distance at most $d - 1$ in $L(G)$ (since $d \geq 3$), so S is a clique in $L(G)^{d-1}$. But $|S| = d + 1$, contrary to $\omega(L(G)^{d-1}) \leq d$. Therefore, $G = C_g$.

Write $g = dq + r$ with $q = \lfloor g/d \rfloor$ and $0 \leq r \leq d - 1$. By Theorem 11, $lrc_d(G) = d + \lceil r/q \rceil$. So $r = 0$, and $g = dq$ with $q > 2$. ∎

We always have $g(G) \leq 2\,diam(G) + 1$. Applying the theorem with $d = diam(G)$, we get the following.

Corollary 16. *Let G be a connected graph which is not a path. If $rc(G) = diam(G)$, then $g(G) \leq 2\,diam(G)$.*

4. Existence of Graphs with Prescribed Values of Local RC and SRC

The authors of [2] asked whether any pair of positive integers $a \leq b$ can be realized as $a = rc(G)$ and $b = src(G)$ for some connected graph G.

Theorem 17 [5]**.** *Let a, b be positive integers. Then there is a connected graph G such that $a = rc(G)$ and $src(G) = b$ if and only if one of the following holds.*

1. $a = b \in \{1, 2\}$.
2. $3 \leq a \leq b$.

There is an analogous result for local rc and src. In part of the proof, we will use the following.

Theorem 18 [11]**.** *Let $m_1, \ldots, m_t, t \geq 2$ be positive integers. If G is the vertex amalgamation of $K_{m_1}, \ldots, K_{m_t}$, then $src(G) = t$ and*

$$rc(G) = \begin{cases} 2, & \text{if } t = 2, \\ \max\{3, u\}, & \text{otherwise,} \end{cases}$$

where u is the number of $i \in \{1, \ldots, t\}$ with $m_i = 2$.

Now, we solve the realization problem for local rc and src: given positive integers d, a, b, is there a connected graph G with $lrc_d(G) = a$ and $lsrc_d(G) = b$?

Theorem 19. *Let d, a, b be positive integers. Then there is a graph G such that $a = lrc_d(G)$ and $b = lsrc_d(G)$ if and only if one of the following holds.*

1. $d = a = b \in \{1, 2\}$.
1. $d = 2$ and $3 \leq a \leq b$.
1. $3 \leq d \leq a \leq b$.

Moreover, we can always choose G to be connected and $d = diam(G)$.

Proof. The forward direction follows from the results in Section 2. It remains to prove the converse. If $d = a = b$, then we can choose $G = P_{d+1}$. If $d = 2$ and $3 \leq a \leq b$, then we can choose G to be the vertex amalgamation of a copies of K_2 and $b - a$ copies of K_3, because $diam(G) = 2$ and $lrc_2(G) = \max\{3, a\} = a$ and $lsrc_2(G) = b$ by Theorem 18.

Suppose $3 \leq d = a < b$. Let G be a connected graph with $diam(G) = rc(G) = a$ and $src(G) = b$ (see Theorem 17). Then $lrc_d(G) = a$ and $lsrc_d(G) = b$.

Now suppose $3 \leq d < a \leq b$. Let G be the graph in Figure 4.
Since $diam(G) = d$, it remains to show $rc(G) = a$ and $src(G) = b$. Let H be the subgraph of G induced by $x_1, \ldots, x_{a+1}$. Any path in G between two vertices in H must be contained in H. Thus, any rainbow coloring of G restricts to a rainbow coloring on H. Moreover, H is a tree. So $rc(G) \geq rc(H) = |E(H)| = a$.
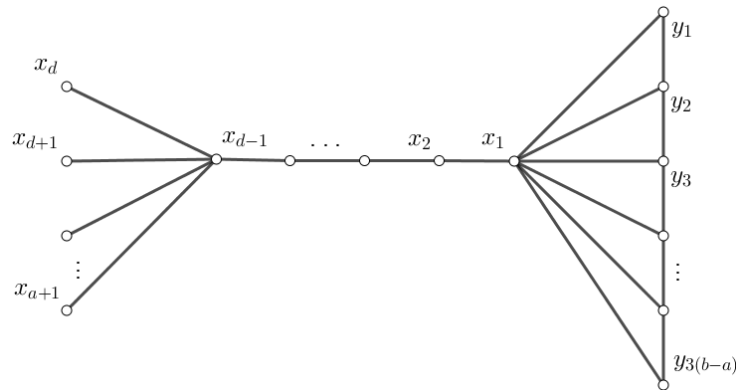
Figure 4. A construction for the case $3 \le d < a \le b$.

On the other hand, the coloring c_1 below is a rainbow coloring,

$$(18) \qquad c_1(e) = \begin{cases} i, & \text{if } e = x_i x_{i+1} \text{ with } 1 \le i \le d-2, \\ i, & \text{if } e = x_{d-1} x_{i+1} \text{ with } d-1 \le i \le a, \\ d-1, & \text{if } e = x_1 y_i \text{ or } e = y_{i+1} y_{i+2} \text{ with } i \equiv 1 \, (\text{mod } 3), \\ d, & \text{if } e = x_1 y_i \text{ or } e = y_{i+1} y_{i+2} \text{ with } i \equiv 2 \, (\text{mod } 3), \\ d+1, & \text{if } e = x_1 y_i \text{ or } e = y_{i+1} y_{i+2} \text{ with } i \equiv 0 \, (\text{mod } 3), \end{cases}$$

where the indices in y_i are read modulo $3(b-a)$.

Let K be the subgraph of G induced by H and y_i for all $i \equiv 1 \, (\text{mod } 3)$. Any geodesic in G between two vertices in K must be contained in K. Thus, any strong rainbow coloring of G restricts to a strong rainbow coloring on K. Moreover, K is a tree. So $src(G) \ge src(K) = |E(K)| = b$. On the other hand, the coloring c_2 below is a strong rainbow coloring,

$$(19) \qquad c_2(e) = \begin{cases} c_1(e), & \text{if } e \in E(H) \cup \{y_i y_{i+1} : 1 \le i \le 3(b-a) - 1\}, \\ a + \lceil \frac{i}{3} \rceil, & \text{if } e = x_1 y_i \text{ with } 0 \le i \le 3(b-a). \end{cases} \qquad \blacksquare$$

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