# A NEW UPPER BOUND FOR THE PERFECT ITALIAN DOMINATION NUMBER OF A TREE 

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#### Abstract

A perfect Italian dominating function (PIDF) on a graph $G$ is a function $f: V(G) \rightarrow\{0,1,2\}$ satisfying the condition that for every vertex $u$ with $f(u)=0$, the total weight of $f$ assigned to the neighbors of $u$ is exactly two. The weight of a PIDF is the sum of its functions values over all vertices. The perfect Italian domination number of $G$, denoted $\gamma_{I}^{p}(G)$, is the minimum weight of a PIDF of $G$. In this paper, we show that for every tree $T$ of order $n \geq 3$, with $\ell(T)$ leaves and $s(T)$ support vertices, $\gamma_{I}^{p}(T) \leq \frac{4 n-\ell(T)+2 s(T)-1}{5}$, improving a previous bound given by T.W. Haynes and M.A. Henning in [Perfect Italian domination in trees, Discrete Appl. Math. 260 (2019) 164177].


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## 1. Introduction

Throughout this paper, $G$ is a simple graph with vertex set $V(G)$ and edge set $E(G)$ (briefly $V, E)$. The order $|V|$ of $G$ is denoted by $n=n(G)$. For every vertex $v \in V(G)$, the open neighborhood of $v$ is the set $N_{G}(v)=N(v)=\{u \in V(G) \mid$ $u v \in E(G)\}$ and its closed neighborhood is the set $N_{G}[v]=N[v]=N(v) \cup\{v\}$. The degree of a vertex $v \in V$ is $\operatorname{deg}_{G}(v)=|N(v)|$. A leaf of $G$ is a vertex of degree one and a support vertex is a vertex adjacent to a leaf. An end support vertex is a support vertex having at most one non-leaf neighbor. For every vertex $v \in V$, the set of all leaves adjacent to $v$ is denoted by $L(v)$ and $L[v]=L(v) \cup\{v\}$. We denote the set of leaves of a graph $G$ by $L(G)$ and the set of support vertices by $S(G)$. We also let $|S(G)|=s(G)$ and $|L(G)|=\ell(T)$. A double star $D S_{q, p}$, with $q \geq p \geq 1$, is a graph consisting of the union of two stars $K_{1, q}$ and $K_{1, p}$ together with an edge joining their centers. The subdivision graph $S_{b}(G)$ of a graph $G$ is that graph obtained from $G$ by replacing each edge $u v$ of $G$ by a vertex $w$ and edges $u w$ and $v w$. A healthy spider $S_{k}(G)$ is the subdivision graph of a star $K_{1, k}$ for $k \geq 2$. A wounded spider $S_{k, t}$ is a graph obtained from a star $K_{1, k}$ by subdividing $t$ edges exactly once, where $1 \leq t \leq k-1$. We denote by $P_{n}$ the path on $n$ vertices. The distance $d_{G}(u, v)$ between two vertices $u$ and $v$ in a connected graph $G$ is the length of a shortest $u-v$ path in $G$. The diameter of a graph $G$, denoted by $\operatorname{diam}(G)$, is the greatest distance between two vertices of $G$. For a vertex $v$ in a rooted tree $T$, let $C(v)$ denote the set of children of $v$, $D(v)$ denotes the set of descendants of $v$ and $D[v]=D(v) \cup\{v\}$. Also, the depth of $v$, depth $(v)$, is the largest distance from $v$ to a vertex in $D(v)$. The maximal subtree at $v$ is the subtree of $T$ induced by $D[v]$, and is denoted by $T_{v}$.

For a real-valued function $f: V \longrightarrow \mathbb{R}$, the weight of $f$ is $\omega(f)=\sum_{v \in V} f(v)$, and for $S \subseteq V$ we define $f(S)=\sum_{v \in S} f(v)$. So $w(f)=f(V)$.

A Roman dominating function on $G$, abbreviated $\operatorname{RDF}$, is a function $f$ : $V \rightarrow\{0,1,2\}$ such that every vertex $u \in V$ for which $f(u)=0$ is adjacent to at least one vertex $v$ for which $f(v)=2$. Roman domination was introduced by Cockayne et al. in [7] and was inspired by the work of ReVelle and Rosing [12] and Stewart [13]. Several new varieties of Roman domination have been introduced since 2004 , among them, we quote the Italian domination originally published in [1] and called Roman $\{2\}$-domination. Further results on Roman domination and its variant can be found in $[2-6]$.

An Italian dominating function on $G$, abbreviated IDF, is a function $f$ : $V \rightarrow\{0,1,2\}$ satisfying the condition that for every vertex $v \in V$ with $f(v)=0$, $\sum_{u \in N(v)} f(u) \geq 2$, that is either $v$ is adjacent to a vertex $u$ with $f(u)=2$, or to at least two vertices $x$ and $y$ with $f(x)=f(y)=1$. The Italian domination number, denoted $\gamma_{I}(G)$, is the minimum weight of an IDF in $G$.

The concept of perfect dominating sets introduced by Livingston and Stout
in [11] has been extended to Roman and Italian dominating functions in [10] and [9], respectively. An RDF $f$ is called perfect if for every vertex $v$ with $f(v)=0$, there is exactly one vertex $u \in N(v)$ with $f(u)=2$, while a IDF $g$ is perfect if for every vertex $w$ with $g(w)=0, g(N(v))=2$. The perfect Roman domination number (respectively, perfect Italian domination number) of $G$, denoted $\gamma_{R}^{p}(G)$ (respectively, $\gamma_{I}^{p}(G)$ ), is the minimum weight of a perfect RDF (respectively, perfect IDF) in $G$. A perfect IDF on $G$ will be abbreviated PIDF. A PIDF $f$ is called a $\gamma_{I}^{p}(G)$-function if $\omega(f)=\gamma_{I}^{p}(G)$.

It was shown in [10] that every tree $T$ of order $n \geq 3$ satisfies $\gamma_{R}^{p}(T) \leq \frac{4}{5} n$. However, this upper bound has recently been improved by Darkooti et al. [8] for trees $T$ with $\ell(T) \geq 2 s(T)-2$, by showing that for any tree $T$ of order $n \geq 3$ with $\ell(T)$ leaves and $s(T)$ support vertices, $\gamma_{R}^{p}(T) \leq(4 n-\ell(T)+2 s(T)-2) / 5$. Moreover, Henning and Haynes showed in [9] that $\frac{4}{5} n$ is also an upper bound of the prefect Italian domination number for any tree of order $n \geq 3$.

In this paper, we shall show that for any tree $T$ of order $n \geq 3$ with $\ell(T)$ leaves and $s(T)$ support vertices, $\gamma_{R}^{p}(T) \leq(4 n-\ell(T)+2 s(T)-1) / 5$. But first let us point out that for both parameters $\gamma_{R}^{p}(G)$ and $\gamma_{I}^{p}(G)$, one may be larger or smaller than the other even for trees. Indeed, for the path $P_{5}$ we have $\gamma_{R}^{p}\left(P_{5}\right)=4$ and $\gamma_{I}^{p}\left(P_{5}\right)=3$ while for the double star $D S_{3,1}$ we have $\gamma_{R}^{p}\left(D S_{3,1}\right)=3$ and $\gamma_{I}^{p}\left(D S_{3,1}\right)=4$. The next result shows that the differences $\gamma_{I}^{p}(G)-\gamma_{R}^{p}(G)$ and $\gamma_{R}^{p}(G)-\gamma_{I}^{p}(G)$ can be arbitrarily large.

Observation 1. For any integer $k \geq 1$, there exist trees $T_{k}$ and $H_{k}$ such that $\gamma_{I}^{p}\left(T_{k}\right)-\gamma_{R}^{p}\left(T_{k}\right)=k$ and $\gamma_{R}^{p}\left(H_{k}\right)-\gamma_{I}^{p}\left(H_{k}\right)=k$.
Proof. Let $T_{k}$ be the tree formed by $k$ double stars $D S_{3,1}$ by adding a new vertex attached to every support vertex of degree four. One can easily see that $\gamma_{I}^{p}\left(T_{k}\right)=4 k+1$ while $\gamma_{R}^{p}\left(T_{k}\right)=3 k+1$.

Now, let $H_{k}$ be the tree formed by $k$ paths $P_{5}$ by adding a new vertex attached to all center vertices of the paths. Then $\gamma_{I}^{p}\left(H_{k}\right)=3 k+1$ while $\gamma_{R}^{p}\left(H_{k}\right)=4 k+1$.

## 2. New Upper Bound

In this section, we present our main result which is an upper bound on the perfect Italian domination number of a tree.

Theorem 2. If $T$ is a tree of order $n \geq 3$ with $\ell(T)$ leaves and $s(T)$ support vertices, then

$$
\gamma_{I}^{p}(T) \leq \frac{4 n-\ell(T)+2 s(T)-1}{5}
$$

Proof. We proceed by induction on the order $n$. If $n \in\{3,4\}$, then clearly $\gamma_{I}^{p}(T) \leq \frac{4 n-\ell(T)+2 s(T)-1}{5}$, establishing the base case. Let $n \geq 5$ and assume that
any tree $T^{\prime}$ of order $n^{\prime}$, with $3 \leq n^{\prime}<n$ satisfies $\gamma_{I}^{p}\left(T^{\prime}\right) \leq \frac{4 n-\ell\left(T^{\prime}\right)+2 s\left(T^{\prime}\right)-1}{5}$. Let $T$ be a tree of order $n$. If $\operatorname{diam}(T)=2$, then $T$ is a star, where $\gamma_{I}^{p}(T)=2<$ $\frac{4 n-\ell(T)+2 s(T)-1}{5}$. If $\operatorname{diam}(T)=3$, then $T$ is a double star, and since $n \geq 5$ we have $\gamma_{I}^{p}(T)=4 \leq \frac{4 n-\ell(T)+2 s(T)-1}{5}$. Hence, we may assume that $T$ has diameter at least 4. If $n=5$, then $T$ is a path $P_{5}$, where $\gamma_{I}^{p}\left(P_{5}\right)=3 \leq \frac{4 n-\ell(T)+2 s(T)-1}{5}$. Hence let $n \geq 6$.

Suppose $v_{1} v_{2} \cdots v_{k}(k \geq 5)$ is a diametral path in $T$ such that $\operatorname{deg}_{T}\left(v_{2}\right)$ is as large as possible. Root $T$ at $v_{k}$. First, assume that $T$ has an end support vertex $y$ of degree three. Without loss of generality, assume that $y=v_{2}$. Let $T^{\prime}=T-T_{v_{2}}$ and $f^{\prime}$ be a $\gamma_{I}^{p}\left(T^{\prime}\right)$-function. If $f^{\prime}\left(v_{3}\right)=0$, then $f^{\prime}$ can be extended to a PIDF of $T$ by assigning a 0 to $v_{2}$ and a 1 to the two leaves of $v_{2}$. If $f^{\prime}\left(v_{3}\right) \geq 1$, then $f^{\prime}$ can be extended to a PIDF of $T$ by assigning a 2 to $v_{2}$ and a 0 to the leaves of $v_{2}$. In either case, $\gamma_{I}^{p}(T) \leq \gamma_{I}^{p}\left(T^{\prime}\right)+2$, and by the induction hypothesis we obtain

$$
\begin{aligned}
\gamma_{I}^{p}(T) & \leq \gamma_{I}^{p}\left(T^{\prime}\right)+2 \leq \frac{4 n^{\prime}-\ell\left(T^{\prime}\right)+2 s\left(T^{\prime}\right)-1}{5}+2 \\
& \leq \frac{4(n-3)-\ell(T)+2+2 s(T)-1}{5}+2 \\
& \leq \frac{4 n-\ell(T)+2 s(T)-1}{5}
\end{aligned}
$$

Hence we can assume that $T$ has no end support vertex of degree three, in particular we have $\operatorname{deg}_{T}\left(v_{2}\right) \neq 3$. Next, suppose that $\operatorname{deg}_{T}\left(v_{3}\right)=2$. If $\operatorname{deg}_{T}\left(v_{2}\right)=2$, then let $T^{\prime}=T-T_{v_{3}}$ and $f^{\prime}$ be a $\gamma_{I}^{p}\left(T^{\prime}\right)$-function. Note that $n^{\prime}=n-3$, $s\left(T^{\prime}\right) \leq s(T)$ and $\ell\left(T^{\prime}\right) \geq \ell(T)-1$. Now if $f^{\prime}\left(v_{4}\right)=0$, then the function $f$ defined by $f\left(v_{2}\right)=2, f\left(v_{1}\right)=f\left(v_{3}\right)=0$ and $f(x)=f^{\prime}(x)$ for $x \in V(T) \backslash\left\{v_{1}, v_{2}, v_{3}\right\}$ is a PIDF of $T$. If $f^{\prime}\left(v_{4}\right) \geq 1$, then the function $f$ defined by $f\left(v_{1}\right)=f\left(v_{3}\right)=1$, $f\left(v_{2}\right)=0$ and $f(x)=f^{\prime}(x)$ for $x \in V(T) \backslash\left\{v_{1}, v_{2}, v_{3}\right\}$ is a PIDF of $T$. In either case, $\gamma_{I}^{p}(T) \leq \gamma_{I}^{p}\left(T^{\prime}\right)+2$, and by the induction hypothesis we obtain

$$
\begin{aligned}
\gamma_{I}^{p}(T) & \leq \gamma_{I}^{p}\left(T^{\prime}\right)+2 \leq \frac{4(n-3)-\ell(T)+1+2 s(T)-1}{5}+2 \\
& <\frac{4 n-\ell(T)+2 s(T)-1}{5}
\end{aligned}
$$

Suppose now that $\operatorname{deg}_{T}\left(v_{2}\right) \geq 4$. Let $T^{\prime}=T-T_{v_{3}}$ and $f^{\prime}$ be a $\gamma_{I}^{p}$-function of $T^{\prime}$. Note that $T^{\prime}$ has order $n^{\prime} \geq 2$. Clearly if $n^{\prime}=2$, then $\gamma_{I}^{p}(T)=4<$ $\frac{4 n-\ell(T)+2 s(T)-1}{5}$. Hence we assume that $n^{\prime} \geq 3$. If $f^{\prime}\left(v_{4}\right)=0$, then we can extend $f^{\prime}$ to a PIDF of $T$ by assigning a 2 to $v_{2}$ and a 0 to every neighbor of $v_{2}$. If $f^{\prime}\left(v_{4}\right) \geq 1$, then we can extend $f^{\prime}$ to a PIDF $f$ of $T$ by assigning a 2 to $v_{2}$, a 1 to $v_{3}$, and a 0 to all leaves of $v_{2}$. In either case, $\gamma_{I}^{p}(T) \leq \gamma_{I}^{p}\left(T^{\prime}\right)+3$ and by the induction hypothesis we obtain

$$
\begin{aligned}
\gamma_{I}^{p}(T) & \leq \gamma_{I}^{p}\left(T^{\prime}\right)+3 \leq \frac{4 n^{\prime}-\ell\left(T^{\prime}\right)+2 s\left(T^{\prime}\right)-1}{5}+3 \\
& \leq \frac{4\left(n-\left|L\left(v_{2}\right)\right|-2\right)-\left(\ell(T)-\left|L\left(v_{2}\right)\right|\right)+2 s(T)-1}{5}+3 \\
& =\frac{4 n-\ell(T)+2 s(T)-1-3 L\left(v_{2}\right)-8}{5}+3<\frac{4 n-\ell(T)+2 s(T)-1}{5} .
\end{aligned}
$$

From now on, we can assume that $\operatorname{deg}_{T}\left(v_{3}\right) \geq 3$ and $\operatorname{deg}_{T}\left(v_{2}\right) \neq 3$. Note that often in our proof a subtree $T^{\prime}$ of $T$ is considered, and so in either case, let $f^{\prime}$ be a $\gamma_{I}^{p}\left(T^{\prime}\right)$-function. Consider the following cases.

Case 1. $\operatorname{deg}_{T}\left(v_{2}\right) \geq 4$ and $T_{v_{3}} \neq D S_{3,1}$. Let us examine the following situations.

Subcase 1.1. $v_{3}$ has at least two leaves. Let $T^{\prime}$ be the tree of order $n^{\prime}$ obtained from $T$ by removing all leaves of $v_{2}$. Note that $n^{\prime}=n-\left|L\left(v_{2}\right)\right|, s\left(T^{\prime}\right)=s(T)-1$ and $\ell\left(T^{\prime}\right)=\ell(T)-\left|L\left(v_{2}\right)\right|+1$. Since $v_{3}$ has at least three leaves in $T^{\prime}$, we conclude that $f^{\prime}\left(v_{3}\right) \geq 1$. Hence the function $f$ defined by $f\left(v_{2}\right)=2, f(x)=0$ for all $x \in L\left(v_{2}\right)$ and $f(x)=f^{\prime}(x)$ for $x \in V(T) \backslash L\left[v_{2}\right]$ is a PIDF of $T$. It follows that $\gamma_{I}^{p}(T) \leq \gamma_{I}^{p}\left(T^{\prime}\right)+2$, and by the induction hypothesis we obtain

$$
\begin{aligned}
\gamma_{I}^{p}(T) & \leq \gamma_{I}^{p}\left(T^{\prime}\right)+2 \leq \frac{4\left(n-\left|L\left(v_{2}\right)\right|\right)-\ell(T)+\left|L\left(v_{2}\right)\right|-1+2 s(T)-3}{5}+2 \\
& <\frac{4 n-\ell(T)+2 s(T)-1}{5} .
\end{aligned}
$$

Subcase 1.2. $v_{3}$ has exactly one leaf, say $v^{\prime}$. If $v_{2}$ is the unique child of $v_{3}$ with depth 1 , then let $T^{\prime}$ be the tree of order $n^{\prime}$ obtained from $T$ by removing all vertices in $T_{v_{2}}$ and adding two new vertices $x_{1}, x_{2}$ attached at $v_{3}$. Since $v_{3}$ has at least three leaves, we have $f^{\prime}\left(v_{3}\right) \geq 1$, and thus the function $f$ defined by $f\left(v_{2}\right)=2, f(x)=0$ for $x \in L\left(v_{2}\right)$ and $f(x)=f^{\prime}(x)$ for $x \in V(T) \backslash L\left[v_{2}\right]$ is a PIDF of $T$. Hence $\gamma_{I}^{p}(T) \leq \gamma_{I}^{p}\left(T^{\prime}\right)+2$, and since $T_{v_{3}} \neq D S_{3,1}$, we must have $\left|L\left(v_{2}\right)\right| \geq 4$. It follows from the induction hypothesis that

$$
\begin{aligned}
\gamma_{I}^{p}(T) & \leq \gamma_{I}^{p}\left(T^{\prime}\right)+2 \leq \frac{4\left(n+1-\left|L\left(v_{2}\right)\right|\right)-\ell(T)+\left|L\left(v_{2}\right)\right|-2+2 s(T)-3}{5}+2 \\
& <\frac{4 n-\ell(T)+2 s(T)-1}{5}
\end{aligned}
$$

Suppose that $v_{3}$ has (at least) two children with depth 1 , say $a$ and $b$ such that $\operatorname{deg}_{T}(a) \geq 4$ and $\operatorname{deg}_{T}(b) \geq 4$. Let $T^{\prime}$ be the tree formed from $T$ by deleting all leaves of $a$ and $b$. Note that $n^{\prime}=n-|L(a)|-|L(b)|, s\left(T^{\prime}\right)=s(T)-2$ and $\ell\left(T^{\prime}\right)=\ell(T)-|L(a)|-|L(b)|+2$. Clearly, $f^{\prime}\left(v_{3}\right) \geq 1$ since $v_{3}$ has three leaves in $T^{\prime}$. Thus the function $f$ defined by $f(a)=f(b)=2, f(x)=0$ for all
$x \in L(a) \cup L(b)$ and $f(x)=f^{\prime}(x)$ for all $x \in V(T) \backslash(L[a] \cup L[b])$ is a PIDF of $T$, and so $\gamma_{I}^{p}(T) \leq \gamma_{I}^{p}\left(T^{\prime}\right)+4$. Using the fact $|L(a)| \geq 3$ and $|L(b)| \geq 3$ and the induction hypothesis we obtain

$$
\begin{aligned}
\gamma_{I}^{p}(T) & \leq \gamma_{I}^{p}\left(T^{\prime}\right)+4 \\
& \leq \frac{4(n-|L(a)|-|L(b)|)-\ell(T)+|L(a)|+|L(b)|-2+2 s(T)-5}{5}+4 \\
& <\frac{4 n-\ell(T)+2 s(T)-1}{5}
\end{aligned}
$$

Hence we can assume now that $v_{2}$ is the unique child of $v_{3}$ with depth one and degree at least 4 . Recall that since $\operatorname{deg}_{T}\left(v_{2}\right) \neq 3$, we may assume that every child of $v_{3}$ with depth 1 that is different from $v_{2}$ has degree two. Note that $\left|C\left(v_{3}\right)\right| \geq 3$. Assume first that $\left|C\left(v_{3}\right)\right| \geq 4$, and let $T^{\prime}$ be the tree of order $n^{\prime}$ obtained from $T-T_{v_{3}}$ by adding three new vertices $x_{1}, x_{2}, x_{3}$ attached at $v_{4}$. Note that $n^{\prime}=$ $n-\left|C\left(v_{3}\right)\right|-\left|L\left(T_{v_{3}}\right)\right|+3, \ell\left(T^{\prime}\right)=\ell(T)-L\left(T_{v_{3}}\right)+3, s\left(T^{\prime}\right) \leq s(T)-\left|C\left(v_{3}\right)\right|+1$. Now, since $v_{4}$ has three leaves in $T^{\prime}$, we must have $f^{\prime}\left(v_{4}\right) \geq 1$, and thus the function $f$ defined by $f\left(v_{2}\right)=2, f(x)=1$ for $x \in\left\{v^{\prime}, v_{3}\right\} \cup\left(L\left(T_{v_{3}}\right) \backslash L\left(v_{2}\right)\right)$, $f(x)=0$ for all $x \in\left(C\left(v_{3}\right) \backslash\left\{v_{2}, v^{\prime}\right\}\right) \cup L\left(v_{2}\right)$ and $f(x)=f^{\prime}(x)$ for otherwise, is a PIDF of $T$. Hence $\gamma_{I}^{p}(T) \leq \gamma_{I}^{p}\left(T^{\prime}\right)+\left|C\left(v_{3}\right)\right|+2$, and by the induction hypothesis it follows that

$$
\begin{aligned}
& \gamma_{I}^{p}(T) \\
& \leq \gamma_{I}^{p}\left(T^{\prime}\right)+\left|C\left(v_{3}\right)\right|+2 \\
& \leq \frac{4\left(n-\left|C\left(v_{3}\right)\right|+3-\left|L\left(T_{v_{3}}\right)\right|\right)-\ell(T)+\left|L\left(T_{v_{3}}\right)\right|-3+2 s(T)-2\left|C\left(v_{3}\right)\right|+1}{5} \\
& +\left|C\left(v_{3}\right)\right|+2 \leq \frac{4 n-\ell(T)+2 s(T)-1}{5}+\frac{-\left|C\left(v_{3}\right)\right|-3\left|L\left(T_{v_{3}}\right)\right|+21}{5}
\end{aligned}
$$

Moreover, since $\left|L\left(T_{v_{3}}\right)\right| \geq\left|C\left(v_{3}\right)\right|+2$, we have $\gamma_{I}^{p}(T) \leq \frac{4 n-\ell(T)+2 s(T)-1}{5}+$ $\frac{-4\left|C\left(v_{3}\right)\right|+15}{5}<\frac{4 n-\ell(T)+2 s(T)-1}{5}$ because of $\left|C\left(v_{3}\right)\right| \geq 4$. Next, we can assume that $\left|C\left(v_{3}\right)\right|=3$, that is $T_{v_{3}}$ is isomorphic to $H_{1}$ in Figure 1. In this case, let $T^{\prime}$ be the tree formed from $T$ by removing all vertices of $T_{v_{3}}$ except $v_{3}$. Clearly $v_{3}$ is a leaf in $T^{\prime}$. If $f^{\prime}\left(v_{3}\right)=0$, then $f\left(v_{4}\right)=2$ and so the function $f$ defined by $f\left(v_{3}\right)=f\left(v^{\prime}\right)=f\left(u_{1}\right)=1, f\left(v_{2}\right)=2, f(x)=0$ for all $x \in L\left(v_{2}\right) \cup\left\{u_{2}\right\}$ and $f(x)=f^{\prime}(x)$ for otherwise is a PIDF of $T$. If $f^{\prime}\left(v_{3}\right)=1$, then we can extend $f^{\prime}$ to be a PIDF of $T$ as above when $f^{\prime}\left(v_{3}\right)=0$, except that we do not assign a 1 to $v_{3}$. In either case, $\gamma_{I}^{p}(T) \leq \gamma_{I}^{p}\left(T^{\prime}\right)+5$. It follows from the induction hypothesis that

$$
\begin{aligned}
\gamma_{I}^{p}(T) & \leq \gamma_{I}^{p}\left(T^{\prime}\right)+5 \leq \frac{4\left(n-4-\left|L\left(v_{2}\right)\right|\right)-\ell(T)+\left|L\left(v_{2}\right)\right|+1+2 s(T)-5}{5}+5 \\
& <\frac{4 n-\ell(T)+2 s(T)-1}{5}
\end{aligned}
$$

Finally, if $f^{\prime}\left(v_{3}\right)=2$, then the function $f$ defined by $f\left(v_{2}\right)=f\left(u_{2}\right)=2, f(x)=0$ for all $x \in L\left(v_{2}\right) \cup\left\{u_{1}, v^{\prime}\right\}$ and $f(x)=f^{\prime}(x)$ for otherwise is a PIDF of $T$. Using the induction hypothesis we obtain

$$
\begin{aligned}
\gamma_{I}^{p}(T) & \leq \gamma_{I}^{p}\left(T^{\prime}\right)+4 \leq \frac{4\left(n-4-\left|L\left(v_{2}\right)\right|\right)-\ell(T)+\left|L\left(v_{2}\right)\right|+1+2 s(T)-5}{5}+4 \\
& <\frac{4 n-\ell(T)+2 s(T)-1}{5}
\end{aligned}
$$



Figure 1. The trees.
Subcase 1.3. $v_{3}$ is not a support vertex. Suppose that $v_{3}$ has at least three children of degree at least 4 , say $a, b$ and $c$. Let $T^{\prime}$ be the tree obtained from $T$ by removing all leaves of $a, b$ and $c$. Note that $n^{\prime}=n-|L(a)|-|L(b)|-|L(c)|$, $s\left(T^{\prime}\right)=s(T)-2$ and $\ell\left(T^{\prime}\right)=\ell(T)-|L(a)|-|L(b)|-|L(c)|+3$. Clearly, since $v_{3}$ has three leaves in $T^{\prime}, f^{\prime}\left(v_{3}\right) \geq 1$, and thus the function $f$ defined by $f(a)=$ $f(b)=f(c)=2, f(x)=0$ for all $x \in L(a) \cup L(b) \cup L(c)$ and $f(x)=f^{\prime}(x)$ for all $x \in V(T) \backslash(L[a] \cup L[b] \cup L[c])$ is a PIDF of $T$. By the induction hypothesis, it follows that

$$
\begin{aligned}
& \gamma_{I}^{p}(T) \leq \gamma_{I}^{p}\left(T^{\prime}\right)+6 \\
& \leq \frac{4(n-|L(a)|-|L(b)|-|L(c)|)-\ell(T)+|L(a)|+|L(b)|+|L(c)|-3+2 s(T)-5}{5}+6 \\
& <\frac{4 n-\ell(T)+2 s(T)-1}{5}
\end{aligned}
$$

Hence, $v_{3}$ has at most two children of degree at least 4 , say $v_{3}$ and $u$ (if any). Let $T^{\prime}$ be the tree of order $n^{\prime}$ obtained from $T-T_{v_{3}}$ by adding three new vertices attached at $v_{4}$. Note that $n^{\prime}=n-\left|C\left(v_{3}\right)\right|-\left|L\left(T_{v_{3}}\right)\right|+2, s\left(T^{\prime}\right) \leq s(T)-\left|C\left(v_{3}\right)\right|+1$ and $\ell\left(T^{\prime}\right)=\ell(T)-\left|L\left(T_{v_{3}}\right)\right|+3$. Clearly, $f^{\prime}\left(v_{4}\right) \geq 1$. Hence the function $f$ defined by $f(x)=2$ for $x \in\left\{v_{2}, u\right\}, f(x)=1$ for $x \in\left(L\left(T_{v_{3}}\right) \cup\left\{v_{3}\right\}\right) \backslash\left(L\left(v_{2}\right) \cup L(u)\right)$, $f(x)=0$ for $x \in\left(C\left(v_{3}\right) \backslash\left\{v_{2}, u\right\}\right) \cup\left(L\left(v_{2}\right) \cup L(u)\right)$ and $f(x)=f^{\prime}(x)$ for otherwise is a PIDF of $T$. By the induction hypothesis we obtain

$$
\begin{aligned}
\gamma_{I}^{p}(T) & \leq \gamma_{I}^{p}\left(T^{\prime}\right)+\left|C\left(v_{3}\right)\right|+3 \\
& \leq \frac{4\left(n-\left|C\left(v_{3}\right)\right|-\left|L\left(T_{v_{3}}\right)\right|+2\right)-\ell(T)+\left|L\left(T_{v_{3}}\right)\right|-3+2 s(T)-2\left|C\left(v_{3}\right)\right|+1}{5} \\
& +\left|C\left(v_{3}\right)\right|+3 \leq \frac{4 n-\ell(T)+2 s(T)-1}{5}+\frac{-\left|C\left(v_{3}\right)\right|-3\left|\ell\left(T_{v_{3}}\right)\right|+22}{5} .
\end{aligned}
$$

Since $\left|L\left(T_{v_{3}}\right)\right| \geq\left|C\left(v_{3}\right)\right|+2$, we have $\gamma_{I}^{p}(T) \leq \frac{4 n-\ell(T)+2 s(T)-1}{5}+\frac{-4\left|C\left(v_{3}\right)\right|+16}{5}$. If $\left|C\left(v_{3}\right)\right| \geq 4$, then $\gamma_{I}^{p}(T) \leq \frac{4 n-\ell(T)+2 s(T)-1}{5}$. Hence, $2 \leq\left|C\left(v_{3}\right)\right| \leq 3$. If $\left|C\left(v_{3}\right)\right|=3$ and $v_{3}$ has two children of degree at least 4 , then one can easily see that $\gamma_{I}^{p}(T) \leq \frac{4 n-\ell(T)+2 s(T)-1}{5}\left(\right.$ since $\left.\left|L\left(T_{v_{3}}\right)\right| \geq\left|C\left(v_{3}\right)\right|+4\right)$. In the sequel, we can assume that $T_{v_{3}}$ is isomorphic to one of $H_{2}, H_{3}, H_{4}$ depicted in Figure 1. In that case, let $T^{\prime \prime}$ be the tree formed from $T$ by removing all vertices of $T_{v_{3}}$ except $v_{3}$. Clearly $v_{3}$ is a leaf in $T^{\prime \prime}$. Let $f^{\prime \prime}$ be a $\gamma_{I}^{p}\left(T^{\prime \prime}\right)$-function. If $f^{\prime \prime}\left(v_{3}\right)=0$, then $f^{\prime \prime}\left(v_{4}\right)=2$ and so let $f$ be a PIDF of $T$ defined as follows: $f(x)=f^{\prime \prime}(x)$ for all $x \in V\left(T^{\prime}\right) \backslash\left\{v_{3}\right\}$ and $f\left(v_{3}\right)=1$. Moreover, every child of $v_{3}$ of degree 2 is assigned a 0 and its unique leaf a 1 ; every child of $v_{3}$ of degree at least 4 is assigned a 2 and its leaves a 0 . If $f^{\prime \prime}\left(v_{3}\right)=1$, then $f^{\prime \prime}$ will be extended to a PIDF of $T$ as above when $f^{\prime}(x)=0$, except we do not assign a 1 to $v_{3}$. Finally, if $f^{\prime \prime}\left(v_{3}\right)=2$, then we use the following assignment for vertices of $T_{v_{3}}$ : assign a 2 to each child of $v_{3}$ and a 0 to each of their leaves. Now, if $T_{v_{3}}=H_{2}$, then in either case described above, we have $\gamma_{I}^{p}(T) \leq \gamma_{I}^{p}\left(T^{\prime \prime}\right)+4$. By the induction hypothesis we obtain

$$
\begin{aligned}
\gamma_{I}^{p}(T) & \leq \gamma_{I}^{p}\left(T^{\prime \prime}\right)+4 \leq \frac{4\left(n-3-\left|L\left(v_{2}\right)\right|\right)-\ell(T)+\left|L\left(v_{2}\right)\right|+1+2 s(T)-3}{5}+4 \\
& <\frac{4 n-\ell(T)+2 s(T)-1}{5}
\end{aligned}
$$

If $T_{v_{3}}=H_{3}$, then $\gamma_{I}^{p}(T) \leq \gamma_{I}^{p}\left(T^{\prime \prime}\right)+5$, and by the induction hypothesis we obtain

$$
\begin{aligned}
\gamma_{I}^{p}(T) & \leq \gamma_{I}^{p}\left(T^{\prime \prime}\right)+5 \\
& \leq \frac{4\left(n-2-\left|L\left(v_{2}\right)\right|-|L(u)|\right)-\ell(T)+\left|L\left(v_{2}\right)\right|+|L(u)|+2 s(T)-3}{5}+5 \\
& <\frac{4 n-\ell(T)+2 s(T)-1}{5}
\end{aligned}
$$

Moreover, if $T_{v_{3}}=H_{4}$, then $\gamma_{I}^{p}(T) \leq \gamma_{I}^{p}\left(T^{\prime \prime}\right)+6$, and by the induction hypothesis it follows that

$$
\begin{aligned}
\gamma_{I}^{p}(T) & \leq \gamma_{I}^{p}\left(T^{\prime \prime}\right)+6 \leq \frac{4\left(n-5-\left|L\left(v_{2}\right)\right|\right)-\ell(T)+2+\left|L\left(v_{2}\right)\right|+2 s(T)-5}{5}+6 \\
& <\frac{4 n-\ell(T)+2 s(T)-1}{5}
\end{aligned}
$$

Before discussing Case 2, we will need the following claim.
Claim. Let $T$ be a wounded spider of order $n$ different from $D S_{2,1}$, with $s(T)$ support vertices and $\ell(T)$ leaves. Then we have the following.
(i) If $6 s(T)-2 \ell(T) \geq 11$, then $\gamma_{I}^{p}(T) \leq \frac{4 n-\ell(T)+2 s(T)-6}{5}$.
(ii) If $6 s(T)-2 \ell(T) \leq 11$, then $\gamma_{I}^{p}(T) \leq \frac{4 n-\ell(T)+2 s(T)-3}{5}$.

Proof. Let $v$ be the center vertex of $T$.
(i) If $6 s(T)-2 \ell(T) \geq 11$, then the function $f$ defined by assigning a 1 to $v$ and every leaf of $T$, and a 0 to remaining vertices of $T$, is a PIDF of $T$ and so

$$
\gamma_{I}^{p}(T) \leq \omega(f)=\ell(T)+1 \leq \frac{4 n-\ell(T)+2 s(T)-6}{5}
$$

(ii) Let $t=|L(v)|-1$. Clearly, $\ell(T)=s(T)+t$ and since $6 s(T)-2 \ell(T) \leq 11$, then $T$ is a double star and since $T$ is not a $D S_{2,1}$, we can see that we have $4 s(T)-2 t \leq 11$ and thus $t \geq 2 s(T)-\frac{11}{2}$. Now if $s(T)=2$, then $T$ is a double star and since $T$ is not a $D S_{2,1}$, we can see that $\gamma_{I}^{p}(T) \leq \frac{4 n-\ell(T)+2 s(T)-3}{5}$. Hence, let $s(T) \geq 3$. Then the function $f$ defined by assigning a 2 to the support vertices of $T$ and a 0 to remaining vertices of $T$ is a PIDF of $T$ of weight $2 s(T)$. Since, $n=s(T)+\ell(T)$ and $\ell(T)=s(T)+t$, it follows that $\frac{4 n-\ell(T)+2 s(T)-3}{5}=\frac{9 s(T)+3 t-3}{5}$. Moreover, since $t \geq 2 s(T)-\frac{11}{2}$ we obtain

$$
\frac{9 s(T)+3 t-3}{5} \geq \frac{9 s(T)+6 s(T)-\frac{33}{2}-3}{5}=3 s(T)-\frac{39}{10}
$$

Now, if $s(T) \geq 4$, then $3 s(T)-\frac{39}{10} \geq 2 s(T) \geq \gamma_{I}^{p}(T)$ and so the desired result follows. Thus we assume that $s(T)=3$. If $t \geq 2 s(T)-\frac{7}{2}$, then as above we have $\frac{9 s(T)+3 t-3}{5} \geq 3 s(T)-\frac{27}{10} \geq 2 s(T) \geq \gamma_{I}^{p}(T)$. Hence, let $t \leq 2 s(T)-\frac{7}{2}=2.5$. Note that in this case $\ell(T) \in\{3,4,5\}$. Then assigning a 1 to $v$ and the leaves of $T$ and a 0 to remaining vertices of $T$ provides a PIDF of $T$ of weight $\ell(T)+1 \leq$ $\frac{4 n-\ell(T)+2 s(T)-3}{5}$, which completes the proof of the claim.

We note from the proof of the claim that there exist PIDFs of $T$ of weight at most $\frac{4\left|V\left(T_{v_{3}}\right)\right|-\ell\left(T_{v_{3}}\right)+2 s\left(T_{v_{3}}\right)-3}{5}$ that assign to the center vertex a 1 or 2.

Now we are ready to examine the next case.
Case 2. $\operatorname{deg}_{T}\left(v_{2}\right)=2$ or $T_{v_{3}}=D S_{3,1}$. From Case 1 and since $v_{2}$ was chosen having a maximum degree, we conclude that $T_{v_{3}}$ is a spider. Assume first that $T_{v_{3}}$ is a healthy spider. If $\left|C\left(v_{3}\right)\right| \geq 3$, then let $T^{\prime}$ be the tree obtained by removing $T_{v_{3}}$ and adding three new vertices attached at $v_{4}$. Note that $n^{\prime}=n-2\left|C\left(v_{3}\right)\right|+2$, $s\left(T^{\prime}\right) \leq s(T)-\left|C\left(v_{3}\right)\right|+1$ and $\ell\left(T^{\prime}\right)=\ell(T)-\left|C\left(v_{3}\right)\right|+3$. Clearly, $f^{\prime}\left(v_{4}\right) \geq 1$ (since $v_{4}$ has three leaves in $T^{\prime}$ ). Thus the function $f$ defined by $f(x)=1$ for $x \in L\left(T_{v_{3}}\right) \cup\left\{v_{3}\right\}, f(x)=0$ for $x \in C\left(v_{3}\right)$ and $f(x)=f^{\prime}(x)$ for $x \in V(T) \backslash V\left(T_{v_{3}}\right)$ is a PIDF of $T$. Hence $\gamma_{I}^{p}(T) \leq \gamma_{I}^{p}\left(T^{\prime}\right)+\left|C\left(v_{3}\right)\right|+1$, and by the induction hypothesis we obtain

$$
\begin{aligned}
& \gamma_{I}^{p}(T) \\
& \leq \gamma_{I}^{p}\left(T^{\prime}\right)+\left|C\left(v_{3}\right)\right|+1 \\
& \leq \frac{4\left(n-2\left|C\left(v_{3}\right)\right|+2\right)-\ell(T)+\left|C\left(v_{3}\right)\right|-3+2 s(T)-2\left|C\left(v_{3}\right)\right|+1}{5}+\left|C\left(v_{3}\right)\right|+1 \\
& \leq \frac{4 n-\ell(T)+2 s(T)-1-4\left|C\left(v_{3}\right)\right|+12}{5} \leq \frac{4 n-\ell(T)+2 s(T)-1}{5} .
\end{aligned}
$$

Now, assume that $\left|C\left(v_{3}\right)\right|=2$, and let $T^{\prime}=T-T_{v_{3}}$. If $f^{\prime}\left(v_{4}\right) \geq 1$, then the function $f$ defined by $f(x)=1$ for $x \in L\left(T_{v_{3}}\right) \cup\left\{v_{3}\right\}, f(x)=0$ for every $x \in C\left(v_{3}\right)$ and $f(x)=f^{\prime}(x)$ for all $x \in V(T) \backslash V\left(T_{v_{3}}\right)$ is a PIDF of $T$ of weight $\gamma_{I}^{p}\left(T^{\prime}\right)+3$. If $f^{\prime}\left(v_{4}\right)=0$, then the function $f$ defined by $f(x)=1$ for $x \in V\left(T_{v_{3}}\right) \backslash\left\{v_{3}\right\}$, $f\left(v_{3}\right)=0$ and $f(x)=f^{\prime}(x)$ for all $x \in V(T) \backslash V\left(T_{v_{3}}\right)$ is a PIDF of $T$ of weight $\gamma_{I}^{p}\left(T^{\prime}\right)+4$. In either case, $\gamma_{I}^{p}(T) \leq \gamma_{I}^{p}\left(T^{\prime}\right)+4$ and by the induction hypothesis we obtain

$$
\begin{aligned}
\gamma_{I}^{p}(T) & \leq \gamma_{I}^{p}\left(T^{\prime}\right)+4 \leq \frac{4(n-5)-\ell(T)+2+2 s(T)-3}{5}+4 \\
& =\frac{4 n-\ell(T)+2 s(T)-1}{5}
\end{aligned}
$$

Suppose now that $T_{v_{3}}$ is a wounded spider $S_{k, t}$. If $T_{v_{3}}=D S_{2,1}$, then let $T^{\prime}=T-T_{v_{3}}$. Clearly $n^{\prime} \geq 2$. If $n^{\prime}=2$, then $\gamma_{i}^{p}\left(T^{\prime}\right)=5<\frac{4 n-\ell(T)+2 s(T)-1}{5}$. Hence we assume that $n^{\prime} \geq 3$. If $f^{\prime}\left(v_{4}\right) \geq 1$, then the function $f$ defined by $f\left(v_{2}\right)=$ $f\left(v_{3}\right)=2, f(x)=0$ for $x \in L\left(T_{v_{3}}\right)$ and $f(x)=f^{\prime}(x)$ for $x \in V(T) \backslash V\left(T_{v_{3}}\right)$ is a PIDF of $T$. If $f^{\prime}\left(v_{4}\right)=0$, then the function $f$ defined by $f\left(v_{1}\right)=2, f(x)=1$ for $x \in L\left(v_{3}\right), f\left(v_{2}\right)=f\left(v_{3}\right)=0$ and $f(x)=f^{\prime}(x)$ for $x \in V(T) \backslash V\left(T_{v_{3}}\right)$ is a PIDF of $T$. In either case, $\gamma_{I}^{p}(T) \leq \gamma_{I}^{p}\left(T^{\prime}\right)+4$. If $\operatorname{deg}_{T}\left(v_{4}\right) \geq 3$, then $s\left(T^{\prime}\right)=s(T)-2$ and $\ell\left(T^{\prime}\right)=\ell(T)-3$ and by the induction hypothesis we obtain

$$
\begin{aligned}
\gamma_{I}^{p}(T) & \leq \gamma_{I}^{p}\left(T^{\prime}\right)+4 \leq \frac{4(n-5)-\ell(T)+3+2 s(T)-5}{5}+4 \\
& <\frac{4 n-\ell(T)+2 s(T)-1}{5}
\end{aligned}
$$

If $\operatorname{deg}_{T}\left(v_{4}\right)=2$, then $s\left(T^{\prime}\right) \leq s(T)-1$ and $\ell\left(T^{\prime}\right)=\ell(T)-2$ and by the induction hypothesis we obtain

$$
\begin{aligned}
\gamma_{I}^{p}(T) & \leq \gamma_{I}^{p}\left(T^{\prime}\right)+4 \leq \frac{4(n-5)-\ell(T)+2+2 s(T)-3}{5}+4 \\
& =\frac{4 n-\ell(T)+2 s(T)-1}{5}
\end{aligned}
$$

From now on we may assume that $v_{4}$ has no child $x$ such that $T_{x}=D S_{2,1}$.
Let $s_{1}$ be the number of children of $v_{4}$ that are leaves and for $i \geq 2$, let $s_{i}$ be the number of children of $v_{4}$ of degree $i$ whose children are all leaves. As we assumed at the beginning of the proof, $T$ has no end support vertex with degree three, it follows that $s_{3}=0$. Let $s \geq 4$ be the number of children of $v_{4}$ of degree at least 4 having no grandchild. Thus

$$
s_{\geq 4}=\sum_{i \geq 4} s_{i} .
$$

Adopting our earlier notation, for each child $v$ of $v_{4}$ with depth 2 , let $n_{v}$ denote the number of children in the subtree $T_{v}$ of $T$. Furthermore, let $n^{*}$ denote the sum of the number of vertices in all such trees $T_{v}$. Also, let $s^{*}$ and $\ell^{*}$ denote the sum of the number of support vertices and leaves vertices in all such trees $T_{v}$, respectively. Note that every child of $v_{4}$ is one of the following four types: (1) a leaf; (2) a support vertex of degree 2 ; (3) a vertex with depth 2 ; (4) a support vertex of degree at least 4 whose children are all leaves. For ease of discussion, we sometimes refer to these children as Type-1, Type-2, Type-3, or Type-4, respectively. Moreover, let $m$ be the number of leaves of all Type-4 children. Consider now the following subcases.

Subcase 2.1. $s_{1}+s_{\geq 4} \geq 3$. Let $T^{\prime}=T-T_{v_{3}}$ be a tree of order $n^{\prime}$. We claim that $f^{\prime}\left(v_{4}\right) \geq 1$. Suppose to the contrary that $f^{\prime}\left(v_{4}\right)=0$. This implies that at most two children of $v_{4}$ in $T^{\prime}$ are assigned positive values under $f^{\prime}$. But since every Type- 1 and Type- 4 child of $v_{4}$ must be assigned a positive value by $f^{\prime}$ when $f^{\prime}\left(v_{4}\right)=0$, this implies that $s_{1}+s_{\geq 4} \leq 2$, a contradiction. Hence, $f^{\prime}\left(v_{4}\right) \geq 1$. Consequently, we can extend $f^{\prime}$ to a PIDF $f$ by adding to it any PIDF of $T_{v_{3}}$ of weight at most $\frac{4 n_{v_{3}}-\ell\left(T_{v_{3}}\right)+2 s\left(T_{v_{3}}\right)-3}{5}$ assigning a 1 or 2 to $v_{3}$ (as claimed above). By the induction hypothesis we obtain

$$
\begin{aligned}
\gamma_{I}^{p}(T) & \leq \gamma_{I}^{p}\left(T^{\prime}\right)+\frac{4 n_{v_{3}}-\ell\left(T_{v_{3}}\right)+2 s\left(T_{v_{3}}\right)-3}{5} \\
& \leq \frac{4\left(n-n_{v_{3}}\right)-\ell(T)+\ell\left(T_{v_{3}}\right)+2 s(T)-2 s\left(T_{v_{3}}\right)-1}{5} \\
& +\frac{4 n_{v_{3}}-\ell\left(T_{v_{3}}\right)+2 s\left(T_{v_{3}}\right)-3}{5}<\frac{4 n-\ell(T)+2 s(T)-1}{5} .
\end{aligned}
$$

In the sequel, we may assume that $s_{1}+s_{\geq 4} \leq 2$.
Subcase 2.2. $s_{1}=2$. Since $s_{1}+s_{\geq 4} \leq 2$, we deduce that $s_{\geq 4}=0$. Let $F$ be the forest formed by the Type- 3 children of $v_{4}$ and their descendants. We note any component of $F$ is a wounded spider including $T_{v_{3}}$ and different from $D S_{2,1}$. Let $T^{\prime}$ be the tree obtained from $T$ by deleting all vertices in $V(F)$ and adding a new vertex $a$ attached at $v_{4}$. Since $v_{4}$ has three leaf neighbors in $T^{\prime}$, we have $f^{\prime}\left(v_{4}\right) \geq 1$. Let $f$ be the PIDF of $T$ defined as follows: $f(x)=f^{\prime}(x)$ for all $x \in V\left(T^{\prime}\right) \backslash\{a\}$ and let the restriction of $f$ to each component, say $T_{v}$, in $F$ be any PIDF of that component of weight at most $\frac{4 n_{v}-\ell\left(T_{v}\right)+2 s\left(T_{v}\right)-3}{5}$. By our earlier observations, the total weight assigned to $F$ is at most $\frac{4 n^{*}-\ell^{*}+2 s^{*}-3}{5}$. Now, by the induction hypothesis we obtain

$$
\begin{aligned}
\gamma_{I}^{p}(T) & \leq \gamma_{I}^{p}\left(T^{\prime}\right)+\frac{4 n^{*}-\ell^{*}+2 s^{*}-3}{5} \\
& \leq \frac{4\left(n-n^{*}+1\right)-\ell(T)+\ell^{*}-1+2 s(T)-2 s^{*}-1}{5}+\frac{4 n^{*}-\ell^{*}+2 s^{*}-3}{5} \\
& \leq \frac{4 n-\ell(T)+2 s(T)-1}{5} .
\end{aligned}
$$

Hence, in the next we may assume that $s_{1} \in\{0,1\}$.
Subcase 2.3. $s_{2} \geq 3$. Let $T^{\prime}$ be the tree of order $n^{\prime}$ obtained from $T-T_{v_{4}}$ by adding three new vertices $x_{1}, x_{2}, x_{3}$ attached at $v_{5}$. Note that $n^{\prime}=n-$ $n^{*}-s_{1}-2 s_{2}-s_{\geq 4}-m+2, \ell\left(T^{\prime}\right)=\ell(T)-\ell^{*}-s_{1}-s_{2}-m+3$ and $s\left(T^{\prime}\right) \leq$ $s(T)-s^{*}-s_{1}-s_{2}-s_{\geq 4}+1$. Clearly, $f^{\prime}\left(v_{5}\right) \geq 1$ (since $v_{5}$ has three leaves in $T^{\prime}$ ). Let $f$ be the PIDF of $T$ defined by $f(x)=f^{\prime}(x)$ for all $x \in V\left(T^{\prime}\right) \backslash\left\{x_{1}, x_{2}, x_{3}\right\}$ and let $f\left(v_{4}\right)=1$. Then assign the weights to the descendants of $v_{4}$ in $T$ as follows: assign a 1 to each Type-1 (leaf) child of $v_{4}$ (recall that $s_{1} \in\{0,1\}$ ); assign a 0 to each Type- 2 child of $v_{4}$ and a 1 to its leaf neighbor; assign a 2 to each Type- 4 child of $v_{4}$ and a 0 to each of its leaves. Finally, for each Type-3 child $v$, assign a PIDF to the subtree $T_{v}$ rooted at $v$ of weight at most $\frac{4 n_{v}-\ell\left(T_{v}\right)+2 s\left(T_{v}\right)-3}{5}$ so that $f(v) \geq 1$. By our earlier observations, the total weight assigned to all Type-3 children of $v$ and their descendants is at most $\frac{4 n^{*}-\ell^{*}+2 s^{*}-3}{5}$. It follows from the induction hypothesis that

$$
\begin{aligned}
\gamma_{I}^{p}(T) & \leq \gamma_{I}^{p}\left(T^{\prime}\right)+\frac{4 n^{*}-\ell^{*}+2 s^{*}-3}{5}+s_{1}+s_{2}+2 s_{\geq 4}+1 \\
& \leq \frac{4 n^{\prime}-\ell\left(T^{\prime}\right)+2 s\left(T^{\prime}\right)-1}{5}+\frac{4 n^{*}-\ell^{*}+2 s^{*}-3}{5}+s_{1}+s_{2}+2 s_{\geq 4}+1 \\
& \leq \frac{4\left(n-n^{*}-s_{1}-2 s_{2}-m-s_{\geq 4}+2\right)-\ell(T)+\ell^{*}+s_{1}+s_{2}+m-3}{5} \\
& +\frac{2 s(T)-2 s^{*}-2 s_{1}-2 s_{2}-2 s_{\geq 4}+1}{5}+\frac{4 n^{*}-\ell^{*}+2 s^{*}-3}{5}+s_{1}+s_{2}+2 s_{\geq 4}+1 \\
& =\frac{4 n-\ell(T)+2 s(T)-1}{5}+\frac{9-3 m-4 s_{2}+4 s_{\geq 4}}{5} .
\end{aligned}
$$

Using the fact that $m \geq 3 s_{\geq 4}$, it follows that $\gamma_{I}^{p}(T) \leq \frac{4 n-\ell(T)+2 s(T)-1}{5}+\frac{9-4 s_{2}-5 s \geq 4}{5}$. Now since $s_{2} \geq 3$, we deduce that $\gamma_{I}^{p}(T) \leq \frac{4 n-\ell(T)+2 s(T)-1}{5}$.

By Subcase 2.3, we can assume that $s_{2} \leq 2$.
Subcase 2.4. $s_{2}+s_{\geq 4} \geq 1$. Let $T^{\prime}$ be the tree of order $n^{\prime}$ obtained by deleting all vertices of $T_{v_{4}}$ except $v_{4}$. Note that $n^{\prime}=n-n^{*}-s_{1}-2 s_{2}-s \geq 4-m$, $s\left(T^{\prime}\right) \leq s(T)-s^{*}-s_{1}-s_{2}-s_{\geq 4}+1$ and $\ell\left(T^{\prime}\right)=\ell(T)-\ell^{*}-s_{1}-s_{2}-m+1$ (since $v_{4}$ is a leaf vertex in $\left.T^{\prime}\right)$. First, let $f^{\prime}\left(v_{4}\right)=2$ and $f$ be a PIDF of $T$ defined by $f(x)=f^{\prime}(x)$ for all $x \in V\left(T^{\prime}\right)$; and then assign the weights to the descendants of $v_{4}$ in $T$ as follows: assign a 0 to each Type- 1 (leaf) child of $v_{4}$, assign a 2 to each Type- 2 child of $v_{4}$ and a 0 to its leaf, and assign a 2 to each Type- 4 child of $v_{4}$ and a 0 to its leaves. Finally, for each Type-3 child $v$, assign a PIDF to the subtree $T_{v}$ rooted at $v$. By our earlier observations, the total weight assigned to all Type-3 children of $v$ and their descendants is at most $\frac{4 n^{*}-\ell^{*}+2 s^{*}-3}{5}$. By the induction hypothesis it follows that

$$
\begin{aligned}
\gamma_{I}^{p}(T) & \leq \gamma_{I}^{p}\left(T^{\prime}\right)+\frac{4 n^{*}-\ell^{*}+2 s^{*}-3}{5}+2 s_{2}+2 s_{\geq 4} \\
& \leq \frac{4 n^{\prime}-\ell\left(T^{\prime}\right)+2 s\left(T^{\prime}\right)-1}{5}+\frac{4 n^{*}-\ell^{*}+2 s^{*}-3}{5}+2 s_{2}+2 s_{\geq 4} \\
& \leq \frac{4\left(n-n^{*}-s_{1}-2 s_{2}-m-s_{\geq 4}\right)-\ell(T)+\ell^{*}+s_{1}+s_{2}+m-1}{5} \\
& +\frac{2 s(T)-2 s^{*}-2 s_{1}-2 s_{2}-2 s_{\geq 4}+1}{5}+\frac{4 n^{*}-\ell^{*}+2 s^{*}-3}{5}+2 s_{2}+2 s_{\geq 4} \\
& \leq \frac{4 n-\ell(T)+2 s(T)-1}{5}+\frac{-5 s_{1}+s_{2}-3 m+4 s_{\geq 4}-2}{5}
\end{aligned}
$$

Now since $m \geq 3 s_{\geq 4}$ and $s_{2} \leq 2$, we get
$\gamma_{I}^{p}(T) \leq \frac{4 n-\ell(T)+2 s(T)-1}{5}+\frac{-5 s_{1}+s_{2}-5 s_{\geq 4}-2}{5}<\frac{4 n-\ell(T)+2 s(T)-1}{5}$.

Suppose now that $f^{\prime}\left(v_{4}\right) \in\{0,1\}$, and let $f$ be a PIDF of $T$ defined by $f(x)=$ $f^{\prime}(x)$ for all $x \in V\left(T^{\prime}\right)$ and let $f\left(v_{4}\right)=1$. Then assign the weights to the descendants of $v_{4}$ in $T$ as follows: assign a 1 to each Type- 1 (leaf) child of $v_{4}$; assign a 0 to each Type- 2 child of $v_{4}$ and a 1 to its leaf neighbor and assign a 2 to each Type- 4 child of $v_{4}$ and 0 to its leaves. Finally, for each Type- 3 child $v$, assign a PIDF of weight at most $\frac{4 n_{v}-\ell\left(T_{v}\right)+2 s\left(T_{v}\right)-3}{5}$ to vertices of $T_{v}$ rooted at $v$ so that $f(v) \geq 1$. By our earlier observations, the total weight assigned to all Type-3 children of $v$ and their descendants is at most $\frac{4 n^{*}-\ell^{*}+2 s^{*}-3}{5}$. By the induction hypothesis we obtain

$$
\begin{aligned}
\gamma_{I}^{p}(T) & \leq \gamma_{I}^{p}\left(T^{\prime}\right)+\frac{4 n^{*}-\ell^{*}+2 s^{*}-3}{5}+s_{1}+s_{2}+2 s_{\geq 4}+1 \\
& \leq \frac{4 n^{\prime}-\ell\left(T^{\prime}\right)+2 s\left(T^{\prime}\right)-1}{5}+\frac{4 n^{*}-\ell^{*}+2 s^{*}-3}{5}+s_{1}+s_{2}+2 s_{\geq 4}+1 \\
& \leq \frac{4\left(n-n^{*}-s_{1}-2 s_{2}-m-s_{\geq 4}\right)-\ell(T)+\ell^{*}+s_{1}+s_{2}+m-1}{5} \\
& +\frac{2 s(T)-2 s^{*}-2 s_{1}-2 s_{2}-2 s_{\geq 4}+1}{5}+\frac{4 n^{*}-\ell^{*}+2 s^{*}-3}{5}+s_{1}+s_{2}+2 s_{\geq 4}+1 \\
& \leq \frac{4 n-\ell(T)+2 s(T)-1}{5}+\frac{-4 s_{2}-3 m+4 s_{\geq 4}+3}{5}
\end{aligned}
$$

Now since $m \geq 3 s_{\geq 4}$, it follows that $\gamma_{I}^{p}(T) \leq \frac{4 n-\ell(T)+2 s(T)-1}{5}+\frac{-4 s_{2}-5 s_{\geq 4}+3}{5}$, and since $s_{2}+s_{\geq 4} \geq 1$, the result follows.

Subcase 2.5. $s_{2}+s_{\geq 4}=0$. Recall that $s_{1} \in\{0,1\}$. Let $v^{\prime}$ be the leaf neighbor of $v_{4}$ (if any). First, let $v_{4}$ has at least two children of Type-3. Let $T^{\prime}$ be the tree of order $n^{\prime}$ obtained by deleting all vertices of $T_{v_{4}}$ except $v_{4}$. Note that $n^{\prime}=n-n^{*}-s_{1}, s\left(T^{\prime}\right) \leq s(T)-s^{*}-s_{1}+1$ and $\ell\left(T^{\prime}\right)=\ell(T)-\ell^{*}-s_{1}+1$ (since $v_{4}$ is a leaf vertex in $T^{\prime}$ ). We also note that if $f^{\prime}\left(v_{4}\right)=0$, then since $v_{4}$ is a leaf in $T^{\prime}$, we must have $f^{\prime}\left(v_{5}\right)=2$. Now, we define a PIDF $f$ of $T$ by $f(x)=f^{\prime}(x)$ for all $x \in V\left(T^{\prime}\right) \backslash\left\{v_{4}\right\}$. Moreover, $f\left(v^{\prime}\right)=1, f\left(v_{4}\right)=1$ if $f^{\prime}\left(v_{4}\right)=0$ and $f\left(v_{4}\right)=f^{\prime}\left(v_{4}\right)$ if $f^{\prime}\left(v_{4}\right) \geq 1$. Also, for each other child $v$ of $v_{4}$, assign a PIDF to the subtree $T_{v}$ of weight at most $\frac{4 n_{v}-\ell\left(T_{v}\right)+2 s\left(T_{v}\right)-3}{5}$. Since there are at least two Type- 3 children of $v_{4}$, the total weight assigned to such subtree $T_{v}$ is $\frac{4 n^{*}-\ell^{*}+2 s^{*}-2 \cdot 3}{5}$. Hence in either case, $\gamma_{I}^{p}(T) \leq \gamma_{I}^{p}\left(T^{\prime}\right)+\frac{4 n^{*}-\ell^{*}+2 s^{*}-6}{5}+s_{1}+1$. Using the induction hypothesis we obtain

$$
\begin{aligned}
\gamma_{I}^{p}(T) & \leq \gamma_{I}^{p}\left(T^{\prime}\right)+\frac{4 n^{*}-\ell^{*}+2 s^{*}-6}{5}+s_{1}+1 \\
& \leq \frac{4 n^{\prime}-\ell\left(T^{\prime}\right)+2 s\left(T^{\prime}\right)-1}{5}+\frac{4 n^{*}-\ell^{*}+2 s^{*}-6}{5}+s_{1}+1 \\
& \leq \frac{4\left(n-n^{*}-s_{1}\right)-\ell(T)+\ell^{*}+s_{1}-1+2 s(T)-2 s^{*}-2 s_{1}+1}{5} \\
& +\frac{4 n^{*}-\ell^{*}+2 s^{*}-6}{5}+s_{1}+1 \leq \frac{4 n-\ell(T)+2 s(T)-1}{5}
\end{aligned}
$$

In the sequel, $v_{3}$ is the only child of $v_{4}$ of Type-3. We distinguish the following.
(i) $T_{v_{3}}=D S_{1,3}$. Consider two situations depending on whether $s_{1}=0$ or $s_{1}=1$.
(a) $s_{1}=0$. Hence $\operatorname{deg}_{T}\left(v_{4}\right)=2$. Let $T^{\prime}=T-T_{v_{4}}$. Clearly, $n^{\prime} \geq 1$. If $n^{\prime}=1$, then $T$ is a wounded spider and by the claim the result follows, and if $n^{\prime}=2$, then
one can easily see that $\gamma_{I}^{p}(T)=6<\frac{4 n-\ell(T)+2 s(T)-1}{5}=7.2$. So let $n^{\prime} \geq 3$. Note that $n^{\prime}=n-7, \ell\left(T^{\prime}\right) \geq \ell(T)-4$ and $s\left(T^{\prime}\right) \leq s(T)-1$. Any $\gamma_{I}^{p}\left(T^{\prime}\right)$-function can be extended to a PIDF of $T$ by assigning a 2 to $v_{2}, v_{3}$ and a 0 to remaining vertices of $T_{v_{4}}$ except $v_{4}$ which will be assigned a 0 if $f^{\prime}\left(v_{5}\right)=0$ and a 1 if $f^{\prime}\left(v_{5}\right) \geq 1$. In either case, $\gamma_{I}^{p}(T) \leq \gamma_{I}^{p}\left(T^{\prime}\right)+5$. By the induction hypothesis we obtain

$$
\begin{aligned}
\gamma_{I}^{p}(T) & \leq \frac{4 n^{\prime}-\ell\left(T^{\prime}\right)+s\left(T^{\prime}\right)-1}{5}+5 \leq \frac{4(n-7)-\ell(T)+4+2 s(T)-3}{5}+5 \\
& <\frac{4 n-\ell(T)+2 s(T)-1}{5}
\end{aligned}
$$

(b) $s_{1}=1$. Let $T^{\prime}$ be the tree obtained from $T$ by removing all vertices $T_{v_{3}}$ except $v_{3}$. If $f^{\prime}\left(v_{3}\right)=0$, then $f^{\prime}\left(v_{4}\right)=2$, and so $f^{\prime}$ can be extended to a PIDF of $T$ by assigning a 2 to $v_{2}, v_{3}$ and a 0 to remaining vertices of $T_{v_{3}}$. Hence $\gamma_{I}^{p}(T) \leq \gamma_{I}^{p}\left(T^{\prime}\right)+4$. If $f^{\prime}\left(v_{3}\right)=2$, then $f^{\prime}\left(v_{4}\right)=0$ and so the other leaf neighbor of $v_{4}$ is assigned a 1 , which is a contradiction. Hence, $f^{\prime}\left(v_{3}\right)=1$. Now, if $\left|L\left(v_{3}\right)\right|=1$, then we extend $f^{\prime}$ to a PIDF of $T$ by assigning a 2 to $v_{2}$, a 1 to $L\left(v_{3}\right)$ and a 0 to the remaining vertices of $T_{v_{3}}$. If $\left|L\left(v_{3}\right)\right|=3$, then we extend $f^{\prime}$ to a PID-function of $T$ by assigning a 1 to $L\left(T_{v_{3}}\right)$ and a 0 to $v_{2}$. In either case, $\gamma_{I}^{p}(T) \leq \gamma_{I}^{p}\left(T^{\prime}\right)+4$. By the induction hypothesis we obtain

$$
\begin{aligned}
\gamma_{I}^{p}(T) & \leq \frac{4 n^{\prime}-\ell\left(T^{\prime}\right)+s\left(T^{\prime}\right)-1}{5}+4 \leq \frac{4(n-5)-\ell(T)+3+2 s(T)-5}{5}+4 \\
& <\frac{4 n-\ell(T)+2 s(T)-1}{5}
\end{aligned}
$$

(ii) $T_{v_{3}}=S_{k, t} \neq D S_{3,1}$. We recall that $T_{v_{3}}$ is different from $D S_{2,1}$. First let $6 s\left(T_{v_{3}}\right)-2 \ell\left(T_{v_{3}}\right) \geq 11$. By our Claim, $\gamma_{I}^{p}\left(T_{v_{3}}\right) \leq \frac{4 n_{v_{3}}-\ell\left(T_{v_{3}}\right)+2 s\left(T_{v_{3}}\right)-6}{5}$. Let $T^{\prime}$ be the tree obtained from $T$ by removing all vertices of $T_{v_{4}}$ except $v_{4}$. Note that $n^{\prime} \geq 2$. Moreover, if $n^{\prime}=2$, then one can see that $\gamma_{I}^{p}(T) \leq \gamma_{I}^{p}\left(T_{v_{3}}\right)+2<$ $\frac{4 n-\ell(T)+2 s(T)-1}{5}$. Hence let $n^{\prime} \geq 3$. Note that $n^{\prime}=n-n_{v_{3}}-s_{1}, \ell\left(T^{\prime}\right)=\ell(T)-$ $\ell\left(T_{v_{3}}\right)-s_{1}+1$ and $s\left(T^{\prime}\right) \leq s(T)-s\left(T_{v_{3}}\right)-s_{1}+1$. Then any $\gamma_{I}^{p}\left(T^{\prime}\right)$-function $f^{\prime}$ can be extended to a PIDF of $T$ by adding to it a PIDF of $T_{v_{3}}$ of weight $\frac{4 n_{v_{3}}-\ell\left(T_{v_{3}}\right)+2 s\left(T_{v_{3}}\right)-6}{5}$ that assigns a 1 to $v_{3}$. Moreover, the leaf neighbor of $v_{4}$ (if any) is assigned a 1 , while $v_{4}$ will be assigned a 1 if $f^{\prime}\left(v_{4}\right)=0$ (note that in that case $f^{\prime}\left(v_{5}\right)=2$ ) or $v_{4}$ will keep the same assignment under $f^{\prime}$ if $f^{\prime}\left(v_{4}\right) \geq 1$. In either case, $\gamma_{I}^{p}(T) \leq \gamma_{I}^{p}\left(T^{\prime}\right)+\gamma_{I}^{p}\left(T_{v_{3}}\right)+s_{1}+1$. Using the induction, we obtain

$$
\begin{aligned}
\gamma_{I}^{p}(T) & \leq \frac{4 n^{\prime}-\ell\left(T^{\prime}\right)+s\left(T^{\prime}\right)-1}{5}+\frac{4 n_{v_{3}}-\ell\left(T_{v_{3}}\right)+2 s\left(T_{v_{3}}\right)-6}{5}+s_{1}+1 \\
& \leq \frac{4\left(n-n_{v_{3}}-s_{1}\right)-\ell(T)+\ell\left(T_{v_{3}}\right)+s_{1}-1+2 s(T)-2 s\left(T_{v_{3}}\right)-2 s_{1}+1}{5} \\
& +\frac{4 n_{v_{3}}-\ell\left(T_{v_{3}}\right)+2 s\left(T_{v_{3}}\right)-6}{5}+s_{1}+1=\frac{4 n-\ell(T)+2 s(T)-1}{5} .
\end{aligned}
$$

Therefore, we can now assume that $6 s\left(T_{v_{3}}\right)-2 \ell\left(T_{v_{3}}\right) \leq 11$. Recall that (by the proof of the Claim) there exists PIDF, say $g$, of $T_{v_{3}}$ of weight at most $\frac{4 n_{v_{3}}-\ell\left(T_{v_{3}}\right)+2 s\left(T_{v_{3}}\right)-3}{5}$ assigning a 2 to $v_{3}$. We now consider two situations depending on whether $s_{1}=0$ or $s_{1}=1$.
(a) $s_{1}=0$. Then $\operatorname{deg}_{T}\left(v_{4}\right)=2$. Let $T^{\prime}=T-T_{v_{4}}$. If $n^{\prime}=1$, then $T$ is a wounded spider and by the claim the result follows, and if $n^{\prime}=2$, then one can easily see that $g$ can be extended to a PIDF of $T$ by assigning a 2 to $v_{6}$ and a 0 to both $v_{4}$ and $v_{5}$, and thus $\gamma_{I}^{p}(T) \leq \frac{4 n_{v_{3}}-\ell\left(T_{v_{3}}\right)+2 s\left(T_{v_{3}}\right)-3}{5}+2 \leq \frac{4 n-\ell(T)+2 s(T)-1}{5}$. So let $n^{\prime} \geq 3$. In this case, any $\gamma_{I}^{p}\left(T^{\prime}\right)$-function can be extended to a PIDF of $T$ by adding to it the PIDF $g$ of $T_{v_{3}}$. Moreover, $v_{4}$ will be assigned a 0 if $f^{\prime}\left(v_{5}\right)=0$ and a 1 if $f^{\prime}\left(v_{5}\right) \geq 1$. In either case, $\gamma_{I}^{p}(T) \leq \gamma_{I}^{p}\left(T^{\prime}\right)+\frac{4 n_{v_{3}}-\ell\left(T_{v_{3}}\right)+2 s\left(T_{v_{3}}\right)-3}{5}+1$. Using the fact that $n^{\prime}=n-n_{v_{3}}-1, \ell\left(T^{\prime}\right) \geq \ell(T)-\ell\left(T_{v_{3}}\right), s\left(T^{\prime}\right) \leq s(T)-s\left(T_{v_{3}}\right)+1$, it follows from the induction hypothesis that

$$
\begin{aligned}
\gamma_{I}^{p}(T) & \leq \frac{4 n^{\prime}-\ell\left(T^{\prime}\right)+s\left(T^{\prime}\right)-1}{5}+\frac{4 n_{v_{3}}-\ell\left(T_{v_{3}}\right)+2 s\left(T_{v_{3}}\right)-3}{5}+1 \\
& \leq \frac{4\left(n-n_{v_{3}}-1\right)-\ell(T)+\ell\left(T_{v_{3}}\right)+2 s(T)-2 s\left(T_{v_{3}}\right)+1}{5} \\
& +\frac{4 n_{v_{3}}-\ell\left(T_{v_{3}}\right)+2 s\left(T_{v_{3}}\right)-3}{5}+1=\frac{4 n-\ell(T)+2 s(T)-1}{5}
\end{aligned}
$$

(b) $s_{1}=1$. Assume first that $v_{3}$ has at least four leaves, and let $T^{\prime}=$ $T \backslash\left\{w, v_{1}, v_{2}\right\}$, where $w \in L\left(v_{3}\right)$. Since $v_{3}$ has at least three leaves we have $f^{\prime}\left(v_{3}\right) \geq 1$. If $f^{\prime}\left(v_{3}\right)=2$, then $f^{\prime}$ is extended to a PIDF of $T$ by assigning a 2 to $v_{2}$ and a 0 to $w, v_{1}$. If $f^{\prime}\left(v_{3}\right)=1$, then $f^{\prime}$ to a PIDF of $T$ by assigning a 1 to $v_{1}, w$ and 0 to $v_{2}$. In either case, $\gamma_{I}^{p}(T) \leq \gamma_{I}^{p}\left(T^{\prime}\right)+2$. By the induction hypothesis we get

$$
\begin{aligned}
\gamma_{I}^{p}(T) & \leq \frac{4 n^{\prime}-\ell\left(T^{\prime}\right)+s\left(T^{\prime}\right)-1}{5}+2 \leq \frac{4(n-3)-\ell(T)+2+2 s(T)-3}{5}+2 \\
& <\frac{4 n-\ell(T)+2 s(T)-1}{5}
\end{aligned}
$$

Hence, we can assume that $v_{3}$ has at most three leaves and thus $\ell\left(T_{v_{3}}\right) \leq s\left(T_{v_{3}}\right)+$ 2. Let $T^{\prime}$ be the tree obtained from $T$ by removing all vertices of $T_{v_{3}}$ except $v_{3}$. Then $n^{\prime}=n-n_{v_{3}}+1, \ell\left(T^{\prime}\right)=\ell(T)-\ell\left(T_{v_{3}}\right)+1$ and $s\left(T^{\prime}\right)=s(T)-s\left(T_{v_{3}}\right)$. If $f^{\prime}\left(v_{3}\right)=0$, then $f^{\prime}\left(v_{4}\right)=2$, and $f^{\prime}$ can be extended to a PIDF of $T$ by adding to it the PIDF $g$ of $T_{v_{3}}$, where $v_{3}$ is reassigned $g\left(v_{3}\right)$ instead of $f^{\prime}\left(v_{3}\right)$. Applying our induction hypothesis, we obtain

$$
\begin{aligned}
\gamma_{I}^{p}(T) & \leq \frac{4 n^{\prime}-\ell\left(T^{\prime}\right)+s\left(T^{\prime}\right)-1}{5}+\frac{4 n_{v_{3}}-\ell\left(T_{v_{3}}\right)+2 s\left(T_{v_{3}}\right)-3}{5} \\
& \leq \frac{4\left(n-n_{v_{3}}+1\right)-\ell(T)+\ell\left(T_{v_{3}}\right)-1+2 s(T)-2 s\left(T_{v_{3}}\right)-1}{5} \\
& +\frac{4 n_{v_{3}}-\ell\left(T_{v_{3}}\right)+2 s\left(T_{v_{3}}\right)-3}{5}=\frac{4 n-\ell(T)+2 s(T)-1}{5} .
\end{aligned}
$$

If $f^{\prime}\left(v_{3}\right)=2$, then $f^{\prime}\left(v_{4}\right)=0$ and the other leaf neighbor of $v_{4}$ in $T^{\prime}$ is assigned a 1 , which provides a contradiction. Hence let $f^{\prime}\left(v_{3}\right)=1$. Then we extend $f^{\prime}$ to a PIDF of $T$ by assigning a 1 to all leaves vertices of $T_{v_{3}}$ and a 0 to remaining vertices of $T_{v_{3}}$ but $v_{3}$. Using the fact that $\ell\left(T_{v_{3}}\right) \leq s\left(T_{v_{3}}\right)+2, n_{v_{3}}=\ell\left(T_{v_{3}}\right)+s\left(T_{v_{3}}\right)$ and the induction hypothesis, we obtain

$$
\begin{aligned}
\gamma_{I}^{p}(T) & \leq \frac{4 n^{\prime}-\ell\left(T^{\prime}\right)+s\left(T^{\prime}\right)-1}{5}+\ell\left(T_{v_{3}}\right) \\
& \leq \frac{4\left(n-n_{v_{3}}+1\right)-\ell(T)+\ell\left(T_{v_{3}}\right)-1+2 s(T)-2 s\left(T_{v_{3}}\right)-1}{5}+\ell\left(T_{v_{3}}\right) \\
& \leq \frac{4 n-\ell(T)+2 s(T)-1}{5}
\end{aligned}
$$

This completes the proof.

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