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A NEW UPPER BOUND FOR THE PERFECT ITALIAN DOMINATION NUMBER OF A TREE

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Abstract

A perfect Italian dominating function (PIDF) on a graph G is a function $f : V(G) \to \{0,1,2\}$ satisfying the condition that for every vertex u with $f(u) = 0$, the total weight of f assigned to the neighbors of u is exactly two. The weight of a PIDF is the sum of its functions values over all vertices. The perfect Italian domination number of G, denoted $\gamma_I^p(G)$, is the minimum weight of a PIDF of G. In this paper, we show that for every tree T of order $n \geq 3$, with $\ell(T)$ leaves and $s(T)$ support vertices, $\gamma_I^p(T) \leq \frac{4n-\ell(T)+2s(T)-1}{5}$, improving a previous bound given by T.W. Haynes and M.A. Henning in [*Perfect Italian domination in trees*, Discrete Appl. Math. 260 (2019) 164–177].

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1. Introduction

Throughout this paper, G is a simple graph with vertex set $V(G)$ and edge set $E(G)$ (briefly V, E). The order $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V(G)$, the *open neighborhood* of v is the set $N_G(v) = N(v) = \{u \in V(G) \mid uv \in E(G)\}$ and its *closed neighborhood* is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = |N(v)|$. A *leaf* of G is a vertex of degree one and a *support vertex* is a vertex adjacent to a leaf. An *end support vertex* is a support vertex having at most one non-leaf neighbor. For every vertex $v \in V$, the set of all leaves adjacent to v is denoted by $L(v)$ and $L[v] = L(v) \cup \{v\}$. We denote the set of leaves of a graph G by $L(G)$ and the set of support vertices by $S(G)$. We also let $|S(G)| = s(G)$ and $|L(G)| = \ell(T)$. A *double star* $DS_{q,p}$, with $q \geq p \geq 1$, is a graph consisting of the union of two stars $K_{1,q}$ and $K_{1,p}$ together with an edge joining their centers. The *subdivision graph* $S_b(G)$ of a graph G is that graph obtained from G by replacing each edge uv of G by a vertex w and edges uw and vw. A *healthy spider* $S_k(G)$ is the subdivision graph of a star $K_{1,k}$ for $k \geq 2$. A *wounded spider* $S_{k,t}$ is a graph obtained from a star $K_{1,k}$ by subdividing t edges exactly once, where $1 \leq t \leq k-1$. We denote by P_n the path on n vertices. The *distance* $d_G(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest $u-v$ path in G. The *diameter* of a graph G, denoted by $\operatorname{diam}(G)$, is the greatest distance between two vertices of G. For a vertex v in a rooted tree T, let $C(v)$ denote the set of children of v, $D(v)$ denotes the set of descendants of v and $D[v] = D(v) \cup \{v\}$. Also, the *depth* of v, $\operatorname{depth}(v)$, is the largest distance from v to a vertex in $D(v)$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v.

For a real-valued function $f : V \longrightarrow \mathbb{R}$, the weight of f is $\omega(f) = \sum_{v \in V} f(v)$, and for $S \subseteq V$ we define $f(S) = \sum_{v \in S} f(v)$. So $w(f) = f(V)$.

A *Roman dominating function* on G, abbreviated RDF, is a function $f : V \to \{0,1,2\}$ such that every vertex $u \in V$ for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. Roman domination was introduced by Cockayne *et al.* in [7] and was inspired by the work of ReVelle and Rosing [12] and Stewart [13]. Several new varieties of Roman domination have been introduced since 2004, among them, we quote the Italian domination originally published in [1] and called Roman $\{2\}$-domination. Further results on Roman domination and its variant can be found in [2–6].

An *Italian dominating function* on G, abbreviated IDF, is a function $f : V \to \{0,1,2\}$ satisfying the condition that for every vertex $v \in V$ with $f(v) = 0$, $\sum_{u \in N(v)} f(u) \geq 2$, that is either v is adjacent to a vertex u with $f(u) = 2$, or to at least two vertices x and y with $f(x) = f(y) = 1$. The Italian domination number, denoted $\gamma_I(G)$, is the minimum weight of an IDF in G.

The concept of perfect dominating sets introduced by Livingston and Stout in [11] has been extended to Roman and Italian dominating functions in [10] and [9], respectively. An RDF f is called *perfect* if for every vertex v with $f(v) = 0$, there is exactly one vertex $u \in N(v)$ with $f(u) = 2$, while a IDF g is perfect if for every vertex w with $g(w) = 0$, $g(N(v)) = 2$. The *perfect Roman domination number* (respectively, *perfect Italian domination number*) of G, denoted $\gamma_R^p(G)$ (respectively, $\gamma_I^p(G)$), is the minimum weight of a perfect RDF (respectively, perfect IDF) in G. A perfect IDF on G will be abbreviated PIDF. A PIDF f is called a $\gamma_I^p(G)$-function if $\omega(f) = \gamma_I^p(G)$.

It was shown in [10] that every tree T of order $n \geq 3$ satisfies $\gamma_R^p(T) \leq \frac{4}{5}n$. However, this upper bound has recently been improved by Darkooti *et al.* [8] for trees T with $\ell(T) \geq 2s(T) - 2$, by showing that for any tree T of order $n \geq 3$ with $\ell(T)$ leaves and $s(T)$ support vertices, $\gamma_R^p(T) \leq (4n- \ell(T) + 2s(T) - 2)/5$. Moreover, Henning and Haynes showed in [9] that $\frac{4}{5}n$ is also an upper bound of the prefect Italian domination number for any tree of order $n \geq 3$.

In this paper, we shall show that for any tree T of order $n \geq 3$ with $\ell(T)$ leaves and $s(T)$ support vertices, $\gamma_R^p(T) \leq (4n- \ell(T) + 2s(T) - 1)/5$. But first let us point out that for both parameters $\gamma_R^p(G)$ and $\gamma_I^p(G)$, one may be larger or smaller than the other even for trees. Indeed, for the path P_5 we have $\gamma_R^p(P_5) = 4$ and $\gamma_I^p(P_5) = 3$ while for the double star $DS_{3,1}$ we have $\gamma_R^p(DS_{3,1}) = 3$ and $\gamma_I^p(DS_{3,1}) = 4$. The next result shows that the differences $\gamma_I^p(G) - \gamma_R^p(G)$ and $\gamma_R^p(G) - \gamma_I^p(G)$ can be arbitrarily large.

Observation 1. *For any integer $k \geq 1$, there exist trees T_k and H_k such that $\gamma_I^p(T_k) - \gamma_R^p(T_k) = k$ and $\gamma_R^p(H_k) - \gamma_I^p(H_k) = k$.*

Proof. Let T_k be the tree formed by k double stars $DS_{3,1}$ by adding a new vertex attached to every support vertex of degree four. One can easily see that $\gamma_I^p(T_k) = 4k + 1$ while $\gamma_R^p(T_k) = 3k + 1$.

Now, let H_k be the tree formed by k paths P_5 by adding a new vertex attached to all center vertices of the paths. Then $\gamma_I^p(H_k) = 3k+1$ while $\gamma_R^p(H_k) = 4k+1$. ■

2. New Upper Bound

In this section, we present our main result which is an upper bound on the perfect Italian domination number of a tree.

Theorem 2. *If T is a tree of order $n \geq 3$ with $\ell(T)$ leaves and $s(T)$ support vertices, then*

$$\gamma_I^p(T) \leq \frac{4n - \ell(T) + 2s(T) - 1}{5}.$$

Proof. We proceed by induction on the order n. If $n \in \{3, 4\}$, then clearly $\gamma_I^p(T) \leq \frac{4n-\ell(T)+2s(T)-1}{5}$, establishing the base case. Let $n \geq 5$ and assume that any tree T' of order n', with $3 \leq n' < n$ satisfies $\gamma_I^p(T') \leq \frac{4n-\ell(T')+2s(T')-1}{5}$. Let T be a tree of order n. If $\mathrm{diam}(T) = 2$, then T is a star, where $\gamma_I^p(T) = 2 < \frac{4n-\ell(T)+2s(T)-1}{5}$. If $\mathrm{diam}(T) = 3$, then T is a double star, and since $n \geq 5$ we have $\gamma_I^p(T) = 4 \leq \frac{4n-\ell(T)+2s(T)-1}{5}$. Hence, we may assume that T has diameter at least 4. If $n = 5$, then T is a path P_5, where $\gamma_I^p(P_5) = 3 \leq \frac{4n-\ell(T)+2s(T)-1}{5}$. Hence let $n \geq 6$.

Suppose $v_1 v_2 \cdots v_k$ $(k \geq 5)$ is a diametral path in T such that $\deg_T(v_2)$ is as large as possible. Root T at v_k. First, assume that T has an end support vertex y of degree three. Without loss of generality, assume that $y = v_2$. Let $T' = T - T_{v_2}$ and f' be a $\gamma_I^p(T')$-function. If $f'(v_3) = 0$, then f' can be extended to a PIDF of T by assigning a 0 to v_2 and a 1 to the two leaves of v_2. If $f'(v_3) \geq 1$, then f' can be extended to a PIDF of T by assigning a 2 to v_2 and a 0 to the leaves of v_2. In either case, $\gamma_I^p(T) \leq \gamma_I^p(T') + 2$, and by the induction hypothesis we obtain

$$\begin{aligned}
\gamma_I^p(T) &\leq \gamma_I^p(T') + 2 \leq \frac{4n' - \ell(T') + 2s(T') - 1}{5} + 2 \\
&\leq \frac{4(n-3) - \ell(T) + 2 + 2s(T) - 1}{5} + 2 \\
&\leq \frac{4n - \ell(T) + 2s(T) - 1}{5}.
\end{aligned}$$

Hence we can assume that T has no end support vertex of degree three, in particular we have $\deg_T(v_2) \neq 3$. Next, suppose that $\deg_T(v_3) = 2$. If $\deg_T(v_2) = 2$, then let $T' = T - T_{v_3}$ and f' be a $\gamma_I^p(T')$-function. Note that $n' = n - 3$, $s(T') \leq s(T)$ and $\ell(T') \geq \ell(T) - 1$. Now if $f'(v_4) = 0$, then the function f defined by $f(v_2) = 2$, $f(v_1) = f(v_3) = 0$ and $f(x) = f'(x)$ for $x \in V(T) \setminus \{v_1, v_2, v_3\}$ is a PIDF of T. If $f'(v_4) \geq 1$, then the function f defined by $f(v_1) = f(v_3) = 1$, $f(v_2) = 0$ and $f(x) = f'(x)$ for $x \in V(T) \setminus \{v_1, v_2, v_3\}$ is a PIDF of T. In either case, $\gamma_I^p(T) \leq \gamma_I^p(T') + 2$, and by the induction hypothesis we obtain

$$\begin{aligned}
\gamma_I^p(T) &\leq \gamma_I^p(T') + 2 \leq \frac{4(n-3) - \ell(T) + 1 + 2s(T) - 1}{5} + 2 \\
&< \frac{4n - \ell(T) + 2s(T) - 1}{5}.
\end{aligned}$$

Suppose now that $\deg_T(v_2) \geq 4$. Let $T' = T - T_{v_3}$ and f' be a γ_I^p-function of T'. Note that T' has order $n' \geq 2$. Clearly if $n' = 2$, then $\gamma_I^p(T) = 4 < \frac{4n-\ell(T)+2s(T)-1}{5}$. Hence we assume that $n' \geq 3$. If $f'(v_4) = 0$, then we can extend f' to a PIDF of T by assigning a 2 to v_2 and a 0 to every neighbor of v_2. If $f'(v_4) \geq 1$, then we can extend f' to a PIDF f of T by assigning a 2 to v_2, a 1 to v_3, and a 0 to all leaves of v_2. In either case, $\gamma_I^p(T) \leq \gamma_I^p(T') + 3$ and by the induction hypothesis we obtain

$$\begin{aligned}\gamma_I^p(T) &\le \gamma_I^p(T') + 3 \le \frac{4n' - \ell(T') + 2s(T') - 1}{5} + 3 \\ &\le \frac{4(n - |L(v_2)| - 2) - (\ell(T) - |L(v_2)|) + 2s(T) - 1}{5} + 3 \\ &= \frac{4n - \ell(T) + 2s(T) - 1 - 3L(v_2) - 8}{5} + 3 < \frac{4n - \ell(T) + 2s(T) - 1}{5}.\end{aligned}$$

From now on, we can assume that $\deg_T(v_3) \ge 3$ and $\deg_T(v_2) \ne 3$. Note that often in our proof a subtree T' of T is considered, and so in either case, let f' be a $\gamma_I^p(T')$-function. Consider the following cases.

Case 1. $\deg_T(v_2) \ge 4$ and $T_{v_3} \ne DS_{3,1}$. Let us examine the following situations.

Subcase 1.1. v_3 has at least two leaves. Let T' be the tree of order n' obtained from T by removing all leaves of v_2. Note that $n' = n - |L(v_2)|$, $s(T') = s(T) - 1$ and $\ell(T') = \ell(T) - |L(v_2)| + 1$. Since v_3 has at least three leaves in T', we conclude that $f'(v_3) \ge 1$. Hence the function f defined by $f(v_2) = 2$, $f(x) = 0$ for all $x \in L(v_2)$ and $f(x) = f'(x)$ for $x \in V(T) \setminus L[v_2]$ is a PIDF of T. It follows that $\gamma_I^p(T) \le \gamma_I^p(T') + 2$, and by the induction hypothesis we obtain

$$\begin{aligned}\gamma_I^p(T) &\le \gamma_I^p(T') + 2 \le \frac{4(n - |L(v_2)|) - \ell(T) + |L(v_2)| - 1 + 2s(T) - 3}{5} + 2 \\ &< \frac{4n - \ell(T) + 2s(T) - 1}{5}.\end{aligned}$$

Subcase 1.2. v_3 has exactly one leaf, say v'. If v_2 is the unique child of v_3 with depth 1, then let T' be the tree of order n' obtained from T by removing all vertices in T_{v_2} and adding two new vertices x_1, x_2 attached at v_3. Since v_3 has at least three leaves, we have $f'(v_3) \ge 1$, and thus the function f defined by $f(v_2) = 2$, $f(x) = 0$ for $x \in L(v_2)$ and $f(x) = f'(x)$ for $x \in V(T) \setminus L[v_2]$ is a PIDF of T. Hence $\gamma_I^p(T) \le \gamma_I^p(T') + 2$, and since $T_{v_3} \ne DS_{3,1}$, we must have $|L(v_2)| \ge 4$. It follows from the induction hypothesis that

$$\begin{aligned}\gamma_I^p(T) &\le \gamma_I^p(T') + 2 \le \frac{4(n + 1 - |L(v_2)|) - \ell(T) + |L(v_2)| - 2 + 2s(T) - 3}{5} + 2 \\ &< \frac{4n - \ell(T) + 2s(T) - 1}{5}.\end{aligned}$$

Suppose that v_3 has (at least) two children with depth 1, say a and b such that $\deg_T(a) \ge 4$ and $\deg_T(b) \ge 4$. Let T' be the tree formed from T by deleting all leaves of a and b. Note that $n' = n - |L(a)| - |L(b)|$, $s(T') = s(T) - 2$ and $\ell(T') = \ell(T) - |L(a)| - |L(b)| + 2$. Clearly, $f'(v_3) \ge 1$ since v_3 has three leaves in T'. Thus the function f defined by $f(a) = f(b) = 2$, $f(x) = 0$ for all

$x \in L(a) \cup L(b)$ and $f(x) = f'(x)$ for all $x \in V(T) \setminus (L[a] \cup L[b])$ is a PIDF of T, and so $\gamma_I^p(T) \le \gamma_I^p(T') + 4$. Using the fact $|L(a)| \ge 3$ and $|L(b)| \ge 3$ and the induction hypothesis we obtain

$$\begin{aligned}\gamma_I^p(T) &\le \gamma_I^p(T') + 4\\ &\le \frac{4(n - |L(a)| - |L(b)|) - \ell(T) + |L(a)| + |L(b)| - 2 + 2s(T) - 5}{5} + 4\\ &< \frac{4n - \ell(T) + 2s(T) - 1}{5}.\end{aligned}$$

Hence we can assume now that v_2 is the unique child of v_3 with depth one and degree at least 4. Recall that since $\deg_T(v_2) \ne 3$, we may assume that every child of v_3 with depth 1 that is different from v_2 has degree two. Note that $|C(v_3)| \ge 3$. Assume first that $|C(v_3)| \ge 4$, and let T' be the tree of order n' obtained from $T - T_{v_3}$ by adding three new vertices x_1, x_2, x_3 attached at v_4. Note that $n' = n - |C(v_3)| - |L(T_{v_3})| + 3$, $\ell(T') = \ell(T) - L(T_{v_3}) + 3$, $s(T') \le s(T) - |C(v_3)| + 1$. Now, since v_4 has three leaves in T', we must have $f'(v_4) \ge 1$, and thus the function f defined by $f(v_2) = 2$, $f(x) = 1$ for $x \in \{v', v_3\} \cup (L(T_{v_3}) \setminus L(v_2))$, $f(x) = 0$ for all $x \in (C(v_3) \setminus \{v_2, v'\}) \cup L(v_2)$ and $f(x) = f'(x)$ for otherwise, is a PIDF of T. Hence $\gamma_I^p(T) \le \gamma_I^p(T') + |C(v_3)| + 2$, and by the induction hypothesis it follows that

$$\begin{aligned}&\gamma_I^p(T)\\ &\le \gamma_I^p(T') + |C(v_3)| + 2\\ &\le \frac{4(n - |C(v_3)| + 3 - |L(T_{v_3})|) - \ell(T) + |L(T_{v_3})| - 3 + 2s(T) - 2|C(v_3)| + 1}{5}\\ &+ |C(v_3)| + 2 \le \frac{4n - \ell(T) + 2s(T) - 1}{5} + \frac{-|C(v_3)| - 3|L(T_{v_3})| + 21}{5}.\end{aligned}$$

Moreover, since $|L(T_{v_3})| \ge |C(v_3)| + 2$, we have $\gamma_I^p(T) \le \frac{4n-\ell(T)+2s(T)-1}{5} + \frac{-4|C(v_3)|+15}{5} < \frac{4n-\ell(T)+2s(T)-1}{5}$ because of $|C(v_3)| \ge 4$. Next, we can assume that $|C(v_3)| = 3$, that is T_{v_3} is isomorphic to H_1 in Figure 1. In this case, let T' be the tree formed from T by removing all vertices of T_{v_3} except v_3. Clearly v_3 is a leaf in T'. If $f'(v_3) = 0$, then $f(v_4) = 2$ and so the function f defined by $f(v_3) = f(v') = f(u_1) = 1$, $f(v_2) = 2$, $f(x) = 0$ for all $x \in L(v_2) \cup \{u_2\}$ and $f(x) = f'(x)$ for otherwise is a PIDF of T. If $f'(v_3) = 1$, then we can extend f' to be a PIDF of T as above when $f'(v_3) = 0$, except that we do not assign a 1 to v_3. In either case, $\gamma_I^p(T) \le \gamma_I^p(T') + 5$. It follows from the induction hypothesis that

$$\begin{aligned}\gamma_I^p(T) &\le \gamma_I^p(T') + 5 \le \frac{4(n - 4 - |L(v_2)|) - \ell(T) + |L(v_2)| + 1 + 2s(T) - 5}{5} + 5\\ &< \frac{4n - \ell(T) + 2s(T) - 1}{5}.\end{aligned}$$

Finally, if $f'(v_3) = 2$, then the function f defined by $f(v_2) = f(u_2) = 2$, $f(x) = 0$ for all $x \in L(v_2) \cup \{u_1, v'\}$ and $f(x) = f'(x)$ for otherwise is a PIDF of T. Using the induction hypothesis we obtain

$$\begin{aligned}\gamma_I^p(T) &\le \gamma_I^p(T') + 4 \le \frac{4(n - 4 - |L(v_2)|) - \ell(T) + |L(v_2)| + 1 + 2s(T) - 5}{5} + 4 \\ &< \frac{4n - \ell(T) + 2s(T) - 1}{5}.\end{aligned}$$

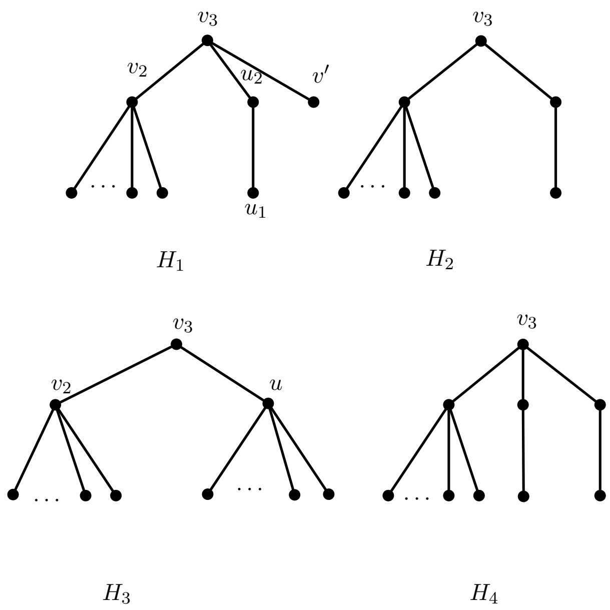

Figure 1. The trees.

Subcase 1.3. v_3 is not a support vertex. Suppose that v_3 has at least three children of degree at least 4, say a, b and c. Let T' be the tree obtained from T by removing all leaves of a, b and c. Note that $n' = n - |L(a)| - |L(b)| - |L(c)|$, $s(T') = s(T) - 2$ and $\ell(T') = \ell(T) - |L(a)| - |L(b)| - |L(c)| + 3$. Clearly, since v_3 has three leaves in T', $f'(v_3) \ge 1$, and thus the function f defined by $f(a) = f(b) = f(c) = 2$, $f(x) = 0$ for all $x \in L(a) \cup L(b) \cup L(c)$ and $f(x) = f'(x)$ for all $x \in V(T) \setminus (L[a] \cup L[b] \cup L[c])$ is a PIDF of T. By the induction hypothesis, it follows that

$$\begin{aligned}&\gamma_I^p(T) \le \gamma_I^p(T') + 6 \\ &\le \frac{4(n - |L(a)| - |L(b)| - |L(c)|) - \ell(T) + |L(a)| + |L(b)| + |L(c)| - 3 + 2s(T) - 5}{5} + 6 \\ &< \frac{4n - \ell(T) + 2s(T) - 1}{5}.\end{aligned}$$

Hence, v_3 has at most two children of degree at least 4, say v_3 and u (if any). Let T' be the tree of order n' obtained from $T - T_{v_3}$ by adding three new vertices attached at v_4. Note that $n' = n - |C(v_3)| - |L(T_{v_3})| + 2$, $s(T') \leq s(T) - |C(v_3)| + 1$ and $\ell(T') = \ell(T) - |L(T_{v_3})| + 3$. Clearly, $f'(v_4) \geq 1$. Hence the function f defined by $f(x) = 2$ for $x \in \{v_2, u\}$, $f(x) = 1$ for $x \in (L(T_{v_3}) \cup \{v_3\}) \setminus (L(v_2) \cup L(u))$, $f(x) = 0$ for $x \in (C(v_3) \setminus \{v_2, u\}) \cup (L(v_2) \cup L(u))$ and $f(x) = f'(x)$ for otherwise is a PIDF of T. By the induction hypothesis we obtain

$$\begin{aligned}\gamma_I^p(T) &\leq \gamma_I^p(T') + |C(v_3)| + 3\\ &\leq \frac{4(n - |C(v_3)| - |L(T_{v_3})| + 2) - \ell(T) + |L(T_{v_3})| - 3 + 2s(T) - 2|C(v_3)| + 1}{5}\\ &+ |C(v_3)| + 3 \leq \frac{4n - \ell(T) + 2s(T) - 1}{5} + \frac{-|C(v_3)| - 3|\ell(T_{v_3})| + 22}{5}.\end{aligned}$$

Since $|L(T_{v_3})| \geq |C(v_3)| + 2$, we have $\gamma_I^p(T) \leq \frac{4n-\ell(T)+2s(T)-1}{5} + \frac{-4|C(v_3)|+16}{5}$. If $|C(v_3)| \geq 4$, then $\gamma_I^p(T) \leq \frac{4n-\ell(T)+2s(T)-1}{5}$. Hence, $2 \leq |C(v_3)| \leq 3$. If $|C(v_3)| = 3$ and v_3 has two children of degree at least 4, then one can easily see that $\gamma_I^p(T) \leq \frac{4n-\ell(T)+2s(T)-1}{5}$ (since $|L(T_{v_3})| \geq |C(v_3)| + 4$). In the sequel, we can assume that T_{v_3} is isomorphic to one of H_2, H_3, H_4 depicted in Figure 1. In that case, let T'' be the tree formed from T by removing all vertices of T_{v_3} except v_3. Clearly v_3 is a leaf in T''. Let f'' be a $\gamma_I^p(T'')$-function. If $f''(v_3) = 0$, then $f''(v_4) = 2$ and so let f be a PIDF of T defined as follows: $f(x) = f''(x)$ for all $x \in V(T') \setminus \{v_3\}$ and $f(v_3) = 1$. Moreover, every child of v_3 of degree 2 is assigned a 0 and its unique leaf a 1; every child of v_3 of degree at least 4 is assigned a 2 and its leaves a 0. If $f''(v_3) = 1$, then f'' will be extended to a PIDF of T as above when $f'(x) = 0$, except we do not assign a 1 to v_3. Finally, if $f''(v_3) = 2$, then we use the following assignment for vertices of T_{v_3}: assign a 2 to each child of v_3 and a 0 to each of their leaves. Now, if $T_{v_3} = H_2$, then in either case described above, we have $\gamma_I^p(T) \leq \gamma_I^p(T'') + 4$. By the induction hypothesis we obtain

$$\begin{aligned}\gamma_I^p(T) &\leq \gamma_I^p(T'') + 4 \leq \frac{4(n - 3 - |L(v_2)|) - \ell(T) + |L(v_2)| + 1 + 2s(T) - 3}{5} + 4\\ &< \frac{4n - \ell(T) + 2s(T) - 1}{5}.\end{aligned}$$

If $T_{v_3} = H_3$, then $\gamma_I^p(T) \leq \gamma_I^p(T'') + 5$, and by the induction hypothesis we obtain

$$\begin{aligned}\gamma_I^p(T) &\leq \gamma_I^p(T'') + 5\\ &\leq \frac{4(n - 2 - |L(v_2)| - |L(u)|) - \ell(T) + |L(v_2)| + |L(u)| + 2s(T) - 3}{5} + 5\\ &< \frac{4n - \ell(T) + 2s(T) - 1}{5}.\end{aligned}$$

Moreover, if $T_{v_3} = H_4$, then $\gamma_I^p(T) \le \gamma_I^p(T'')+6$, and by the induction hypothesis it follows that

$$\gamma_I^p(T) \le \gamma_I^p(T'') + 6 \le \frac{4(n-5-|L(v_2)|) - \ell(T) + 2 + |L(v_2)| + 2s(T) - 5}{5} + 6$$
$$< \frac{4n - \ell(T) + 2s(T) - 1}{5}.$$

Before discussing Case 2, we will need the following claim.

Claim. *Let T be a wounded spider of order n different from $DS_{2,1}$, with $s(T)$ support vertices and $\ell(T)$ leaves. Then we have the following.*

(i) *If $6s(T) - 2\ell(T) \ge 11$, then $\gamma_I^p(T) \le \frac{4n-\ell(T)+2s(T)-6}{5}$.*

(ii) *If $6s(T) - 2\ell(T) \le 11$, then $\gamma_I^p(T) \le \frac{4n-\ell(T)+2s(T)-3}{5}$.*

Proof. Let v be the center vertex of T.

(i) If $6s(T) - 2\ell(T) \ge 11$, then the function f defined by assigning a 1 to v and every leaf of T, and a 0 to remaining vertices of T, is a PIDF of T and so

$$\gamma_I^p(T) \le \omega(f) = \ell(T) + 1 \le \frac{4n - \ell(T) + 2s(T) - 6}{5}.$$

(ii) Let $t = |L(v)| - 1$. Clearly, $\ell(T) = s(T) + t$ and since $6s(T) - 2\ell(T) \le 11$, then T is a double star and since T is not a $DS_{2,1}$, we can see that we have $4s(T) - 2t \le 11$ and thus $t \ge 2s(T) - \frac{11}{2}$. Now if $s(T) = 2$, then T is a double star and since T is not a $DS_{2,1}$, we can see that $\gamma_I^p(T) \le \frac{4n-\ell(T)+2s(T)-3}{5}$. Hence, let $s(T) \ge 3$. Then the function f defined by assigning a 2 to the support vertices of T and a 0 to remaining vertices of T is a PIDF of T of weight $2s(T)$. Since, $n = s(T) + \ell(T)$ and $\ell(T) = s(T) + t$, it follows that $\frac{4n-\ell(T)+2s(T)-3}{5} = \frac{9s(T)+3t-3}{5}$. Moreover, since $t \ge 2s(T) - \frac{11}{2}$ we obtain

$$\frac{9s(T) + 3t - 3}{5} \ge \frac{9s(T) + 6s(T) - \frac{33}{2} - 3}{5} = 3s(T) - \frac{39}{10}.$$

Now, if $s(T) \ge 4$, then $3s(T) - \frac{39}{10} \ge 2s(T) \ge \gamma_I^p(T)$ and so the desired result follows. Thus we assume that $s(T) = 3$. If $t \ge 2s(T) - \frac{7}{2}$, then as above we have $\frac{9s(T)+3t-3}{5} \ge 3s(T) - \frac{27}{10} \ge 2s(T) \ge \gamma_I^p(T)$. Hence, let $t \le 2s(T) - \frac{7}{2} = 2.5$. Note that in this case $\ell(T) \in \{3, 4, 5\}$. Then assigning a 1 to v and the leaves of T and a 0 to remaining vertices of T provides a PIDF of T of weight $\ell(T) + 1 \le \frac{4n-\ell(T)+2s(T)-3}{5}$, which completes the proof of the claim. □

We note from the proof of the claim that there exist PIDFs of T of weight at most $\frac{4|V(T_{v_3})|-\ell(T_{v_3})+2s(T_{v_3})-3}{5}$ that assign to the center vertex a 1 or 2.

Now we are ready to examine the next case.

Case 2. $\deg_T(v_2) = 2$ or $T_{v_3} = DS_{3,1}$. From Case 1 and since v_2 was chosen having a maximum degree, we conclude that T_{v_3} is a spider. Assume first that T_{v_3} is a healthy spider. If $|C(v_3)| \geq 3$, then let T' be the tree obtained by removing T_{v_3} and adding three new vertices attached at v_4. Note that $n' = n-2|C(v_3)|+2$, $s(T') \leq s(T) - |C(v_3)| + 1$ and $\ell(T') = \ell(T) - |C(v_3)| + 3$. Clearly, $f'(v_4) \geq 1$ (since v_4 has three leaves in T'). Thus the function f defined by $f(x) = 1$ for $x \in L(T_{v_3}) \cup \{v_3\}$, $f(x) = 0$ for $x \in C(v_3)$ and $f(x) = f'(x)$ for $x \in V(T) \setminus V(T_{v_3})$ is a PIDF of T. Hence $\gamma_I^p(T) \leq \gamma_I^p(T') + |C(v_3)| + 1$, and by the induction hypothesis we obtain

$$\begin{aligned}
&\gamma_I^p(T)\\
&\leq \gamma_I^p(T') + |C(v_3)| + 1\\
&\leq \frac{4(n-2|C(v_3)|+2) - \ell(T) + |C(v_3)| - 3 + 2s(T) - 2|C(v_3)|+1}{5} + |C(v_3)|+1\\
&\leq \frac{4n - \ell(T) + 2s(T) - 1 - 4|C(v_3)| + 12}{5} \leq \frac{4n - \ell(T) + 2s(T) - 1}{5}.
\end{aligned}$$

Now, assume that $|C(v_3)| = 2$, and let $T' = T - T_{v_3}$. If $f'(v_4) \geq 1$, then the function f defined by $f(x) = 1$ for $x \in L(T_{v_3}) \cup \{v_3\}$, $f(x) = 0$ for every $x \in C(v_3)$ and $f(x) = f'(x)$ for all $x \in V(T) \setminus V(T_{v_3})$ is a PIDF of T of weight $\gamma_I^p(T') + 3$. If $f'(v_4) = 0$, then the function f defined by $f(x) = 1$ for $x \in V(T_{v_3}) \setminus \{v_3\}$, $f(v_3) = 0$ and $f(x) = f'(x)$ for all $x \in V(T) \setminus V(T_{v_3})$ is a PIDF of T of weight $\gamma_I^p(T') + 4$. In either case, $\gamma_I^p(T) \leq \gamma_I^p(T') + 4$ and by the induction hypothesis we obtain

$$\begin{aligned}
\gamma_I^p(T) &\leq \gamma_I^p(T') + 4 \leq \frac{4(n-5) - \ell(T) + 2 + 2s(T) - 3}{5} + 4\\
&= \frac{4n - \ell(T) + 2s(T) - 1}{5}.
\end{aligned}$$

Suppose now that T_{v_3} is a wounded spider $S_{k,t}$. If $T_{v_3} = DS_{2,1}$, then let $T' = T - T_{v_3}$. Clearly $n' \geq 2$. If $n' = 2$, then $\gamma_i^p(T') = 5 < \frac{4n-\ell(T)+2s(T)-1}{5}$. Hence we assume that $n' \geq 3$. If $f'(v_4) \geq 1$, then the function f defined by $f(v_2) = f(v_3) = 2$, $f(x) = 0$ for $x \in L(T_{v_3})$ and $f(x) = f'(x)$ for $x \in V(T) \setminus V(T_{v_3})$ is a PIDF of T. If $f'(v_4) = 0$, then the function f defined by $f(v_1) = 2$, $f(x) = 1$ for $x \in L(v_3)$, $f(v_2) = f(v_3) = 0$ and $f(x) = f'(x)$ for $x \in V(T) \setminus V(T_{v_3})$ is a PIDF of T. In either case, $\gamma_I^p(T) \leq \gamma_I^p(T') + 4$. If $\deg_T(v_4) \geq 3$, then $s(T') = s(T) - 2$ and $\ell(T') = \ell(T) - 3$ and by the induction hypothesis we obtain

$$\begin{aligned}
\gamma_I^p(T) &\leq \gamma_I^p(T') + 4 \leq \frac{4(n-5) - \ell(T) + 3 + 2s(T) - 5}{5} + 4\\
&< \frac{4n - \ell(T) + 2s(T) - 1}{5}.
\end{aligned}$$

If $\deg_T(v_4) = 2$, then $s(T') \leq s(T) - 1$ and $\ell(T') = \ell(T) - 2$ and by the induction hypothesis we obtain

$$\begin{aligned}\gamma_I^p(T) &\leq \gamma_I^p(T') + 4 \leq \frac{4(n-5) - \ell(T) + 2 + 2s(T) - 3}{5} + 4 \\ &= \frac{4n - \ell(T) + 2s(T) - 1}{5}.\end{aligned}$$

From now on we may assume that v_4 has no child x such that $T_x = DS_{2,1}$.

Let s_1 be the number of children of v_4 that are leaves and for $i \geq 2$, let s_i be the number of children of v_4 of degree i whose children are all leaves. As we assumed at the beginning of the proof, T has no end support vertex with degree three, it follows that $s_3 = 0$. Let $s_{\geq 4}$ be the number of children of v_4 of degree at least 4 having no grandchild. Thus

$$s_{\geq 4} = \sum_{i \geq 4} s_i.$$

Adopting our earlier notation, for each child v of v_4 with depth 2, let n_v denote the number of children in the subtree T_v of T. Furthermore, let n^* denote the sum of the number of vertices in all such trees T_v. Also, let s^* and ℓ^* denote the sum of the number of support vertices and leaves vertices in all such trees T_v, respectively. Note that every child of v_4 is one of the following four types: (1) a leaf; (2) a support vertex of degree 2; (3) a vertex with depth 2; (4) a support vertex of degree at least 4 whose children are all leaves. For ease of discussion, we sometimes refer to these children as Type-1, Type-2, Type-3, or Type-4, respectively. Moreover, let m be the number of leaves of all Type-4 children. Consider now the following subcases.

Subcase 2.1. $s_1 + s_{\geq 4} \geq 3$. Let $T' = T - T_{v_3}$ be a tree of order n'. We claim that $f'(v_4) \geq 1$. Suppose to the contrary that $f'(v_4) = 0$. This implies that at most two children of v_4 in T' are assigned positive values under f'. But since every Type-1 and Type-4 child of v_4 must be assigned a positive value by f' when $f'(v_4) = 0$, this implies that $s_1 + s_{\geq 4} \leq 2$, a contradiction. Hence, $f'(v_4) \geq 1$. Consequently, we can extend f' to a PIDF f by adding to it any PIDF of T_{v_3} of weight at most $\frac{4n_{v_3} - \ell(T_{v_3}) + 2s(T_{v_3}) - 3}{5}$ assigning a 1 or 2 to v_3 (as claimed above). By the induction hypothesis we obtain

$$\begin{aligned}\gamma_I^p(T) &\leq \gamma_I^p(T') + \frac{4n_{v_3} - \ell(T_{v_3}) + 2s(T_{v_3}) - 3}{5} \\ &\leq \frac{4(n - n_{v_3}) - \ell(T) + \ell(T_{v_3}) + 2s(T) - 2s(T_{v_3}) - 1}{5} \\ &+ \frac{4n_{v_3} - \ell(T_{v_3}) + 2s(T_{v_3}) - 3}{5} < \frac{4n - \ell(T) + 2s(T) - 1}{5}.\end{aligned}$$

In the sequel, we may assume that $s_1 + s_{\geq 4} \leq 2$.

Subcase 2.2. $s_1 = 2$. Since $s_1 + s_{\geq 4} \leq 2$, we deduce that $s_{\geq 4} = 0$. Let F be the forest formed by the Type-3 children of v_4 and their descendants. We note any component of F is a wounded spider including T_{v_3} and different from $DS_{2,1}$. Let T' be the tree obtained from T by deleting all vertices in $V(F)$ and adding a new vertex a attached at v_4. Since v_4 has three leaf neighbors in T', we have $f'(v_4) \geq 1$. Let f be the PIDF of T defined as follows: $f(x) = f'(x)$ for all $x \in V(T') \setminus \{a\}$ and let the restriction of f to each component, say T_v, in F be any PIDF of that component of weight at most $\frac{4n_v - \ell(T_v) + 2s(T_v) - 3}{5}$. By our earlier observations, the total weight assigned to F is at most $\frac{4n^* - \ell^* + 2s^* - 3}{5}$. Now, by the induction hypothesis we obtain

$$\begin{aligned}\gamma_I^p(T) &\leq \gamma_I^p(T') + \frac{4n^* - \ell^* + 2s^* - 3}{5}\\ &\leq \frac{4(n - n^* + 1) - \ell(T) + \ell^* - 1 + 2s(T) - 2s^* - 1}{5} + \frac{4n^* - \ell^* + 2s^* - 3}{5}\\ &\leq \frac{4n - \ell(T) + 2s(T) - 1}{5}.\end{aligned}$$

Hence, in the next we may assume that $s_1 \in \{0, 1\}$.

Subcase 2.3. $s_2 \geq 3$. Let T' be the tree of order n' obtained from $T - T_{v_4}$ by adding three new vertices x_1, x_2, x_3 attached at v_5. Note that $n' = n - n^* - s_1 - 2s_2 - s_{\geq 4} - m + 2$, $\ell(T') = \ell(T) - \ell^* - s_1 - s_2 - m + 3$ and $s(T') \leq s(T) - s^* - s_1 - s_2 - s_{\geq 4} + 1$. Clearly, $f'(v_5) \geq 1$ (since v_5 has three leaves in T'). Let f be the PIDF of T defined by $f(x) = f'(x)$ for all $x \in V(T') \setminus \{x_1, x_2, x_3\}$ and let $f(v_4) = 1$. Then assign the weights to the descendants of v_4 in T as follows: assign a 1 to each Type-1 (leaf) child of v_4 (recall that $s_1 \in \{0, 1\}$); assign a 0 to each Type-2 child of v_4 and a 1 to its leaf neighbor; assign a 2 to each Type-4 child of v_4 and a 0 to each of its leaves. Finally, for each Type-3 child v, assign a PIDF to the subtree T_v rooted at v of weight at most $\frac{4n_v - \ell(T_v) + 2s(T_v) - 3}{5}$ so that $f(v) \geq 1$. By our earlier observations, the total weight assigned to all Type-3 children of v and their descendants is at most $\frac{4n^* - \ell^* + 2s^* - 3}{5}$. It follows from the induction hypothesis that

$$\begin{aligned}\gamma_I^p(T) &\leq \gamma_I^p(T') + \frac{4n^* - \ell^* + 2s^* - 3}{5} + s_1 + s_2 + 2s_{\geq 4} + 1\\ &\leq \frac{4n' - \ell(T') + 2s(T') - 1}{5} + \frac{4n^* - \ell^* + 2s^* - 3}{5} + s_1 + s_2 + 2s_{\geq 4} + 1\\ &\leq \frac{4(n - n^* - s_1 - 2s_2 - m - s_{\geq 4} + 2) - \ell(T) + \ell^* + s_1 + s_2 + m - 3}{5}\\ &+ \frac{2s(T) - 2s^* - 2s_1 - 2s_2 - 2s_{\geq 4} + 1}{5} + \frac{4n^* - \ell^* + 2s^* - 3}{5} + s_1 + s_2 + 2s_{\geq 4} + 1\\ &= \frac{4n - \ell(T) + 2s(T) - 1}{5} + \frac{9 - 3m - 4s_2 + 4s_{\geq 4}}{5}.\end{aligned}$$

Using the fact that $m \geq 3s_{\geq 4}$, it follows that $\gamma_I^p(T) \leq \frac{4n-\ell(T)+2s(T)-1}{5} + \frac{9-4s_2-5s_{\geq 4}}{5}$. Now since $s_2 \geq 3$, we deduce that $\gamma_I^p(T) \leq \frac{4n-\ell(T)+2s(T)-1}{5}$.

By Subcase 2.3, we can assume that $s_2 \leq 2$.

Subcase 2.4. $s_2 + s_{\geq 4} \geq 1$. Let T' be the tree of order n' obtained by deleting all vertices of T_{v_4} except v_4. Note that $n' = n - n^* - s_1 - 2s_2 - s_{\geq 4} - m$, $s(T') \leq s(T) - s^* - s_1 - s_2 - s_{\geq 4} + 1$ and $\ell(T') = \ell(T) - \ell^* - s_1 - s_2 - m + 1$ (since v_4 is a leaf vertex in T'). First, let $f'(v_4) = 2$ and f be a PIDF of T defined by $f(x) = f'(x)$ for all $x \in V(T')$; and then assign the weights to the descendants of v_4 in T as follows: assign a 0 to each Type-1 (leaf) child of v_4, assign a 2 to each Type-2 child of v_4 and a 0 to its leaf, and assign a 2 to each Type-4 child of v_4 and a 0 to its leaves. Finally, for each Type-3 child v, assign a PIDF to the subtree T_v rooted at v. By our earlier observations, the total weight assigned to all Type-3 children of v and their descendants is at most $\frac{4n^*-\ell^*+2s^*-3}{5}$. By the induction hypothesis it follows that

$$\begin{aligned}
\gamma_I^p(T) &\leq \gamma_I^p(T') + \frac{4n^* - \ell^* + 2s^* - 3}{5} + 2s_2 + 2s_{\geq 4} \\
&\leq \frac{4n' - \ell(T') + 2s(T') - 1}{5} + \frac{4n^* - \ell^* + 2s^* - 3}{5} + 2s_2 + 2s_{\geq 4} \\
&\leq \frac{4(n - n^* - s_1 - 2s_2 - m - s_{\geq 4}) - \ell(T) + \ell^* + s_1 + s_2 + m - 1}{5} \\
&+ \frac{2s(T) - 2s^* - 2s_1 - 2s_2 - 2s_{\geq 4} + 1}{5} + \frac{4n^* - \ell^* + 2s^* - 3}{5} + 2s_2 + 2s_{\geq 4} \\
&\leq \frac{4n - \ell(T) + 2s(T) - 1}{5} + \frac{-5s_1 + s_2 - 3m + 4s_{\geq 4} - 2}{5}.
\end{aligned}$$

Now since $m \geq 3s_{\geq 4}$ and $s_2 \leq 2$, we get

$$\gamma_I^p(T) \leq \frac{4n - \ell(T) + 2s(T) - 1}{5} + \frac{-5s_1 + s_2 - 5s_{\geq 4} - 2}{5} < \frac{4n - \ell(T) + 2s(T) - 1}{5}.$$

Suppose now that $f'(v_4) \in \{0, 1\}$, and let f be a PIDF of T defined by $f(x) = f'(x)$ for all $x \in V(T')$ and let $f(v_4) = 1$. Then assign the weights to the descendants of v_4 in T as follows: assign a 1 to each Type-1 (leaf) child of v_4; assign a 0 to each Type-2 child of v_4 and a 1 to its leaf neighbor and assign a 2 to each Type-4 child of v_4 and 0 to its leaves. Finally, for each Type-3 child v, assign a PIDF of weight at most $\frac{4n_v-\ell(T_v)+2s(T_v)-3}{5}$ to vertices of T_v rooted at v so that $f(v) \geq 1$. By our earlier observations, the total weight assigned to all Type-3 children of v and their descendants is at most $\frac{4n^*-\ell^*+2s^*-3}{5}$. By the induction hypothesis we obtain

$$\begin{aligned}
\gamma_I^p(T) &\le \gamma_I^p(T') + \frac{4n^* - \ell^* + 2s^* - 3}{5} + s_1 + s_2 + 2s_{\ge 4} + 1 \\
&\le \frac{4n' - \ell(T') + 2s(T') - 1}{5} + \frac{4n^* - \ell^* + 2s^* - 3}{5} + s_1 + s_2 + 2s_{\ge 4} + 1 \\
&\le \frac{4(n - n^* - s_1 - 2s_2 - m - s_{\ge 4}) - \ell(T) + \ell^* + s_1 + s_2 + m - 1}{5} \\
&+ \frac{2s(T) - 2s^* - 2s_1 - 2s_2 - 2s_{\ge 4} + 1}{5} + \frac{4n^* - \ell^* + 2s^* - 3}{5} + s_1 + s_2 + 2s_{\ge 4} + 1 \\
&\le \frac{4n - \ell(T) + 2s(T) - 1}{5} + \frac{-4s_2 - 3m + 4s_{\ge 4} + 3}{5}.
\end{aligned}$$

Now since $m \ge 3s_{\ge 4}$, it follows that $\gamma_I^p(T) \le \frac{4n-\ell(T)+2s(T)-1}{5} + \frac{-4s_2-5s_{\ge 4}+3}{5}$, and since $s_2 + s_{\ge 4} \ge 1$, the result follows.

Subcase 2.5. $s_2 + s_{\ge 4} = 0$. Recall that $s_1 \in \{0, 1\}$. Let v' be the leaf neighbor of v_4 (if any). First, let v_4 has at least two children of Type-3. Let T' be the tree of order n' obtained by deleting all vertices of T_{v_4} except v_4. Note that $n' = n - n^* - s_1$, $s(T') \le s(T) - s^* - s_1 + 1$ and $\ell(T') = \ell(T) - \ell^* - s_1 + 1$ (since v_4 is a leaf vertex in T'). We also note that if $f'(v_4) = 0$, then since v_4 is a leaf in T', we must have $f'(v_5) = 2$. Now, we define a PIDF f of T by $f(x) = f'(x)$ for all $x \in V(T') \setminus \{v_4\}$. Moreover, $f(v') = 1$, $f(v_4) = 1$ if $f'(v_4) = 0$ and $f(v_4) = f'(v_4)$ if $f'(v_4) \ge 1$. Also, for each other child v of v_4, assign a PIDF to the subtree T_v of weight at most $\frac{4n_v-\ell(T_v)+2s(T_v)-3}{5}$. Since there are at least two Type-3 children of v_4, the total weight assigned to such subtree T_v is $\frac{4n^*-\ell^*+2s^*-2\cdot 3}{5}$. Hence in either case, $\gamma_I^p(T) \le \gamma_I^p(T') + \frac{4n^*-\ell^*+2s^*-6}{5} + s_1 + 1$. Using the induction hypothesis we obtain

$$\begin{aligned}
\gamma_I^p(T) &\le \gamma_I^p(T') + \frac{4n^* - \ell^* + 2s^* - 6}{5} + s_1 + 1 \\
&\le \frac{4n' - \ell(T') + 2s(T') - 1}{5} + \frac{4n^* - \ell^* + 2s^* - 6}{5} + s_1 + 1 \\
&\le \frac{4(n - n^* - s_1) - \ell(T) + \ell^* + s_1 - 1 + 2s(T) - 2s^* - 2s_1 + 1}{5} \\
&+ \frac{4n^* - \ell^* + 2s^* - 6}{5} + s_1 + 1 \le \frac{4n - \ell(T) + 2s(T) - 1}{5}.
\end{aligned}$$

In the sequel, v_3 is the only child of v_4 of Type-3. We distinguish the following.

(i) $T_{v_3} = DS_{1,3}$. Consider two situations depending on whether $s_1 = 0$ or $s_1 = 1$.

(a) $s_1 = 0$. Hence $\deg_T(v_4) = 2$. Let $T' = T - T_{v_4}$. Clearly, $n' \ge 1$. If $n' = 1$, then T is a wounded spider and by the claim the result follows, and if $n' = 2$, then one can easily see that $\gamma_I^p(T) = 6 < \frac{4n-\ell(T)+2s(T)-1}{5} = 7.2$. So let $n' \geq 3$. Note that $n' = n-7$, $\ell(T') \geq \ell(T)-4$ and $s(T') \leq s(T)-1$. Any $\gamma_I^p(T')$-function can be extended to a PIDF of T by assigning a 2 to v_2, v_3 and a 0 to remaining vertices of T_{v_4} except v_4 which will be assigned a 0 if $f'(v_5) = 0$ and a 1 if $f'(v_5) \geq 1$. In either case, $\gamma_I^p(T) \leq \gamma_I^p(T') + 5$. By the induction hypothesis we obtain

$$\begin{aligned}\gamma_I^p(T) &\leq \frac{4n'-\ell(T')+s(T')-1}{5} + 5 \leq \frac{4(n-7)-\ell(T)+4+2s(T)-3}{5} + 5\\ &< \frac{4n-\ell(T)+2s(T)-1}{5}.\end{aligned}$$

(b) $s_1 = 1$. Let T' be the tree obtained from T by removing all vertices T_{v_3} except v_3. If $f'(v_3) = 0$, then $f'(v_4) = 2$, and so f' can be extended to a PIDF of T by assigning a 2 to v_2, v_3 and a 0 to remaining vertices of T_{v_3}. Hence $\gamma_I^p(T) \leq \gamma_I^p(T') + 4$. If $f'(v_3) = 2$, then $f'(v_4) = 0$ and so the other leaf neighbor of v_4 is assigned a 1, which is a contradiction. Hence, $f'(v_3) = 1$. Now, if $|L(v_3)| = 1$, then we extend f' to a PIDF of T by assigning a 2 to v_2, a 1 to $L(v_3)$ and a 0 to the remaining vertices of T_{v_3}. If $|L(v_3)| = 3$, then we extend f' to a PID-function of T by assigning a 1 to $L(T_{v_3})$ and a 0 to v_2. In either case, $\gamma_I^p(T) \leq \gamma_I^p(T') + 4$. By the induction hypothesis we obtain

$$\begin{aligned}\gamma_I^p(T) &\leq \frac{4n'-\ell(T')+s(T')-1}{5} + 4 \leq \frac{4(n-5)-\ell(T)+3+2s(T)-5}{5} + 4\\ &< \frac{4n-\ell(T)+2s(T)-1}{5}.\end{aligned}$$

(ii) $T_{v_3} = S_{k,t} \neq DS_{3,1}$. We recall that T_{v_3} is different from $DS_{2,1}$. First let $6s(T_{v_3}) - 2\ell(T_{v_3}) \geq 11$. By our Claim, $\gamma_I^p(T_{v_3}) \leq \frac{4n_{v_3}-\ell(T_{v_3})+2s(T_{v_3})-6}{5}$. Let T' be the tree obtained from T by removing all vertices of T_{v_4} except v_4. Note that $n' \geq 2$. Moreover, if $n' = 2$, then one can see that $\gamma_I^p(T) \leq \gamma_I^p(T_{v_3}) + 2 < \frac{4n-\ell(T)+2s(T)-1}{5}$. Hence let $n' \geq 3$. Note that $n' = n - n_{v_3} - s_1, \ell(T') = \ell(T) - \ell(T_{v_3}) - s_1 + 1$ and $s(T') \leq s(T) - s(T_{v_3}) - s_1 + 1$. Then any $\gamma_I^p(T')$-function f' can be extended to a PIDF of T by adding to it a PIDF of T_{v_3} of weight $\frac{4n_{v_3}-\ell(T_{v_3})+2s(T_{v_3})-6}{5}$ that assigns a 1 to v_3. Moreover, the leaf neighbor of v_4 (if any) is assigned a 1, while v_4 will be assigned a 1 if $f'(v_4) = 0$ (note that in that case $f'(v_5) = 2$) or v_4 will keep the same assignment under f' if $f'(v_4) \geq 1$. In either case, $\gamma_I^p(T) \leq \gamma_I^p(T') + \gamma_I^p(T_{v_3}) + s_1 + 1$. Using the induction, we obtain

$$\begin{aligned}\gamma_I^p(T) &\leq \frac{4n'-\ell(T')+s(T')-1}{5} + \frac{4n_{v_3}-\ell(T_{v_3})+2s(T_{v_3})-6}{5} + s_1 + 1\\ &\leq \frac{4(n-n_{v_3}-s_1)-\ell(T)+\ell(T_{v_3})+s_1-1+2s(T)-2s(T_{v_3})-2s_1+1}{5}\\ &+ \frac{4n_{v_3}-\ell(T_{v_3})+2s(T_{v_3})-6}{5} + s_1 + 1 = \frac{4n-\ell(T)+2s(T)-1}{5}.\end{aligned}$$

Therefore, we can now assume that $6s(T_{v_3}) - 2\ell(T_{v_3}) \le 11$. Recall that (by the proof of the Claim) there exists PIDF, say g, of T_{v_3} of weight at most $\frac{4n_{v_3}-\ell(T_{v_3})+2s(T_{v_3})-3}{5}$ assigning a 2 to v_3. We now consider two situations depending on whether $s_1 = 0$ or $s_1 = 1$.

(a) $s_1 = 0$. Then $\deg_T(v_4) = 2$. Let $T' = T - T_{v_4}$. If $n' = 1$, then T is a wounded spider and by the claim the result follows, and if $n' = 2$, then one can easily see that g can be extended to a PIDF of T by assigning a 2 to v_6 and a 0 to both v_4 and v_5, and thus $\gamma_I^p(T) \le \frac{4n_{v_3}-\ell(T_{v_3})+2s(T_{v_3})-3}{5} + 2 \le \frac{4n-\ell(T)+2s(T)-1}{5}$. So let $n' \ge 3$. In this case, any $\gamma_I^p(T')$-function can be extended to a PIDF of T by adding to it the PIDF g of T_{v_3}. Moreover, v_4 will be assigned a 0 if $f'(v_5) = 0$ and a 1 if $f'(v_5) \ge 1$. In either case, $\gamma_I^p(T) \le \gamma_I^p(T') + \frac{4n_{v_3}-\ell(T_{v_3})+2s(T_{v_3})-3}{5} + 1$. Using the fact that $n' = n - n_{v_3} - 1$, $\ell(T') \ge \ell(T) - \ell(T_{v_3})$, $s(T') \le s(T) - s(T_{v_3}) + 1$, it follows from the induction hypothesis that

$$\begin{aligned}
\gamma_I^p(T) &\le \frac{4n' - \ell(T') + s(T') - 1}{5} + \frac{4n_{v_3} - \ell(T_{v_3}) + 2s(T_{v_3}) - 3}{5} + 1 \\
&\le \frac{4(n - n_{v_3} - 1) - \ell(T) + \ell(T_{v_3}) + 2s(T) - 2s(T_{v_3}) + 1}{5} \\
&+ \frac{4n_{v_3} - \ell(T_{v_3}) + 2s(T_{v_3}) - 3}{5} + 1 = \frac{4n - \ell(T) + 2s(T) - 1}{5}.
\end{aligned}$$

(b) $s_1 = 1$. Assume first that v_3 has at least four leaves, and let $T' = T \setminus \{w, v_1, v_2\}$, where $w \in L(v_3)$. Since v_3 has at least three leaves we have $f'(v_3) \ge 1$. If $f'(v_3) = 2$, then f' is extended to a PIDF of T by assigning a 2 to v_2 and a 0 to w, v_1. If $f'(v_3) = 1$, then f' to a PIDF of T by assigning a 1 to v_1, w and 0 to v_2. In either case, $\gamma_I^p(T) \le \gamma_I^p(T') + 2$. By the induction hypothesis we get

$$\begin{aligned}
\gamma_I^p(T) &\le \frac{4n' - \ell(T') + s(T') - 1}{5} + 2 \le \frac{4(n-3) - \ell(T) + 2 + 2s(T) - 3}{5} + 2 \\
&< \frac{4n - \ell(T) + 2s(T) - 1}{5}.
\end{aligned}$$

Hence, we can assume that v_3 has at most three leaves and thus $\ell(T_{v_3}) \le s(T_{v_3}) + 2$. Let T' be the tree obtained from T by removing all vertices of T_{v_3} except v_3. Then $n' = n - n_{v_3} + 1$, $\ell(T') = \ell(T) - \ell(T_{v_3}) + 1$ and $s(T') = s(T) - s(T_{v_3})$. If $f'(v_3) = 0$, then $f'(v_4) = 2$, and f' can be extended to a PIDF of T by adding to it the PIDF g of T_{v_3}, where v_3 is reassigned $g(v_3)$ instead of $f'(v_3)$. Applying our induction hypothesis, we obtain

$$\begin{aligned}\gamma_I^p(T) &\leq \frac{4n' - \ell(T') + s(T') - 1}{5} + \frac{4n_{v_3} - \ell(T_{v_3}) + 2s(T_{v_3}) - 3}{5} \\ &\leq \frac{4(n - n_{v_3} + 1) - \ell(T) + \ell(T_{v_3}) - 1 + 2s(T) - 2s(T_{v_3}) - 1}{5} \\ &+ \frac{4n_{v_3} - \ell(T_{v_3}) + 2s(T_{v_3}) - 3}{5} = \frac{4n - \ell(T) + 2s(T) - 1}{5}.\end{aligned}$$

If $f'(v_3) = 2$, then $f'(v_4) = 0$ and the other leaf neighbor of v_4 in T' is assigned a 1, which provides a contradiction. Hence let $f'(v_3) = 1$. Then we extend f' to a PIDF of T by assigning a 1 to all leaves vertices of T_{v_3} and a 0 to remaining vertices of T_{v_3} but v_3. Using the fact that $\ell(T_{v_3}) \leq s(T_{v_3})+2$, $n_{v_3} = \ell(T_{v_3})+s(T_{v_3})$ and the induction hypothesis, we obtain

$$\begin{aligned}\gamma_I^p(T) &\leq \frac{4n' - \ell(T') + s(T') - 1}{5} + \ell(T_{v_3}) \\ &\leq \frac{4(n - n_{v_3} + 1) - \ell(T) + \ell(T_{v_3}) - 1 + 2s(T) - 2s(T_{v_3}) - 1}{5} + \ell(T_{v_3}) \\ &\leq \frac{4n - \ell(T) + 2s(T) - 1}{5}.\end{aligned}$$

This completes the proof. ■

References

[1] M. Chellali, T.W. Haynes, S.T. Hedetniemi and A. McRae, *Roman* {2}*-domination*, Discrete Appl. Math. **204** (2016) 22–28. https://doi.org/10.1016/j.dam.2015.11.013

[2] M. Chellali, N. Jafari Rad, S.M. Sheikholeslami and L. Volkmann, *Roman domination in graphs*, in: Topics in Domination in Graphs, T.W. Haynes, S.T. Hedetniemi and M.A. Henning (Ed(s)), Springer (2020) 365–409.

[3] M. Chellali, N. Jafari Rad, S.M. Sheikholeslami and L. Volkmann, *Varieties of Roman domination*, in: Structures of Domination in Graphs, T.W. Haynes, S.T. Hedetniemi and M.A. Henning (Ed(s)) 273–307. https://doi.org/10.1007/978-3-030-58892-2_10

[4] M. Chellai, N. Jafari Rad, S.M. Sheikholeslami and L. Volkmann, *Varieties of Roman dominationm* II, AKCE Int. J. Graphs Comb. **17** (2020) 966–984. https://doi.org/10.1016/j.akcej.2019.12.001

[5] M. Chellai, N. Jafari Rad, S.M. Sheikholeslami and L. Volkmann, *A survey on Roman domination parameters in directed graphs*, J. Combin. Math. Combin. Comput. **115** (2020) 141–171.

[6] M. Chellai, N. Jafari Rad, S.M. Sheikholeslami and L. Volkmann, *The Roman domatic problem in graphs and digraphs: A survey*, Discuss. Math. Graph Theory, in-press.
https://doi.org/10.7151/dmgt.2313

[7] E.J. Cockayne, P.A. Dreyer Jr., S.M. Hedetniemi and S.T. Hedetniemi, *Roman domination in graphs*, Discrete Math. **278** (2004) 11–22.
https://doi.org/10.1016/j.disc.2003.06.004

[8] M. Darkooti, A. Alhevaz, S. Rahimi and H. Rahbani, *On perfect Roman domination number in trees: complexity and bounds*, J. Comb. Optim. **38** (2019) 712–720.
https://doi.org/10.1007/s10878-019-00408-y

[9] T.W. Haynes and M.A. Henning, *Perfect Italian domination in trees*, Discrete Appl. Math. **260** (2019) 164–177.
https://doi.org/10.1016/j.dam.2019.01.038

[10] M.A. Henning, W.F. Klostermeyer and G. MacGillivray, *Perfect Roman domination in trees*, Discrete Appl. Math. **236** (2018) 235–245.
https://doi.org/10.1016/j.dam.2017.10.027

[11] M. Livingston and Q.F. Stout, *Perfect dominating sets*, Congr. Numer. **79** (1990) 187–203.

[12] C.S. ReVelle and K.E. Rosing, *Defendens Imperium Romanum: A classical problem in military strategy*, Amer. Math. Monthly **107** (2000) 585–594.
https://doi.org/10.1080/00029890.2000.12005243

[13] I. Stewart, *Defend the Roman Empire*, Sci. Amer. **281** (1999) 136–139.
https://doi.org/10.1038/scientificamerican1299-136

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