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DESCRIBING MINOR 5-STARS IN 3-POLYTOPES WITH MINIMUM DEGREE 5 AND NO VERTICES OF DEGREE 6 OR 7

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Abstract

In 1940, in attempts to solve the Four Color Problem, Henry Lebesgue gave an approximate description of the neighborhoods of 5-vertices in the class $\mathbf{P_5}$ of 3-polytopes with minimum degree 5. This description depends on 32 main parameters.

$$\begin{gathered}(6,6,7,7,7),\ (6,6,6,7,9),\ (6,6,6,6,11),\\ (5,6,7,7,8),\ (5,6,6,7,12),\ (5,6,6,8,10),\ (5,6,6,6,17),\\ (5,5,7,7,13),\ (5,5,7,8,10),\ (5,5,6,7,27),\\ (5,5,6,6,\infty),\ (5,5,6,8,15),\ (5,5,6,9,11),\\ (5,5,5,7,41),\ (5,5,5,8,23),\ (5,5,5,9,17),\\ (5,5,5,10,14),\ (5,5,5,11,13).\end{gathered}$$

Not many precise upper bounds on these parameters have been obtained as yet, even for restricted subclasses in $\mathbf{P_5}$. In 2018, Borodin, Ivanova, Kazak proved that every forbidding vertices of degree from 7 to 11 results in a tight description $(5,5,6,6,\infty)$, $(5,6,6,6,15)$, $(6,6,6,6,6)$. Recently, Borodin, Ivanova, and Kazak proved every 3-polytope in $\mathbf{P_5}$ with no vertices of degrees 6, 7, and 8 has a 5-vertex whose neighborhood is majorized by one of the sequences $(5,5,5,5,\infty)$ and $(5,5,10,5,12)$, which is tight and improves a corresponding description $(5,5,5,5,\infty)$, $(5,5,9,5,17)$, $(5,5,10,5,14)$, $(5,5,11,5,13)$ that follows from the Lebesgue Theorem.

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The purpose of this paper is to prove that every 3-polytope with minimum degree 5 and no vertices of degree 6 or 7 has a 5-vertex whose neighborhood is majorized by one of the ordered sequences $(5,5,5,5,\infty)$, $(5,5,8,5,14)$, or $(5,5,10,5,12)$.

Keywords: planar graph, structural properties, 3-polytope, 5-star, neighborhood.

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1. Introduction

By a 3-*polytope* P we mean a finite 3-connected plane graph. The *degree* $d(v)$ of a vertex v ($d(f)$ of a face f) in P is the number of edges incident with it. Let $\mathbf{P}_5$ denote the class of 3-polytopes with minimum degree 5. A k-*vertex* (k-*face*) is a vertex (face) of degree k; a k^+-*vertex* has degree at least k, etc.

By a *minor* k-*star* $S_k^{(m)}$ we mean a star with k rays centered at a 5^--vertex. The *weight* (*height*) of an $S_k^{(m)}$ in P is the degree sum (maximum degree) of its boundary vertices, and $w_k(P)$ ($h_k(P)$) denotes the minimum weight (height) of minor k-stars in P.

In 1904, Wernicke [27] proved that every 3-polytope in $\mathbf{P}_5$ has a 5-vertex adjacent to a 6^--vertex, which was strengthened by Franklin [16] in 1922 by proving that in fact there is a 5-vertex adjacent to two 6^--vertices. Recently, Borodin and Ivanova [2] proved that every 3-polytope in $\mathbf{P}_5$ has also a vertex of degree at most 6 adjacent to a 5-vertex and another vertex of degree at most 6, which description is tight.

We say that a 5-vertex v is *of type* $(k_1,\dots,k_5)$ or simply a $(k_1,\dots,k_5)$-*vertex* if the ordered sequence of degrees of its neighbors is majorized by the vector $(k_1,\dots,k_5)$. If the order of certain entries in the type is irrelevant, then we put a line over them.

In 1940, the following description of the neighborhoods of 5-vertices in $\mathbf{P}_5$ was given by Lebesgue [24, p. 36], which absorbs the results of Wernicke [27] and Franklin [16].

Theorem 1 (Lebesgue [24])**.** *Every triangulated* 3*-polytope with minimum degree* 5 *contains a* 5*-vertex of one of the following types:*

$$(\overline{6,6,7,7,7}),\ (\overline{6,6,6,7,9}),\ (6,6,6,6,11),$$
$$(\overline{5,6,7,7,8}),\ (5,6,\overline{6,7},11),\ (5,6,\overline{6,8},8),$$
$$(5,6,\overline{6,9},7),\ (5,7,6,6,12),\ (5,8,6,6,10),\ (5,6,6,6,17),$$
$$(5,5,\overline{7,7,8}),\ (5,13,5,7,7),\ (5,10,5,7,8),$$
$$(5,8,5,7,9),\ (5,7,5,7,10),\ (5,7,5,8,8),$$

$$(5,5,7,6,12),\ (5,5,8,6,10),\ (5,6,5,7,12),$$
$$(5,6,5,8,10),\ (5,17,5,6,7),\ (5,11,5,6,8),$$
$$(5,11,5,6,9),\ (5,7,5,6,13),\ (5,8,5,6,11),\ (5,9,5,6,10),\ (5,6,6,5,\infty),$$
$$(5,5,7,5,41),\ (5,5,8,5,23),\ (5,5,9,5,17),\ (5,5,10,5,14),\ (5,5,11,5,13).$$

In particular, Theorem 1 implies that there is a 5-vertex with three 7^--neighbors, which means that $h\left(S_3^{(m)}\right) \le 7$. Another corollary of Theorem 1 is that $w\left(S_3^{(m)}\right) \le 24$, which was improved in 1996 by Jendrol' and Madaras [21] to the sharp bound $w\left(S_3^{(m)}\right) \le 23$. Furthermore, Jendrol' and Madaras [21] gave a tight description of minor 3-stars in $\mathbf{P}_5$: there is a $(6,6,6)$- or $(5,6,7)$-star. Recently, Borodin and Ivanova [1], using the sharp bound $w\left(S_4^{(m)}\right) \le 30$ by Borodin and Woodall [14], obtained a tight description of minor 4-stars in $\mathbf{P}_5$.

Jendrol' and Madaras [21] also showed that if a polytope P in $\mathbf{P_5}$ is allowed to have a 5-vertex adjacent to four 5-vertices (such a 5-vertex is also called a *minor* $(5,5,5,5,\infty)$*-star*), then $h_5(P)$ (and hence $w_5(P)$) can be arbitrarily large. In 2014, Borodin, Ivanova, and Jensen [7] showed that the same phenomenon holds under a weaker assumption that 5-vertices are allowed to have two 5-neighbors and two 6-neighbors. Thus, the term $(5,6,6,5,\infty)$ in Theorem 1 is necessary.

Some recent sharp bounds on the height and weight of minor 5-stars in various subclasses of $\mathbf{P}_5$, along with several related results, can be found in [1–9, 11–15, 17, 20, 22] and surveys [4, 23].

In particular, Borodin, Ivanova and Nikiforov [13] obtained a sharp bound $h\left(S_5^{(m)}\right) \le 17$ under the absence of 6-vertices, which improves the upper bound 41 that follows from Theorem 1.

In 2013, Ivanova and Nikiforov [18] corrected two misprints in the statement of Theorem 1: 11 in $(5,11,5,6,8)$ should be replaced by 15, and in $(5,17,5,6,7)$ there should be 27 instead of 17. Later on, they improved [19, 26] thus corrected version of Theorem 1 by replacing 41 and 23 in the types $(5,5,7,5,41)$ and $(5,5,8,5,23)$ to 31 and 22, respectively.

Theorem 2 (Ivanova, Nikiforov [18, 19, 26])**.** *Every* 3*-polytope with minimum degree* 5 *contains a* 5*-vertex of one of the following types:*

$$(\overline{6,6,7,7,7}),\ (\overline{6,6,6,7,9}),\ (6,6,6,6,11),$$
$$(5,8,\overline{6,7,7}),\ (5,7,6,8,7),\ (5,6,\overline{6,7},11),\ (5,6,\overline{6,8},8),$$
$$(5,7,6,6,12),\ (5,8,6,6,10),\ (5,6,6,6,17),$$
$$(5,5,7,\overline{7,8}),\ (5,13,5,7,7),\ (5,10,5,7,8),\ (5,8,5,7,9),$$
$$(5,7,5,7,10),(5,7,5,8,8),\ (5,5,7,6,12),\ (5,5,8,6,10),$$
$$(5,6,5,7,12),\ (5,6,5,8,10),\ (5,27,5,6,7),\ (5,15,5,6,8),$$
$$(5,11,5,6,9),\ (5,7,5,6,13),\ (5,8,5,6,11),\ (5,9,5,6,10),$$

$$(5,6,6,5,\infty),$$
$$(5,5,7,5,31),\ (5,5,8,5,22),\ (5,5,9,5,17),\ (5,5,10,5,14),\ (5,5,11,5,13).$$

Recently, Li, Rao, and Wang [25] obtained two descriptions of minor 5-stars in plane graphs with minimum degree 5, in which some parameters are better and some are worse than in Theorems 1 and 2.

Recently, Borodin, Ivanova, and Kazak proved in [8] that forbidding vertices of degree from 7 to 11 in $\mathbf{P}_5$ results in a tight description $(\overline{5,5,6,6,\infty})$, $(\overline{5,6,6,6,15})$, $(6,6,6,6,6)$, which improves a description $(\overline{5,5,6,6,\infty})$, $(\overline{5,6,6,6,17})$, $(6,6,6,6,6)$ that follows from Theorem 1.

If vertices of degrees 6, 7, and 8 are forbidden, then Theorem 1 implies a 5-vertex of one of the following types: $(5,5,5,5,\infty)$, $(5,5,9,5,17)$, $(5,5,10,5,14)$, $(5,5,11,5,13)$. Recently, Borodin, Ivanova, and Kazak [10] proved a precise description of 5-stars in this subclass of $\mathbf{P}_5$: $(5,5,5,5,\infty)$ and $(5,5,10,5,12)$, where all parameters are best possible.

The purpose of this paper is to extend and strengthen the description in [10] as follows.

Theorem 3. *Every 3-polytope with minimum degree 5 and without vertices of degrees of 6 or 7 has a 5-vertex of one of the following types:* $(5,5,5,5,\infty)$, $(5,5,8,5,14)$, *or* $(5,5,10,5,12)$.

2. PROOF OF THEOREM 3

2.1. The tightness

To confirm the tightness of the term $(5,5,10,5,12)$, we start with the $(5,6,6)$-Archimedean solid, which is a cubic 3-polytope whose each vertex is incident with a 5-face and two 6-faces, replace all its vertices by small 3-faces, and cap each 10^+-face obtained.

The resulting 3-polytope has only 5-vertices, 10-vertices, and 12-vertices, and all 5-vertices are of type $(5,5,10,5,12)$ or $(5,5,12,5,12)$, as desired.

The construction confirming the tightness of $(5,5,5,5,\infty)$ is due to Jendrol' and Madaras [21].

To confirm the tightness of the term $(5,5,8,5,14)$ we start with the $(3,4,4,4)$ Archimedean solid $A(3,4,4,4)$, which is a 4-regular 3-polytope whose each vertex is incident with a 3-face and three 4-faces. Now cap each 4-face of $A(3,4,4,4)$ to obtain a triangulation T whose each face is incident with a 4-vertex and two 7^+-vertices. The dual D of T is a cubic 3-polytope, and we replace all its vertices by small 3-faces.

The resulting 3-polytope R is cubic and such that each vertex is incident with a 3-face, 8-face, and 14^+-face. Capping all 8^+-faces of R yields a desired

3-polytope in which every 5-vertex has a 14^+-neighbor and another 8^+-neighbor, where these two major neighbors are non-consecutive.

2.2. Discharging

Suppose that a 3-polytope P_5' is a counterexample to the main statement of Theorem 3. In particular, each 5-vertex in P_5' has at most three 5-neighbors and is adjacent either to at most two 5-vertices, or otherwise to two consecutive 8^+-vertices, or a 8-vertex non-consecutive with a 15^+-vertex, or a vertex of degree 9 or 10 non-consecutive with a 13+-vertex, or two non-consecutive 11^+-vertices.

Let P_5 be a counterexample with the most edges on the same vertices as P_5'.

Remark 4. P_5 has no 4^+-face with two non-consecutive 8^+-vertices along the boundary, for otherwise adding a diagonal between these vertices would result in a counterexample with a greater number of edges.

Let V, E, and F be the sets of vertices, edges, and faces of P_5. Euler's formula $|V| - |E| + |F| = 2$ implies

$$\sum_{v \in V} (d(v) - 6) + \sum_{f \in F} (2d(f) - 6) = -12. \tag{1}$$

We assign an *initial charge* $\mu(v) = d(v) - 6$ to each $v \in V$ and $\mu(f) = 2d(f) - 6$ to each $f \in F$, so that only 5-vertices have negative initial charge. Using the properties of P_5 as a counterexample to Theorem 3, we define a local redistribution of charges, preserving their sum, such that the final charge $\mu(x)$ is non-negative for all $x \in V \cup F$. This will contradict the fact that the sum of the final charges is, by (1), equal to -12.

The *final charge* $\mu'(x)$ whenever $x \in V \cup F$ is defined by applying the rules R1–R8 below (see Figure 1).

For a vertex v, let $v_1, \ldots, v_{d(v)}$ be the vertices adjacent to v in a cyclic order. A vertex is *simplicial* if it is completely surrounded by 3-faces. A 5-vertex v is *strong* if $d(v_1) = d(v_2) = 5$, $d(v_3) \ge 8$, $d(v_4) \ge 8$, $d(v_5) \ge 8$, and there is a 3-face vv_1v_2. Note that v also is incident to 3-faces v_3vv_4 and v_4vv_5 due to Remark 4.

A simplicial 5-vertex v such that $d(v_1) = d(v_2) = d(v_4) = 5$, $8 \le d(v_3) \le 10$, and hence $d(v_5) \ge 13$ is *poor*, and v_1 is *paired* with v.

We note that the poor and paired neighbors in the neighborhood of each 13^+-vertex w are in one-to-one correspondence with each other. Indeed, if w_2 were paired with two poor vertices w_1 and w_3, then w_2 would have four 5-neighbors, a contradiction. On the other hand, if w_1, w_2, w_3 are poor neighbors of w, where w_1 and w_2 have a common neighbor of degree from 8 to 10, then w_2 is paired with w_3, but not with w_1 due to a unique 3-face incident with three 5-vertices at a poor vertex. We also see that a paired vertex v_1 is poor itself if and only if v_2 is strong.

A simplicial 5-vertex v such that $d(v_1) = d(v_2) = d(v_3) = 5$, $d(v_4) = 8$, and hence $d(v_5) \geq 8$ is *bad*, and v_3 is *conjugate* with v. By symmetry, v_1 is also conjugate with v if $d(v_5) = 8$.

R1. *A* 4^+*-face* $f = v_1 \cdots v_{d(f)}$ *gives each incident* 5*-vertex* v_2:

(a) $\frac{1}{2}$ *if* $d(v_1) = d(v_3) = 5$, *or*

(b) $\frac{3}{4}$ *if* $d(v_1) \geq 8$ *and* $d(v_3) = 5$.

R2. *A* 5*-vertex* v *with* $d(v_1) \geq 8$ *receives the following charge from its* 8^+*-neighbor* v_2:

(a) *if* $d(v_3) = 5$, *then* $\frac{3}{8}$, $\frac{1}{2}$, $\frac{7}{12}$, *or* $\frac{3}{4}$ *in the cases* $d(v_2) = 8$, $9 \leq d(v_2) \leq 12$, $13 \leq d(v_2) \leq 14$, *or* $d(v_2) \geq 15$, *respectively, and*

(b) $\frac{1}{2}$ *if* $d(v_3) \geq 8$.

R3. *A non-simplicial* 5*-vertex* v *with* $d(v_1) = d(v_3) = d(v_4) = 5$ *receives* $\frac{1}{4}$ *from each of its* 8^+*-neighbors* v_2 *and* v_5.

R4. *A strong* 5*-vertex* v *with* $d(v_1) = d(v_2) = 5$ *gives* $\frac{1}{8}$ *or* $\frac{1}{6}$ *to* v_1 *if* $d(v_5) = 8$ *or* $d(v_5) \geq 9$, *respectively, and the same is valid for* v_2 *depending on* $d(v_3)$ *by symmetry.*

R5. *A simplicial* 5*-vertex* v *with* $d(v_1) = d(v_2) = d(v_4) = 5$ *receives from* v_5:

(a) $\frac{1}{4}$ *if* $d(v_5) = 8$,

(b) $\frac{1}{3}$ *if* $9 \leq d(v_5) \leq 10$, *and*

(c) $\frac{1}{2}$ *if* $11 \leq d(v_5) \leq 12$.

R6. *If a simplicial vertex* v *satisfies* $d(v_1) = d(v_2) = d(v_4) = 5$, $d(v_3) \geq 8$, *and* $d(v_5) \geq 13$, *then* v_5 *gives* $\frac{1}{2}$ *or* $\frac{5}{8}$ *to* v *if* $13 \leq d(v_5) \leq 14$ *or* $d(v_5) \geq 15$, *respectively, with the following two exceptions:*

(ex1) *if* $13 \leq d(v_5) \leq 14$, $9 \leq d(v_3) \leq 10$ *and* v_2 *is not strong* (*hence* v_2 *has three* 5*-neighbors and a* 13^+*-neighbor*), *then* v_5 *gives* $\frac{7}{12}$ *to* v;

(ex2) *if* $13 \leq d(v_5) \leq 14$, v_1 *is a poor vertex paired with* v, *and* v_2 *is not strong* (*so* v_2 *has three* 5*-neighbors*), *then* v_5 *also gives* $\frac{7}{12}$ *to* v.

R7. *Every poor* 5*-vertex* v *with a non-strong neighbor* v_2 *receives from its paired vertex* v_1:

(a) $\frac{1}{8}$ *if* v *has an* 8*-neighbor* v_3, *or*

(b) $\frac{1}{12}$ *if* $9 \leq d(v_3) \leq 10$.

R8. *If vertex* v *satisfies* $d(v_1) = d(v_3) = 5$, $d(v_2) \geq 8$, $d(v_4) \geq 8$ *and* $d(v_5) = 8$, *then* v *receives* $\frac{1}{4}$ *from* v_2.

R9. *A bad* 5*-vertex* v *receives* $\frac{1}{8}$ *from each conjugate vertex that is neither strong nor simplicial.*

R10. *If a bad 5-vertex v has a conjugate neighbor v_3 that is simplicial and non-strong (so v_3 is poor with a 15^+-neighbor), then v receives $\frac{1}{8}$ from the 5-vertex v_2 across the face v_2vv_3. By symmetry, the same holds for v_1 and v_1vv_2 if $d(v_5) = 8$.*

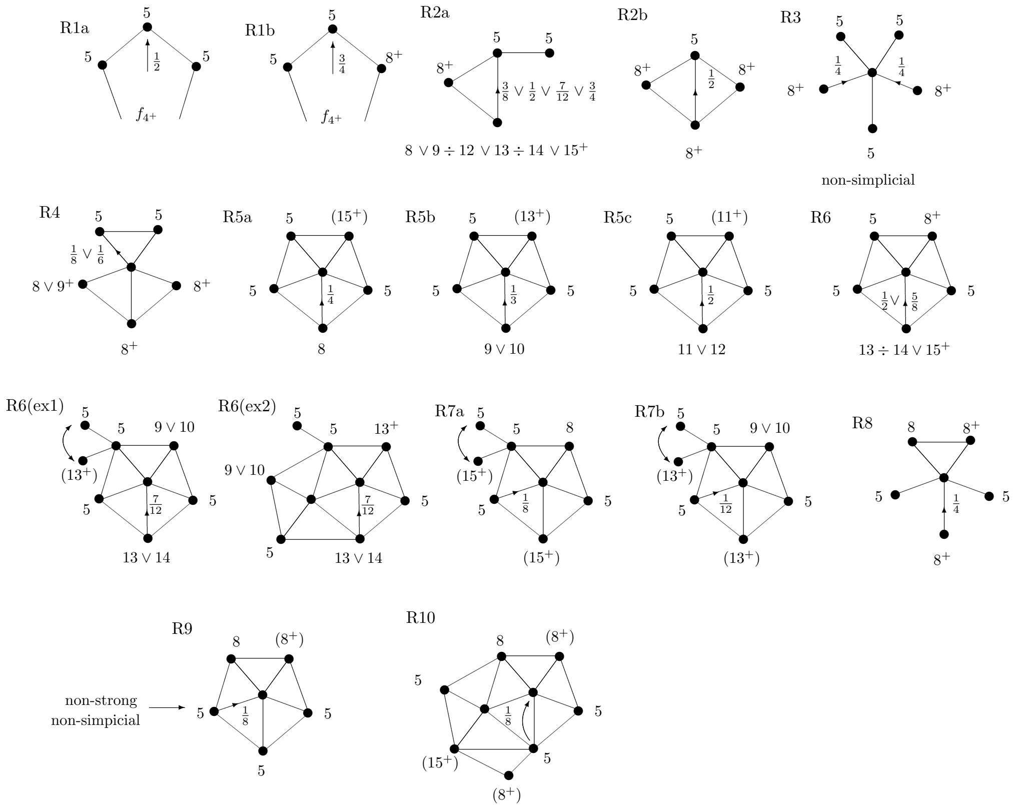

Figure 1. Rules of discharging.

2.3. Checking $\mu'(x) \geq 0$ whenever $x \in V \cup F$

If f is a 4^+-face, then the donation of $\frac{3}{4}$ by R1b may be interpreted as giving $\frac{1}{2}$ to the 5-vertex and $\frac{1}{4}$ to the neighbor 8^+-vertex along the boundary $\partial(f)$ of f. As a result, each vertex in $\partial(f)$ receives at most $2 \times \frac{1}{4}$ from f after this averaging, so we have $\mu'(f) \geq 2d(f) - 6 - d(f) \times \frac{1}{2} = \frac{3(d(f)-4)}{2} \geq 0$.

Now suppose $v \in V$.

Case 1. $d(v) = 5$. If v is adjacent to at least four 8^+-vertices, then $\mu'(v) \geq 5 - 6 + 4 \times \frac{3}{8} > 0$ by R2, since v does not give charge away by R4, R7, R9 or R10.

Suppose v has precisely three 8^+-neighbors. If they are consecutive round v, say v_1, v_2, v_3, then v receives at least $\frac{1}{2} + 2 \times \frac{3}{8} > 1$ from them by R2 in view of

Remark 4. Also, v can give $\frac{1}{8}$ or $\frac{1}{6}$ to each of the two 5-neighbors v_4 and v_5 by R4, and $\frac{1}{8}$ or $\frac{1}{12}$ to one of v_4 and v_5 by R7, if v is strong.

More specifically, if $d(v_3) = 8$ then v_4 receives $\frac{1}{8}$ from v while v receives $\frac{3}{8}$ from v_3 by R2a, so v_3 brings v the total of $\frac{1}{4} = \frac{3}{8} - \frac{1}{8}$. If $d(v_3) \geq 9$, then v_4 receives $\frac{1}{6}$ from v by R4 while v receives at least $\frac{1}{2}$ from v_3 by R2a, so v_3 actually brings at least $\frac{1}{3} = \frac{1}{2} - \frac{1}{6}$ to v.

Thus each of v_1 and v_3 thus brings v the total of at least $\frac{1}{4}$ by R2 combined with R4, while v_2 brings $\frac{1}{2}$ to v by R2b, so $\mu'(v) \geq -1 + \frac{1}{2} + 2 \times \frac{1}{4} = 0$ if v does not give charge by R7.

If v gives $\frac{1}{8}$ by R7a, then v receives $\frac{3}{4}$ from each of v_1, v_3 by R2a, so $\mu'(v) \geq -1 + \frac{1}{2} + 2 \times \frac{3}{4} - \frac{1}{8} - 2 \times \frac{1}{6} > 0$ in view of R2 and R4. If v gives $\frac{1}{12}$ by R7b, then v receives $\frac{7}{12}$ from each of v_1, v_3 by R2a, so $\mu'(v) \geq -1 + \frac{1}{2} + 2 \times \frac{7}{12} - \frac{1}{12} - 2 \times \frac{1}{6} > 0$ in view of R2 and R4.

Now suppose $d(v_1) = d(v_3) = 5$. Here, v does not give charge to v_1 and v_3 by R4 or R7, so it suffices for v to collect the total of at least 1 from its three 8^+-neighbors. If $d(v_4) \geq 9$ and $d(v_5) \geq 9$, then $\mu'(v) \geq -1 + 2 \times \frac{1}{2} = 0$ by R2a in view of Remark 4; otherwise, we have $d(v_4) = 8$ and $d(v_5) \geq 8$ by symmetry, which yields $\mu'(v) \geq -1 + 2 \times \frac{3}{8} + \frac{1}{4} = 0$ by R2a combined with R8, as desired.

It remains to assume that v has precisely two 8^+-neighbors due to the absence of $(5, 5, 5, 5, \infty)$-vertex. First suppose $d(v_4) \geq 8$ and $d(v_5) \geq 8$. If v is not simplicial, then $\mu'(v) \geq -1 + \frac{1}{2} + 2 \times \frac{3}{8} - 2 \times \frac{1}{8} = 0$ by R1, R2a, R4, R7 and R10. So suppose v is simplicial.

We next show that the total balance of v caused by donations from v_4 according to R2a, from v_3 due to R9, and from v_2 across the face v_2vv_3 by R10, in view of possible giving charge from v to a poor vertex v_3 by R7 and, when $d(v_4) \geq 15$, to a bad vertex v_2 by R10. By symmetry between v_4 and v_5 this will result in $\mu'(v) \geq -1 + 2 \times \frac{1}{2} = 0$.

First suppose $d(v_4) = 8$. Now v receives $\frac{3}{8}$ from v_4 by R2a and does not loose charge by R7, but can gives $\frac{1}{8}$ by R10. If v gives $\frac{1}{8}$ by R10, then $d(v_5) \geq 15$ and v receives $\frac{3}{4}$ from v_5 by R2a, so $\mu'(v) \geq -1 + \frac{3}{8} + \frac{3}{4} - \frac{1}{8} = 0$. If v does not give $\frac{1}{8}$ by R10, then the required $\frac{1}{8}$ comes from v_3 either by R4 if v_3 is strong, or by R9 if v_3 is not simplicial, or by R10 (the same is true for v_1), hence $\mu'(v) \geq -1 + 2 \times \frac{3}{8} + 2 \times \frac{1}{8} = 0$.

If $9 \leq d(v_4) \leq 12$, then it suffices to observe that v receives $\frac{1}{2}$ by R2a and does not give charge away by R7. If v gives $\frac{1}{8}$ by R10, then v receives $\frac{3}{4}$ from 15^+-neighbor by R2a, and we have $\mu'(v) \geq -1 + \frac{1}{2} + \frac{3}{4} - \frac{1}{8} > 0$.

When $13 \leq d(v_4) \leq 14$, our v receives $\frac{7}{12}$ by R2a and can give away $\frac{1}{12}$ by R7b if v is paired with a poor vertex v_3 or $\frac{1}{8}$ to v_2 by R10.

Finally, if $d(v_4) \geq 15$ then v receives $\frac{3}{4}$ by R2a and can give away $\frac{1}{8}$ to a poor vertex v_3 by R7b and also $\frac{1}{8}$ to a bad vertex v_2 by R10. So again the balance of v_3 is at least $\frac{1}{2} = \frac{3}{4} - 2 \times \frac{1}{8}$, as desired.

From now on suppose $d(v_1) \geq 8$ and $d(v_3) \geq 8$. If v is not simplicial, then v receives $2 \times \frac{1}{4}$ from v_1 and v_3 by R3 and at least $\frac{1}{2}$ from an incident 4^+-face by R1. Thus we are done unless v gives $\frac{1}{12}$ or $\frac{1}{8}$ to at least one of v_4 and v_5 by R7 or R9, which can happen only if the face $f = \cdots v_4 v v_5$ is a triangle. However, then v actually receives $\frac{3}{4}$ by R1b at least once, and we have $\mu'(v) \geq -1 + \frac{3}{4} + 2 \times \frac{1}{4} - 2 \times \frac{1}{8} = 0$.

Finally, suppose v is simplicial. Now v does not give charge by R9. If v gives $\frac{1}{8}$ or $\frac{1}{12}$ to v_5 by R7, so that v is paired with a poor vertex v_5, then $d(v_1) \geq 15$ or $d(v_1) \geq 13$, respectively, due to the absence $(5,5,5,8,14)$- and $(5,5,5,10,12)$-vertex by assumption. (Hereafter, we consider two possibilities in parallel, depending on whether v_5 has an 8-neighbor or a neighbor of degree 9 or 10.) Furthermore, v_4 is not strong, which implies that v_4 has a 5-neighbor different from v and v_5. In turn, this means that $d(v_3) \geq 15$ or $d(v_3) \geq 13$, respectively, since otherwise we would have a $(5,5,5,8,14)$-vertex or $(5,5,10,5,12)$-vertex, a contradiction.

Thus v receives from v_1 either $\frac{5}{8}$ by R6 or $\frac{7}{12}$ by R6ex2, respectively, and hence v_1 brings the total of $\frac{1}{2} = \frac{5}{8} - \frac{1}{8} = \frac{7}{12} - \frac{1}{12}$ to v. By symmetry, the same is true for v_3: no matter whether it is paired with v_4 or not, it brings $\frac{1}{2}$ either by R6 or by R6ex2 combined with R7.

Thus we have $\mu'(v) = -1 + 2 \times \frac{1}{2} = 0$ when v gives away $\frac{1}{8}$ or $\frac{1}{12}$ at least once to a poor neighbor according to R7, so from now we can assume that v is not a donator of charge by R7.

We know that each 11^+-neighbor gives v at least $\frac{1}{2}$ by R5c and R6, so it remains to assume that $d(v_1) \leq 10$, which means that v is poor.

First suppose $d(v_1) = 8$; then $d(v_3) \geq 15$ since we have no $(5,5,8,5,14)$-vertex by assumption. No matter whether v_5 is strong or otherwise, our v receives $\frac{1}{8}$ either from v_5 by R4 or from its paired vertex v_4 by R7a, respectively. Also, v receives $\frac{1}{4}$ from v_1 by R5a and $\frac{5}{8}$ from v_3 by R6a, so we have $\mu'(v) = 0$ in both options.

Now, if $9 \leq d(v_1) \leq 10$ then $d(v_3) \geq 13$ due to the absence $(5,5,10,5,12)$-vertex. Now if v_5 is strong, then v receives $\frac{1}{6}$ from v_5 by R4, $\frac{1}{3}$ from v_1 by R5b, and $\frac{1}{2}$ from v_3 by R6a, so we have $\mu'(v) = 0$. Otherwise, v receives $\frac{1}{12}$ from v_4 by R7b and $\frac{1}{3}$ from v_1. Also, v receives from v_3 either $\frac{7}{12}$ by R6ex1 if $d(v_3) \leq 14$ or $\frac{5}{8}$ (which is greater than $\frac{7}{12}$) by R6 if $d(v_3) \geq 15$. This again makes $\mu'(v) \geq 0$, as desired.

Finally, if $11 \leq d(v_1) \leq 12$ and $11 \leq d(v_3) \leq 12$, then $\mu'(v) = 0$ by R5c.

Case 2. $d(v) = 8$. We can average donations of v to its 5-neighbors according to R2, R3, R5a, and R8 as follows. If $d(v_1) = d(v_2) = 5$ and $d(v_3) \geq 8$, which is the situation of R2a, then v instead gives $\frac{1}{4}$ to v_2 and $\frac{1}{8}$ to v_3. Similarly, instead of giving $\frac{1}{2}$ to a 5-neighbor v_2 by R2b, our v now gives $\frac{1}{4}$ to v_2 and $\frac{1}{8}$ to each of the 8^+-vertices v_1 and v_3. As a result, each neighbor receives at most

$\frac{1}{4} = \frac{1}{8}+\frac{1}{8} = \frac{3}{8}-\frac{1}{8}$ from v after averaging, so $\mu'(v) \geq d(v)-6-\frac{d(v)}{4} = \frac{3(d(v)-8)}{4} \geq 0$, as desired.

Case 3. $9 \leq d(v) \leq 10$. We now average donations of v to its 5-neighbors according to R2, R3, R5b, and R8 in the same fashion. Instead of giving $\frac{1}{2}$ to a 5-neighbor v_2 by R2b, our v gives $\frac{1}{6}$ to each of the vertices v_1, v_2, and v_3. If $d(v_1) = d(v_2) = 5$ and $d(v_3) \geq 9$, which happens in R2a, then v rather gives $\frac{1}{3}$ to v_2 and $\frac{1}{6}$ to v_3. As a result, each neighbor receives at most $\frac{1}{3} = \frac{1}{6} + \frac{1}{6} = \frac{1}{2} - \frac{1}{6}$ from v, so $\mu'(v) \geq d(v) - 6 - \frac{d(v)}{3} = \frac{2(d(v)-9)}{3} \geq 0$, and we are done.

Case 4. $11 \leq d(v) \leq 12$. We note that v gives each neighbor at most $\frac{1}{2}$ by R2, R3, R5c, and R8, so $\mu'(v) \geq d(v) - 6 - \frac{d(v)}{2} = \frac{d(v)-12}{2}$, which settles the case $d(v) = 12$.

So suppose $d(v) = 11$. If v has an 8^+-neighbor, then $\mu'(v) \geq 11-6-10\times\frac{1}{2} = 0$. Thus we can assume that v is completely surrounded by 5-vertices. If v is incident with a 4^+-face $\cdots v_1 v v_2$, then each of v_1 and v_2 is non-simplicial and hence can only receive $\frac{1}{4}$ from v by R3 or R8. Indeed, if the neighbors of v_1 in a cyclic order are $\ldots, x_1, v, y_1$, then $d(x_1) = d(y_1) = 5$ due to Remark 1, and the same argument works for v_2. This implies $\mu'(v) \geq 5 - 2 \times \frac{1}{4} - (11-2) \times \frac{1}{2} = 0$.

Therefore, it remains to assume in addition that v is simplicial. Now if there is a 4^+-face $\cdots v_1' v_1 v_2 v_2'$, then each of v_1 and v_2 receives at most $\frac{1}{4}$ from v: either by R3, which happens when v_1 has three 5-neighbors, or possibly by R8, otherwise. So again $\mu'(v) \geq 0$.

Thus we are done unless there are vertices $w_1, \ldots, w_{11}$ lying in 3-faces $w_k v_k v_{k+1}$ whenever $1 \leq k \leq 11$ (addition mod 11 throughout proving Case 4). If so, then we cannot have $d(w_k) \leq 8 \geq d(w_{k+1})$ for any k, for otherwise $w(S_5(v_{k+1})) \leq 3 \times 5 + 2 \times 8 + 11 = 42$, which is impossible. By the oddness of 11, this implies that, say, $d(w_1) \geq 9$ and $d(w_2) \geq 9$. It follows from Remark 1 that there is a 3-face $w_1 v_2 w_2$, and it suffices to observe that v gives no charge to v_2 by R8 or any other our rule. Hence we have $\mu'(v) \geq 5 - 10 \times \frac{1}{2} = 0$.

Case 5. $13 \leq d(v) \leq 14$. We know that v gives at most $\frac{7}{12}$ to each adjacent 5-vertex by R1–R8. Since $\mu(v) = d(v) - 6 - \frac{7d(v)}{12} = \frac{5d(v)-72}{12}$, it follows that $\mu'(v) \geq -\frac{2}{12}$ for $d(v) = 14$, and $\mu'(v) \geq -\frac{7}{12}$ for $d(v) = 13$. Therefore, we use some additional reasons to improve these rough estimations in order to prove $\mu'(v) \geq 0$.

First of all, we can assume that v is completely surrounded by 5-vertices, for otherwise $\mu'(v) \geq d(v) - 6 - \frac{7(d(v)-1)}{12} = \frac{5(d(v)-13)}{12} \geq 0$, as desired.

Secondly, if v is not simplicial then v gives at most $\frac{1}{4}$ to each of at least two vertices incident with a common 4^+-face with v due to the argument used in Case 4, which means that in fact $\mu'(v) \geq d(v) - 6 - 2 \times \frac{1}{4} - \frac{7(d(v)-2)}{12} \geq \frac{5(d(v)-13)}{12} + \frac{1}{12} > 0$.

Thus we are done unless v is simplicial and completely surrounded by 5-vertices. Furthermore, if there is a 4^+-face $\cdots v_1'v_1v_2v_2'$, then we similarly have $\mu'(v) \geq \frac{1}{12}$.

So again there is a cyclic sequence $W_{d(v)} = w_1, \ldots, w_{d(v)}$ such that there are 3-faces $w_kv_kv_{k+1}$ whenever $1 \leq k \leq d(v)$ (addition mod $d(v)$). As before, there are no two consecutive 5-vertices in $W_{d(v)}$ since each v_k must have an 8^+-neighbor other than v.

If there is an 8-vertex in $W_{d(v)}$, say w_2, then $d(w_1) \geq 8$ and $d(w_3) \geq 8$, since $43-3\times5-13-8 = 7$. Thus, in fact each of v_2 and v_3 receives at most $\frac{1}{4}$ from v by R3, R8 rather than $\frac{7}{12}$, and we again have $\mu'(v) \geq d(v)-6-2\times\frac{1}{4}-\frac{7(d(v)-2)}{12} > 0$, as above. In what follows, we can assume that $d(w_i) \geq 9$ or $d(w_i) = 5$ whenever $1 \leq k \leq d(v)$.

If there are two consecutive 9^+-vertices in $W_{d(v)}$, say w_1 and w_2, then v_2 receives no charge from v by R1–R8, so we can improve our rough estimation $\mu'(v) \geq -\frac{7}{12}$ to $\mu'(v) \geq -\frac{7}{12} + \frac{7}{12} \geq 0$, as desired. This completes the proof for $d(v) = 13$ due to the oddness of 13.

So suppose $d(v) = 14$, all neighbors of v are simplicial, and $d(w_1) = d(w_3) = \cdots = d(w_{13}) = 5$, for otherwise v gives at most $\frac{1}{4}$ to one of its neighbors, and we already have $\mu'(v) \geq -\frac{2}{12} + \frac{7}{12} - \frac{1}{4} > 0$.

Now if at least one of 5-vertices in W_{14}, say w_1, is strong, that is w_1 has an 8^+-neighbor outside W_{14}, then each of v_1 and v_2 receives $\frac{1}{2}$ by R6a rather than $\frac{7}{12}$ by R6ex1 or R6ex2, which yields $\mu'(v) \geq 8 - 2\times\frac{1}{2} - 12\times\frac{7}{12} = 0$.

Thus we can assume that all $w_1, w_3, \ldots, w_{13}$ are non-strong, that is each of them has a 5-neighbor outside W_{14}. Among the seven 9^+-vertices $w_2, w_4, \ldots, w_{14}$, there are no two consecutive (cyclically) 10^--vertices, for otherwise we would have a minor 5-star with weight at most 40, which is impossible.

By parity reasons and symmetry, we can assume that $d(w_{14}) \geq 11$ and $d(w_2) \geq 11$. So each of v_1 and v_2 obeys the general rule R6 rather than its exceptions R6ex1 or R6ex2. This means that again $\mu'(v) \geq 14-6-2\times\frac{1}{2}-12\times\frac{7}{12} = 0$, as desired.

Case 6. $d(v) \geq 15$. We know that v gives at most $\frac{5}{8}$ to each adjacent 5-vertex by R1–R8, except for giving $\frac{3}{4}$ in R2a.

We now average these donations so that each 8^+-neighbor will receive at most $2\times\frac{1}{8}$ from v, while each 5-neighbor will receive at most $\frac{5}{8}$. To this end, it suffices to switch $\frac{1}{8}$ from the donation of $\frac{3}{4}$ to a 5-vertex v_2 by R2a to the neighbor 8^+-vertex v_1.

Since $\mu(v) = d(v)-6-\frac{5d(v)}{8} = \frac{3(d(v)-16)}{8}$, it follows that our averaging results in $\mu'(v) \geq 0$ for $d(v) \geq 16$.

Finally, suppose $d(v) = 15$. If v has an 8^+-neighbor or a non-simplicial 5-neighbor, then $\mu'(v) \geq 15 - 6 - \frac{1}{4} - 14\times\frac{5}{8} = 0$ by R1–R8.

Thus we can assume that v is completely surrounded by simplicial 5-vertices, which means that the sequence W_{15} introduced in Case 5 is actually a 15-cycle. Again, W_{15} has no two consecutive 5-vertices, which implies by parity reasons and symmetry that $d(w_1) \geq 8$ and $d(w_2) \geq 8$. Since v_2 receives $\frac{1}{4}$ from v by R8 and nothing by any other our rule, we are done.

Thus we have proved $\mu'(x) \geq 0$ whenever $x \in V \cup F$, which contradicts (1) and completes the proof of Theorem 3.

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