

## TREES WHOSE EVEN-DEGREE VERTICES INDUCE A PATH ARE ANTIMAGIC

ANTONI LOZANO

*Computer Science Department*  
*Universitat Politècnica de Catalunya, Spain*

**e-mail:** antoni@cs.upc.edu

MERCÈ MORA, CARLOS SEARA

*Mathematics Department*  
*Universitat Politècnica de Catalunya, Spain*

**e-mail:** merce.mora@upc.edu  
carlos.seara@upc.edu

AND

JOAQUÍN TEY

*Mathematics Department*  
*Universidad Autónoma Metropolitana-Iztapalapa, México*

**e-mail:** jtey@xanum.uam.mx

### Abstract

An *antimagic labeling* of a connected graph  $G$  is a bijection from the set of edges  $E(G)$  to  $\{1, 2, \dots, |E(G)|\}$  such that all vertex sums are pairwise distinct, where the *vertex sum* at vertex  $v$  is the sum of the labels assigned to edges incident to  $v$ . A graph is called *antimagic* if it has an antimagic labeling. In 1990, Hartsfield and Ringel conjectured that every simple connected graph other than  $K_2$  is antimagic; however the conjecture remains open, even for trees. In this note we prove that trees whose vertices of even degree induce a path are antimagic, extending a result given by Liang, Wong, and Zhu [*Anti-magic labeling of trees*, *Discrete Math.* 331 (2014) 9–14].

**Keywords:** antimagic labeling, tree.

**2010 Mathematics Subject Classification:** 05C78, 05C85, 68R10.

## 1. INTRODUCTION

All graphs considered in this work are finite, undirected and simple. Given a graph  $G = (V(G), E(G))$  and a vertex  $v \in V(G)$ , we denote by  $E_G(v)$  the set of edges incident to  $v$  and by  $d_G(v) = |E_G(v)|$  the degree of  $v$  in  $G$ . We will just write  $E(v)$  and  $d(v)$  when  $G$  is clear from the context. A tree is a connected and acyclic graph, and a forest is a disjoint union of trees. Undefined terms in this work can be found in [2].

An (*edge*) *labeling* of a graph  $G$  is a mapping from  $E(G)$  to the set of non-negative integers. A labeling  $\phi$  of a connected graph  $G$  is called *antimagic* if it is a bijection  $\phi : E(G) \rightarrow \{1, 2, \dots, |E(G)|\}$  such that all vertex sums are pairwise distinct, where the *vertex sum*  $s(v)$  at vertex  $v \in V(G)$  is defined as  $s(v) = \sum_{e \in E(v)} \phi(e)$ . A graph is called *antimagic* if it has an antimagic labeling.

In 1990, Hartsfield and Ringel [5] conjectured that every simple connected graph other than  $K_2$  is antimagic. The conjecture has received much attention (see [4]); but it is widely open in general, even for trees. Nevertheless, several classes of trees are known to be antimagic (see [1, 3, 5–10]).

Given a tree  $T$ ,  $V_{\text{even}}(T)$  (respectively,  $V_{\text{odd}}(T)$ ) denotes the set of even (respectively, odd) degree vertices of  $T$ . Regarding trees such that  $V_{\text{even}}$  induces a path, Liang, Wong, and Zhu [7] proved the following two theorems.

**Theorem 1** [7]. *If  $T$  is a tree such that  $V_{\text{even}}(T)$  induces a path and  $|V_{\text{even}}(T)|$  is odd, then  $T$  is antimagic.*

**Theorem 2** [7]. *Let  $T$  be a tree such that  $V_{\text{even}}(T)$  induces a path of order  $2p$ ,  $(v_1, \dots, v_{2p})$ . Let  $v_0$  (respectively,  $v_{2p+1}$ ) be a neighbor of  $v_1$  (respectively,  $v_{2p}$ ) different from  $v_2$  (respectively,  $v_{2p-1}$ ). If  $d(v_p) \neq d(v_{2p+1}) + 1$  or  $d(v_{p+1}) \neq d(v_0) + 1$ , then  $T$  is antimagic.*

The aim of this note is to extend Theorem 2 to all cases, that is, to prove the antimagicness of trees such that  $V_{\text{even}}(T)$  induces a path whenever  $|V_{\text{even}}(T)|$  is even, obtaining as a consequence that trees whose even-degree vertices induce a path are antimagic.

## 2. CONSTRUCTING AN ANTIMAGIC LABELING

In the proof of the next theorem we follow and extend the main idea developed by Liang, Wong, and Zhu in [7]. We denote by  $[a, b]$  the set of consecutive integers  $\{a, a + 1, \dots, b\}$ , where  $a \leq b$ .

**Theorem 3.** *If  $T$  is a tree such that  $V_{\text{even}}(T)$  induces a path and  $|V_{\text{even}}(T)|$  is even, then  $T$  is antimagic.*

**Proof.** Let  $|V_{\text{even}}(T)| = 2p$ , it is known that trees without vertices of degree 2 are antimagic [6, 7], hence we may assume  $p \geq 1$ . Let  $P = (v_0, v_1, v_2, \dots, v_{2p}, v_{2p+1})$  be a path induced by  $V_{\text{even}}(T)$  extended with endpoints in  $V_{\text{odd}}(T)$ , that is,  $V_{\text{even}}(T) = \{v_1, \dots, v_{2p}\}$  and  $\{v_0, v_{2p+1}\} \subseteq V_{\text{odd}}(T)$ . For every  $1 \leq i \leq 2p + 1$ , we denote by  $e_i$  the edge  $v_{i-1}v_i$ .

We prove the theorem by constructing an antimagic labeling  $\phi$  of  $T$  in two steps. The first step produces a labeling of a subtree of  $T$  containing the path  $P$  and satisfying a particular additional condition. This labeling will be extended to an antimagic labeling of  $T$  at the second step.

Let  $m = |E(T)|$ . We will use the residues modulo  $m + 2$  to compare vertex sums: since vertex sums are distinct if they are distinct modulo  $m + 2$ , it is enough to compare vertex sums whenever they are equal modulo  $m + 2$  in order to check that they all are pairwise distinct.

For each (not necessarily connected) subgraph  $T'$  of  $T$ , we set  $L_\phi(T') = \{\phi(e) : e \in E(T')\}$  and  $s_{T'}(v) = \sum_{e \in E_{T'}(v)} \phi(e)$  for every  $v \in V(T')$  such that  $d_{T'}(v) \geq 1$ . Obviously, if  $T' = T$ , then  $s_{T'} = s$ . The set of all vertex sums modulo  $m + 2$  in  $T'$  will be denoted by  $R_{m+2}(T')$ , that is,

$$R_{m+2}(T') = \{s_{T'}(v) \pmod{m+2} : v \in V(T') \text{ and } d_{T'}(v) \geq 1\} \subseteq \{0, 1, \dots, m+1\}.$$

**Step I.** The labeling of the tree  $T_1$  constructed at this step will satisfy the following condition: all vertex sums in  $T_1$  will be pairwise distinct modulo  $m + 2$  with at most one exception; moreover, if the vertex sums are equal modulo  $m + 2$  for a pair of vertices, then exactly one of them will be a leaf in  $T$ , and the vertex sums in  $T_1$  for both vertices in the pair will be different.

As a starting point, let  $T_1 := P$  and define

$$\begin{aligned} \phi(e_{2i+1}) &:= i + 1, & \text{for } 0 \leq i \leq p; \\ \phi(e_{2i}) &:= m - p + i, & \text{for } 1 \leq i \leq p. \end{aligned}$$

Hence,

$$(1) \quad L_\phi(T_1) = [1, p + 1] \cup [m - p + 1, m].$$

Moreover,  $s_{T_1}(v_0) = 1$ ,  $s_{T_1}(v_{2p+1}) = p + 1$ , and for  $1 \leq i \leq 2p$ ,  $s_{T_1}(v_i) = m - p + i + 1$ . Next, we calculate the set  $R_{m+2}(T_1)$  according to the values of  $p$  and to the degrees of  $v_0$  and  $v_{2p+1}$ .

*Case 1.*  $p = 1$ . In this case,  $|V(T_1)| = 4$  and

$$(2) \quad R_{m+2}(T_1) = \{0, 1, 2, m + 1\}.$$

Hence, vertex sums at the vertices of  $T_1$  are distinct.

Case 2.  $p > 1$ . In such a case,

$$(3) \quad R_{m+2}(T_1) = ([0, p + 1] \setminus \{p\}) \cup [m - p + 2, m + 1],$$

and only the residues of vertex sums at vertices  $v_0$  and  $v_{p+2}$  are equal. In fact, we have that  $s_{T_1}(v_0) \equiv 1 \equiv s_{T_1}(v_{p+2}) \pmod{m + 2}$  (see an example in Figures 1(a) and 2(a)).

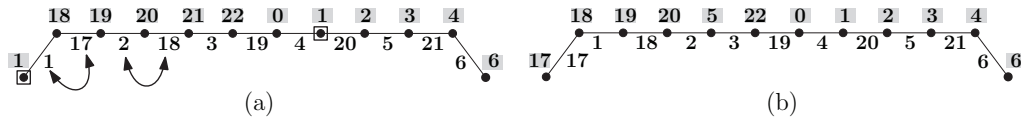


Figure 1. Labeling of  $T_1$  for  $p = 5$  and  $m = 21$ ; (a) before the swaps, and (b) after the swaps. The shadowed label at each vertex is the vertex sum modulo 23. Squared vertices have the same label.

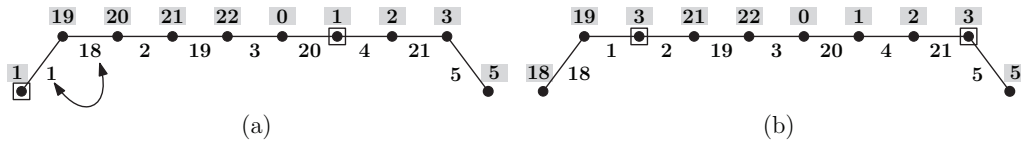


Figure 2. Labeling of  $T_1$  for  $p = 4$  and  $m = 21$ ; (a) before the swap, and (b) after the swap. The shadowed label at each vertex is the vertex sum modulo 23. Squared vertices have the same label.

Now we distinguish two subcases.

Subcase 2.1. At least one of the vertices  $v_0$  or  $v_{2p+1}$  is a leaf in  $T$ . Notice that by properly relabeling the vertices of  $T_1$ , we may assume  $d_T(v_0) = 1$ . Then,  $s_{T_1}(v_0) \equiv 1 \equiv s_{T_1}(v_{p+2}) \pmod{m + 2}$ , but  $s_{T_1}(v_0) = 1 < m + 3 = s_{T_1}(v_{p+2})$ .

Subcase 2.2. Neither  $v_0$  nor  $v_{2p+1}$  are leaves in  $T$ . In this case, for  $1 \leq i \leq \lfloor \frac{p-1}{2} \rfloor$ , we swap the labels of the edges  $e_{2i-1}$  and  $e_{2i}$ , that is,

$$\begin{aligned} \phi(e_{2i-1}) &:= m - p + i, & \text{for } 1 \leq i \leq \lfloor \frac{p-1}{2} \rfloor; \\ \phi(e_{2i}) &:= i, & \text{for } 1 \leq i \leq \lfloor \frac{p-1}{2} \rfloor. \end{aligned}$$

Notice that the endpoints of the subpath of  $T_1$  involved in the swaps are  $v_0$  and  $v_k$ , where  $k = p - 2$ , if  $p$  is even; and  $k = p - 1$ , if  $p$  is odd. After the swaps,  $s_{T_1}(v_0) = m - p + 1$ ; also it can be easily checked that  $s_{T_1}(v_k) = p - 1$ , if  $p$  is even; and  $s_{T_1}(v_k) = p$ , if  $p$  is odd; and the vertex sum at any other vertex in  $T_1$  remains unchanged. We distinguish cases depending on the parity of  $p$ .

(a)  $p$  odd. In this case we have that  $s_{T_1}(v_k) = p$ , implying that the residues modulo  $m + 2$  at the vertices of  $T_1$  are pairwise distinct. Concretely,

$$(4) \quad R_{m+2}(T_1) = [0, p + 1] \cup ([m - p + 1, m + 1] \setminus \{m\}).$$

(b) *p even*. In this case, even if  $p = 2$  (in which case no swaps take place), we have that  $s_{T_1}(v_k) = p - 1 \equiv s_{T_1}(v_{2p}) \pmod{m + 2}$ , and thus

$$(5) \quad R_{m+2}(T_1) = ([0, p + 1] \setminus \{p\}) \cup ([m - p + 1, m + 1] \setminus \{m - 1\}).$$

Notice that only  $s_{T_1}(v_k)$  and  $s_{T_1}(v_{2p})$  have the same residue in  $T_1$  (see an example in Figure 2(b)). Now, let  $x_0 = v_{2p+1}$  and let  $P' = (x_0, x_1, \dots, x_\ell)$  be a maximal subpath of  $T$  starting at  $x_0$  and with vertices in  $V_{\text{odd}}(T)$ . Observe that, in such a case,  $x_\ell$  is a leaf in  $T$  and there exist vertices  $y_0, \dots, y_{\ell-1} \in V_{\text{odd}}(T)$  such that  $x_i y_i \in E(T)$  (see an example in Figure 3). We update  $T_1$  as the tree induced by the set of vertices of the paths  $P$  and  $P'$ , and  $\{y_0, \dots, y_{\ell-1}\}$ :

$$T_1 := T[\{v_0, \dots, v_{2p}\} \cup \{x_0, \dots, x_\ell\} \cup \{y_0, \dots, y_{\ell-1}\}].$$

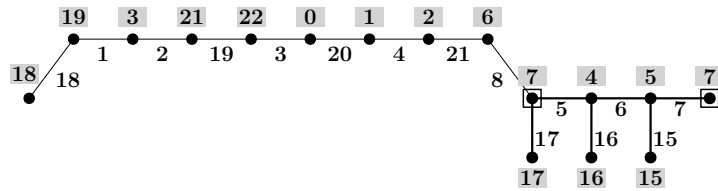


Figure 3. Labeling of  $T_1$  with  $p = 4$ ,  $\ell = 3$ , and  $m = 21$ . The shadowed label at each vertex is the vertex sum modulo 23. Squared vertices have the same label.

We define the labels of the new edges and update the label of the edge  $e_{2p+1}$  as follows:

$$\begin{aligned} \phi(e_{2p+1}) &:= p + \ell + 1; \\ \phi(x_i x_{i+1}) &:= p + i + 1, \quad \text{for } 0 \leq i \leq \ell - 1; \\ \phi(x_i y_i) &:= m - p - i, \quad \text{for } 0 \leq i \leq \ell - 1. \end{aligned}$$

Thus, we have

$$(6) \quad L_\phi(T_1) = [1, p + \ell + 1] \cup [m - p - \ell + 1, m],$$

and

$$\begin{aligned} s_{T_1}(x_i) &= m + p + i + 1 \equiv p + i - 1 \pmod{m + 2}, \quad \text{for } 1 \leq i \leq \ell - 1, \\ s_{T_1}(v_{2p}) &= m + p + \ell + 1 \equiv p + \ell - 1 \pmod{m + 2}, \\ s_{T_1}(x_0) &= m + p + \ell + 2 \equiv p + \ell \pmod{m + 2}, \\ s_{T_1}(x_\ell) &= p + \ell \equiv p + \ell \pmod{m + 2}, \\ s_{T_1}(y_i) &= m - p - i \equiv m - p - i \pmod{m + 2}, \quad \text{for } 0 \leq i \leq \ell - 1. \end{aligned}$$

Therefore, taking into account equality (5);

$$(7) \quad R_{m+2}(T_1) = [0, p + \ell] \cup ([m - p - \ell + 1, m + 1] \setminus \{m - 1\}),$$

where  $m \geq 2p + 2\ell + 1$ . So, only  $s_{T_1}(x_0)$  and  $s_{T_1}(x_\ell)$  have the same residue, concretely  $p + \ell$ . However,  $s_{T_1}(x_0) = m + p + \ell + 2 > p + \ell = s_{T_1}(x_\ell)$ , and hence all vertex sums in  $T_1$  are different (see an example in Figure 3).

Notice that, in each of the above cases,  $|L_\phi(T_1)| = |E(T_1)|$ . Hence  $\phi$ , restricted to  $E(T_1)$ , is a bijection from  $E(T_1)$  to  $L_\phi(T_1)$ .

**Step II.** Now, let  $T_2$  be the forest obtained by removing all the edges of  $T_1$ . Each component of  $T_2$  has exactly one vertex in  $T_1$ . Therefore, if  $T_2(v)$  denotes the component of  $T_2$  containing  $v$ ,

$$T_2 := T - E(T_1) = \bigcup_{v \in V(T_1)} T_2(v).$$

Clearly,  $T_2(v)$  can be viewed as a directed rooted tree with root at  $v$ , where every edge is directed away from the root. Moreover, since every vertex of  $T_2(v)$  different from  $v$  has odd degree in  $T$ , each vertex in  $T_2(v)$  has an even number of children in this rooted tree and, therefore,  $|E(T_2(v))|$  is even for every  $v \in V(T_1)$ . Hence,  $|E(T_2)|$  is even. If we set  $\ell = 0$  whenever  $T_1 = P$ , then, by equalities (1) and (6), the available labels for the edges of  $T_2$  are

$$(8) \quad L_\phi(T_2) := [p + \ell + 2, m - p - \ell].$$

As  $L_\phi(T_2) = [a, b]$ , where  $a + b = m + 2$ , and each  $w \in V(T_2)$  has an even number of children, we can label the edges of  $T_2$  with integers in  $L_\phi(T_2)$  fulfilling the following additional condition: if a vertex  $w$  has an outgoing edge with label  $t$  in the corresponding rooted tree of  $T_2$ , then  $w$  has another outgoing edge with label  $m + 2 - t$  (see an example in Figure 4).

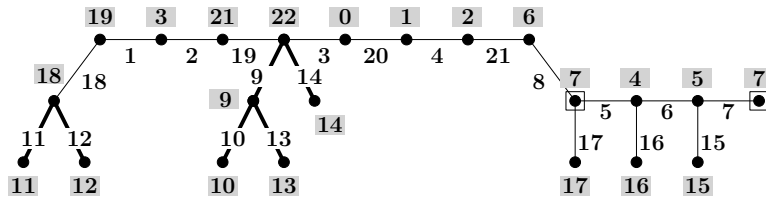


Figure 4. An antimagic labeling of a tree with  $m = 21$ . Thicker edges correspond to the forest  $T_2$  and are labeled in Step II (in this example, the forest  $T_2$  has two nontrivial components). The shadowed label at each vertex is the vertex sum modulo 23. Squared vertices have the same label, but different vertex sums.

Clearly, by the previous discussion, the labeling  $\phi$  already constructed is a bijection from  $E(T)$  to  $[1, m]$ . Finally, we just need to show that the vertex sums defined by  $\phi$  in  $T$  are pairwise distinct.

Observe that in  $T_2$ , the sum of the labels of the outgoing edges of  $v$  is a multiple of  $m + 2$ . Thus, the following two conditions hold.

1. For every  $v \in V(T_1)$ ,  $s(v) \equiv s_{T_1}(v) \pmod{m+2}$ .
2. For every  $v \in V(T) \setminus V(T_1)$ ,  $s(v) \equiv \phi(f) \pmod{m+2}$ , where  $f$  is the incoming edge of  $v$  in  $T_2$ .

Let  $u, v \in V(T)$ . We consider the following cases.

(a)  $u, v \in V(T) \setminus V(T_1)$ . Since  $\phi : E(T) \rightarrow [1, m]$  is a bijection, condition 2 implies that  $s(u) \neq s(v)$ .

(b)  $u, v \in V(T_1)$ . If  $s_{T_1}(u) \not\equiv s_{T_1}(v) \pmod{m+2}$ , then by condition 1 we have that  $s(u) \neq s(v)$ . Otherwise, as we have seen in Cases 2.1 and 2.2(b) of Step I, just one of these vertices, say  $u$ , is a leaf in  $T$  and  $s_{T_1}(u) < s_{T_1}(v)$ . Therefore, we have that  $s(u) = s_{T_1}(u) < s_{T_1}(v) \leq s(v)$ , as we wanted to prove.

(c) One of the vertices belongs to  $V(T_1)$  and the other to  $V(T) \setminus V(T_1)$ . We can assume without loss of generality that  $u \in V(T_1)$  and  $v \in V(T) \setminus V(T_1)$ . By condition 1,  $s(u) \pmod{m+2} \in R_{m+2}(T_1)$ . Moreover, by condition 2,  $s(v) \pmod{m+2} \in L_\phi(T_2)$ . By equalities (2), (3), (4), (7), and (8), we have  $R_{m+2}(T_1) \cap L_\phi(T_2) = \emptyset$ . Hence,  $s(u) \neq s(v)$ .

Thus, the theorem holds. ■

The next result follows from Theorems 1 and 3.

**Corollary 4.** *If  $T$  is a tree such that  $V_{\text{even}}(T)$  induces a path, then  $T$  is antimagic.*

### Acknowledgments

A. Lozano is supported by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement ERC-2014-CoG 648276 AUTAR); M. Mora is supported by projects Gen. Cat. DGR 2017SGR1336, MINECO MTM2015-63791-R, and H2020-MSCA-RISE project 734922-CONNECT; and C. Seara is supported by projects Gen. Cat. DGR 2017SGR1640, MINECO MTM2015-63791-R, and H2020-MSCA-RISE project 734922-CONNECT.

### REFERENCES

- [1] F.H. Chang, P. Chin, W.T. Li and Z. Pan, *The strongly antimagic labelings of double spiders* (2017).  
arXiv:1712.09477
- [2] G. Chartrand, L. Lesniak and P. Zhang, *Graphs and Digraphs* (CRC Press Boca Raton, 2011).
- [3] K. Deng and Y. Li, *Caterpillars with maximum degree 3 are antimagic*, *Discrete Math.* **342** (2019) 1799–1801.  
<https://doi.org/10.1016/j.disc.2019.02.021>

- [4] J.A. Gallian, *Graph labeling*, Electron. J. Combin. **2018** (#DS6)  
[.https://doi.org/10.37236/27](https://doi.org/10.37236/27)
- [5] N. Hartsfield and G. Ringel, *Pearls in Graph Theory* (Academic Press INC., Boston, 1990 (revised version, 1994).
- [6] G. Kaplan, A. Lev and Y. Roditty, *On zero-sum partitions and anti-magic trees*, Discrete Math. **309** (2009) 2010–2014.  
<https://doi.org/10.1016/j.disc.2008.04.012>
- [7] Y.-Ch. Liang, T.-L. Wong and X. Zhu, *Anti-magic labeling of trees*, Discrete Math. **331** (2014) 9–14.  
<https://doi.org/10.1016/j.disc.2014.04.021>
- [8] A. Lozano, M. Mora and C. Seara, *Antimagic labelings of caterpillars*, Appl. Math. Comput. **347** (2019) 734–740.  
<https://doi.org/10.1016/j.amc.2018.11.043>
- [9] A. Lozano, M. Mora, C. Seara and J. Tey, *Caterpillars are antimagic* (2018).  
arXiv:1812.06715
- [10] J.L. Shang, *Spiders are antimagic*, Ars Combin. **118** (2015) 367–372.

Received 30 August 2019

Revised 21 March 2020

Accepted 25 March 2020



This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 734922.