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# EXTREMAL DIGRAPHS AVOIDING DISTINCT WALKS OF LENGTH 4 WITH THE SAME ENDPOINTS

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#### Abstract

Let  $n \ge 8$  be an integer. We characterize the extremal digraphs of order n with the maximum number of arcs avoiding distinct walks of length 4 with the same endpoints.

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### 1. INTRODUCTION

The Turán-type problem is one of the hottest topics in extremal graph theory, which concerns the number of edges in graphs containing no given subgraphs and the extremal graphs achieving this maximum. Most of the previous results of Turán problems concern undirected graphs and only a few Turán problems on digraphs have been investigated; see [1-6].

In this paper, we consider simple digraphs, i.e., digraphs without multiple arcs but allowing loops. We abbreviate directed walks and directed cycles as walks and cycles, respectively. The number of vertices in a digraph is called its *order* and the number of arcs its *size*. Denote by  $\mathscr{F}_k$  the family of digraphs consisting of two different walks of length k with the same initial vertex and the same terminal vertex. A digraph D is said to be  $\mathscr{F}_k$ -free if D contains no subgraph from  $\mathscr{F}_k$ . Let  $ex(n, \mathscr{F}_k)$  be the maximum size of  $\mathscr{F}_k$ -free digraphs of order n, and let  $EX(n, \mathscr{F}_k)$  be the set of  $\mathscr{F}_k$ -free digraphs of order n with size  $ex(n, \mathscr{F}_k)$ .

In 2007, Zhan posed the following Turán-type problem on digraphs at a seminar, and later this problem was listed as an open question in [7].

**Problem 1.** Given positive integers n and k, determine  $ex(n, \mathscr{F}_k)$  and  $EX(n, \mathscr{F}_k)$ .

In 2010, Wu [6] solved the case k = 2. In 2011, Huang, Zhan [5] solved the case  $k \ge n-1$  and determined the extremal numbers for the cases k = n-2 and k = n-3. In 2019, Huang, Lyu and Qiao [4] characterized the extremal digraphs for the cases k = n-2 and k = n-3. And they gave the solutions to the case  $5 \le k \le n-4$ . They also determined the extremal number for the case k = 4, but the structures of the extremal digraphs are not clear. In this paper, we characterize the structures of the extremal digraphs when k = 4.

For a digraph  $D = (\mathcal{V}, \mathcal{A})$ , we denote by  $a(D) = |\mathcal{A}|$  the size of D. Given  $X \subset \mathcal{V}$ , we denote by D[X] and D - X the subgraphs of D induced by X and  $\mathcal{V} \setminus X$ , respectively. For convenience, a set X will be abbreviated as x if it is a singleton  $\{x\}$ . For  $i, j \in \mathcal{V}$ , if there is an arc from i to j, then we say j is a successor of i, and i is a predecessor of j. The notation  $i \to j$  means there is an arc from i to j;  $i \to j$  means there exists no arc from i to j. We denote by (i, j) the arc from i to j. For  $u \in \mathcal{V}$  and  $S \subset \mathcal{V}$ , the notation  $u \to S$  means there exists an arc from u to each vertex of S;  $u \to S$  means there is no arc from u to any vertex of S. Analogously, we define  $S \to u$  and  $S \to u$ . For  $S, T \subset \mathcal{V}$ , we denote by  $\mathcal{A}(S,T)$  the set of arcs from S to T. The cardinalities of  $\mathcal{A}(S,T)$  is denoted by a(S,T). For convenience, S (Respectively, T) will be abbreviated as s (Respectively, t) if it is a singleton  $\{s\}$  (Respectively,  $\{t\}$ ).

A digraph  $D = (\mathcal{V}, \mathcal{A})$  is said to be *transitive* if for any three vertices  $x, y, z \in \mathcal{V}, (x, y) \in \mathcal{A}$  and  $(y, z) \in \mathcal{A}$  imply  $(x, z) \in \mathcal{A}$ . Recall that a tournament is an orientation of the complete graph. We denote by  $T_n$  the transitive tournament on vertices  $\{1, 2, \ldots, n\}$  such that  $i \to j$  if and only if i < j. A *c*-partite transitive tournament is an orientation of a complete *c*-partite graph which is also transitive. A *c*-partite transitive tournament is called balanced if any pair partite sets differ by at most one in size. We present the diagram of the *k*-partite transitive tournaments as follows.



Let M be the digraph as follows.



## EXTREMAL DIGRAPHS AVOIDING DISTINCT WALKS OF LENGTH 4 WITH ...987

We define H(n) as a family of digraphs of order n, each of whose elements has a vertex partition  $\mathcal{V} = V_1 \cup V_2 \cup V_3$ , where  $|V_1| = \lfloor (n-5)/2 \rfloor$ ,  $|V_2| = 5$  and  $|V_3| = \lceil (n-5)/2 \rceil$ , such that  $D[V_1]$  and  $D[V_3]$  are empty, and  $D[V_2]$  is isomorphic to M. Moreover, for  $x \in V_i$ ,  $y \in V_j$  and  $i \neq j$ ,  $x \to y$  if and only if i < j. For convenience, we also use H(n) to indicate a special digraph with the same structure as we define above if it makes no confusion. In the following, we present two examples in H(9) and H(10), respectively.



Now we state our main result as follows.

**Theorem 2.** Let  $n \ge 12$  be an integer. Then  $D \in EX(n, \mathscr{F}_4)$  if and only if D is a balanced 4-partite transitive tournament.

Remark that the cases  $n \in \{5, 6, 7\}$  have been solved in [4, 5]. In this paper, we also characterize the extremal digraphs for  $n \in \{8, 9, 10, 11\}$ .

## 2. Lemmas

We always use  $\langle n \rangle = \{1, 2, ..., n\}$  to denote the vertex set of a digraph D of order n unless otherwise stated. Let  $D = (\mathcal{V}, \mathcal{A})$  be a digraph with vertex set  $\mathcal{V}$  and arc set  $\mathcal{A}$ . Denote by  $\overleftarrow{D}$  the reverse of D, which is obtained by reversing the directions of all arcs of D. Denote by

$$N^+(u) = \{x \in \mathcal{V} | (u, x) \in \mathcal{A}\}$$
 and  $N^-(u) = \{x \in \mathcal{V} | (x, u) \in \mathcal{A}\}$ 

the sets of successors and predecessors of a vertex u. The *out-degree* and *in-degree* of u are  $d^+(u) = |N^+(u)|$  and  $d^-(u) = |N^-(u)|$ , respectively. Let  $d_D(u) = a(D) - b(u)$ 

a(D-u) be the number of arcs incident with u. We abbreviate it as d(u) if no confusion rises. We have

$$d(u) = \begin{cases} d^+(u) + d^-(u) - 1, & \text{if } u \to u; \\ d^+(u) + d^-(u), & \text{otherwise.} \end{cases}$$

Let d be the number of loops in D. Since  $a(D) = \sum_{u \in \mathcal{V}} d^+(u) = \sum_{u \in \mathcal{V}} d^-(u)$ , then

(1) 
$$2a(D) = \sum_{u \in \mathcal{V}} d(u) + d.$$

To determine  $EX(n, \mathscr{F}_4)$  for  $n \geq 8$ , we need the following technical lemmas.

**Lemma 3** [4]. Let  $n \ge 3$  and p be nonnegative integers, and let D be a digraph on n vertices. Given  $q \ge 0$  such that p(n-1)/2 + q is a positive integer, if

$$a(D-i) \leq \frac{(n-1)(n-2)}{2} - \frac{p(n-1)}{2} - q \quad for \ all \ i \in \langle n \rangle,$$

then

$$a(D) \le \frac{n(n-1)}{2} - \frac{p(n+1)}{2} - q - 1.$$

**Lemma 4.** Let s, k, t, n be positive integers with  $t \ge 3$ ,  $s + k \ge 3$  and n = s + k + t + 1. Suppose  $D = (\mathcal{V}, \mathcal{A})$  is a digraph of order n such that D - n is a blow-up of  $T_{t+2}$  with vertex partition  $\mathcal{V} \setminus \{n\} = \bigcup_{i=1}^{t+2} V_i$ , where

$$V_1 = \{1, \dots, k\}, V_2 = \{k+1\}, \dots, V_{t+1} = \{k+t\}, V_{t+2} = \{k+t+1, \dots, k+t+s\},$$

and  $(x, y) \in \mathcal{A}$  for  $x \in V_i$ ,  $y \in V_j$  if and only if i < j. If D is  $\mathscr{F}_{t+1}$ -free and

(2) 
$$d(n) \ge \max\{s, k\} + 2,$$

then

$$n \not\rightarrow n, \quad n \not\rightarrow V_1, \quad V_{t+2} \not\rightarrow n,$$

and

$$\{(i,n),(n,j)\} \not\subseteq \mathcal{A} \quad for \ all \quad 1 \le i,j \le n-1 \ and \ j \le i+2.$$

**Proof.** It is sufficient to prove that  $n \not\rightarrow n$  because the other results could be obtained by adopting the same augments as in the proof of [4, Lemma 4]. We let  $V'_1 = V_1, V'_2 = \mathcal{V} \setminus (V_1 \cup V_{t+2} \cup \{n\})$  and  $V'_3 = V_{t+2}$ . Note that we already have  $n \not\rightarrow V'_1$  and  $V'_3 \not\rightarrow n$ . To the contrary, suppose  $n \rightarrow n$ .

Suppose a vertex  $u \in V'_1$  is a predecessor of n. If a vertex  $u_1 \in V'_2$  is a predecessor of n, there are the following two distinct walks of length t + 1:

$$\left\{ \begin{array}{l} u \to u_1 \to n \to \dots \to n, \\ u \to n \to n \to \dots \to n, \end{array} \right.$$

a contradiction. If a vertex  $u_2 \in V'_2$  is a successor of n, there are the following two distinct walks of length t + 1:

$$\begin{cases} u \to k+1 \to k+2 \to \dots \to k+t+1, \\ u \to n \to \dots \to n \to u_2 \to k+t+1, \end{cases}$$

a contradiction. If a vertex  $u_3 \in V'_3$  is a successor of n, there are the following two distinct walks of length t + 1:

$$\begin{cases} u \to k+1 \to k+2 \to \dots \to k+t \to u_3, \\ u \to n \to \dots \to n \to u_3, \end{cases}$$

a contradiction. Now we have  $d(n) \leq |V'_1| + 1 = k + 1$ , which contradicts (2). Hence,  $V'_1 \not\rightarrow n$ . Similarly, we obtain  $n \not\rightarrow V'_3$ .

Suppose  $u_1, u_2 \in V'_2$  are two predecessors of n with  $u_1 \to u_2$ . Then there are the following two distinct walks of length t + 1:

$$\left\{\begin{array}{l} u_1 \to n \to \dots \to n, \\ u_1 \to u_2 \to n \to \dots \to n, \end{array}\right.$$

a contradiction. Hence,  $V'_2$  contains at most one predecessor of n. Similarly,  $V'_2$  contains at most one successor of n. It follows that  $d(n) \leq 3$ , which contradicts (2). Hence,  $n \not\rightarrow n$ .

**Lemma 5.** Let  $k_1, k_2, n$  be positive integers with  $n = k_1 + k_2 + 4$  and  $k_1 + k_2 \ge 3$ . Suppose  $D = (\mathcal{V}, \mathcal{A})$  is an  $\mathscr{F}_4$ -free n-vertex digraph such that D - n is a blow-up of  $T_5$  with vertex partition  $\mathcal{V} \setminus \{n\} = \bigcup_{i=1}^5 V_i$ , where

$$V_1 = \{1, \dots, k_1\}, V_2 = \{k_1 + 1\}, V_3 = \{k_1 + 2\}, V_4 = \{k_1 + 3\}, V_5 = \{k_1 + 4, \dots, k_1 + k_2 + 3\},$$

and  $(x, y) \in \mathcal{A}$  for  $x \in V_i$ ,  $y \in V_j$  if and only if i < j. If  $d(n) = k_1 + k_2 + 1$  and n has both predecessors and successors, then

- (1)  $V_1 \rightarrow n \text{ and } n \rightarrow V_5;$
- (2)  $k_1 + 1 \to n \text{ or } n \to k_1 + 3.$

**Proof.** Let  $s \in \langle n-1 \rangle$  be the largest integer with  $s \to n, t \in \langle n-1 \rangle$  be the smallest integer with  $n \to t$ . By Lemma 4, we have t > s + 2 and  $n \not\rightarrow n$ . It follows that

$$d(n) \le s + (n-t) \le n-3 = k_1 + k_2 + 1.$$

Since the above equalities hold, we get t - s = 3,  $\{1, \ldots, s\} \rightarrow n$  and  $n \rightarrow \{t, \ldots, n-1\}$ . If  $s < k_1$ , there are the following two distinct walks of length 4 from s to n - 1:

$$\begin{cases} s \to k_1 + 1 \to k_1 + 2 \to k_1 + 3 \to n - 1, \\ s \to n \to s + 3 \to k_1 + 3 \to n - 1, \end{cases}$$

a contradiction. Hence,  $s \ge k_1$ . Similarly,  $t \le k_1 + 4$ . Then (1) follows. Note that t = s + 3, we have  $s = k_1$  or  $s = k_1 + 1$ , then (2) follows immediately.

**Lemma 6.** Let  $n \ge 9$  and  $D \in EX(n, \mathscr{F}_4)$ . If there is  $i_0 \in \mathcal{V}$  such that  $D-i_0$  is a balanced 4-partite transitive tournament, then D is a balanced 4-partite transitive tournament or isomorphic to H(n) or its reverse. Moreover, if  $n \ge 12$ , then D is a balanced 4-partite transitive tournament.

**Proof.** We assume that  $D-i_0$  is a balanced 4-partite transitive tournament with vertex partition  $\mathcal{V}(D-i_0) = V_1 \cup V_2 \cup V_3 \cup V_4$ , and  $i \to j$  if and only if s(i) < s(j). Here s(i) is the index of the set i belongs to. We let s and t be nonnegative integers such that n = 4s + t + 1 and t < 4. We assume that

$$V_1 = \{u_1, u_2, \dots, u_{|V_1|}\}, V_2 = \{v_1, v_2, \dots, v_{|V_2|}\}, V_3 = \{w_1, w_2, \dots, w_{|V_3|}\}, V_4 = \{x_1, x_2, \dots, x_{|V_4|}\}.$$

Let  $\{j_1, j_2, ..., j_t\}$  be an arbitrary *t*-subset of  $\{1, 2, 3, 4\}$ . If  $i \in \{j_1, j_2, ..., j_t\}$ , we let  $|V_i| = s + 1$ ; else, we let  $|V_i| = s$ . It is clear that

$$(3) s \ge 2,$$

and

(4) 
$$d(i_0) = 3s + t.$$

Suppose that  $i_0 \rightarrow i_0$ . Then any pair of the predecessors or successors of  $i_0$  are not adjacent. It follows that

(5) 
$$d(i_0) = d^+(i_0) + d^-(i_0) - 1 \le 2s + \bar{t} + 1,$$

where  $\bar{t} = \min\{t, 2\}$ , which contradicts (4). Hence  $i_0 \not\rightarrow i_0$ .

Let  $s' \in \langle 4 \rangle$  be the largest integer such that there is a vertex  $i \in V_{s'}$  with  $i \to i_0$  and  $t' \in \langle 4 \rangle$  be the smallest integer such that there is a vertex  $j \in V_{t'}$  with  $i_0 \to j$ . Here we let s' = 0 if  $\mathcal{V} \not\rightarrow i_0$  and t' = 5 if  $i_0 \not\rightarrow \mathcal{V}$ .

We assert that

$$(6) t' \ge 2.$$

Otherwise,  $i_0$  has a successor in  $V_1$ , say  $u_1$ . Then there are the following two distinct walks of length 4:

$$\left\{ \begin{array}{l} i_0 \rightarrow u_1 \rightarrow v_1 \rightarrow w_1 \rightarrow x_1, \\ i_0 \rightarrow u_1 \rightarrow v_2 \rightarrow w_1 \rightarrow x_1, \end{array} \right.$$

a contradiction. Similarly, we have

$$(7) s' \le 3.$$

Suppose s' = t' = 2. Without loss of generality, we let  $v_1, v_2 \in V_2$  with  $v_1 \to i_0$  and  $i_0 \to v_2$ . There are the following two distinct walks of length 4:

$$\left\{ \begin{array}{l} v_1 \to i_0 \to v_2 \to w_1 \to x_1, \\ v_1 \to i_0 \to v_2 \to w_2 \to x_1, \end{array} \right.$$

a contradiction. Similarly, for the case s' = t' = 3, we also get a contradiction. Combining with (6) and (7) we get

$$(8) s' \neq t'$$

Suppose s' > t'. In fact, we only need to consider the case s' = 3 and t' = 2. Let  $v_1 \in V_2$  be a successor of  $i_0$  and  $w_1 \in V_3$  be a predecessor of  $i_0$ . If  $i_0$  has another successor in  $V_2$ , say  $v_2$ , there are the following two distinct walks of length 4:

$$\begin{cases} u_1 \to w_1 \to i_0 \to v_1 \to x_1, \\ u_1 \to w_1 \to i_0 \to v_2 \to x_1, \end{cases}$$

a contradiction. Hence,  $i_0$  has exactly one successor in  $V_2$ . Similarly,  $i_0$  has exactly one predecessor in  $V_3$ . It follows from (4) and (6) that  $i_0$  has a predecessor in  $V_1$ , say  $u_1$ . Then there are the following two distinct walks of length 4:

$$\left\{ \begin{array}{l} u_1 \to i_0 \to v_1 \to w_1 \to x_1, \\ u_1 \to w_1 \to i_0 \to v_1 \to x_1, \end{array} \right.$$

a contradiction. Therefore, we get

$$(9) t' \ge s' + 1$$

Suppose t' = s' + 1. If s' = 1, without loss of generality, we let  $u_1 \to i_0$  and  $i_0 \to v_1$ . There are the following two distinct walks of length 4:

$$\begin{cases} u_1 \to i_0 \to v_1 \to w_1 \to x_1, \\ u_1 \to i_0 \to v_1 \to w_2 \to x_1, \end{cases}$$

a contradiction. Hence, we get  $s' \neq 1$ . Similarly, we get  $s' \neq 3$ . Combining with (6) and (7), we get s' = 2. Without loss of generality, we let  $v_1 \rightarrow i_0$  and  $i_0 \rightarrow w_1$ . If  $i_0$  has another successor in  $V_3$ , say  $w_2$ , there are the following two distinct walks of length 4:

$$\begin{cases} u_1 \to v_1 \to i_0 \to w_1 \to x_1, \\ u_1 \to v_1 \to i_0 \to w_2 \to x_1, \end{cases}$$

a contradiction. Thus,  $i_0$  has exactly one successor in  $V_3$ . Similarly,  $i_0$  has exactly one predecessor in  $V_2$ . Combining with (3) and (4), we get  $|V_2| = |V_3| = 2$ ,  $V_1 \rightarrow i_0$ and  $i_0 \rightarrow V_4$ . It follows that D is H(n) or its reverse.

Suppose t' = s' + 2. By (3) and (4), we get  $V_i \to n$  for  $i \leq s'$  and  $n \to V_j$  for  $j \geq t'$ . Moreover,  $|V_{s'+1}| = s$ . We can conclude that D is a balanced 4-partite transitive tournament.

For the case  $n \ge 12$ , it is sufficient to notice that  $a(H(n)) < ex(n, \mathscr{F}_4)$ .

# 3. The Structures of the Extremal Digraphs for $n \ge 8$

In this section, we characterize the extremal digraphs for  $n \ge 8$ . Denote by  $F_1$  the digraph  $T_8 - \{(1,2), (1,3), (2,3), (7,8)\}$ ,  $F_2$  the digraph  $T_8 - \{(1,2), (4,5), (4,6), (7,8)\}$ . In case the readers are not familiar with the structures of  $F_1$  and  $F_2$ , we present them as follows.



**Theorem 7.** A digraph  $D \in EX(8, \mathscr{F}_4)$  if and only if D is a balanced 4-partite transitive tournament or isomorphic to one of  $\{F_1, F_1, F_2, F_2\}$ .

EXTREMAL DIGRAPHS AVOIDING DISTINCT WALKS OF LENGTH 4 WITH ...993

**Proof.** The sufficiency of this theorem is obvious. It is sufficient to show the necessity part. Suppose  $D \in Ex(8, \mathscr{F}_4)$ . It follows that  $a(D) = ex(8, \mathscr{F}_4) = 24$  from [4, Theorem 1]. By [5, Corollary 11], we have  $a(D-i) \leq 19$  for all  $i \in \langle 8 \rangle$ . We distinguish two cases.

Case 1. a(D-i) = 19 for some  $i \in \langle 8 \rangle$ . By [4, Theorem 9], we assume without loss of generality that  $D-8 = T_7 - \{(1,2), (6,7)\}$ . Since d(8) = a(D) - a(D-8) = 5, applying Lemma 4, we obtain  $\{6,7,8\} \not\rightarrow 8, 8 \not\rightarrow \{1,2\}$  and  $\{(i,8), (8,j)\} \not\subset \mathcal{A}$  for  $j \leq i+2$ .

If 8 has no predecessor, then D is isomorphic to  $F_1$ . If 8 has no successor, then D is isomorphic to the reverse of  $F_1$ . If 8 has both predecessors and successors, applying Lemma 5, we obtain D is isomorphic to  $F_2$  or its reverse.

Case 2.  $a(D-i) \leq 18$  for all  $i \in \langle 8 \rangle$ . By the definition of d(i) and (1), we obtain

(10) 
$$d(i) = 6 \text{ for all } i \in \langle 8 \rangle.$$

Moreover, there exists no loops in D, i.e.,

(11) 
$$i \not\rightarrow i \text{ for all } i \in \langle 8 \rangle.$$

Claim 1. There exists no 2-cycles in D.

Otherwise, without loss of generality we assume  $7 \leftrightarrow 8$ . By (10) we get  $a(D - \{7,8\}) = a(D) - d(7) - d(8) + 2 = 14$ . Applying [4, Theorem 8] to  $D - \{7,8\}$ , we have  $D - \{7,8\}$  is isomorphic to  $T_6 - \{(1,2)\}$  or  $T_6 - \{(5,6)\}$ . Suppose  $D - \{7,8\}$  is isomorphic to  $T_6 - \{(1,2)\}$ . Without loss of generality, we let  $D - \{7,8\} = T_6 - \{(1,2)\}$ . If  $8 \to 1$ , there are the following two distinct walks of length 4:

$$\begin{cases} 8 \to 1 \to 3 \to 4 \to 6, \\ 8 \to 1 \to 3 \to 5 \to 6, \end{cases}$$

a contradiction. Similarly, we get  $8 \not\rightarrow \{1, 2\}$  and  $7 \not\rightarrow \{1, 2\}$ .

Combining with (10) we have  $1 \to \{7, 8\}$ . If there is  $u \in \{3, 4, 5\}$  with  $8 \to u$ , there are the following two distinct walks of length 4:

$$\begin{cases} 1 \to 3 \to 4 \to 5 \to 6, \\ 1 \to 7 \to 8 \to u \to 6; \end{cases}$$

a contradiction. If  $8 \rightarrow 6$ , there are the following two distinct walks of length 4:

$$\left\{ \begin{array}{l} 1 \to 3 \to 4 \to 5 \to 6, \\ 1 \to 8 \to 7 \to 8 \to 6, \end{array} \right.$$

a contradiction. By (10), 8 has at least 2 predecessors in  $\{3, 4, 5, 6\}$ . We assume  $u_1, u_2$  are two predecessors of 8 with  $3 \le u_1 < u_2 \le 6$ . Then there are the following two distinct walks of length 4:

$$\left\{ \begin{array}{l} 1 \rightarrow u_1 \rightarrow 8 \rightarrow 7 \rightarrow 8, \\ 1 \rightarrow u_2 \rightarrow 8 \rightarrow 7 \rightarrow 8, \end{array} \right.$$

a contradiction. For the case  $D - \{7, 8\}$  is isomorphic to  $T_6 - \{(5, 6)\}$ , we also get a contradiction. This completes the proof of Claim 1.

It follows that

(12) 
$$a(i,j) + a(j,i) \le 1$$
 for distinct  $i, j \in \langle 8 \rangle$ .

For each pair  $\{i, j\} \subset \langle 8 \rangle$ , we have  $a(D - \{i, j\}) = a(D) - 12 + a(i, j) + a(j, i) \leq 13$ . Applying Lemma 3 to D - 8, there exists  $i \in \langle 7 \rangle$  such that  $a(D - \{i, 8\}) \geq 13$ . Thus  $a(D - \{i, 8\}) = 13$ . Without loss of generality, we assume

(13) 
$$a(D - \{7, 8\}) = 13.$$

Moreover,

(14) 
$$a(7,8) + a(8,7) = 1.$$

By (10) and (12), we have  $d_{D-\{7,8\}}(i) \ge 4$  for  $i \in \langle 6 \rangle$ . It follows from (13) that  $a(D-\{i,7,8\}) \le 9$  for  $i \in \langle 6 \rangle$ . Applying Lemma 3 to  $D-\{7,8\}$ , there exists  $i \in \langle 6 \rangle$  such that  $a(D-\{i,7,8\}) = 9$ . Without loss of generality, we assume

(15) 
$$a(D - \{6, 7, 8\}) = 9.$$

Moreover,

(16) 
$$a(6,i) + a(i,6) = 1 \text{ for } i \in \{7,8\}.$$

By (10) and (12), we have  $d_{D-\{6,7,8\}}(i) \ge 3$  for  $i \in \langle 5 \rangle$ . It follows from (15) that  $a(D - \{i, 6, 7, 8\}) \le 6$  for  $i \in \langle 5 \rangle$ . Applying Lemma 3 to  $D - \{6, 7, 8\}$ , there is  $i \in \langle 5 \rangle$  such that  $a(D - \{i, 6, 7, 8\}) \ge 6$ . Hence, the equality holds for some  $i \in \langle 5 \rangle$ . Without loss of generality, we assume

(17) 
$$a(D - \{5, 6, 7, 8\}) = 6.$$

Moreover,

(18) 
$$a(5,i) + a(i,5) = 1 \text{ for } i \in \{6,7,8\}.$$

Let  $\alpha = \{1, 2, 3, 4\}$ . It is clear that

(19) 
$$a(D[\alpha]) = a(D - \alpha) = 6.$$

We have the following claim.

**Claim 2.**  $D[\alpha]$  and  $D - \alpha$  are both transitive tournaments.

It is sufficient to show that they both have no cycles. Suppose  $D[\alpha]$  has a cycle. By Claim 1, (11) and [3, Lemma 2.2],  $D[\alpha]$  has a 3-cycle. Without loss of generality, we let  $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$ . By Claim 1, (11) and (17), we get a(i, 4) + a(4, i) = 1 for  $i \in \{1, 2, 3\}$ . If 4 has both predecessors and successors in  $\{1, 2, 3\}$ , assume without loss of generality that  $1 \rightarrow 4$  and  $4 \rightarrow \{2, 3\}$ , then there are the following two distinct walks of length 4:

$$\begin{cases} 4 \to 2 \to 3 \to 1 \to 2, \\ 4 \to 3 \to 1 \to 4 \to 2, \end{cases}$$

a contradiction. Hence,  $4 \to \{1, 2, 3\}$  or  $\{1, 2, 3\} \to 4$ . Let *B* be a digraph on  $\{1, 2, 3, 4\}$  with arc set  $\{(1, 2), (2, 3), (3, 1), (1, 4), (2, 4), (3, 4)\}$ . Then  $D[\alpha]$  is isomorphic to *B* or its reverse. Applying the same arguments as above,  $D - \alpha$  is a transitive tournament or isomorphic to one of  $\{B, \overline{B}\}$ .

Suppose  $D[\alpha]$  is isomorphic to B or  $\overleftarrow{B}$  and so is  $D - \alpha$ . Without loss of generality, let  $\{(1,2), (2,3), (3,1)\} \subset \mathcal{A}(D[\alpha])$  and  $\{(5,6), (6,7), (7,5)\} \subset \mathcal{A}(D - \alpha)$ . If  $1 \to 5$ , there are the following two distinct walks of length 4:

$$\begin{cases} 1 \to 2 \to 3 \to 1 \to 5, \\ 1 \to 5 \to 6 \to 7 \to 5, \end{cases}$$

a contradiction. Hence, we have  $1 \neq 5$ . Similarly, we have  $\{1, 2, 3\} \neq 5$  and  $5 \neq \{1, 2, 3\}$ . It follows that  $d(5) \leq 4$ , which contradicts (10). Hence,  $D - \alpha$  is a transitive tournament. Without loss of generality, let

$$D - \alpha = T_8 - \alpha$$

Suppose  $D[\alpha] = B$ . We can obtain

(20)  $a(\{1,2,3\},\{5\}) \le 1.$ 

Otherwise, there are the following distinct walks of length 4:

$$\left\{\begin{array}{l} 1 \to 5 \to 6 \to 7 \to 8, \\ 1 \to 2 \to 5 \to 7 \to 8, \\ 1 \to 2 \to 3 \to 5 \to 8, \end{array}\right.$$

a contradiction. We obtain  $a(\{1,2\},\{6\}) \leq 1$ , otherwise there are the following distinct walks of length 4:

$$\left\{ \begin{array}{l} 3 \rightarrow 1 \rightarrow 6 \rightarrow 7 \rightarrow 8, \\ 3 \rightarrow 1 \rightarrow 2 \rightarrow 6 \rightarrow 8, \end{array} \right.$$

a contradiction. Similarly, we get  $a(\{1,3\},\{6\}) \leq 1$  and  $a(\{2,3\},\{6\}) \leq 1$ , which implies that

(21) 
$$a(\{1,2,3\},\{6\}) \le 1.$$

By (10) and (20), 5 has one successor in  $\{1, 2, 3\}$ , say  $u_1$ . Similarly, 6 has one successor in  $\{1, 2, 3\}$ , say  $v_1$ . We let  $u_1 \rightarrow u_2 \rightarrow u_3$  be the walk along  $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$ , and  $v_2 \in \{1, 2, 3\}$  be the successor of  $v_1$ . Then there are the following distinct walks of length 4:

$$\begin{cases} 5 \to u_1 \to u_2 \to u_3 \to 4, \\ 5 \to 6 \to v_1 \to v_2 \to 4, \end{cases}$$

a contradiction.

Suppose  $D[\alpha] = \overleftarrow{B}$ . Note that the reverse of a transitive tournament is still a transitive tournament. Applying the same arguments, we could get a contradiction.

Now we have proved  $D[\alpha]$  is a transitive tournament. Exchanging the roles of  $D[\alpha]$  and  $D-\alpha$  and repeating the same arguments as above, we get that  $D-\alpha$  is also a transitive tournament. This completes Claim 2.

Next, we show D is a balanced 4-partite transitive tournament. Without loss of generality, let

$$D[\alpha] = T_4$$
 and  $D - \alpha = T_8 - \alpha$ .

We obtain  $6 \rightarrow 1$ , otherwise there are the following two distinct walks of length 4:

$$\begin{cases} 5 \to 6 \to 1 \to 2 \to 4, \\ 5 \to 6 \to 1 \to 3 \to 4. \end{cases}$$

Similarly, we get  $\{2,3,4\} \not\rightarrow 5, 8 \not\rightarrow \{1,2,3\}, \{3,4\} \not\rightarrow 6, 7 \not\rightarrow \{1,2\}$  and  $4 \not\rightarrow 7$ .

Suppose  $1 \to 5$ . We obtain that  $i \not\rightarrow i+4$  for  $i \in \{2, 3, 4\}$ , otherwise there are two distinct walks of length 4 from 1 to 8. If  $1 \not\rightarrow 6$ , by (10), we get  $6 \rightarrow \{2, 3, 4\}$ . Then there are the following two distinct walks of length 4:

$$\begin{cases} 1 \to 5 \to 6 \to 2 \to 4, \\ 1 \to 5 \to 6 \to 3 \to 4, \end{cases}$$

a contradiction. Hence,  $1 \rightarrow 6$ . Then  $6 \not\rightarrow 2$ , otherwise there are the following two distinct walks of length 4:

$$\left\{ \begin{array}{l} 1 \rightarrow 5 \rightarrow 6 \rightarrow 2 \rightarrow 4, \\ 1 \rightarrow 6 \rightarrow 2 \rightarrow 3 \rightarrow 4, \end{array} \right.$$

a contradiction. By (10), we get  $6 \rightarrow \{3, 4\}$ . It follows that  $7 \not\rightarrow \{3, 4\}$ , otherwise there are the following distinct walks of length 4:

$$\left\{ \begin{array}{l} 1 \to 5 \to 6 \to 3 \to 4, \\ 1 \to 6 \to 7 \to 3 \to 4, \end{array} \right. \left\{ \begin{array}{l} 1 \to 5 \to 6 \to 3 \to 4, \\ 1 \to 5 \to 6 \to 7 \to 4. \end{array} \right.$$

Then  $d(7) \leq 5$ , which contradicts (10). Therefore, we get  $1 \rightarrow 5$ . Applying the same arguments, we get  $5 \rightarrow 1$ . It follows from (10) that

$$1 \to \{6, 7, 8\}, \quad 5 \to \{2, 3, 4\}.$$

Suppose  $2 \to 6$ . We obtain that  $i \not\rightarrow i + 4$  for  $i \in \{3, 4\}$ , otherwise there are two distinct walks of length 4 from 1 to 8. By Claim 1 we have  $6 \not\rightarrow 2$ . If  $2 \not\rightarrow 7$ , by (10), we have  $7 \rightarrow \{3, 4\}$ . Then there are the following two distinct walks of length 4:

$$\begin{cases} 1 \to 2 \to 6 \to 7 \to 4, \\ 1 \to 6 \to 7 \to 3 \to 4, \end{cases}$$

a contradiction. Hence  $2 \rightarrow 7$ . By (10) we have  $2 \rightarrow 8$ . Moreover, we get  $3 \rightarrow 8$  and  $8 \rightarrow 4$ . There are the following two distinct walks of length 4:

$$\begin{cases} 5 \to 6 \to 7 \to 8 \to 4, \\ 5 \to 2 \to 6 \to 8 \to 4, \end{cases}$$

a contradiction. Therefore,  $2 \rightarrow 6$ . Similarly, we get  $6 \rightarrow 2$ . By (10), we get

$$2 \to \{7, 8\}, \quad 6 \to \{3, 4\}.$$

Suppose  $3 \to 7$ . It is clear that  $4 \not\rightarrow 8$ . It follows from (10) that  $3 \not\rightarrow 8$ . Moreover, we get  $8 \to 4$ . Then there are the following two distinct walks of length 4:

$$\begin{cases} 5 \to 6 \to 7 \to 8 \to 4, \\ 5 \to 2 \to 7 \to 8 \to 4, \end{cases}$$

a contradiction. Hence,  $3 \rightarrow 7$ . Similarly, we get  $7 \rightarrow 3$ . It follows from (10) that  $4 \rightarrow 8, 8 \rightarrow 4$  and

$$3 \rightarrow 8, \quad 7 \rightarrow 4.$$

Then D is a balanced 4-partite transitive tournament. This completes the proof.

Denote by  $F_3$  the digraph  $T_9 - \{(1,2), (1,3), (2,3), (7,8), (7,9), (8,9)\}, F_4$  the digraph  $T_9 - \{(1,2), (1,3), (2,3), (5,6), (5,7), (8,9)\}, F_5$  the digraph  $T_9 - \{(1,2), (1,3), (2,3), (4,5), (4,6), (8,9)\}$ . We present these digraphs as follows.



We give the structures of the extremal digraphs for n = 9 as follows.

**Theorem 8.** A digraph  $D \in EX(9, \mathscr{F}_4)$  if and only if D is a balanced 4-partite transitive tournament or isomorphic to one of  $\{F_3, F_4, F_5, F_5, H(9)\}$ .

**Proof.** The sufficiency of this theorem is obvious. It is sufficient to show the necessity part. Suppose  $D \in Ex(9, \mathscr{F}_4)$ . By [4, Theorem 1] we have  $a(D) = ex(9, \mathscr{F}_4) = 30$  and

(22) 
$$d(i) \ge 6 \text{ for all } i \in \mathcal{V}.$$

Combining with Lemma 3, there exists  $i \in \mathcal{V}$  such that a(D-i) = 24. By Theorem 7, D-i or its reverse is a balanced 4-partite transitive tournament or isomorphic to one of  $\{F_1, F_2\}$ . Without loss of generality, we let i = 9. It follows that

(23) 
$$d(9) = 6$$
.

We consider the following cases.

Case 1.  $D-9 = F_1$ . By Lemma 4, we get  $\{7, 8, 9\} \not\rightarrow 9$  and  $9 \not\rightarrow \{1, 2, 3\}$ . If 9 has no successor, then  $\langle 6 \rangle \rightarrow 9$ . Moreover,  $D = F_3$ . Now assume 9 has at least one successor. Since  $9 \not\rightarrow \{1, 2, 3, 9\}$  and (23), 9 has a predecessor. By Lemma 5, D is isomorphic to  $F_4$  or  $F_5$ . For the case  $D - \{9\} = \overleftarrow{F_1}$ , we get that D is isomorphic to one of  $\{F_3, F_4, F_5\}$ .

Case 2.  $D-9 = F_2$ . By (22) we get  $d_{D-4}(9) \ge 4$ . Applying Lemma 4 to D-4 we have

(24) 
$$9 \not\rightarrow \{1,2\} \text{ and } \{7,8,9\} \not\rightarrow 9.$$

We claim that

(25) 
$$a(4,9) + a(9,4) = 1.$$

Recalling the structure of  $F_2$  and (22), we have  $a(4,9) + a(9,4) \ge 1$ . If  $4 \leftrightarrow 9$ , we have  $\{1,2\} \rightarrow 9$ . Otherwise a(D-i) = 24 for  $i \in \{1,2\}$ . Applying Theorem 7 to D-i, we get that D has no cycle, a contradiction. Similarly, we get  $\{7,8\} \rightarrow 9$ . Combining with (23), we get  $6 \rightarrow 9$ . Applying Theorem 7 to D-6, we have D-6 has no cycles, a contradiction.

Suppose 9 has no predecessor in  $\{1, 2, 3\} \cup \{5, 6, 7, 8\}$ . By (23) (24) and (25), we obtain  $9 \rightarrow \{3, 5, 6, 7, 8\}$ . If a(4, 9) = 1, there are the following two distinct walks of length 4:

$$\left\{ \begin{array}{l} 4 \rightarrow 9 \rightarrow 3 \rightarrow 5 \rightarrow 7, \\ 4 \rightarrow 9 \rightarrow 3 \rightarrow 6 \rightarrow 7, \end{array} \right.$$

a contradiction. Hence,  $4 \rightarrow 9$  and  $9 \rightarrow 4$ . Then *D* is isomorphic to  $F_4$ . Suppose 9 has no successors in  $\{1, 2, 3\} \cup \{5, 6, 7, 8\}$ . Similarly, we get that *D* is isomorphic to  $\overleftarrow{F_5}$ .

Suppose 9 has both predecessors and successors in  $\{1,2,3\} \cup \{5,6,7,8\}$ . By (23) and (25), we get  $d_{D-4}(9) = 5$ . Applying Lemma 5 to D-4, we obtain  $\{1,2\} \rightarrow 9, 9 \rightarrow \{7,8\}, a(3,9) + a(9,6) = 1$ . If  $4 \rightarrow 9$ , then there are the following two distinct walks of length 4:

$$\left\{ \begin{array}{l} 1 \rightarrow 3 \rightarrow 5 \rightarrow 6 \rightarrow 7, \\ 1 \rightarrow 3 \rightarrow 4 \rightarrow 9 \rightarrow 7, \end{array} \right.$$

a contradiction. Hence, it follows  $9 \rightarrow 4$  from (25). Then we get  $3 \rightarrow 9$ , otherwise there are the following walks of length 4:

$$\left\{ \begin{array}{l} 1 \rightarrow 3 \rightarrow 5 \rightarrow 6 \rightarrow 7, \\ 1 \rightarrow 3 \rightarrow 9 \rightarrow 4 \rightarrow 7, \end{array} \right.$$

a contradiction. Hence, D is isomorphic to H(9).

For the case  $D - 9 = \overleftarrow{F_2}$ , applying the same arguments as above, we can conclude that D is isomorphic to one of  $\{\overleftarrow{F_4}, F_5, H(9)\}$ .

Case 3. D-9 is a balanced 4-partite transitive tournament. By Lemma 6, D is a balanced 4-partite transitive tournament or isomorphic to H(9). This completes the proof.

Denote by  $F_6$  the digraph  $T_{10} - \{(1, 2), (1, 3), (2, 3), (5, 6), (5, 7), (8, 9), (8, 10), (9, 10)\}$ , whose structure is presented as follows.



We give the structures of the extremal digraphs for n = 10 as follows.

**Theorem 9.** A digraph  $D \in EX(10, \mathscr{F}_4)$  if and only if D is a balanced 4-partite transitive tournament or isomorphic to one of  $\{F_6, F_6, H(10), H(10)\}$ .

**Proof.** The sufficiency of this theorem is obvious. It is sufficient to show the necessity part. Suppose  $D \in Ex(10, \mathscr{F}_4)$ . By [4, Theorem 1] we have  $a(D) = ex(10, \mathscr{F}_4) = 37$  and

(26) 
$$d(i) \ge 7 \text{ for all } i \in \mathcal{V}.$$

Combining with Lemma 3, there exists  $i \in \mathcal{V}$  such that a(D-i) = 30. By Theorem 8, D-i or its reverse is a balanced 4-partite transitive tournament or isomorphic to one of  $\{F_3, F_4, F_5, H(9)\}$ . Without loss of generality, we let i = 10. It is clear that

(27) 
$$d(10) = 7$$

We consider the following cases.

Case 1.  $D - 10 = F_3$ . By Lemma 4,  $10 \neq \{1, 2, 3\}$  and  $\{7, 8, 9, 10\} \neq 10$ . It follows from (27) that 10 has both successors and predecessors in  $\langle 9 \rangle$ . By Lemma 5, D is isomorphic to  $F_6$  or its reverse.

Case 2.  $D - 10 = F_4$ . It follows from (27) that  $d_{D-5}(10) \ge 5$ . Applying Lemma 4 to D - 5, we have  $10 \Rightarrow 1$ . Combining with (26) we obtain that  $1 \Rightarrow 10$ and a(D-1) = 30. Applying the same arguments as in the proof of Case 2 of Theorem 8, D - 1 or its reverse is isomorphic to one of  $\{F_4, F_5, H(9)\}$ . We can conclude that D or its reverse is isomorphic to one of  $\{F_6, H(10)\}$ . For the case  $D - 10 = F_4$ , we get the same result.

Case 3.  $D - 10 = F_5$ . Using the same arguments as in the above case, we get that D or its reverse is isomorphic to one of  $\{F_6, H(10)\}$ . For the case  $D - 10 = \overleftarrow{F_5}$ , we get the same result.

Case 4. D - 10 = H(9). Without loss of generality, we let

$$H(9) = T_9 - \{(1,2), (3,4), (3,5), (5,7), (6,7), (8,9)\}.$$

By [5, Lemma 1(iv)],  $a(\{3,7\},10) + a(10,\{3,7\}) \leq 3$ . Combing with (27), we get  $d_{D-\{3,7\}}(10) \geq 4$ . Applying Lemma 4 to  $D - \{3,7\}$ , we obtain  $10 \not\rightarrow \{1,2\}$ ,  $\{8,9,10\} \not\rightarrow 10$ . It follows from (26) that  $a(3,10) + a(10,3) \geq 1$  and  $a(7,10) + a(10,7) \geq 1$ . Moreover, at least one equality holds. Without loss of generality, we let a(3,10) + a(10,3) = 1. Then a(D-3) = 30. By Theorem 8, D-3 contains no cycle, which implies that a(7,10) + a(10,7) = 1. Now we have  $a(D - \{3,7\}) = 24$  and  $d_{D-\{3,7\}}(10) = 5$ .

If 10 has no predecessor, we easily get D is isomorphic to the reverse of H(10). Similarly, if 10 has no successor, D is isomorphic to H(10). Now assume 10 has both predecessors and successors. Applying Lemma 5 to  $D - \{3, 7\}$ , we get  $\{1, 2\} \rightarrow 10, 10 \rightarrow \{8, 9\}$ , and a(4, 10) + a(10, 6) = 1. Without loss of generality, we assume  $4 \rightarrow 10$ . Since a(7, 10) + a(10, 7) = 1, we may assume  $7 \rightarrow 10$ , otherwise we can rename the vertices 7 and 10. Then there are the following walks of length 4:

$$\begin{cases} 1 \to 3 \to 7 \to 10 \to 8, \\ 1 \to 4 \to 5 \to 6 \to 8, \end{cases}$$

a contradiction.

Case 5. D - 10 is a balanced 4-partite transitive tournament. By Lemma 6, D is a balanced 4-partite transitive tournament or isomorphic to one of  $\{H(10), H(10)\}$ . This completes the proof.

**Theorem 10.** A digraph  $D \in EX(11, \mathscr{F}_4)$  if and only if D is a balanced 4-partite transitive tournament or isomorphic to H(11).

**Proof.** The sufficiency of this theorem is obvious. It is sufficient to show the necessity part. Suppose  $D \in Ex(11, \mathscr{F}_4)$ . By [4, Theorem 1], we get  $a(D) = ex(11, \mathscr{F}_4) = 45$  and

(28) 
$$d(i) \ge 8 \text{ for all } i \in \mathcal{V}$$

By Lemma 3 and Theorem 9, there exists  $i \in \mathcal{V}$  such that D - i or its reverse is a balanced 4-partite transitive tournament or isomorphic to one of  $\{F_6, H(10)\}$ . Without loss of generality, let i = 11. It is clear that

(29) 
$$d(11) = 8$$

We consider the following cases.

Case 1.  $D - 11 = F_6$ . By (29) we get  $d_{D-5}(11) \ge 6$ . Applying Lemma 5 to D - 5, we obtain  $11 \nleftrightarrow \{1, 2, 3\}$  and  $\{8, 9, 10, 11\} \nleftrightarrow 11$ . By (28) we get  $1 \to 11$  and d(1) = 8. Applying the same arguments as in Case 2 of Theorem 9, we get that D - 1 is isomorphic to one of  $\{F_6, H(10)\}$ . Similarly, we get D - 10 is isomorphic to one of  $\{F_6, \overline{H}(10)\}$ . We can conclude that D is isomorphic to H(11).

Case 2. D - 11 = H(10). Without loss of generality, we let

$$H(10) = T_{10} - \{(1,2), (3,4), (3,5), (6,7), (5,7), (8,9), (8,10), (9,10)\}.$$

By [5, Lemma 1(iv)],  $a(\{3,7\},11) + a(11,\{3,7\}) \leq 3$ . Combing with (29), we get  $d_{D-\{3,7\}}(11) \geq 5$ . Applying Lemma 4 to  $D - \{3,7\}$ , we get  $10 \not\rightarrow 11$ . It follows from (28) that  $11 \rightarrow 10$  and a(D-10) = 37. By Theorem 9, we get a(3,11) + a(11,3) = 1 and a(7,11) + a(11,7) = 1. Applying Theorem 9 to D-3 and D-7, respectively, we get D-3 is isomorphic to one of  $\{F_6, H(10), \overline{H}(10)\}$ , and so is D-7. We can conclude that D is isomorphic to H(11). Similarly, for the case  $D-11 = \overline{H}(10)$ , we also get D is isomorphic to H(11).

Case 3. D - 11 is a balanced 4-partite transitive tournament. By Lemma 6, D is a balanced 4-partite transitive tournament or isomorphic to H(11).

Now we give the proof of Theorem 2.

Extremal Digraphs Avoiding Distinct Walks of Length 4 with ...1003

**Proof of Theorem 2.** The sufficiency of this theorem is obvious. It is sufficient to show the necessity part. We first consider the case n = 12. Suppose  $D \in Ex(12, \mathscr{F}_4)$ . By [4, Theorem 1], we get  $a(D) = ex(12, \mathscr{F}_4) = 54$  and

(30) 
$$d(i) \ge 9 \text{ for all } i \in \mathcal{V}.$$

Combining with (1) we get

(31) 
$$d(i) = 9 \text{ for all } i \in \mathcal{V}.$$

By Theorem 10, there exists  $i \in \mathcal{V}$  such that D-i is a balanced 4-partite transitive tournament or isomorphic to H(11). Without loss of generality, let i = 12. We consider the following cases.

Case 1. 
$$D - 12 = H(11)$$
. Here we let  
 $H(11) = T_{11} - \{(1, 2), (1, 3), (2, 3), (4, 5), (4, 6), (6, 8), (7, 8), (9, 10), (9, 11), (10, 11)\}.$ 

By (31) we get  $d_{D-\{4,8\}}(12) \ge 5$ . Applying Lemma 4 to  $D-\{4,8\}$ , we get  $12 \nrightarrow 1$ . Combining with (31) and the structure of H(11), we get a(1,12) = 1 and D-1 is isomorphic to H(11). It follows that 12 and 1 share the same predecessors and successors. We may assume  $1 \rightarrow 12$ , then there are the following two walks of length 4 with the same endpoints:

$$\left\{ \begin{array}{l} 1 \rightarrow 12 \rightarrow 5 \rightarrow 6 \rightarrow 9, \\ 1 \rightarrow 5 \rightarrow 6 \rightarrow 7 \rightarrow 9, \end{array} \right.$$

a contradiction.

Case 2. D - 12 is a balanced 4-partite transitive tournament. By Lemma 6, D is a balanced 4-partite transitive tournament.

For the case  $n \ge 13$ , Lemma 6 guarantees our result. This completes the proof.

#### 4. Conclusion

In this paper, we characterize the structures of the digraphs in  $EX(n, \mathscr{F}_4)$  by analyzing the detailed structures of its subgraph of order n-1. There exists at least one walk of length 4 in some digraphs of  $Ex(n, \mathscr{F}_4)$  when  $n \in \{5, 6, 7, 8, 9, 10, 11\}$ , while for  $n \geq 12$  there is not any walk of length 4 in the digraphs belonging to  $Ex(n, \mathscr{F}_4)$ . As far as we know, for any fixed  $k \geq 5$  and sufficiently large n, there is no walk of length k in the digraphs in  $EX(n, \mathscr{F}_k)$ . So it is interesting to figure out: what will happen to the maximum size of  $\mathscr{F}_k$ -free digraphs when there exists a walk of length k? We pose a problem as follows. **Problem 11.** Given positive integers n and k, determine the maximum size of  $\mathscr{F}_k$ -free digraphs in which there exists a walk of length k as well as the structures of the extremal digraphs attaining this maximum.

For other extremal problems on digraphs, the techniques we used may still be valid. In our opinion, these techniques might be effective when the target digraphs contain enough arcs. But in most situations the detailed arguments are very different for different digraphs.

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