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EXTREMAL DIGRAPHS AVOIDING DISTINCT WALKS OF LENGTH 4 WITH THE SAME ENDPOINTS

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Abstract

Let $n \geq 8$ be an integer. We characterize the extremal digraphs of order n with the maximum number of arcs avoiding distinct walks of length 4 with the same endpoints.

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1. Introduction

The Turán-type problem is one of the hottest topics in extremal graph theory, which concerns the number of edges in graphs containing no given subgraphs and the extremal graphs achieving this maximum. Most of the previous results of Turán problems concern undirected graphs and only a few Turán problems on digraphs have been investigated; see [1–6].

In this paper, we consider simple digraphs, i.e., digraphs without multiple arcs but allowing loops. We abbreviate directed walks and directed cycles as walks and cycles, respectively. The number of vertices in a digraph is called its *order* and the number of arcs its *size*. Denote by $\mathscr{F}_k$ the family of digraphs consisting of two different walks of length k with the same initial vertex and the same terminal vertex. A digraph D is said to be $\mathscr{F}_k$*-free* if D contains no subgraph from $\mathscr{F}_k$. Let $ex(n, \mathscr{F}_k)$ be the maximum size of $\mathscr{F}_k$-free digraphs of order n, and let $EX(n, \mathscr{F}_k)$ be the set of $\mathscr{F}_k$-free digraphs of order n with size $ex(n, \mathscr{F}_k)$.

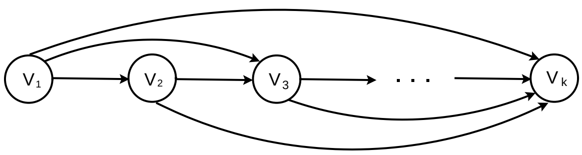
In 2007, Zhan posed the following Turán-type problem on digraphs at a seminar, and later this problem was listed as an open question in [7].

Problem 1. Given positive integers n and k, determine $ex(n, \mathscr{F}_k)$ and $EX(n, \mathscr{F}_k)$.

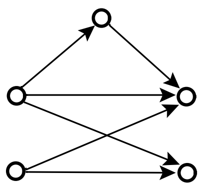
In 2010, Wu [6] solved the case $k = 2$. In 2011, Huang, Zhan [5] solved the case $k \geq n-1$ and determined the extremal numbers for the cases $k = n-2$ and $k = n-3$. In 2019, Huang, Lyu and Qiao [4] characterized the extremal digraphs for the cases $k = n-2$ and $k = n-3$. And they gave the solutions to the case $5 \leq k \leq n-4$. They also determined the extremal number for the case $k = 4$, but the structures of the extremal digraphs are not clear. In this paper, we characterize the structures of the extremal digraphs when $k = 4$.

For a digraph $D = (\mathcal{V}, \mathcal{A})$, we denote by $a(D) = |\mathcal{A}|$ the size of D. Given $X \subset \mathcal{V}$, we denote by $D[X]$ and $D - X$ the subgraphs of D induced by X and $\mathcal{V} \backslash X$, respectively. For convenience, a set X will be abbreviated as x if it is a singleton $\{x\}$. For $i, j \in \mathcal{V}$, if there is an arc from i to j, then we say j is a *successor* of i, and i is a *predecessor* of j. The notation $i \to j$ means there is an arc from i to j; $i \nrightarrow j$ means there exists no arc from i to j. We denote by (i, j) the arc from i to j. For $u \in \mathcal{V}$ and $S \subset \mathcal{V}$, the notation $u \to S$ means there exists an arc from u to each vertex of S; $u \nrightarrow S$ means there is no arc from u to any vertex of S. Analogously, we define $S \to u$ and $S \nrightarrow u$. For $S, T \subset \mathcal{V}$, we denote by $\mathcal{A}(S, T)$ the set of arcs from S to T. The cardinalities of $\mathcal{A}(S, T)$ is denoted by $a(S, T)$. For convenience, S (Respectively, T) will be abbreviated as s (Respectively, t) if it is a singleton $\{s\}$ (Respectively, $\{t\}$).

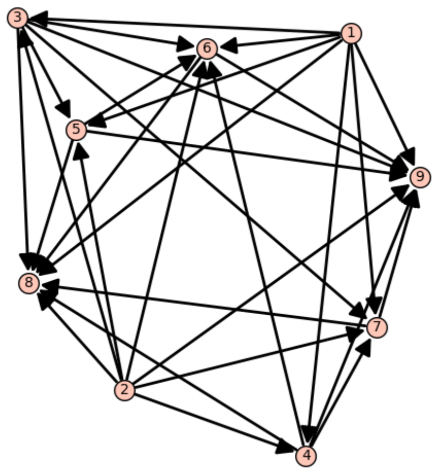
A digraph $D = (\mathcal{V}, \mathcal{A})$ is said to be *transitive* if for any three vertices $x, y, z \in \mathcal{V}$, $(x, y) \in \mathcal{A}$ and $(y, z) \in \mathcal{A}$ imply $(x, z) \in \mathcal{A}$. Recall that a tournament is an orientation of the complete graph. We denote by T_n the transitive tournament on vertices $\{1, 2, \ldots, n\}$ such that $i \to j$ if and only if $i < j$. A c-partite transitive tournament is an orientation of a complete c-partite graph which is also transitive. A c-partite transitive tournament is called balanced if any pair partite sets differ by at most one in size. We present the diagram of the k-partite transitive tournaments as follows.



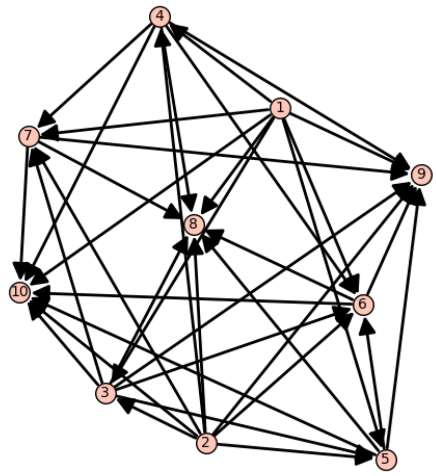
Let M be the digraph as follows.



We define $H(n)$ as a family of digraphs of order n, each of whose elements has a vertex partition $\mathcal{V} = V_1 \cup V_2 \cup V_3$, where $|V_1| = \lfloor (n-5)/2 \rfloor$, $|V_2| = 5$ and $|V_3| = \lceil (n-5)/2 \rceil$, such that $D[V_1]$ and $D[V_3]$ are empty, and $D[V_2]$ is isomorphic to M. Moreover, for $x \in V_i$, $y \in V_j$ and $i \neq j$, $x \to y$ if and only if $i < j$. For convenience, we also use $H(n)$ to indicate a special digraph with the same structure as we define above if it makes no confusion. In the following, we present two examples in $H(9)$ and $H(10)$, respectively.



$H(9)$



$H(10)$

Now we state our main result as follows.

Theorem 2. *Let $n \geq 12$ be an integer. Then $D \in EX(n, \mathscr{F}_4)$ if and only if D is a balanced 4-partite transitive tournament.*

Remark that the cases $n \in \{5, 6, 7\}$ have been solved in [4,5]. In this paper, we also characterize the extremal digraphs for $n \in \{8, 9, 10, 11\}$.

2. Lemmas

We always use $\langle n \rangle = \{1, 2, \ldots, n\}$ to denote the vertex set of a digraph D of order n unless otherwise stated. Let $D = (\mathcal{V}, \mathcal{A})$ be a digraph with vertex set $\mathcal{V}$ and arc set $\mathcal{A}$. Denote by $\overleftarrow{D}$ the reverse of D, which is obtained by reversing the directions of all arcs of D. Denote by

$$N^+(u) = \{x \in \mathcal{V} | (u, x) \in \mathcal{A}\} \quad \text{and} \quad N^-(u) = \{x \in \mathcal{V} | (x, u) \in \mathcal{A}\}$$

the sets of successors and predecessors of a vertex u. The *out-degree* and *in-degree* of u are $d^+(u) = |N^+(u)|$ and $d^-(u) = |N^-(u)|$, respectively. Let $d_D(u) = a(D) -$

$a(D-u)$ be the number of arcs incident with u. We abbreviate it as $d(u)$ if no confusion rises. We have

$$d(u) = \begin{cases} d^+(u) + d^-(u) - 1, & \text{if } u \to u; \\ d^+(u) + d^-(u), & \text{otherwise.} \end{cases}$$

Let d be the number of loops in D. Since $a(D) = \sum_{u\in\mathcal{V}} d^+(u) = \sum_{u\in\mathcal{V}} d^-(u)$, then

$$2a(D) = \sum_{u\in\mathcal{V}} d(u) + d. \tag{1}$$

To determine $EX(n, \mathscr{F}_4)$ for $n \geq 8$, we need the following technical lemmas.

Lemma 3 [4]**.** *Let $n \geq 3$ and p be nonnegative integers, and let D be a digraph on n vertices. Given $q \geq 0$ such that $p(n-1)/2 + q$ is a positive integer, if*

$$a(D-i) \leq \frac{(n-1)(n-2)}{2} - \frac{p(n-1)}{2} - q \quad for\ all\ i \in \langle n\rangle,$$

then

$$a(D) \leq \frac{n(n-1)}{2} - \frac{p(n+1)}{2} - q - 1.$$

Lemma 4. *Let s, k, t, n be positive integers with $t \geq 3$, $s + k \geq 3$ and $n = s+k+t+1$. Suppose $D = (\mathcal{V}, \mathcal{A})$ is a digraph of order n such that $D - n$ is a blow-up of T_{t+2} with vertex partition $\mathcal{V} \setminus \{n\} = \bigcup_{i=1}^{t+2} V_i$, where*

$$V_1 = \{1, \ldots, k\},\ V_2 = \{k+1\}, \ldots, V_{t+1} = \{k+t\},\ V_{t+2} = \{k+t+1, \ldots, k+t+s\},$$

and $(x, y) \in \mathcal{A}$ for $x \in V_i$, $y \in V_j$ if and only if $i < j$. If D is $\mathscr{F}_{t+1}$-free and

$$d(n) \geq \max\{s, k\} + 2, \tag{2}$$

then

$$n \nrightarrow n, \quad n \nrightarrow V_1, \quad V_{t+2} \nrightarrow n,$$

and

$$\{(i, n), (n, j)\} \nsubseteq \mathcal{A} \quad for\ all \quad 1 \leq i, j \leq n-1 \text{ and } j \leq i+2.$$

Proof. It is sufficient to prove that $n \nrightarrow n$ because the other results could be obtained by adopting the same augments as in the proof of [4, Lemma 4]. We let $V_1' = V_1$, $V_2' = \mathcal{V} \setminus (V_1 \cup V_{t+2} \cup \{n\})$ and $V_3' = V_{t+2}$. Note that we already have $n \nrightarrow V_1'$ and $V_3' \nrightarrow n$. To the contrary, suppose $n \to n$.

Suppose a vertex $u \in V_1'$ is a predecessor of n. If a vertex $u_1 \in V_2'$ is a predecessor of n, there are the following two distinct walks of length $t+1$:

$$\begin{cases} u \to u_1 \to n \to \cdots \to n, \\ u \to n \to n \to \cdots \to n, \end{cases}$$

a contradiction. If a vertex $u_2 \in V_2'$ is a successor of n, there are the following two distinct walks of length $t+1$:

$$\begin{cases} u \to k+1 \to k+2 \to \cdots \to k+t+1, \\ u \to n \to \cdots \to n \to u_2 \to k+t+1, \end{cases}$$

a contradiction. If a vertex $u_3 \in V_3'$ is a successor of n, there are the following two distinct walks of length $t+1$:

$$\begin{cases} u \to k+1 \to k+2 \to \cdots \to k+t \to u_3, \\ u \to n \to \cdots \to n \to u_3, \end{cases}$$

a contradiction. Now we have $d(n) \leq |V_1'| + 1 = k+1$, which contradicts (2). Hence, $V_1' \nrightarrow n$. Similarly, we obtain $n \nrightarrow V_3'$.

Suppose $u_1, u_2 \in V_2'$ are two predecessors of n with $u_1 \to u_2$. Then there are the following two distinct walks of length $t+1$:

$$\begin{cases} u_1 \to n \to \cdots \to n, \\ u_1 \to u_2 \to n \to \cdots \to n, \end{cases}$$

a contradiction. Hence, V_2' contains at most one predecessor of n. Similarly, V_2' contains at most one successor of n. It follows that $d(n) \leq 3$, which contradicts (2). Hence, $n \nrightarrow n$. ■

Lemma 5. *Let k_1, k_2, n be positive integers with $n = k_1 + k_2 + 4$ and $k_1 + k_2 \geq 3$. Suppose $D = (\mathcal{V}, \mathcal{A})$ is an $\mathscr{F}_4$-free n-vertex digraph such that $D - n$ is a blow-up of T_5 with vertex partition $\mathcal{V} \setminus \{n\} = \bigcup_{i=1}^{5} V_i$, where*

$$V_1 = \{1, \ldots, k_1\},\ V_2 = \{k_1+1\},\ V_3 = \{k_1+2\},\ V_4 = \{k_1+3\},$$
$$V_5 = \{k_1+4, \ldots, k_1+k_2+3\},$$

and $(x, y) \in \mathcal{A}$ for $x \in V_i$, $y \in V_j$ if and only if $i < j$. If $d(n) = k_1 + k_2 + 1$ and n has both predecessors and successors, then

(1) *$V_1 \to n$ and $n \to V_5$;*

(2) *$k_1 + 1 \to n$ or $n \to k_1 + 3$.*

Proof. Let $s \in \langle n-1 \rangle$ be the largest integer with $s \to n$, $t \in \langle n-1 \rangle$ be the smallest integer with $n \to t$. By Lemma 4, we have $t > s+2$ and $n \nrightarrow n$. It follows that

$$d(n) \leq s + (n-t) \leq n - 3 = k_1 + k_2 + 1.$$

Since the above equalities hold, we get $t - s = 3$, $\{1, \ldots, s\} \to n$ and $n \to \{t, \ldots, n-1\}$. If $s < k_1$, there are the following two distinct walks of length 4 from s to $n-1$:

$$\begin{cases} s \to k_1 + 1 \to k_1 + 2 \to k_1 + 3 \to n - 1, \\ s \to n \to s + 3 \to k_1 + 3 \to n - 1, \end{cases}$$

a contradiction. Hence, $s \geq k_1$. Similarly, $t \leq k_1 + 4$. Then (1) follows. Note that $t = s + 3$, we have $s = k_1$ or $s = k_1 + 1$, then (2) follows immediately. ∎

Lemma 6. *Let $n \geq 9$ and $D \in EX(n, \mathscr{F}_4)$. If there is $i_0 \in \mathcal{V}$ such that $D - i_0$ is a balanced 4-partite transitive tournament, then D is a balanced 4-partite transitive tournament or isomorphic to $H(n)$ or its reverse. Moreover, if $n \geq 12$, then D is a balanced 4-partite transitive tournament.*

Proof. We assume that $D - i_0$ is a balanced 4-partite transitive tournament with vertex partition $\mathcal{V}(D - i_0) = V_1 \cup V_2 \cup V_3 \cup V_4$, and $i \to j$ if and only if $s(i) < s(j)$. Here $s(i)$ is the index of the set i belongs to. We let s and t be nonnegative integers such that $n = 4s + t + 1$ and $t < 4$. We assume that

$$V_1 = \{u_1, u_2, \ldots, u_{|V_1|}\},\ V_2 = \{v_1, v_2, \ldots, v_{|V_2|}\},$$
$$V_3 = \{w_1, w_2, \ldots, w_{|V_3|}\},\ V_4 = \{x_1, x_2, \ldots, x_{|V_4|}\}.$$

Let $\{j_1, j_2, \ldots, j_t\}$ be an arbitrary t-subset of $\{1, 2, 3, 4\}$. If $i \in \{j_1, j_2, \ldots, j_t\}$, we let $|V_i| = s + 1$; else, we let $|V_i| = s$. It is clear that

$$s \geq 2, \tag{3}$$

and

$$d(i_0) = 3s + t. \tag{4}$$

Suppose that $i_0 \to i_0$. Then any pair of the predecessors or successors of i_0 are not adjacent. It follows that

$$d(i_0) = d^+(i_0) + d^-(i_0) - 1 \leq 2s + \bar{t} + 1, \tag{5}$$

where $\bar{t} = \min\{t, 2\}$, which contradicts (4). Hence $i_0 \nrightarrow i_0$.

Let $s' \in \langle 4 \rangle$ be the largest integer such that there is a vertex $i \in V_{s'}$ with $i \to i_0$ and $t' \in \langle 4 \rangle$ be the smallest integer such that there is a vertex $j \in V_{t'}$ with $i_0 \to j$. Here we let $s' = 0$ if $\mathcal{V} \nrightarrow i_0$ and $t' = 5$ if $i_0 \nrightarrow \mathcal{V}$.

We assert that

$$t' \geq 2. \tag{6}$$

Otherwise, i_0 has a successor in V_1, say u_1. Then there are the following two distinct walks of length 4:

$$\begin{cases} i_0 \to u_1 \to v_1 \to w_1 \to x_1, \\ i_0 \to u_1 \to v_2 \to w_1 \to x_1, \end{cases}$$

a contradiction. Similarly, we have

$$s' \leq 3. \tag{7}$$

Suppose $s' = t' = 2$. Without loss of generality, we let $v_1, v_2 \in V_2$ with $v_1 \to i_0$ and $i_0 \to v_2$. There are the following two distinct walks of length 4:

$$\begin{cases} v_1 \to i_0 \to v_2 \to w_1 \to x_1, \\ v_1 \to i_0 \to v_2 \to w_2 \to x_1, \end{cases}$$

a contradiction. Similarly, for the case $s' = t' = 3$, we also get a contradiction. Combining with (6) and (7) we get

$$s' \neq t'. \tag{8}$$

Suppose $s' > t'$. In fact, we only need to consider the case $s' = 3$ and $t' = 2$. Let $v_1 \in V_2$ be a successor of i_0 and $w_1 \in V_3$ be a predecessor of i_0. If i_0 has another successor in V_2, say v_2, there are the following two distinct walks of length 4:

$$\begin{cases} u_1 \to w_1 \to i_0 \to v_1 \to x_1, \\ u_1 \to w_1 \to i_0 \to v_2 \to x_1, \end{cases}$$

a contradiction. Hence, i_0 has exactly one successor in V_2. Similarly, i_0 has exactly one predecessor in V_3. It follows from (4) and (6) that i_0 has a predecessor in V_1, say u_1. Then there are the following two distinct walks of length 4:

$$\begin{cases} u_1 \to i_0 \to v_1 \to w_1 \to x_1, \\ u_1 \to w_1 \to i_0 \to v_1 \to x_1, \end{cases}$$

a contradiction. Therefore, we get

$$t' \geq s' + 1. \tag{9}$$

Suppose $t' = s' + 1$. If $s' = 1$, without loss of generality, we let $u_1 \to i_0$ and $i_0 \to v_1$. There are the following two distinct walks of length 4:

$$\begin{cases} u_1 \to i_0 \to v_1 \to w_1 \to x_1, \\ u_1 \to i_0 \to v_1 \to w_2 \to x_1, \end{cases}$$

a contradiction. Hence, we get $s' \neq 1$. Similarly, we get $s' \neq 3$. Combining with (6) and (7), we get $s' = 2$. Without loss of generality, we let $v_1 \to i_0$ and $i_0 \to w_1$. If i_0 has another successor in V_3, say w_2, there are the following two distinct walks of length 4:

$$\begin{cases} u_1 \to v_1 \to i_0 \to w_1 \to x_1, \\ u_1 \to v_1 \to i_0 \to w_2 \to x_1, \end{cases}$$

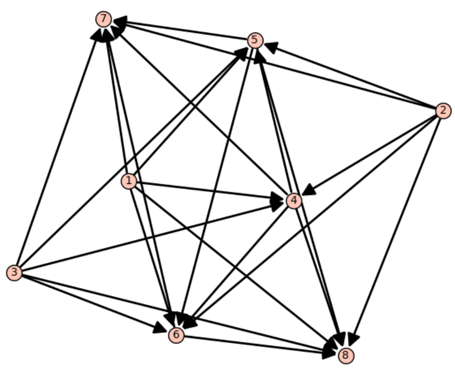
a contradiction. Thus, i_0 has exactly one successor in V_3. Similarly, i_0 has exactly one predecessor in V_2. Combining with (3) and (4), we get $|V_2| = |V_3| = 2$, $V_1 \to i_0$ and $i_0 \to V_4$. It follows that D is $H(n)$ or its reverse.

Suppose $t' = s' + 2$. By (3) and (4), we get $V_i \to n$ for $i \leq s'$ and $n \to V_j$ for $j \geq t'$. Moreover, $|V_{s'+1}| = s$. We can conclude that D is a balanced 4-partite transitive tournament.

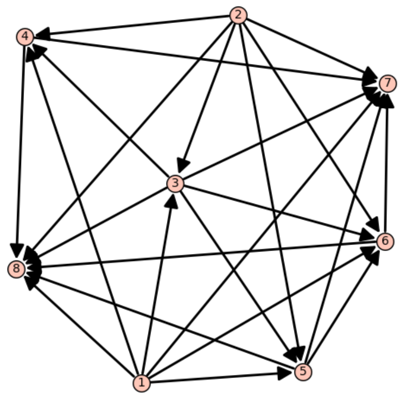
For the case $n \geq 12$, it is sufficient to notice that $a(H(n)) < ex(n, \mathscr{F}_4)$. ■

3. The Structures of the Extremal Digraphs for $n \geq 8$

In this section, we characterize the extremal digraphs for $n \geq 8$. Denote by F_1 the digraph $T_8 - \{(1,2),(1,3),(2,3),(7,8)\}$, F_2 the digraph $T_8 - \{(1,2),(4,5),(4,6),(7,8)\}$. In case the readers are not familiar with the structures of F_1 and F_2, we present them as follows.



F_1



F_2

Theorem 7. *A digraph $D \in EX(8, \mathscr{F}_4)$ if and only if D is a balanced 4-partite transitive tournament or isomorphic to one of* $\{F_1, \overleftarrow{F_1}, F_2, \overleftarrow{F_2}\}$.

Proof. The sufficiency of this theorem is obvious. It is sufficient to show the necessity part. Suppose $D \in Ex(8, \mathscr{F}_4)$. It follows that $a(D) = ex(8, \mathscr{F}_4) = 24$ from [4, Theorem 1]. By [5, Corollary 11], we have $a(D - i) \leq 19$ for all $i \in \langle 8 \rangle$. We distinguish two cases.

Case 1. $a(D-i) = 19$ for some $i \in \langle 8 \rangle$. By [4, Theorem 9], we assume without loss of generality that $D-8 = T_7 - \{(1,2),(6,7)\}$. Since $d(8) = a(D) - a(D-8) = 5$, applying Lemma 4, we obtain $\{6,7,8\} \nrightarrow 8$, $8 \nrightarrow \{1,2\}$ and $\{(i,8),(8,j)\} \not\subset \mathcal{A}$ for $j \leq i+2$.

If 8 has no predecessor, then D is isomorphic to F_1. If 8 has no successor, then D is isomorphic to the reverse of F_1. If 8 has both predecessors and successors, applying Lemma 5, we obtain D is isomorphic to F_2 or its reverse.

Case 2. $a(D-i) \leq 18$ for all $i \in \langle 8 \rangle$. By the definition of $d(i)$ and (1), we obtain

$$d(i) = 6 \quad \text{for all} \quad i \in \langle 8 \rangle. \tag{10}$$

Moreover, there exists no loops in D, i.e.,

$$i \nrightarrow i \quad \text{for all} \quad i \in \langle 8 \rangle. \tag{11}$$

Claim 1. *There exists no* 2*-cycles in* D.

Otherwise, without loss of generality we assume $7 \leftrightarrow 8$. By (10) we get $a(D - \{7,8\}) = a(D) - d(7) - d(8) + 2 = 14$. Applying [4, Theorem 8] to $D - \{7,8\}$, we have $D - \{7,8\}$ is isomorphic to $T_6 - \{(1,2)\}$ or $T_6 - \{(5,6)\}$. Suppose $D - \{7,8\}$ is isomorphic to $T_6 - \{(1,2)\}$. Without loss of generality, we let $D - \{7,8\} = T_6 - \{(1,2)\}$. If $8 \to 1$, there are the following two distinct walks of length 4:

$$\begin{cases} 8 \to 1 \to 3 \to 4 \to 6, \\ 8 \to 1 \to 3 \to 5 \to 6, \end{cases}$$

a contradiction. Similarly, we get $8 \nrightarrow \{1,2\}$ and $7 \nrightarrow \{1,2\}$.

Combining with (10) we have $1 \to \{7,8\}$. If there is $u \in \{3,4,5\}$ with $8 \to u$, there are the following two distinct walks of length 4:

$$\begin{cases} 1 \to 3 \to 4 \to 5 \to 6, \\ 1 \to 7 \to 8 \to u \to 6, \end{cases}$$

a contradiction. If $8 \to 6$, there are the following two distinct walks of length 4:

$$\begin{cases} 1 \to 3 \to 4 \to 5 \to 6, \\ 1 \to 8 \to 7 \to 8 \to 6, \end{cases}$$

a contradiction. By (10), 8 has at least 2 predecessors in $\{3,4,5,6\}$. We assume u_1, u_2 are two predecessors of 8 with $3 \le u_1 < u_2 \le 6$. Then there are the following two distinct walks of length 4:

$$\begin{cases} 1 \to u_1 \to 8 \to 7 \to 8, \\ 1 \to u_2 \to 8 \to 7 \to 8, \end{cases}$$

a contradiction. For the case $D - \{7,8\}$ is isomorphic to $T_6 - \{(5,6)\}$, we also get a contradiction. This completes the proof of Claim 1. □

It follows that

$$a(i,j) + a(j,i) \le 1 \quad \text{for distinct } i,j \in \langle 8\rangle. \tag{12}$$

For each pair $\{i,j\} \subset \langle 8\rangle$, we have $a(D-\{i,j\}) = a(D)-12+a(i,j)+a(j,i) \le 13$. Applying Lemma 3 to $D-8$, there exists $i \in \langle 7\rangle$ such that $a(D - \{i,8\}) \ge 13$. Thus $a(D - \{i,8\}) = 13$. Without loss of generality, we assume

$$a(D - \{7,8\}) = 13. \tag{13}$$

Moreover,

$$a(7,8) + a(8,7) = 1. \tag{14}$$

By (10) and (12), we have $d_{D-\{7,8\}}(i) \ge 4$ for $i \in \langle 6\rangle$. It follows from (13) that $a(D-\{i,7,8\}) \le 9$ for $i \in \langle 6\rangle$. Applying Lemma 3 to $D-\{7,8\}$, there exists $i \in \langle 6\rangle$ such that $a(D - \{i,7,8\}) = 9$. Without loss of generality, we assume

$$a(D - \{6,7,8\}) = 9. \tag{15}$$

Moreover,

$$a(6,i) + a(i,6) = 1 \quad \text{for } i \in \{7,8\}. \tag{16}$$

By (10) and (12), we have $d_{D-\{6,7,8\}}(i) \ge 3$ for $i \in \langle 5\rangle$. It follows from (15) that $a(D - \{i,6,7,8\}) \le 6$ for $i \in \langle 5\rangle$. Applying Lemma 3 to $D - \{6,7,8\}$, there is $i \in \langle 5\rangle$ such that $a(D - \{i,6,7,8\}) \ge 6$. Hence, the equality holds for some $i \in \langle 5\rangle$. Without loss of generality, we assume

$$a(D - \{5,6,7,8\}) = 6. \tag{17}$$

Moreover,

$$a(5,i) + a(i,5) = 1 \quad \text{for } i \in \{6,7,8\}. \tag{18}$$

Let $\alpha = \{1,2,3,4\}$. It is clear that

$$a(D[\alpha]) = a(D - \alpha) = 6. \tag{19}$$

We have the following claim.

Claim 2. *$D[\alpha]$ and $D-\alpha$ are both transitive tournaments.*

It is sufficient to show that they both have no cycles. Suppose $D[\alpha]$ has a cycle. By Claim 1, (11) and [3, Lemma 2.2], $D[\alpha]$ has a 3-cycle. Without loss of generality, we let $1 \to 2 \to 3 \to 1$. By Claim 1, (11) and (17), we get $a(i,4)+a(4,i)=1$ for $i \in \{1,2,3\}$. If 4 has both predecessors and successors in $\{1,2,3\}$, assume without loss of generality that $1 \to 4$ and $4 \to \{2,3\}$, then there are the following two distinct walks of length 4:

$$\begin{cases} 4 \to 2 \to 3 \to 1 \to 2, \\ 4 \to 3 \to 1 \to 4 \to 2, \end{cases}$$

a contradiction. Hence, $4 \to \{1,2,3\}$ or $\{1,2,3\} \to 4$. Let B be a digraph on $\{1,2,3,4\}$ with arc set $\{(1,2),(2,3),(3,1),(1,4),(2,4),(3,4)\}$. Then $D[\alpha]$ is isomorphic to B or its reverse. Applying the same arguments as above, $D-\alpha$ is a transitive tournament or isomorphic to one of $\{B, \overleftarrow{B}\}$.

Suppose $D[\alpha]$ is isomorphic to B or $\overleftarrow{B}$ and so is $D-\alpha$. Without loss of generality, let $\{(1,2),(2,3),(3,1)\} \subset \mathcal{A}(D[\alpha])$ and $\{(5,6),(6,7),(7,5)\} \subset \mathcal{A}(D-\alpha)$. If $1 \to 5$, there are the following two distinct walks of length 4:

$$\begin{cases} 1 \to 2 \to 3 \to 1 \to 5, \\ 1 \to 5 \to 6 \to 7 \to 5, \end{cases}$$

a contradiction. Hence, we have $1 \nrightarrow 5$. Similarly, we have $\{1,2,3\} \nrightarrow 5$ and $5 \nrightarrow \{1,2,3\}$. It follows that $d(5) \leq 4$, which contradicts (10). Hence, $D-\alpha$ is a transitive tournament. Without loss of generality, let

$$D-\alpha = T_8-\alpha.$$

Suppose $D[\alpha]=B$. We can obtain

$$a(\{1,2,3\},\{5\}) \leq 1. \tag{20}$$

Otherwise, there are the following distinct walks of length 4:

$$\begin{cases} 1 \to 5 \to 6 \to 7 \to 8, \\ 1 \to 2 \to 5 \to 7 \to 8, \\ 1 \to 2 \to 3 \to 5 \to 8, \end{cases}$$

a contradiction. We obtain $a(\{1,2\},\{6\}) \leq 1$, otherwise there are the following distinct walks of length 4:

$$\begin{cases} 3 \to 1 \to 6 \to 7 \to 8, \\ 3 \to 1 \to 2 \to 6 \to 8, \end{cases}$$

a contradiction. Similarly, we get $a(\{1,3\},\{6\}) \le 1$ and $a(\{2,3\},\{6\}) \le 1$, which implies that

$$a(\{1,2,3\},\{6\}) \le 1. \tag{21}$$

By (10) and (20), 5 has one successor in $\{1,2,3\}$, say u_1. Similarly, 6 has one successor in $\{1,2,3\}$, say v_1. We let $u_1 \to u_2 \to u_3$ be the walk along $1 \to 2 \to 3 \to 1$, and $v_2 \in \{1,2,3\}$ be the successor of v_1. Then there are the following distinct walks of length 4:

$$\begin{cases} 5 \to u_1 \to u_2 \to u_3 \to 4, \\ 5 \to 6 \to v_1 \to v_2 \to 4, \end{cases}$$

a contradiction.

Suppose $D[\alpha] = \overleftarrow{B}$. Note that the reverse of a transitive tournament is still a transitive tournament. Applying the same arguments, we could get a contradiction.

Now we have proved $D[\alpha]$ is a transitive tournament. Exchanging the roles of $D[\alpha]$ and $D-\alpha$ and repeating the same arguments as above, we get that $D-\alpha$ is also a transitive tournament. This completes Claim 2. □

Next, we show D is a balanced 4-partite transitive tournament. Without loss of generality, let

$$D[\alpha] = T_4 \quad \text{and} \quad D - \alpha = T_8 - \alpha.$$

We obtain $6 \not\to 1$, otherwise there are the following two distinct walks of length 4:

$$\begin{cases} 5 \to 6 \to 1 \to 2 \to 4, \\ 5 \to 6 \to 1 \to 3 \to 4. \end{cases}$$

Similarly, we get $\{2,3,4\} \not\to 5$, $8 \not\to \{1,2,3\}$, $\{3,4\} \not\to 6$, $7 \not\to \{1,2\}$ and $4 \not\to 7$.

Suppose $1 \to 5$. We obtain that $i \not\to i+4$ for $i \in \{2,3,4\}$, otherwise there are two distinct walks of length 4 from 1 to 8. If $1 \not\to 6$, by (10), we get $6 \to \{2,3,4\}$. Then there are the following two distinct walks of length 4:

$$\begin{cases} 1 \to 5 \to 6 \to 2 \to 4, \\ 1 \to 5 \to 6 \to 3 \to 4, \end{cases}$$

a contradiction. Hence, $1 \to 6$. Then $6 \not\to 2$, otherwise there are the following two distinct walks of length 4:

$$\begin{cases} 1 \to 5 \to 6 \to 2 \to 4, \\ 1 \to 6 \to 2 \to 3 \to 4, \end{cases}$$

a contradiction. By (10), we get $6 \to \{3,4\}$. It follows that $7 \nrightarrow \{3,4\}$, otherwise there are the following distinct walks of length 4:

$$\begin{cases} 1 \to 5 \to 6 \to 3 \to 4, \\ 1 \to 6 \to 7 \to 3 \to 4, \end{cases} \quad \begin{cases} 1 \to 5 \to 6 \to 3 \to 4, \\ 1 \to 5 \to 6 \to 7 \to 4. \end{cases}$$

Then $d(7) \le 5$, which contradicts (10). Therefore, we get $1 \nrightarrow 5$. Applying the same arguments, we get $5 \nrightarrow 1$. It follows from (10) that

$$1 \to \{6,7,8\}, \quad 5 \to \{2,3,4\}.$$

Suppose $2 \to 6$. We obtain that $i \nrightarrow i+4$ for $i \in \{3,4\}$, otherwise there are two distinct walks of length 4 from 1 to 8. By Claim 1 we have $6 \nrightarrow 2$. If $2 \nrightarrow 7$, by (10), we have $7 \to \{3,4\}$. Then there are the following two distinct walks of length 4:

$$\begin{cases} 1 \to 2 \to 6 \to 7 \to 4, \\ 1 \to 6 \to 7 \to 3 \to 4, \end{cases}$$

a contradiction. Hence $2 \to 7$. By (10) we have $2 \nrightarrow 8$. Moreover, we get $3 \to 8$ and $8 \to 4$. There are the following two distinct walks of length 4:

$$\begin{cases} 5 \to 6 \to 7 \to 8 \to 4, \\ 5 \to 2 \to 6 \to 8 \to 4, \end{cases}$$

a contradiction. Therefore, $2 \nrightarrow 6$. Similarly, we get $6 \nrightarrow 2$. By (10), we get

$$2 \to \{7,8\}, \quad 6 \to \{3,4\}.$$

Suppose $3 \to 7$. It is clear that $4 \nrightarrow 8$. It follows from (10) that $3 \nrightarrow 8$. Moreover, we get $8 \to 4$. Then there are the following two distinct walks of length 4:

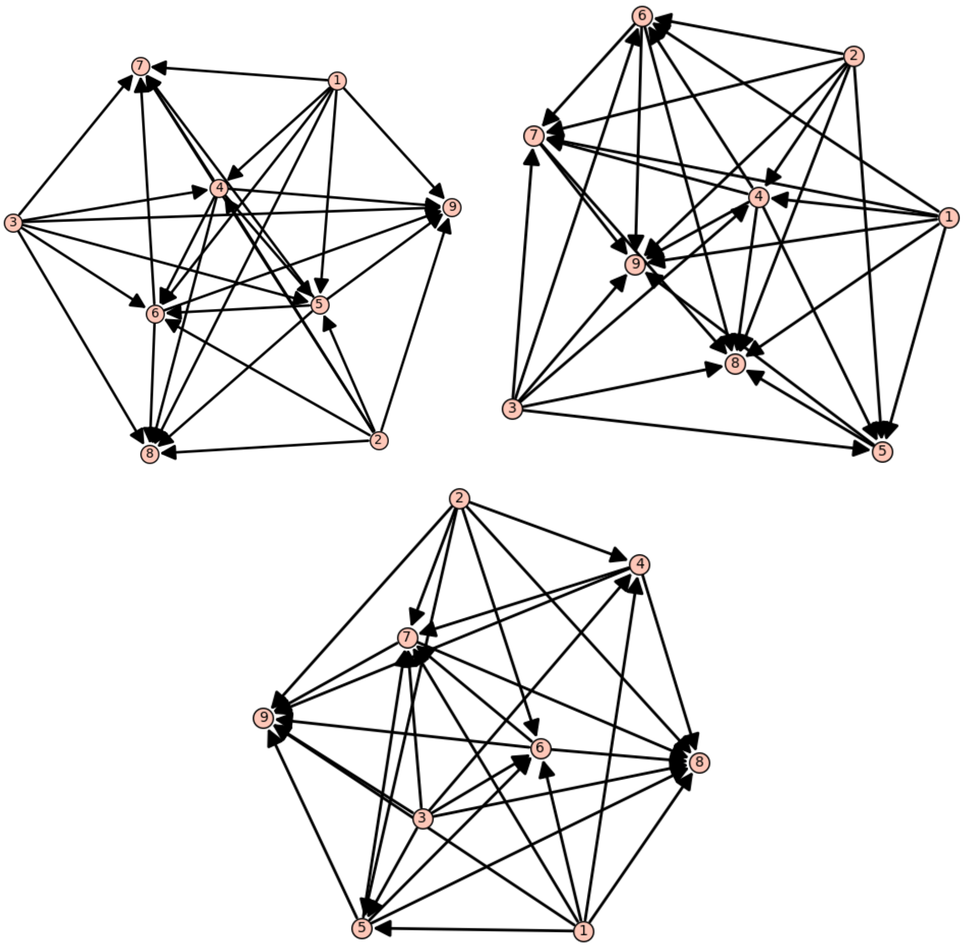
$$\begin{cases} 5 \to 6 \to 7 \to 8 \to 4, \\ 5 \to 2 \to 7 \to 8 \to 4, \end{cases}$$

a contradiction. Hence, $3 \nrightarrow 7$. Similarly, we get $7 \nrightarrow 3$. It follows from (10) that $4 \nrightarrow 8$, $8 \nrightarrow 4$ and

$$3 \to 8, \quad 7 \to 4.$$

Then D is a balanced 4-partite transitive tournament. This completes the proof. ∎

Denote by F_3 the digraph $T_9 - \{(1,2),(1,3),(2,3),(7,8),(7,9),(8,9)\}$, F_4 the digraph $T_9 - \{(1,2),(1,3),(2,3),(5,6),(5,7),(8,9)\}$, F_5 the digraph $T_9 - \{(1,2),(1,3),(2,3),(4,5),(4,6),(8,9)\}$. We present these digraphs as follows.



F_3 F_4 F_5

We give the structures of the extremal digraphs for $n = 9$ as follows.

Theorem 8. *A digraph $D \in EX(9, \mathscr{F}_4)$ if and only if D is a balanced 4-partite transitive tournament or isomorphic to one of $\{F_3, F_4, \overleftarrow{F_4}, F_5, \overleftarrow{F_5}, H(9)\}$.*

Proof. The sufficiency of this theorem is obvious. It is sufficient to show the necessity part. Suppose $D \in Ex(9, \mathscr{F}_4)$. By [4, Theorem 1] we have $a(D) = ex(9, \mathscr{F}_4) = 30$ and

$$d(i) \geq 6 \quad \text{for all} \quad i \in \mathcal{V}. \tag{22}$$

Combining with Lemma 3, there exists $i \in \mathcal{V}$ such that $a(D-i) = 24$. By Theorem 7, $D-i$ or its reverse is a balanced 4-partite transitive tournament or isomorphic to one of $\{F_1, F_2\}$. Without loss of generality, we let $i = 9$. It follows that

$$d(9) = 6. \tag{23}$$

We consider the following cases.

Case 1. $D - 9 = F_1$. By Lemma 4, we get $\{7, 8, 9\} \nrightarrow 9$ and $9 \nrightarrow \{1, 2, 3\}$. If 9 has no successor, then $\langle 6 \rangle \to 9$. Moreover, $D = F_3$. Now assume 9 has at least one successor. Since $9 \nrightarrow \{1, 2, 3, 9\}$ and (23), 9 has a predecessor. By Lemma 5, D is isomorphic to F_4 or F_5. For the case $D - \{9\} = \overleftarrow{F_1}$, we get that D is isomorphic to one of $\{F_3, \overleftarrow{F_4}, \overleftarrow{F_5}\}$.

Case 2. $D - 9 = F_2$. By (22) we get $d_{D-4}(9) \geq 4$. Applying Lemma 4 to $D - 4$ we have

$$9 \nrightarrow \{1, 2\} \text{ and } \{7, 8, 9\} \nrightarrow 9. \tag{24}$$

We claim that

$$a(4, 9) + a(9, 4) = 1. \tag{25}$$

Recalling the structure of F_2 and (22), we have $a(4, 9) + a(9, 4) \geq 1$. If $4 \leftrightarrow 9$, we have $\{1, 2\} \to 9$. Otherwise $a(D - i) = 24$ for $i \in \{1, 2\}$. Applying Theorem 7 to $D - i$, we get that D has no cycle, a contradiction. Similarly, we get $\{7, 8\} \to 9$. Combining with (23), we get $6 \nrightarrow 9$. Applying Theorem 7 to $D - 6$, we have $D - 6$ has no cycles, a contradiction.

Suppose 9 has no predecessor in $\{1, 2, 3\} \cup \{5, 6, 7, 8\}$. By (23) (24) and (25), we obtain $9 \to \{3, 5, 6, 7, 8\}$. If $a(4, 9) = 1$, there are the following two distinct walks of length 4:

$$\begin{cases} 4 \to 9 \to 3 \to 5 \to 7, \\ 4 \to 9 \to 3 \to 6 \to 7, \end{cases}$$

a contradiction. Hence, $4 \nrightarrow 9$ and $9 \to 4$. Then D is isomorphic to F_4. Suppose 9 has no successors in $\{1, 2, 3\} \cup \{5, 6, 7, 8\}$. Similarly, we get that D is isomorphic to $\overleftarrow{F_5}$.

Suppose 9 has both predecessors and successors in $\{1, 2, 3\} \cup \{5, 6, 7, 8\}$. By (23) and (25), we get $d_{D-4}(9) = 5$. Applying Lemma 5 to $D - 4$, we obtain $\{1, 2\} \to 9$, $9 \to \{7, 8\}$, $a(3, 9) + a(9, 6) = 1$. If $4 \to 9$, then there are the following two distinct walks of length 4:

$$\begin{cases} 1 \to 3 \to 5 \to 6 \to 7, \\ 1 \to 3 \to 4 \to 9 \to 7, \end{cases}$$

a contradiction. Hence, it follows $9 \to 4$ from (25). Then we get $3 \to 9$, otherwise there are the following walks of length 4:

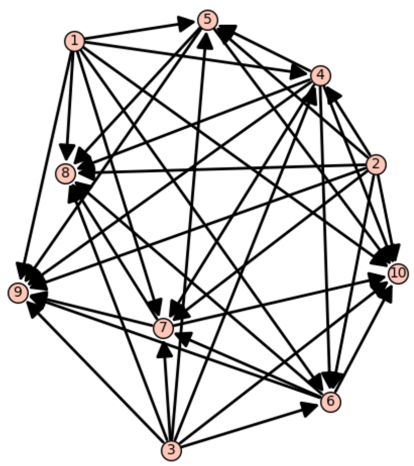
$$\begin{cases} 1 \to 3 \to 5 \to 6 \to 7, \\ 1 \to 3 \to 9 \to 4 \to 7, \end{cases}$$

a contradiction. Hence, D is isomorphic to $H(9)$.

For the case $D - 9 = \overleftarrow{F_2}$, applying the same arguments as above, we can conclude that D is isomorphic to one of $\{\overleftarrow{F_4}, F_5, H(9)\}$.

Case 3. $D - 9$ is a balanced 4-partite transitive tournament. By Lemma 6, D is a balanced 4-partite transitive tournament or isomorphic to $H(9)$. This completes the proof. ■

Denote by F_6 the digraph $T_{10} - \{(1,2),(1,3),(2,3),(5,6),(5,7),(8,9),(8,10),(9,10)\}$, whose structure is presented as follows.



F_6

We give the structures of the extremal digraphs for $n = 10$ as follows.

Theorem 9. *A digraph $D \in EX(10, \mathscr{F}_4)$ if and only if D is a balanced 4-partite transitive tournament or isomorphic to one of $\{F_6, \overleftarrow{F_6}, H(10), \overleftarrow{H}(10)\}$.*

Proof. The sufficiency of this theorem is obvious. It is sufficient to show the necessity part. Suppose $D \in Ex(10, \mathscr{F}_4)$. By [4, Theorem 1] we have $a(D) = ex(10, \mathscr{F}_4) = 37$ and

$$d(i) \geq 7 \quad \text{for all} \quad i \in \mathcal{V}. \tag{26}$$

Combining with Lemma 3, there exists $i \in \mathcal{V}$ such that $a(D - i) = 30$. By Theorem 8, $D - i$ or its reverse is a balanced 4-partite transitive tournament or isomorphic to one of $\{F_3, F_4, F_5, H(9)\}$. Without loss of generality, we let $i = 10$. It is clear that

$$d(10) = 7. \tag{27}$$

We consider the following cases.

Case 1. $D - 10 = F_3$. By Lemma 4, $10 \nrightarrow \{1, 2, 3\}$ and $\{7, 8, 9, 10\} \nrightarrow 10$. It follows from (27) that 10 has both successors and predecessors in $\langle 9 \rangle$. By Lemma 5, D is isomorphic to F_6 or its reverse.

Case 2. $D - 10 = F_4$. It follows from (27) that $d_{D-5}(10) \geq 5$. Applying Lemma 4 to $D - 5$, we have $10 \nrightarrow 1$. Combining with (26) we obtain that $1 \rightarrow 10$ and $a(D - 1) = 30$. Applying the same arguments as in the proof of Case 2 of Theorem 8, $D - 1$ or its reverse is isomorphic to one of $\{F_4, F_5, H(9)\}$. We can conclude that D or its reverse is isomorphic to one of $\{F_6, H(10)\}$. For the case $D - 10 = \overleftarrow{F_4}$, we get the same result.

Case 3. $D - 10 = F_5$. Using the same arguments as in the above case, we get that D or its reverse is isomorphic to one of $\{F_6, H(10)\}$. For the case $D - 10 = \overleftarrow{F_5}$, we get the same result.

Case 4. $D - 10 = H(9)$. Without loss of generality, we let

$$H(9) = T_9 - \{(1, 2), (3, 4), (3, 5), (5, 7), (6, 7), (8, 9)\}.$$

By [5, Lemma 1(iv)], $a(\{3, 7\}, 10) + a(10, \{3, 7\}) \leq 3$. Combing with (27), we get $d_{D-\{3,7\}}(10) \geq 4$. Applying Lemma 4 to $D - \{3, 7\}$, we obtain $10 \nrightarrow \{1, 2\}$, $\{8, 9, 10\} \nrightarrow 10$. It follows from (26) that $a(3, 10) + a(10, 3) \geq 1$ and $a(7, 10) + a(10, 7) \geq 1$. Moreover, at least one equality holds. Without loss of generality, we let $a(3, 10) + a(10, 3) = 1$. Then $a(D - 3) = 30$. By Theorem 8, $D - 3$ contains no cycle, which implies that $a(7, 10) + a(10, 7) = 1$. Now we have $a(D - \{3, 7\}) = 24$ and $d_{D-\{3,7\}}(10) = 5$.

If 10 has no predecessor, we easily get D is isomorphic to the reverse of $H(10)$. Similarly, if 10 has no successor, D is isomorphic to $H(10)$. Now assume 10 has both predecessors and successors. Applying Lemma 5 to $D - \{3, 7\}$, we get $\{1, 2\} \rightarrow 10$, $10 \rightarrow \{8, 9\}$, and $a(4, 10) + a(10, 6) = 1$. Without loss of generality, we assume $4 \rightarrow 10$. Since $a(7, 10) + a(10, 7) = 1$, we may assume $7 \rightarrow 10$, otherwise we can rename the vertices 7 and 10. Then there are the following walks of length 4:

$$\begin{cases} 1 \rightarrow 3 \rightarrow 7 \rightarrow 10 \rightarrow 8, \\ 1 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 8, \end{cases}$$

a contradiction.

Case 5. $D-10$ is a balanced 4-partite transitive tournament. By Lemma 6, D is a balanced 4-partite transitive tournament or isomorphic to one of $\{H(10), \overleftarrow{H}(10)\}$. This completes the proof. ■

Theorem 10. *A digraph $D \in EX(11, \mathscr{F}_4)$ if and only if D is a balanced 4-partite transitive tournament or isomorphic to $H(11)$.*

Proof. The sufficiency of this theorem is obvious. It is sufficient to show the necessity part. Suppose $D \in Ex(11, \mathscr{F}_4)$. By [4, Theorem 1], we get $a(D) = ex(11, \mathscr{F}_4) = 45$ and

$$d(i) \geq 8 \quad \text{for all} \quad i \in \mathcal{V}. \tag{28}$$

By Lemma 3 and Theorem 9, there exists $i \in \mathcal{V}$ such that $D-i$ or its reverse is a balanced 4-partite transitive tournament or isomorphic to one of $\{F_6, H(10)\}$. Without loss of generality, let $i = 11$. It is clear that

$$d(11) = 8. \tag{29}$$

We consider the following cases.

Case 1. $D - 11 = F_6$. By (29) we get $d_{D-5}(11) \geq 6$. Applying Lemma 5 to $D-5$, we obtain $11 \nrightarrow \{1,2,3\}$ and $\{8,9,10,11\} \nrightarrow 11$. By (28) we get $1 \to 11$ and $d(1) = 8$. Applying the same arguments as in Case 2 of Theorem 9, we get that $D-1$ is isomorphic to one of $\{F_6, H(10)\}$. Similarly, we get $D-10$ is isomorphic to one of $\{F_6, \overleftarrow{H}(10)\}$. We can conclude that D is isomorphic to $H(11)$.

Case 2. $D - 11 = H(10)$. Without loss of generality, we let

$$H(10) = T_{10} - \{(1,2), (3,4), (3,5), (6,7), (5,7), (8,9), (8,10), (9,10)\}.$$

By [5, Lemma 1(iv)], $a(\{3,7\}, 11) + a(11, \{3,7\}) \leq 3$. Combing with (29), we get $d_{D-\{3,7\}}(11) \geq 5$. Applying Lemma 4 to $D - \{3,7\}$, we get $10 \nrightarrow 11$. It follows from (28) that $11 \to 10$ and $a(D-10) = 37$. By Theorem 9, we get $a(3,11) + a(11,3) = 1$ and $a(7,11) + a(11,7) = 1$. Applying Theorem 9 to $D-3$ and $D-7$, respectively, we get $D-3$ is isomorphic to one of $\{F_6, H(10), \overleftarrow{H}(10)\}$, and so is $D-7$. We can conclude that D is isomorphic to $H(11)$. Similarly, for the case $D - 11 = \overleftarrow{H}(10)$, we also get D is isomorphic to $H(11)$.

Case 3. $D-11$ is a balanced 4-partite transitive tournament. By Lemma 6, D is a balanced 4-partite transitive tournament or isomorphic to $H(11)$. ■

Now we give the proof of Theorem 2.

Proof of Theorem 2. The sufficiency of this theorem is obvious. It is sufficient to show the necessity part. We first consider the case $n = 12$. Suppose $D \in Ex(12, \mathscr{F}_4)$. By [4, Theorem 1], we get $a(D) = ex(12, \mathscr{F}_4) = 54$ and

$$d(i) \geq 9 \quad \text{for all} \quad i \in \mathcal{V}. \tag{30}$$

Combining with (1) we get

$$d(i) = 9 \quad \text{for all} \quad i \in \mathcal{V}. \tag{31}$$

By Theorem 10, there exists $i \in \mathcal{V}$ such that $D - i$ is a balanced 4-partite transitive tournament or isomorphic to $H(11)$. Without loss of generality, let $i = 12$. We consider the following cases.

Case 1. $D - 12 = H(11)$. Here we let

$$H(11) = T_{11} - \{(1,2),(1,3),(2,3),(4,5),(4,6),(6,8),(7,8),(9,10),\\ (9,11),(10,11)\}.$$

By (31) we get $d_{D-\{4,8\}}(12) \geq 5$. Applying Lemma 4 to $D - \{4,8\}$, we get $12 \nrightarrow 1$. Combining with (31) and the structure of $H(11)$, we get $a(1,12) = 1$ and $D - 1$ is isomorphic to $H(11)$. It follows that 12 and 1 share the same predecessors and successors. We may assume $1 \to 12$, then there are the following two walks of length 4 with the same endpoints:

$$\begin{cases} 1 \to 12 \to 5 \to 6 \to 9, \\ 1 \to 5 \to 6 \to 7 \to 9, \end{cases}$$

a contradiction.

Case 2. $D - 12$ is a balanced 4-partite transitive tournament. By Lemma 6, D is a balanced 4-partite transitive tournament.

For the case $n \geq 13$, Lemma 6 guarantees our result. This completes the proof. ∎

4. CONCLUSION

In this paper, we characterize the structures of the digraphs in $EX(n, \mathscr{F}_4)$ by analyzing the detailed structures of its subgraph of order $n-1$. There exists at least one walk of length 4 in some digraphs of $Ex(n, \mathscr{F}_4)$ when $n \in \{5,6,7,8,9,10,11\}$, while for $n \geq 12$ there is not any walk of length 4 in the digraphs belonging to $Ex(n, \mathscr{F}_4)$. As far as we know, for any fixed $k \geq 5$ and sufficiently large n, there is no walk of length k in the digraphs in $EX(n, \mathscr{F}_k)$. So it is interesting to figure out: what will happen to the maximum size of $\mathscr{F}_k$-free digraphs when there exists a walk of length k? We pose a problem as follows.

Problem 11. Given positive integers n and k, determine the maximum size of $\mathscr{F}_k$-free digraphs in which there exists a walk of length k as well as the structures of the extremal digraphs attaining this maximum.

For other extremal problems on digraphs, the techniques we used may still be valid. In our opinion, these techniques might be effective when the target digraphs contain enough arcs. But in most situations the detailed arguments are very different for different digraphs.

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