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COLORINGS OF PLANE GRAPHS WITHOUT LONG MONOCHROMATIC FACIAL PATHS

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Abstract

Let G be a plane graph. A facial path of G is a subpath of the boundary walk of a face of G. We prove that each plane graph admits a 3-coloring (a 2-coloring) such that every monochromatic facial path has at most 3 vertices (at most 4 vertices). These results are in a contrast with the results of Chartrand, Geller, Hedetniemi (1968) and Axenovich, Ueckerdt, Weiner (2017) which state that for any positive integer t there exists a 4-colorable (a 3-colorable) plane graph G_t such that in any its 3-coloring (2-coloring) there is a monochromatic path of length at least t. We also prove that every plane graph is 2-list-colorable in such a way that every monochromatic facial path has at most 4 vertices.

Keywords: plane graph, facial path, vertex-coloring.

2010 Mathematics Subject Classification: 05C10, 05C15.

1. Introduction and Notations

All graphs considered in this paper are simple connected plane graphs provided that it is not stated otherwise. We use standard graph theory terminology according to [3]. However, the most frequent notions of the paper are defined through it. A *plane graph* is a particular drawing of a planar graph in the Euclidean plane such that no edges intersect except at their endvertices. Let G be a connected plane graph with vertex set $V(G)$, edge set $E(G)$, and face set $F(G)$. The *boundary* of a face f is the boundary in the usual topological sense. It is the collection of all edges and vertices contained in the closure of f that can be organized into a closed walk in G traversing along a simple closed curve lying just inside the face f. This closed walk is unique up to the choice of initial vertex and direction, and is called the *boundary walk* of the face f (see [12], p. 101). The *size* of a face f is the length of its boundary walk. A *k-face* is a face of size k. Let f be a face of size k having the boundary walk $v_0v_1 \cdots v_{k-1}v_0$ with $v_i \in V(G)$ and $v_iv_{i+1} \in E(G)$, $i = 0, \ldots, k-1$, subscripts taken modulo k. A *facial path* of f is a subpath $v_mv_{m+1} \cdots v_n$ (subscripts taken modulo k) of the boundary walk of f (i.e., a facial path is any path which is a consecutive part of the boundary walk of a face). A *k-path*, denoted by P_k, is a path on k vertices.

A *vertex k-coloring* (or, simply, a *k-coloring*) of a graph G is a mapping $\varphi : V(G) \to \{1, \ldots, k\}$. A coloring is *proper* if no two adjacent vertices obtain the same color. A graph is *k-colorable* if it has a proper k-coloring. Unless otherwise stated, the colorings in this paper are not necessarily proper.

The *linear vertex-arboricity* of a graph G is the minimum number of subsets into which the vertex set of G can be partitioned so that every subset induces a linear forest (i.e., a forest in which every component is a path).

Poh [15] and independently Goddard [11] proved that the linear vertex-arboricity of any planar graph is at most three, thus every planar graph can be colored with at most three colors so that each of its monochromatic components is a path. Can these monochromatic paths be short? Chartrand, Geller, and Hedetniemi [7] proved that for every positive integer t, there exists a 4-colorable plane triangulation G_t such that any its 3-coloring involves a monochromatic path of length t. The same authors showed a similar result for 3-colorable graphs. They proved [8] that for every positive integer t, there exists an outerplanar graph G_t such that any its 2-coloring involves a monochromatic path of length t. Recently, Axenovich, Ueckerdt, and Weiner [2] constructed for every $t \geq 2$ a planar graph G_t of girth 4 (triangle-free planar graphs are 3-colorable, see [13]) such that in any 2-coloring of G_t there is a monochromatic path of length at least t. So in planar graphs with small girth the monochromatic paths can be very long.

On the other hand, Borodin, Kostochka, and Yancey [5] proved that every planar graph of girth at least 7 admits a 2-coloring such that each of its monochromatic components has at most 2 vertices. Axenovich, Ueckerdt, and Weiner [2] proved a similar result for planar graphs of girth 6. They showed that every planar graph of girth at least 6 has a 2-coloring such that each of its monochromatic components is a path of length at most 14. The problem whether planar graphs of girth 5 admit a 2-coloring such that the monochromatic components are short paths is open in general. Lovász [14] showed that every subcubic graph admits a

2-coloring such that every vertex has at most one neighbor of the same color.

Broersma *et al.* [6] proved that it is NP-hard to decide whether a planar graph has a 3-coloring (a 2-coloring) without monochromatic path P_n, $n \geq 3$.

In this paper we focus on facial paths of plane graphs. We show that every plane graph has a 3-coloring and also a 2-coloring such that every monochromatic facial path is short. Among others we prove that every plane graph admits a 2-coloring such that each monochromatic facial path has at most 4 vertices and every plane graph has a 3-coloring such that each monochromatic facial path has at most 3 vertices. We also prove that every plane graph is 2-list-colorable without monochromatic facial 5-paths.

2. Results

As every planar graph has a proper 4-coloring [1], i.e., admits a 4-coloring without monochromatic facial 2-paths, we focus here on 2-colorings and 3-colorings.

Czap, Jendroľ, and Valiska [10] proved that every 3-connected plane graph admits a 2-coloring without monochromatic facial 5-paths. First we show that 3-connectedness is not necessary in this assertion.

Theorem 1. *Every plane graph admits a* 2*-coloring without monochromatic facial* 5*-paths.*

Proof. Suppose there is a counterexample to Theorem 1. Let G be a counterexample with the minimum number of vertices.

First we prove that the minimum degree of G is at least two. Suppose that v is a vertex of degree one and let u be its neighbor in G. By the minimality of G, the graph $G - v$ admits a 2-coloring φ without monochromatic facial 5-paths. The coloring of $G - v$ can be easily extended to a coloring of G. It suffices to color v with a color different from $\varphi(u)$.

Now we extend G to a plane (multi)graph by adding some edges. Let f be a face of G of size at least four. Let $v_0 v_1 \cdots v_{k-1} v_0$ be the boundary walk of f. The vertices v_i and v_{i+2} are distinct since G is simple and it has no vertex of degree one. We insert into f the diagonals $v_0v_2, v_2v_4, \ldots, v_{k-2}v_0$ if k is even, and the diagonals $v_0v_2, v_2v_4, \ldots, v_{k-3}v_{k-1}$ if k is odd. In such a way we obtain a new plane (multi)graph H. By the Four Color Theorem [1], H has a proper coloring with at most four colors, say a, b, c, d. Note that every 3-cycle uses three different colors. If we assign 1 to all the vertices colored with a, b and assign 2 to all the vertices colored with c, d, then we obtain a 2-coloring of H such that no 3-cycle is monochromatic. This coloring of H induces a coloring of G. Now suppose that G has a monochromatic facial 5-path $v_iv_{i+1}v_{i+2}v_{i+3}v_{i+4}$ (subscripts taken modulo k). From the subscripts $i+1, i+2, i+3$ at least one is odd. Without loss of generality assume that $i+1$ is odd. Then $v_i v_{i+1} v_{i+2}$ is a 3-face in H, so two colors appear on these three vertices, consequently the path $v_i v_{i+1} v_{i+2} v_{i+3} v_{i+4}$ cannot be monochromatic, a contradiction. ■

From the proof of Theorem 1 it follows that every plane graph G admits a 2-coloring such that G is without monochromatic facial 5-paths, moreover, no monochromatic facial 4-path appears on faces of even size. The following result improves this assertion.

Theorem 2. *Every 2-connected plane graph without 5-faces admits a 2-coloring without monochromatic facial 4-paths.*

Proof. Let G be a 2-connected plane graph without 5-faces. Let f be a face of G of size at least four. Let $v_0 v_1 \cdots v_{k-1} v_0$ be the boundary walk of f. If k is even, then we insert the diagonals $v_0v_2, v_2v_4, \ldots, v_{k-2}v_0$ into f. Assume that k is odd. Now, k is at least 7, since G has no 5-faces. First we add the diagonals $v_0v_2, v_2v_4, \ldots, v_{k-3}v_{k-1}$ into f, then we add v_2v_{k-3} and finally we put a vertex v to the new 4-face $v_0v_2v_{k-3}v_{k-1}$ and join it with the vertices $v_0, v_2, v_{k-3}, v_{k-1}$, see Figure 1 for illustration.

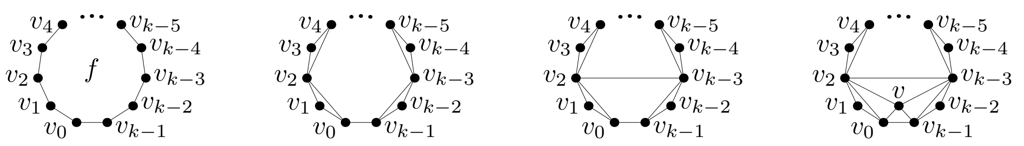

Figure 1. The extension of faces of odd size.

In such a way we obtain a new plane (multi)graph H. The Four Color Theorem implies that H has a 2-coloring such that no 3-cycle is monochromatic. This coloring of H induces a required coloring of G. The facial path $v_{k-2}v_{k-1}v_0v_1$ of f cannot be monochromatic in G, since otherwise the 3-cycle $v_2v_{k-3}v$ is monochromatic in H. Every other facial 4-path of f contains three vertices which form a 3-face in H, therefore two colors appear on them. ■

Conjecture 3. *Every plane graph admits a 2-coloring without monochromatic facial 4-paths.*

If Conjecture 3 is true, then, because of the next theorem, the bound 4 for the number of vertices is best possible. Note that Conjecture 3 is true for outerplane graphs [10].

Theorem 4. *There is an infinite family of plane graphs such that any 2-coloring involves a monochromatic facial 3-path.*

Proof. Consider a plane graph G containing the configuration H depicted in Figure 2. Suppose to the contrary that G admits a 2-coloring φ without monochromatic facial P_3. The face f determined by the vertices v_1, v_2, v_3 has size 3. The fact that φ uses two colors implies that there is a pair of adjacent vertices on the boundary of f that have the same color. Without loss of generality we can assume that $\varphi(v_2) = \varphi(v_3) = 1$. Then necessarily $\varphi(v_4) = \varphi(v_5) = 2$, otherwise G contains a monochromatic facial 3-path. Now, there is no admissible color for the vertex v_6, because $v_4v_6v_5$ and $v_3v_6v_2$ are facial 3-paths, a contradiction.

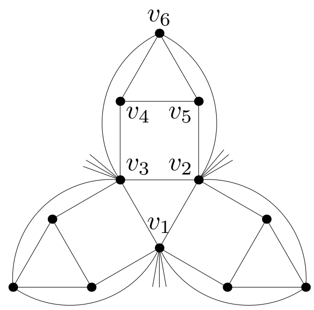

Figure 2. The configuration H.

■

Theorem 5. *Every plane graph admits a* 3*-coloring without monochromatic facial* 4*-paths.*

Proof. We proceed as in the proof of Theorem 1. Let G be a counterexample with the minimum number of vertices. In this case we extend G to a 1-planar graph. A graph is called 1-planar if there exists its drawing in the plane such that each edge is crossed at most once.

Using the same arguments as in the proof of Theorem 1 we can show that G has no vertex of degree one.

Now we extend G to a 1-planar (multi)graph by adding some edges. Let g be a face of G of size at least four. Let $v_0v_1 \cdots v_{k-1}v_0$ be the boundary walk of g. The vertices v_i and v_{i+2} are distinct since G is simple and has no vertex of degree one. If k is even, then we insert the diagonals $v_0v_2, v_2v_4, \ldots, v_{k-2}v_0$ into g. If k is odd, then we insert the diagonals $v_0v_2, v_2v_4, \ldots, v_{k-3}v_{k-1}$ and finally we insert $v_{k-1}v_1$ which crosses v_0v_2. In such a way we obtain a 1-plane (multi)graph H. Every 1-planar graph has a proper coloring with at most six colors [4]. So H has a proper coloring with at most six colors, say a, b, c, d, e, f. Evidently every 3-cycle uses three different colors. If we assign 1 to all the vertices colored with a, b, assign 2 to all the vertices colored with c, d and assign 3 to all the vertices colored with e, f, then we obtain a 3-coloring of H such that no 3-cycle is monochromatic. This coloring of H induces a coloring of G. Now suppose that G has a monochromatic facial 4-path $v_iv_{i+1}v_{i+2}v_{i+3}$ (subscripts taken modulo k).

If $i+1$ or $i+2$ is odd, then $v_iv_{i+1}v_{i+2}$ or $v_{i+1}v_{i+2}v_{i+3}$ is a 3-cycle in H, so two colors appear on these three vertices. If $i+1$ and $i+2$ are even, then necessarily k is odd. In this case $v_iv_{i+1}v_{i+2} = v_{k-1}v_0v_1$ or $v_{i+1}v_{i+2}v_{i+3} = v_{k-1}v_0v_1$. Since $v_{k-1}v_0v_1$ is a 3-cycle in H it cannot be monochromatic, a contradiction. ■

Conjecture 6. *Every plane graph admits a* 3*-coloring without monochromatic facial* 3*-paths.*

Conjecture 6 holds for plane graphs without cycles of length t, for some $t \in \{3,4,5\}$. A graph is $(3,1)$*-colorable* if it has a 3-coloring such that every vertex has at most one neighbor receiving the same color as itself. Clearly, if every vertex has at most one neighbor receiving the same color as itself, then there is no monochromatic 3-path. From Grötzsch's theorem [13] it follows that planar graphs without 3-cycles are $(3,1)$-colorable. Wang and Xu [17] proved that planar graphs without 4-cycles are $(3,1)$-colorable. Finally, Wang and Xu [18] showed that planar graphs without 5-cycles are also $(3,1)$-colorable. Note that the (3,1)-coloring problem is NP-complete even for planar graphs, see [9].

3. List Coloring

A *list assignment* of a graph G is a function L that assigns a list $L(v)$ of possible colors to each vertex $v \in V(G)$. An L*-coloring* is a coloring of G such that each vertex $v \in V(G)$ is assigned a color from $L(v)$. We say that G is 2*-list-colorable* if G admits an L-coloring for every list assignment L with $|L(v)| = 2$ for all $v \in V(G)$.

Now, we show that the list version of Theorem 1 also holds.

Theorem 7. *Every plane graph is* 2*-list-colorable without monochromatic facial* 5*-paths.*

Proof. Suppose there is a counterexample to Theorem 7. Let G be a counterexample with the minimum number of vertices and let L be a list assignment for which G has no L-coloring without monochromatic facial 5-paths. Using the same arguments as in the proof of Theorem 1 we can show that G has no vertex of degree one. Now we extend G to a plane (multi)graph H in the same way as in the proof of Theorem 1. Thomassen [16] proved that every planar graph is 2-list-colorable without monochromatic triangles. Therefore, H has an L-coloring without monochromatic triangles. Using the same arguments as in the proof of Theorem 1 we can show that any such L-coloring of H is also an L-coloring of G without monochromatic facial 5-paths, a contradiction. ■

We finish the paper with conjecture that the list version of Theorem 5 holds as well.

Conjecture 8. *Every plane graph is* 3*-list-colorable without monochromatic facial* 4*-paths.*

Conjecture 8 is closely related to the following question. Is every 1-planar graph 3-list-colorable without monochromatic triangles? If the answer is yes, then Conjecture 8 is true.

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