*Discussiones Mathematicae*
*Graph Theory 42 (2022) 967–984*
*https://doi.org/10.7151/dmgt.2318*

# TOTAL PROTECTION OF LEXICOGRAPHIC PRODUCT GRAPHS

Abel Cabrera Martínez

and

Juan Alberto Rodríguez-Velázquez

*Universitat Rovira i Virgili*
*Departament d'Enginyeria Informàtica i Matemàtiques*
*Av. Països Catalans 26, 43007 Tarragona, Spain*

**e-mail:** abel.cabrera@urv.cat
juanalberto.rodriguez@urv.cat

**Abstract**

Given a graph $G$ with vertex set $V(G)$, a function $f : V(G) \to \{0, 1, 2\}$ is said to be a total dominating function if $\sum_{u\in N(v)} f(u) > 0$ for every $v \in V(G)$, where $N(v)$ denotes the open neighbourhood of $v$. Let $V_i = \{x \in V(G) : f(x) = i\}$. A total dominating function $f$ is a total weak Roman dominating function if for every vertex $v \in V_0$ there exists a vertex $u \in N(v) \cap (V_1 \cup V_2)$ such that the function $f'$, defined by $f'(v) = 1$, $f'(u) = f(u) - 1$ and $f'(x) = f(x)$ whenever $x \in V(G) \setminus \{u, v\}$, is a total dominating function as well. If $f$ is a total weak Roman dominating function and $V_2 = \emptyset$, then we say that $f$ is a secure total dominating function. The weight of a function $f$ is defined to be $\omega(f) = \sum_{v\in V(G)} f(v)$. The total weak Roman domination number (secure total domination number) of a graph $G$ is the minimum weight among all total weak Roman dominating functions (secure total dominating functions) on $G$. In this article, we show that these two parameters coincide for lexicographic product graphs. Furthermore, we obtain closed formulae and tight bounds for these parameters in terms of invariants of the factor graphs involved in the product.

**Keywords:** total weak Roman domination, secure total domination, total domination, lexicographic product.

**2010 Mathematics Subject Classification:** 05C69, 05C76.

## 1. Introduction

It is well known that the theory of domination in graphs can be developed using functions $f : V(G) \to A$, where $V(G)$ is the vertex set of a graph $G$ and $A$ is a set of nonnegative numbers. With this approach, the different types of domination are obtained by imposing certain restrictions on $f$. For instance, $f : V(G) \to \{0, 1, \dots\}$ is said to be a *dominating function* if for every vertex $v$ such that $f(v) = 0$, there exists a vertex $u \in N(v)$ such that $f(u) > 0$, where $N(v)$ denotes the open neighbourhood of $v$. Analogously, $f : V(G) \to \{0, 1, \dots\}$ is said to be a *total dominating function* (TDF) if for every vertex $v$, there exists $u \in N(v)$ such that $f(u) > 0$.

The *weight* of a function $f$ is defined to be $\omega(f) = \sum_{v \in V(G)} f(v)$. The (*total*) *domination number* of $G$, denoted by $(\gamma_t(G))$ $\gamma(G)$, is the minimum weight among all (total) dominating functions. These two parameters have been extensively studied. For instance, we cite the following books [15, 16, 19]. Although the use of functions is not necessary to reach the concept of (total) domination number, later we will see that this idea helps us to easily introduce other more elaborate concepts. Obviously, a set $X \subseteq V(G)$ is a (*total*) *dominating set* if there exists a (total) dominating function $f$ such that such that $X = \{x : f(x) > 0\}$.

From now on, we restrict ourselves to the case of functions $f : V(G) \to \{0, 1, 2\}$, which are related to the following approach to protection of a graph described by Cockayne *et al.* [12]. Suppose that one or more guards are stationed at some of the vertices of a simple graph $G$ and that a guard at a vertex can deal with a problem at any vertex in its closed neighbourhood. Consider a function $f : V(G) \to \{0, 1, 2\}$ where $f(v)$ is the number of guards at $v$, and let $V_i = \{v \in V(G) : f(v) = i\}$ for every $i \in \{0, 1, 2\}$. We will identify $f$ with the partition of $V(G)$ induced by $f$ and write $f(V_0, V_1, V_2)$. Given a set $S \subseteq V(G)$, $f(S) = \sum_{v \in S} f(v)$. In this case, the weight of $f$ is $\omega(f) = f(V(G)) = |V_1| + 2|V_2|$.

We now consider some graph protection approaches. The functions in each approach protect the graph according to a certain strategy.

A *Roman dominating function* (RDF) is a function $f(V_0, V_1, V_2)$ such that for every vertex $v \in V_0$ there exists a vertex $u \in V_2$ which is adjacent to $v$. The *Roman domination number*, denoted by $\gamma_R(G)$, is the minimum weight among all RDFs on $G$. This concept of protection has historical motivation [23] and was formally proposed by Cockayne *et al.* in [9]. Many variations and generalizations of Roman domination number like double Roman domination number [1], Italian domination number [17] (also known as Roman 2-domination number [7]), perfect Italian domination number [14] and weak Roman domination number [18] are available in literature.

A *weak Roman dominating function* (WRDF) is defined to be a dominating function $f(V_0, V_1, V_2)$ satisfying that for every vertex $v \in V_0$ there exists a vertex

$u \in N(v) \cap (V_1 \cup V_2)$ such that the function $f'$, defined by $f'(v) = 1$, $f'(u) = f(u)-1$ and $f'(x) = f(x)$ whenever $x \in V(G) \setminus \{u, v\}$, is a dominating function as well. The *weak Roman domination number*, denoted by $\gamma_r(G)$, is the minimum weight among all weak Roman dominating functions on $G$. This concept of protection was introduced by Henning and Hedetniemi [18] and studied further in [8, 11, 22].

In this paper we will use the following idea of total protection of a vertex. A vertex $v \in V_0$ is said to be *totally protected* under $f(V_0, V_1, V_2)$ if $f$ is a TDF and there exists a vertex $u \in N(v) \cap (V_1 \cup V_2)$ such that the function $f'$, defined by $f'(v) = 1$, $f'(u) = f(u) - 1$ and $f'(x) = f(x)$ whenever $x \in V(G) \setminus \{u, v\}$, is a TDF as well. In such a case, if it is necessary to emphasize the role of $u$, then we will say that $v$ is *totally protected by* $u$ under $f$. In this context, if $V_2 = \emptyset$, then we also say that $v$ is totally protected by $u$ under $V_1$.

The following concept was introduced in [5]. A *total weak Roman dominating function* (TWRDF) is a TDF $f(V_0, V_1, V_2)$ such that every vertex in $V_0$ is totally protected under $f$. The *total weak Roman domination number*, denoted by $\gamma_{tr}(G)$, is the minimum weight among all total weak Roman dominating functions on $G$.

A *secure total dominating function* (STDF) is defined to be a TWRDF $f(V_0, V_1, V_2)$ in which $V_2 = \emptyset$. Obviously, $f(V_0, V_1, \emptyset)$ is a STDF if and only if $V_1$ is a total dominating set and for every vertex $v \in V_0$ there exists $u \in N(v) \cap V_1$ such that $(V_1 \setminus \{u\}) \cup \{v\}$ is a total dominating set as well. In such a case, $V_1$ is said to be a *secure total dominating set* (STDS). The *secure total domination number*, denoted by $\gamma_{st}(G)$, is the minimum cardinality among all secure total dominating sets. This concept was introduced by Benecke *et al.* in [2] and studied further in [3, 4, 6, 13, 20].

Given a graph $G$, the problem of computing $\gamma_{tr}(G)$ is NP-hard [5], and the problem of computing $\gamma_{st}(G)$ is also NP-hard [13]. This suggests finding the total weak Roman domination number and the secure total domination number for special classes of graphs or obtaining good bounds on these invariants. In this article, we show that these two parameters coincide for lexicographic product graphs. Furthermore, we obtain closed formulae and tight bounds for these parameters in terms of invariants of the factor graphs involved in the product.

The lexicographic product of two graphs $G$ and $H$ is the graph $G \circ H$ whose vertex set is $V(G \circ H) = V(G) \times V(H)$ and $(u, v)(x, y) \in E(G \circ H)$ if and only if $ux \in E(G)$ or $u = x$ and $vy \in E(H)$. Notice that for any vertex $u \in V(G)$ the subgraph of $G \circ H$ induced by $\{u\} \times V(H)$ is isomorphic to $H$. For simplicity, we will denote this subgraph by $H_u$.

Throughout the paper, we will use the notation $K_n$, $N_n$, $K_{1,n-1}$, $C_n$ and $P_n$ for complete graphs, empty graphs, star graphs, cycle graphs and path graphs of order $n$, respectively. We will use the notation $G \cong H$ if $G$ and $H$ are isomorphic graphs. For a vertex $v$ of a graph $G$, the *closed neighbourhood*, denoted by $N[v]$, equals $N(v) \cup \{v\}$. A vertex $v \in V(G)$ such that $N[v] = V(G)$ is said to be a *universal vertex.*

A TWRDF of weight $\gamma_{tr}(G)$ will be called a $\gamma_{tr}(G)$-*function.* A similar agreement will be assumed when referring to optimal functions (and sets) associated to other parameters used in the article. For the remainder of the paper, definitions will be introduced whenever a concept is needed.

## 2. Some Tools

In this short section we collect some tools, which are known results on the (total) weak Roman domination number and the secure total domination number.

**Proposition 1** [5]**.** *The following inequalities hold for any graph $G$ with no isolated vertex.*

(i) $\gamma(G) \le \gamma_r(G) \le \gamma_{tr}(G) \le 2\gamma_t(G)$.

(ii) $\gamma_t(G) \le \gamma_{tr}(G) \le \gamma_{st}(G)$.

(iii) $\gamma(G) + 1 \le \gamma_{tr}(G)$.

**Theorem 2** [5]**.** *Let $G$ be a graph. The following statements are equivalent.*

(a) $\gamma_{tr}(G) = \gamma_r(G)$.

(b) *There exists a $\gamma_r(G)$-function $f(V_0, V_1, V_2)$ such that $V_1 = \emptyset$ and $V_2$ is a total dominating set.*

(c) $\gamma_r(G) = 2\gamma_t(G)$.

The problem of characterizing the graphs with $\gamma_{st}(G) = \gamma_t(G)$ was solved by Klostermeyer and Mynhardt [20].

**Theorem 3** [20]**.** *If $G$ is a connected graph, then the following statements are equivalent.*

- $\gamma_{st}(G) = \gamma_t(G)$.
- $\gamma_{st}(G) = 2$.
- *$G$ has at least two universal vertices.*

The following result is a direct consequence of Proposition 1(ii) and Theorem 3.

**Theorem 4.** *Let $G$ be a connected graph. If $G$ does not have two universal vertices, then*

$$\gamma_{st}(G) \ge \gamma_t(G) + 1.$$

**Remark 5.** For any nontrivial path $P_n$ and any cycle $C_n$ of order $n \ge 4$,

(i) $\gamma_{tr}(P_n) \overset{[5]}{=} \gamma_{st}(P_n) \overset{[2]}{=} \left\lceil \frac{5(n-2)}{7} \right\rceil + 2$;

(ii) $\gamma_{tr}(C_n) \overset{[5]}{=} \gamma_{st}(C_n) \overset{[3]}{=} \left\lceil \frac{5n}{7} \right\rceil$.

A set $X \subseteq V(G)$ is called a 2-*packing* if $N[u] \cap N[v] = \emptyset$ for every pair of different vertices $u, v \in X$ [16]. The 2-*packing number* $\rho(G)$ is the maximum cardinality among all 2-packings of $G$. A 2-packing of cardinality $\rho(G)$ is called a $\rho(G)$-*set*.

**Theorem 6** [22]**.** *For any graph $G$ with no isolated vertex and any noncomplete graph $H$,*

$$\gamma_r(G \circ H) \geq \max\{\gamma_r(G), \gamma_t(G), 2\rho(G)\}.$$

*Furthermore, for any graph $G$ and any integer $n \geq 1$,*

$$\gamma_r(G \circ K_n) = \gamma_r(G).$$

**Theorem 7** [22]**.** *Let $n \geq 2$ be an integer and let $H$ be a graph. If $\gamma(H) \geq 4$, then*

$$\gamma_r(P_n \circ H) = \begin{cases} n, & n \equiv 0 \pmod 4, \\ n+2, & n \equiv 2 \pmod 4, \\ n+1, & \text{otherwise.} \end{cases}$$

A *double total dominating set* of a graph $G$ with minimum degree at least two is a set $S$ of vertices of $G$ such that every vertex in $V(G)$ is adjacent to at least two vertices in $S$ [19]. The *double total domination number* of $G$, denoted by $\gamma_{2,t}(G)$, is the minimum cardinality among all double total dominating sets.

**Theorem 8** [22]**.** *If $G$ is a graph with minimum degree at least two, then for any graph $H$,*

$$\gamma_{2,t}(G \circ H) \leq \gamma_{2,t}(G).$$

To conclude this section we would recall the following upper bound on the total domination number.

**Theorem 9** [10]**.** *For any connected graph $G$ of order $n \geq 3$,*

$$\gamma_t(G) \leq \frac{2n}{3}.$$

## 3. Main Results on Lexicographic Product Graphs

The next theorem shows that the total weak Roman domination number and the secure total domination number coincide for all lexicographic product graphs.

**Theorem 10.** *For any graph $G$ with no isolated vertex and any nontrivial graph $H$, i.e., any graph $H$ of order greater than one,*

$$\gamma_{tr}(G \circ H) = \gamma_{st}(G \circ H).$$

***Proof.*** Proposition 1(ii) leads to $\gamma_{tr}(G \circ H) \leq \gamma_{st}(G \circ H)$. Let $f(V_0, V_1, V_2)$ be a $\gamma_{tr}(G \circ H)$-function such that $|V_2|$ is minimum. We suppose that $\gamma_{tr}(G \circ H) < \gamma_{st}(G \circ H)$. In such a case, $V_2 \neq \emptyset$ and we fix a vertex $(u, v) \in V_2$. We differentiate two cases.

*Case* 1. $(N(u) \times V(H)) \cap (V_1 \cup V_2) \neq \emptyset$. If $f(u, v') > 0$ for every $v' \in V(H)$, then the function $g$, defined by $g(u, v) = 1$ and $g(a, b) = f(a, b)$ whenever $(a, b) \neq (u, v)$, is a TWRDF on $G \circ H$ and $\omega(g) = \omega(f) - 1$, which is a contradiction. Hence, there exists $v' \in V(H)$ such that $f(u, v') = 0$. In this case, we define the function $g(V_0', V_1', V_2')$ by $V_0' = V_0 \setminus \{(u, v')\}$, $V_1' = V_1 \cup \{(u, v), (u, v')\}$ and $V_2' = V_2 \setminus \{(u, v)\}$. Now, if a vertex $w \in V_0' \subseteq V_0$ is totally protected by $z \in V_1 \cup V_2 \subseteq V_1' \cup V_2'$ under $f$, then $w$ is also totally protected under $g$ by $z$, which implies that $g$ is a $\gamma_{tr}(G \circ H)$-function. Notice that $|V_2'| = |V_2| - 1$, which is a contradiction again.

*Case* 2. $N(u, v) \cap (V_1 \cup V_2) \subseteq V(H_u)$. In this case, for any $(u', v') \in N(u) \times V(H)$ we define the function $g(V_0', V_1', V_2')$ by $V_0' = V_0 \setminus \{(u', v')\}$, $V_1' = V_1 \cup \{(u, v), (u', v')\}$ and $V_2' = V_2 \setminus \{(u, v)\}$. As above, if a vertex $w \in V_0' \subseteq V_0$ is totally protected by $z \in V_1 \cup V_2 \subseteq V_1' \cup V_2'$ under $f$, then $w$ is also totally protected by $z$ under $g$. Hence, $g$ is a $\gamma_{tr}(G \circ H)$-function and $|V_2'| = |V_2| - 1$, which is a contradiction.

According to the two cases above we conclude that $V_2 = \emptyset$, which implies that $f$ is a $\gamma_{st}(G \circ H)$-function, an so $\gamma_{tr}(G \circ H) = \gamma_{st}(G \circ H)$. ∎

From now on we proceed to express the value of $\gamma_{st}(G \circ H)$ (or its bounds) in terms of several parameters of $G$ and $H$. To this end, we need to introduce the following notation. For a set $S \subseteq V(G \circ H)$ we define the following subsets of $V(G)$:

$$\mathcal{A}_S = \{v \in V(G) : |S \cap V(H_v)| \geq 2\};$$

$$\mathcal{B}_S = \{v \in V(G) : |S \cap V(H_v)| = 1\};$$

$$\mathcal{C}_S = \{v \in V(G) : S \cap V(H_v) = \emptyset\}.$$

Surprisingly, we have not been able to find any reference about the following basic result.

**Theorem 11.** *For any graph $G$ with no isolated vertex and any graph $H$,*

$$\gamma_t(G \circ H) = \gamma_t(G).$$

***Proof.*** Let $D$ be a $\gamma_t(G)$-set and let $v \in V(H)$. Observe that $D' = D \times \{v\}$ is a total dominating set of $G \circ H$. Hence, $\gamma_t(G \circ H) \leq |D'| = |D| = \gamma_t(G)$.

Now, let $S$ be a $\gamma_t(G \circ H)$-set and define $S' \subseteq V(G)$ as follows.

- For every vertex $x \in \mathcal{A}_S \cup \mathcal{B}_S$, set $x \in S'$.
- For every vertex $x \in \mathcal{A}_S$, choose a vertex $x' \in N(x) \setminus (\mathcal{A}_S \cup \mathcal{B}_S)$ (if any) and set $x' \in S'$.

Since $G$ does not have isolated vertices, $S'$ is a total dominating set of $G$. Hence, $\gamma_t(G) \leq |S'| \leq |S| = \gamma_t(G \circ H)$, which completes the proof. ■

**Theorem 12.** *For any graph $G$ with no isolated vertex and any nontrivial graph $H$,*

$$\max\{\gamma_r(G), \gamma_t(G), 2\rho(G)\} \leq \gamma_{st}(G \circ H) \leq 2\gamma_t(G).$$

***Proof.*** By Proposition 1 and Theorems 10 and 11, we have that

$$\gamma_t(G) = \gamma_t(G \circ H) \leq \gamma_{st}(G \circ H) = \gamma_{tr}(G \circ H) \leq 2\gamma_t(G \circ H) = 2\gamma_t(G).$$

Now, by Proposition 1 and Theorems 6 and 11 we have that

$$\gamma_{st}(G \circ H) = \gamma_{tr}(G \circ H) \geq \gamma_r(G \circ H) \geq \gamma_r(G).$$

Finally, for any $\rho(G)$-set $X$ and any $\gamma_{st}(G \circ H)$-set $S$ we have that

$$\gamma_{st}(G \circ H) = |S| = \sum_{u \in V(G)} |S \cap V(H_u)| \geq \sum_{u \in X} \sum_{w \in N[u]} |S \cap V(H_w)| \geq 2|X| = 2\rho(G).$$

Therefore, the result follows. ■

In Theorem 22 we will characterize the graphs satisfying $\gamma_{st}(G \circ H) = \gamma_t(G)$ and later we will give some examples of graphs achieving the remaining bounds established in Theorem 12.

**Corollary 13.** *If $G$ is a nontrivial graph and $\gamma(G) = 1$, then for any nontrivial graph $H$,*

$$\gamma_{st}(G \circ H) \leq 4.$$

In Section 4 we characterize the graphs with $\gamma_{st}(G \circ H) \in \{2, 3\}$. Hence, by Corollary 13 the graphs with $\gamma_{st}(G \circ H) = 4$ will be automatically characterized whenever $\gamma(G) = 1$.

The following result is a direct consequence of Theorems 2 and 12.

**Theorem 14.** *Let $G$ be a graph with no isolated vertex and let $H$ be any graph.*

(i) *If $\gamma_{tr}(G) = \gamma_r(G)$, then $\gamma_{st}(G \circ H) = 2\gamma_t(G)$.*

(ii) *If $\gamma_t(G) = \frac{1}{2}\max\{\gamma_r(G), 2\rho(G)\}$, then $\gamma_{st}(G \circ H) = 2\gamma_t(G)$.*

**Theorem 15.** *For any graph $G$ with no isolated vertex and any nontrivial graph $H$, the following statements are equivalent.*

(i) $\gamma_{st}(G \circ H) = \gamma_r(G \circ H)$.

(ii) $\gamma_r(G \circ H) = 2\gamma_t(G)$.

***Proof.*** The result is obtained by combining Theorems 2, 10 and 11. ∎

We now consider the case where $G$ is a graph of minimum degree at least two.

**Theorem 16.** *Let $G$ be a graph of minimum degree at least two and order $n$. The following statements hold.*

(i) *For any graph $H$, $\gamma_{st}(G \circ H) \leq \gamma_{2,t}(G)$.*

(ii) *For any graph $H$, $\gamma_{st}(G \circ H) \leq n$.*

***Proof.*** Since every $\gamma_{2,t}(G \circ H)$-set is an STDS of $G \circ H$, we deduce that $\gamma_{st}(G \circ H) \leq \gamma_{2,t}(G \circ H)$. Hence, from Theorem 8 we deduce (i). Finally, since $\gamma_{2,t}(G) \leq n$, from (i) we deduce (ii). ∎

Particular cases of graphs where $\gamma_{st}(G \circ H) = \gamma_{2,t}(G)$ will be shown in Theorem 23(iii) and (v). Moreover, an example of graphs where $\gamma_{st}(G \circ H) = \gamma_{2,t}(G) = n$ will be shown in Theorem 31.

As shown in [22] there exists a family $\mathcal{H}_k$ of graphs such that $\gamma_r(G) = \gamma_{2,t}(G)$, for every $G \in \mathcal{H}_k$. Hence, for any $G \in \mathcal{H}_k$ and any graph $H$ we have that $\gamma_{st}(G \circ H) = \gamma_{2,t}(G)$. A graph $G$ belongs to $\mathcal{H}_k$ if and only if it is constructed from a cycle $C_k$ and $k$ empty graphs $N_{s_1}, \ldots, N_{s_k}$ of order $s_1, \ldots, s_k$, respectively, and joining by an edge each vertex from $N_{s_i}$ with the vertices $v_i$ and $v_{i+1}$ of $C_k$. Here we are assuming that $v_i$ is adjacent to $v_{i+1}$ in $C_k$, where the subscripts are taken modulo $k$. Figure 1 shows a graph $G$ belonging to $\mathcal{H}_k$, where $k = 4$, $s_1 = s_3 = 3$ and $s_2 = s_4 = 2$.

Theorems 12 and 9 lead to the following bound which is useful if $G$ has vertices of degree one.

**Theorem 17.** *For any connected graph $G$ of order $n \geq 3$ and any graph $H$,*

$$\gamma_{st}(G \circ H) \leq 2\left\lfloor \frac{2n}{3} \right\rfloor.$$

As shown in [22] there exists a family of trees $T_n$, which we will call *combs*, such that for any graph $H$ with $\gamma(H) \geq 4$ we have that $\gamma_r(T_n \circ H) = 2\left\lfloor \frac{2n}{3} \right\rfloor$. Therefore, for these graphs, $\gamma_{st}(T_n \circ H) = 2\left\lfloor \frac{2n}{3} \right\rfloor$. We now proceed to describe the family of combs. Take a path $P_k$ of length $k = \left\lceil \frac{n}{3} \right\rceil$, with vertices $v_1, \ldots, v_k$, and attach a path $P_3$ to each vertex $v_1, \ldots, v_{k-1}$, by identifying each $v_i$ with a leaf of

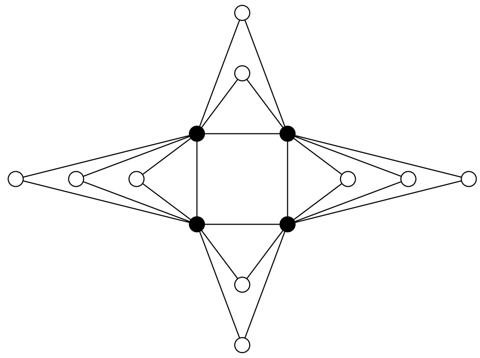

Figure 1. The set of black-coloured vertices is a $\gamma_{2,t}(G)$-set.

its corresponding copy of $P_3$. Finally, we attach a path of length $l = n-3\left\lceil \frac{n}{3}\right\rceil + 2$ to $v_k$. Figure 2 shows the construction of $T_n$ for different values of $n$. Notice that the comb of order six is simply $T_6 \cong P_6$.

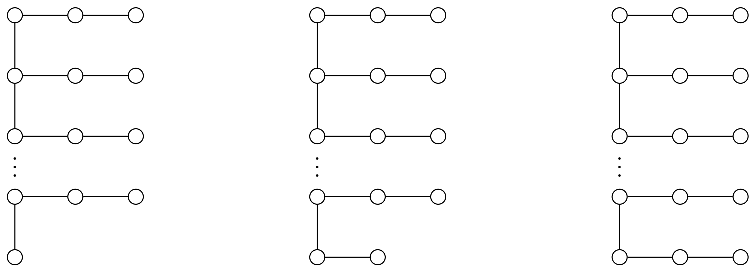

Figure 2. $T_n$ for $l = 0, 1, 2$.

**Lemma 18.** *For any graph $G$ with no isolated vertex and any nontrivial graph $H$, there exists a $\gamma_{st}(G \circ H)$-set $S$ such that $|S \cap V(H_u)| \leq 2$, for every $u \in V(G)$.*

***Proof.*** Given an STDS $S$ of $G \circ H$, we define $S_3 = \{x \in V(G) : |S \cap V(H_x)| \geq 3\}$. Let $S$ be a $\gamma_{st}(G \circ H)$-set such that $|S_3|$ is minimum among all $\gamma_{st}(G \circ H)$-sets. If $|S_3| = 0$, then we are done. Hence, we suppose that there exists $u \in S_3$ and let $(u, v) \in S$. We assume that $|S \cap V(H_u)|$ is minimum among all vertices in $S_3$. It is readily seen that if there exists $u' \in N(u)$ such that $|S \cap V(H_u)| \geq 2$, then $S' = S \setminus \{(u, v)\}$ is an STDS of $G \circ H$, which is a contradiction. Hence, if $u' \in N(u)$, then $|S \cap V(H_{u'})| \leq 1$, and in this case it is not difficult to check that for $(u', v') \notin S$ the set $S'' = (S \setminus \{(u, v)\}) \cup \{(u', v')\}$ is an STDS of $G \circ H$. If $|S''_3| < |S_3|$, then we obtain a contradiction, otherwise we can repeat this process with $S''$, until obtaining a contradiction. Therefore, the result follows. ■

**Theorem 19.** *Let $G$ be a graph with no isolated vertex and let $H$ be a nontrivial graph.*

(i) *If $\gamma(H) = 1$, then $\gamma_{st}(G \circ H) \leq \gamma_{tr}(G)$.*

(ii) *If $H$ has at least two universal vertices, then $\gamma_{st}(G \circ H) \leq 2\gamma(G)$.*

(iii) *If $\gamma(H) > 2$, then $\gamma_{st}(G \circ H) \geq \gamma_{tr}(G)$.*

***Proof.*** Let $f$ be a $\gamma_{tr}(G)$-function and let $v$ be a universal vertex of $H$. Let $f'$ be the function defined by $f'(u, v) = f(u)$ for every $u \in V(G)$ and $f'(x, y) = 0$ whenever $x \in V(G)$ and $y \in V(H) \setminus \{v\}$. It is readily seen that $f'$ is a TWRDF on $G \circ H$. Hence, by Theorem 10 we conclude that $\gamma_{st}(G \circ H) = \gamma_{tr}(G \circ H) \leq \omega(f') = \omega(f) = \gamma_{tr}(G)$ and (i) follows.

Let $D$ be a $\gamma(G)$-set and let $y_1, y_2$ be two universal vertices of $H$. It is not difficult to see that $S = D \times \{y_1, y_2\}$ is an STDS of $G \circ H$. Therefore, $\gamma_{st}(G \circ H) \leq |S| = 2\gamma(G)$ and (ii) follows.

From now on, let $S$ be a $\gamma_{st}(G \circ H)$-set that satisfies Lemma 18 and assume that $\gamma(H) > 2$. Let $g(V_0, V_1, V_2)$ be the function defined by $g(u) = |S \cap V(H_u)|$ for every $u \in V(G)$. We claim that $g$ is a TWRDF on $G$. It is clear that every vertex in $V_1$ has to be adjacent to some vertex in $V_1 \cup V_2$ and, if $\gamma(H) > 2$, then by Theorem 3 we have that $\gamma_{st}(H) > 3$, which implies that every vertex in $V_2$ has to be adjacent to some vertex in $V_1 \cup V_2$. Hence, $V_1 \cup V_2$ is a total dominating set of $G$. Now, if $x \in V_0$, then $S \cap V(H_x) = \emptyset$, and so there exists a vertex $(x_1, y_1) \in N(V(H_x)) \cap S$ which totally protects every vertex in $V(H_x)$. Hence, $x$ is totally protected by $x_1 \in V_1 \cup V_2$ under $g$. Thus, $g$ is a TWRDF on $G$ and so $\gamma_{tr}(G) \leq \omega(g) = |S| = \gamma_{st}(G \circ H)$. Therefore, (iii) follows. ■

The following result is a direct consequence of Theorems 12 and 19. Notice that a graph $H$ has at least two universal vertices if and only if $\gamma_{st}(H) = 2$, by Theorem 3.

**Theorem 20.** *Let $G$ be a graph with no isolated vertex and let $H$ be a nontrivial graph.*

(i) *If $\gamma(G) = \rho(G)$ and $\gamma_{st}(H) = 2$, then $\gamma_{st}(G \circ H) = 2\gamma(G)$.*

(ii) *If $\gamma_{tr}(G) \in \{\gamma_r(G), \gamma_t(G), 2\rho(G)\}$ and $\gamma(H) = 1$, then $\gamma_{st}(G \circ H) = \gamma_{tr}(G)$.*

(iii) *If $\gamma_{tr}(G) = 2\gamma_t(G)$ and $\gamma(H) > 2$, then $\gamma_{st}(G \circ H) = \gamma_{tr}(G)$.*

In general, for a graph $H$ such that $\gamma(H) \geq 2$, the equality $\gamma_{st}(G \circ H) = \gamma_{tr}(G)$ does not imply that $\gamma_{tr}(G) = 2\gamma_t(G)$. For instance, the graph $P_5 \circ P_4$ shown in Figure 3 satisfies $\gamma_{st}(P_5 \circ P_4) = \gamma_{tr}(P_5) = 5 < 6 = 2\gamma_t(P_5)$.

It is well known that $\gamma(T) = \rho(T)$ for any tree $T$. Hence, the following corollary is a direct consequence of Theorem 20.

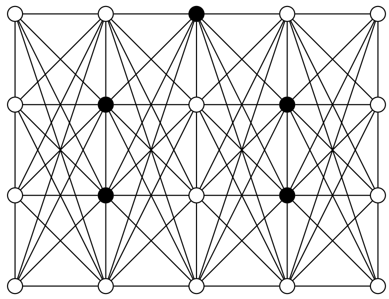

Figure 3. The set of black-coloured vertices is a $\gamma_{st}(P_5 \circ P_4)$-set.

**Corollary 21.** *For any tree $T$ of order at least two and any graph $H$ with $\gamma_{st}(H) = 2$,*

$$\gamma_{st}(T \circ H) = 2\gamma(T).$$

## 4. Small Values of $\gamma_{st}(G \circ H)$

We now characterize the graphs with $\gamma_{st}(G \circ H) \in \{2, 3\}$.

**Theorem 22.** *For any nontrivial connected graph $G$ and any nontrivial graph $H$, the following statements are equivalent.*

(i) $\gamma_{st}(G \circ H) = \gamma_t(G)$.

(ii) $\gamma_{st}(G \circ H) = 2$.

(iii) $\gamma_{st}(G) = \gamma(G) + 1 = \gamma(H) + 1 = 2$ *or* $\gamma_{st}(H) = \gamma(G) + 1 = \gamma(H) + 1 = 2$.

***Proof.*** By Theorems 3 and 11 we conclude that (i) and (ii) are equivalent. Notice that $G \circ H$ has at least two universal vertices if and only if $\gamma(G) = \gamma(H) = 1$, and also $G$ has at least two universal vertices or $H$ has at least two universal vertices. Hence, by Theorem 3 we conclude that (ii) and (iii) are equivalent. ■

**Theorem 23.** *Let $G$ be a nontrivial connected graph and $H$ a graph with no isolated vertex. Then $\gamma_{st}(G \circ H) = 3$ if and only if one of the following conditions is satisfied.*

(i) $G \cong P_2$ *and* $\gamma(H) = 2$.

(ii) *$G$ has exactly one universal vertex and either $\gamma(H) = 2$ or $H$ has exactly one universal vertex.*

(iii) *$G$ has exactly one universal vertex, $\gamma_{2,t}(G) = 3$ and $\gamma(H) \geq 3$.*

(iv) *$G \ncong P_2$ has at least two universal vertices and $\gamma(H) \geq 2$.*

(v) $\gamma(G) = 2$ *and* $\gamma_{2,t}(G) = 3$.

(vi) $\gamma(G) = 2$, $\gamma_{st}(G) = 3 < \gamma_{2,t}(G)$ *and* $\gamma(H) = 1$.

***Proof.*** Let $S$ be a $\gamma_{st}(G \circ H)$-set and assume that $|S| = 3$. By Theorems 4 and 11 we have that $3 = \gamma_{st}(G \circ H) > \gamma_t(G \circ H) = \gamma_t(G) \geq 2$, which implies that $\gamma_t(G) = 2$ and so $\gamma(G) \in \{1, 2\}$. We differentiate two cases.

*Case* 1. $\gamma(G) = 1$. In this case, Theorem 22 leads to $\gamma_{st}(H) \geq 3$. Now, we consider the following subcases.

*Subcase* 1.1. $G \cong P_2$. Notice that Theorem 22 leads to $\gamma(H) \geq 2$. Suppose that $\gamma(H) \geq 3$ and let $V(G) = \{u, w\}$. By Theorem 4 we have $\gamma_{st}(H) \geq 4$ and so $S \cap V(H_u) \neq \emptyset$ and $S \cap V(H_w) \neq \emptyset$. Without loss of generality, let $S \cap V(H_u) = \{(u, v_1), (u, v_2)\}$ and $|S \cap V(H_w)| = 1$. Since $\gamma(H) \geq 3$, we have that $\{v_1, v_2\}$ is not a dominating set of $H$, which implies that no vertex in $\{u\} \times (V(H) \setminus (N(v_1) \cup N(v_2))$ is totally protected under $S$, which is a contradiction. Hence $\gamma(H) = 2$. Therefore, (i) follows.

*Subcase* 1.2. $G$ has exactly one universal vertex. If $\gamma(H) \leq 2$, then by Theorem 22 we deduce that either $\gamma(H) = 2$ or $H$ has exactly one universal vertex, and (ii) follows. Assume that $\gamma(H) \geq 3$. As in Subcase 1.1, we conclude that $\gamma_{st}(H) \geq 4$ and so $|S \cap V(H_x)| \leq 2$ for every $x \in V(G)$. Now, if there exist two vertices $u, w \in V(G)$ and two vertices $v_1, v_2 \in V(H)$ such that $S \cap V(H_u) = \{(u, v_1), (u, v_2)\}$ and $|S \cap V(H_w)| = 1$, then we deduce that no vertex in $\{u\} \times (V(H) \setminus (N(v_1) \cup N(v_2))$ is totally protected under $S$, which is a contradiction. Therefore, $\mathcal{A}_S = \emptyset$ and $\mathcal{B}_S$ has to be a $\gamma_{2,t}(G)$-set, as if there exists $x \in V(G)$ such that $|N(x) \cap \mathcal{B}_S| \leq 1$, then $V(H_x)$ has vertices which are no totally protected under $S$. Therefore, (iii) follows.

*Subcase* 1.3. $G \not\cong P_2$ has at least two universal vertices. In this case, by Theorem 22 we deduce that $\gamma(H) \geq 2$, and so (iv) follows.

*Case* 2. $\gamma(G) = 2$. In this case, Theorem 4 leads to $\gamma_{st}(G) \geq 3$. If there exist two vertices $u, w \in V(G)$ such that $\mathcal{A}_S = \{u\}$ and $\mathcal{B}_S = \{w\}$, then $\{u, w\}$ is a $\gamma_t(G)$-set, and so for any $x \in N(w) \setminus N[u]$ we have that no vertex in $V(H_x)$ is totally protected under $S$, which is a contradiction. Therefore, $\mathcal{A}_S = \emptyset$ and $|\mathcal{B}_S| = 3$, which implies that $\mathcal{B}_S$ is a $\gamma_{st}(G)$-set. Let $\langle \mathcal{B}_S \rangle$ be the subgraph induced by $\mathcal{B}_S$. Notice that either $\langle \mathcal{B}_S \rangle \cong K_3$ or $\langle \mathcal{B}_S \rangle \cong P_3$. In the first case, $\mathcal{B}_S$ is a $\gamma_{2,t}(G)$-set and (v) follows. Now, assume that $\langle \mathcal{B}_S \rangle \cong P_3$. If $\gamma(H) \geq 2$, then for any vertex $x$ of degree one in $\langle \mathcal{B}_S \rangle$ we have that $V(H_x)$ has vertices which are not totally protected under $S$, which is a contradiction. Therefore, $\gamma(H) = 1$ and if $\gamma_{st}(G) = \gamma_{2,t}(G)$, then $G$ satisfies (v), otherwise $G$ satisfies (vi), by Theorem 16.

Conversely, notice that if $G$ and $H$ satisfy one of the six conditions above, then Theorem 22 leads to $\gamma_{st}(G \circ H) \geq 3$. To conclude that $\gamma_{st}(G \circ H) = 3$, we proceed to show how to define an STDS $D$ of $G \circ H$ of cardinality three for each of the six conditions.

(i) Let $\{v_1, v_2\}$ be a $\gamma(H)$-set and $V(G) = \{u, w\}$. In this case, we define $D = \{(u, v_1), (u, v_2), (w, v_1)\}$.

(ii) Let $u$ be a universal vertex of $G$ and $w \in V(G) \setminus \{u\}$. If $\{v_1, v_2\}$ is a $\gamma(H)$-set or $v_1$ is a universal vertex of $H$ and $v_2 \in V(H) \setminus \{v_1\}$, then we set $D = \{(u, v_1), (u, v_2), (w, v_1)\}$.

(iii) Let $X$ be a $\gamma_{2,t}(G)$-set and $v \in V(H)$. In this case, $D = X \times \{v\}$.

(iv) Let $u, w \in V(G)$ be two universal vertices, $z \in V(G) \setminus \{u, w\}$ and $v \in V(H)$. In this case, $D = \{(u, v), (w, v), (z, v)\}$.

(v) Let $X$ be a $\gamma_{2,t}(G)$-set and $v \in V(H)$. In this case, $D = X \times \{v\}$.

(vi) Let $X$ be a $\gamma_{st}(G)$-set and $v$ be a universal vertex of $H$. In this case, $D = X \times \{v\}$.

It is readily seen that in all cases $D$ is an STDS of $G \circ H$. Therefore, $\gamma_{st}(G \circ H) = 3$. ■

**Theorem 24.** *Let $G$ be a nontrivial connected graph and $H$ a nontrivial graph with at least one isolated vertex. Then $\gamma_{st}(G \circ H) = 3$ if and only if at least one of the following conditions is satisfied.*

(i) $\gamma(G) = 1$ *and* $\gamma(H) = 2$.

(ii) $\gamma_{2,t}(G) = 3$.

***Proof.*** Notice that $\gamma(H) \geq 2$, as $H$ is a nontrivial graph with at least one isolated vertex. Let $S$ be a $\gamma_{st}(G \circ H)$-set that satisfies Lemma 18 and assume that $|S| = 3$. Now, we consider two cases.

*Case* 1. $\mathcal{A}_S \neq \emptyset$. In this case we have that $|\mathcal{A}_S| = |\mathcal{B}_S| = 1$. Let $u, w \in V(G)$ such that $\mathcal{A}_S = \{u\}$ and $\mathcal{B}_S = \{w\}$. Notice that $\{u, w\}$ is a $\gamma_t(G)$-set and, if there exists $x \in N(w) \setminus N[u]$, then no vertex in $V(H_x)$ is totally protected under $S$, which is a contradiction. Hence, $\gamma(G) = 1$. Now, since $H$ has at least one isolated vertex, if $\gamma(H) > 2$, then $H_u$ has at least one vertex which is not totally protected under $S$, which is a contradiction. Therefore, $\gamma(H) = 2$ and (i) follows.

*Case* 2. $\mathcal{A}_S = \emptyset$. In this case we have that $|\mathcal{B}_S| = 3$, which implies that $\mathcal{B}_S$ is a $\gamma_{st}(G)$-set. Let $\langle \mathcal{B}_S \rangle$ be the subgraph induced by $\mathcal{B}_S$. Notice that either $\langle \mathcal{B}_S \rangle \cong K_3$ or $\langle \mathcal{B}_S \rangle \cong P_3$. Suppose that $\langle \mathcal{B}_S \rangle \cong P_3$ and let $x$ be a vertex of degree one in $\langle \mathcal{B}_S \rangle$. Since $H$ has at least one isolated vertex, there exists at least one vertex in $V(H_x)$ which is not totally protected under $S$, which is a contradiction. Hence, $\langle \mathcal{B}_S \rangle \cong K_3$, which implies that $\mathcal{B}_S$ is a $\gamma_{2,t}(G)$-set and so (ii) follows.

Conversely, notice that if $G$ and $H$ satisfy one of the two conditions above, then Theorem 22 leads to $\gamma_{st}(G \circ H) \geq 3$. To conclude that $\gamma_{st}(G \circ H) = 3$, we proceed to show how to define an STDS $D$ of $G \circ H$ of cardinality three for each of the two conditions.

(i) Let $\{u\}$ be a $\gamma(G)$-set, $w \in V(G) \setminus \{u\}$ and $\{v_1, v_2\}$ be a $\gamma(H)$-set. In this case, $D = \{(u, v_1), (u, v_2), (w, v_1)\}$.

(ii) Let $X$ be a $\gamma_{2,t}(G)$-set and $v \in V(H)$. In this case, $D = X \times \{v\}$. It is readily seen that in both cases $D$ is an STDS of $G \circ H$. Therefore, $\gamma_{st}(G \circ H) = 3$. ■

The following result, which is a direct consequence of Theorems 12, 22, 23 and 24, shows the cases when $G$ is isomorphic to a complete graph or a star graph.

**Proposition 25.** *For any integer $n \geq 3$, the following statements hold.*

(i) *If $H$ is a graph with no isolated vertex, then*

$$\gamma_{st}(K_n \circ H) = \begin{cases} 2, & \textit{if} \quad \gamma(H) = 1, \\ 3, & \text{otherwise.} \end{cases}$$

*and*

$$\gamma_{st}(K_{1,n-1} \circ H) = \begin{cases} 2, & \textit{if} \quad \gamma_{st}(H) = 2, \\ 3, & \textit{if} \quad \gamma_{st}(H) \geq 3 \quad \textit{and} \quad \gamma(H) \leq 2, \\ 4, & \text{otherwise.} \end{cases}$$

(ii) *If $H$ is a nontrivial graph with at least one isolated vertex, then*

$$\gamma_{st}(K_n \circ H) = 3$$

*and*

$$\gamma_{st}(K_{1,n-1} \circ H) = \begin{cases} 3, & \textit{if} \quad \gamma(H) = 2, \\ 4, & \text{otherwise.} \end{cases}$$

We now consider the cases in which $G$ is a double star graph or a complete bipartite graph. Recall that a double star $S_{n_1,n_2}$ is the graph obtained by joining the center of two stars $K_{1,n_1}$ and $K_{1,n_2}$ with an edge. The following result is a direct consequence of Theorems 12, 22, 23 and and 24.

**Proposition 26.** *Let $H$ be a nontrivial graph. For any integers $n_2 \geq n_1 \geq 2$, the following statements hold.*

$$\gamma_{st}(S_{n_1,n_2} \circ H) = 4$$

*and*

$$\gamma_{st}(K_{n_1,n_2} \circ H) = \begin{cases} 3, & \textit{if} \quad n_1 = 2 \quad \textit{and} \quad \gamma(H) = 1, \\ 4, & \text{otherwise.} \end{cases}$$

## 5. SPECIAL CASES WHERE $G \cong P_n$ AND $G \cong C_n$

First, we analyse the case where $G \cong P_n$ and $\gamma(H) = 1$ or $\gamma(H) \geq 4$.

**Theorem 27.** *Let $n \geq 2$ be an integer and let $H$ be a graph with $\gamma(H) = 1$. If $\gamma_{st}(H) = 2$, then*

$$\gamma_{st}(P_n \circ H) = 2\left\lceil \frac{n}{3} \right\rceil.$$

*Otherwise,* $\gamma_{st}(P_n \circ H \leq 2\left\lceil \frac{5(n-2)}{7} \right\rceil + 2$.

***Proof.*** If $\gamma_{st}(H) = 2$, then by Corollary 21 we deduce that $\gamma_{st}(P_n \circ H) = 2\gamma(P_n)$. Now, if $\gamma_{st}(H) \geq 3$, then by Theorem 19 we deduce $\gamma_{st}(P_n \circ H) \leq \gamma_{tr}(P_n)$. ■

As shown in [22], if $\gamma(H) \geq 4$, then $\gamma_r(P_n \circ H) = 2\gamma_t(P_n)$. Hence, from Proposition 1 and Theorems 12 and 7 we derive the following result.

**Theorem 28.** *Let $n \geq 2$ be an integer and let $H$ be a graph. If $\gamma(H) \geq 4$, then*

$$\gamma_{st}(P_n \circ H) = \gamma_r(P_n \circ H) = \begin{cases} n, & n \equiv 0 \pmod 4, \\ n+2, & n \equiv 2 \pmod 4, \\ n+1, & \text{otherwise.} \end{cases}$$

The following result is a direct consequence of Theorems 19 and 20.

**Theorem 29.** *Let $n \geq 3$ be an integer and let $H$ be a graph.*

- *If $H$ has exactly one universal vertex, then $\gamma_{st}(C_n \circ H) \leq \left\lceil \frac{5n}{7} \right\rceil$.*
- *If $H$ has at least two universal vertices, then $\gamma_{st}(C_n \circ H) \leq 2\left\lceil \frac{n}{3} \right\rceil$, and if $n \equiv 0 \pmod 3$, then the equality holds.*

**Lemma 30.** *Let $G$ be a nontrivial connected graph and let $H$ be any graph. The following statements hold for every $\gamma_{st}(G \circ H)$-set $S$.*

(i) *If $\gamma(H) \geq 2$ and $x \in \mathcal{B}_S$, then $\sum_{u \in N(x)} |S \cap V(H_u)| \geq 2$.*

(ii) *If $\gamma_t(H) \geq 3$ and $x \in \mathcal{A}_S$, then $\sum_{u \in N(x)} |S \cap V(H_u)| \geq 2$.*

***Proof.*** If $\gamma(H) \geq 2$ and there exists a vertex $x \in \mathcal{B}_S$ such that $\sum_{u \in N(x)} |S \cap V(H_u)| \leq 1$, then there exists a vertex in $V(H_x) \setminus S$ which is not totally protected under $S$. Therefore, (i) follows.

Now, assume that $\gamma_t(H) \geq 3$, and notice that Theorem 4 leads to $\gamma_{st}(H) \geq 4$. Suppose that there exists $x \in \mathcal{A}_S$ such that $\sum_{u \in N(x)} |S \cap V(H_u)| \leq 1$. Notice that, in such a case, either $2 \leq |S \cap V(H_x)| \leq 3$ and $S \cap \left(\bigcup_{u \in N(x)} V(H_u)\right) = \emptyset$ or $|S \cap V(H_x)| = 2$ and $\left|S \cap \left(\bigcup_{u \in N(x)} V(H_u)\right)\right| = 1$, which implies that there exists a vertex in $V(H_x) \setminus S$ which is not totally protected under $S$, as $\gamma_{st}(H_x) \geq 4$ and $\gamma_t(H_x) \geq 3$. Therefore, (ii) follows. ■

**Theorem 31.** *Let $n \geq 3$ be an integer and let $H$ be a graph. If $\gamma_t(H) \geq 3$, then*

$$\gamma_{st}(C_n \circ H) = n.$$

***Proof.*** From Theorem 16 we know that $\gamma_{st}(C_n \circ H) \leq n$. We only need to prove that $\gamma_{st}(C_n \circ H) \geq n$. Let $S$ be a $\gamma_{st}(G \circ H)$-set that satisfies Lemma 18. If $\mathcal{C}_S = \emptyset$, then $\gamma_{st}(C_n \circ H) = |S| \geq n$. Thus we assume that $\mathcal{C}_S \neq \emptyset$.

Let $V(C_n) = \{u_i, \ldots, u_n\}$, where the subscripts are taken modulo $n$ and consecutive vertices are adjacent. We differentiate two cases for $u_i \in \mathcal{C}_S$.

*Case* 1. $u_i$ is not adjacent to any vertex in $\mathcal{C}_S$. In this case, by Lemma 30 we have that $u_{i+2} \in \mathcal{A}_S$ and $u_{i+1} \in \mathcal{A}_S \cup \mathcal{B}_S$. Analogously, $u_{i-2} \in \mathcal{A}_S$ and $u_{i-1} \in \mathcal{A}_S \cup \mathcal{B}_S$.

*Case* 2. $u_{i+1} \in \mathcal{C}_S$. Since every vertex in $V(H_{u_i})$ has to be totally protected under $S$, we have that $u_{i-1}, u_{i+2} \in \mathcal{A}_S$ and so Lemma 30(ii) leads to $u_{i-2}, u_{i+3} \in \mathcal{A}_S$.

According to the two cases above, $|\mathcal{A}_S| \geq |\mathcal{C}_S|$, which implies that $\gamma_{st}(C_n \circ H) \geq 2|\mathcal{A}_S| + |\mathcal{B}_S| \geq n$. Therefore, the result follows. ■

## Acknowledgements

This work has been partially supported by the Spanish Ministry of Economy, Industry and Competitiveness, under the grants MTM2016-78227-C2-1-P and MTM2017-90584-REDT.

## References

[1] R.A. Beeler, T.W. Haynes and S.T. Hedetniemi, *Double Roman domination*, Discrete Appl. Math. **211** (2016) 23–29. https://doi.org/10.1016/j.dam.2016.03.017

[2] S. Benecke, E.J. Cockayne and C.M. Mynhardt, *Secure total domination in graphs*, Util. Math. **74** (2007) 247–259.

[3] S. Benecke, E.J. Cockayne and C.M. Mynhardt, *Total protection of a graph*, unpublished manuscript.

[4] R. Brigham, J. Carrington and R. Vitray, *Connected graphs with maximum total domination number*, J. Combin. Math. Combin. Comput. **34** (2000) 81–96.

[5] A. Cabrera Martínez, L.P. Montejano and J.A. Rodríguez-Velázquez, *Total weak Roman domination in graphs*, Symmetry **11** (2019) No. 831. https://doi.org/10.3390/sym11060831

[6] A. Cabrera Martínez and J.A. Rodríguez-Velázquez, *On the secure total domination number of graphs*, Symmetry **11** (2019) No. 1165. https://doi.org/10.3390/sym11091165

[7] M. Chellali, T.W. Haynes, S.T. Hedetniemi and A. McRae, *Roman* $\{2\}$*-domination*, Discrete Appl. Math. **204** (2016) 22–28. https://doi.org/10.1016/j.dam.2015.11.013

[8] M. Chellali, T.W. Haynes and S.T. Hedetniemi, *Bounds on weak Roman and 2-rainbow domination numbers*, Discrete Appl. Math. **178** (2014) 27–32.
https://doi.org/10.1016/j.dam.2014.06.016

[9] E.J. Cockayne, P.A. Dreyer Jr., S.M. Hedetniemi and S.T. Hedetniemi, *Roman domination in graphs*, Discrete Math. **278** (2004) 11–22.
https://doi.org/10.1016/j.disc.2003.06.004

[10] E.J. Cockayne, R.M. Dawes and S.T. Hedetniemi, *Total domination in graphs*, Networks **10** (1980) 211–219.
https://doi.org/10.1002/net.3230100304

[11] E.J. Cockayne, O. Favaron and C.M. Mynhardt, *Secure domination, weak Roman domination and forbidden subgraphs*, Bull. Inst. Combin. Appl. **39** (2003) 87–100.

[12] E.J. Cockayne, P.J.P. Grobler, W.R. Gründlingh, J. Munganga and J.H. van Vuuren, *Protection of a graph*, Util. Math. **67** (2005) 19–32.

[13] O. Duginov, *Secure total domination in graphs: Bounds and complexity*, Discrete Appl. Math. **222** (2017) 97–108.
https://doi.org/10.1016/j.dam.2016.08.018

[14] T.W. Haynes and M.A. Henning, *Perfect Italian domination in trees*, Discrete Appl. Math. **260** (2019) 164–177.
https://doi.org/10.1016/j.dam.2019.01.038

[15] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Domination in Graphs: Advanced Topics (Taylor & Francis, 1998).

[16] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in Graphs (Marcel Dekker, Inc., New York, 1998).

[17] M.A. Henning and W.F. Klostermeyer, *Italian domination in trees*, Discrete Appl. Math. **217** (2017) 557–564.
https://doi.org/10.1016/j.dam.2016.09.035

[18] M.A. Henning and S.T. Hedetniemi, *Defending the Roman Empire—A new strategy*, Discrete Math. **266** (2003) 239–251.
https://doi.org/10.1016/S0012-365X(02)00811-7

[19] M.A. Henning and A. Yeo, Total Domination in Graphs (Springer, New York, 2013).
https://doi.org/10.1007/978-1-4614-6525-6

[20] W.F. Klostermeyer and C.M. Mynhardt, *Secure domination and secure total domination in graphs*, Discuss. Math. Graph Theory **28** (2008) 267–284.
https://doi.org/10.7151/dmgt.1405

[21] A. Meir and J.W. Moon, *Relations between packing and covering numbers of a tree*, Pacific J. Math. **61** (1975) 225–233.
https://doi.org/10.2140/pjm.1975.61.225

[22] M. Valveny, H. Pérez-Rosés and J.A. Rodríguez-Velázquez, *On the weak Roman domination number of lexicographic product graphs*, Discrete Appl. Math. **263** (2019) 257–270.
https://doi.org/10.1016/j.dam.2018.03.039

[23] I. Stewart, *Defend the Roman Empire!*, Sci. Amer. **281** (1999) 136–138.
https://doi.org/10.1038/scientificamerican1299-136

Received 13 June 2019
Revised 25 March 2020
Accepted 26 March 2020