

NEW RESULTS RELATING INDEPENDENCE AND MATCHINGS

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Abstract

In this paper we study relationships between the *matching number*, written $\mu(G)$, and the *independence number*, written $\alpha(G)$. Our first main result is to show

$$\alpha(G) \leq \mu(G) + |X| - \mu(G[N_G[X]]),$$

where X is *any* intersection of maximum independent sets in G . Our second main result is to show

$$\delta(G)\alpha(G) \leq \Delta(G)\mu(G),$$

where $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum vertex degrees of G , respectively. These results improve on and generalize known relations between $\mu(G)$ and $\alpha(G)$. Further, we also give examples showing these improvements.

Keywords: independent sets, independence number, matchings, matching number.

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1. INTRODUCTION

Graphs considered here will be finite, undirected, and with no loops. Let G be a graph with order $n(G) = |V(G)|$ and size $m(G) = |E(G)|$. The open neighborhood of a vertex $v \in V(G)$ is the set of all vertices adjacent to v , written $N_G(v)$, whereas the closed neighborhood of v is $N_G[v] = N_G(v) \cup \{v\}$. The minimum and maximum vertex degrees of G will be denoted $\delta(G)$ and $\Delta(G)$, respectively.

For a subset $X \subseteq V(G)$, we will use the notations $N_G(X) = \bigcup_{v \in X} N_G(v)$ and $N_G[X] = X \cup N_G(X)$, also $G[X]$ will denote the subgraph induced by X . A matching is a subset $M \subseteq E(G)$ of non-adjacent edges. Vertices incident with a matching are called *saturated* by that matching. The *matching number* is the cardinality of a maximum matching in G , and will be denoted by $\mu(G)$. A subset $X \subseteq V(G)$ is *independent* if no edge has both endpoints in X . The cardinality of a maximum independent set in G , written $\alpha(G)$, is the *independence number* of G . The *core* of G , written $\text{core}(G)$, is the intersection of all maximum independent sets in G .

The graph parameters $\alpha(G)$ and $\mu(G)$ are in general negatively correlated (adding edges doesn't increase the independence number and doesn't decrease the matching number) but incomparable as can be seen by the following observations. Namely, if $G = E_n$, the n -vertex empty graph, then $0 = \mu(G) < \alpha(G) = n$. Further, if $G = K_n$, the n -vertex complete graph with $n \geq 3$, then $1 = \alpha(G) < \mu(G) = \lfloor \frac{n}{2} \rfloor$.

However from the point of view of "almost all graphs", random graph theory gives $c_1 \log(\mu(G)) \leq \alpha(G) \leq c_2 \log(\mu(G))$ with high probability [3, 5, 12]. Thus, with high probability in a random graph, $\mu(G)$ is much higher than $\alpha(G)$. That is, $\alpha(G) \leq \mu(G)$ for almost all graphs. In fact, in 1979 Pulleyblank [24] discussed 2-bicritical graphs in the context of a linear programming relaxation of the independence number; proving for these graphs that $\alpha(G) \leq \mu(G)$; for a relevant and more recent discussion of 2-bicritical graphs see [16]. It should be noted that this line of research was developed in part due to the interesting relationships between independence structure and early combinatorial optimization literature; for more on this relationship see [4, 20, 23].

Despite the above observations and examples, there exists many relationships between $\alpha(G)$ and $\mu(G)$. The following inequality is one of the most well known examples.

$$(1) \quad n(G) - 2\mu(G) \leq n(G) - 2\mu^*(G) \leq \alpha(G) \leq n - \mu(G).$$

Here $\mu^*(G)$ denotes the cardinality of a minimum maximal matching in G (this invariant is also called the *edge domination number* of G , see [25] for more on this topic). Graphs that satisfy the righthand side of (1) with equality are called

König-Egerváry, and have been extensively studied; see for example [1, 15, 17, 18]. Boros *et al.* [2] proved $\alpha(G) \leq \mu(G) + |\text{core}(G)| - 1$ whenever G is a graph with $\alpha(G) > \mu(G)$. Recently Levit *et al.* [19] proved a similar result, namely $\alpha(G) \leq \mu(G) + |\text{core}(G)| - |N_G(\text{core}(G))|$ whenever G is a graph with a matching from $N_G(\text{core}(G))$ into $\text{core}(G)$. Non-intersecting maximum independent sets were also studied by Deniz *et al.* [8], who showed $\alpha(G) \leq \mu(G)$, provided G contains two disjoint maximum independent sets. Levit *et al.* [19] also showed that $\alpha(G) \leq \mu(G)$, under the condition that G contains a unique odd cycle.

So the first source of motivation in our paper is to try and obtain a deeper understanding of these kinds of inequalities relating $\alpha(G)$ and $\mu(G)$ via the cardinality of the core of G . Another source of motivation comes from the following example: If G is the bipartite graph $K_{\delta, n-\delta}$, where $n \geq 2\delta \geq 2$, then $\delta(G) = \delta$ and $\Delta(G) = n - \delta$. Clearly $n - \delta = \Delta(G) = \alpha(G)$ and $\delta = \mu(G)$, and so, $\alpha(G) = \frac{\Delta(G)}{\delta(G)} \mu(G)$. Thus, a natural question arises, namely, is this the best possible upper-bound on $\alpha(G)$ in terms of the parameters $\mu(G)$, $\delta(G)$, and $\Delta(G)$?

Our main two theorems supply answers to the problems and motivation mentioned above. These two theorems are shown below.

Theorem 1. *If G is a graph and X is any intersection of maximum independent sets, then*

$$\alpha(G) \leq \mu(G) + |X| - \mu(G[N_G[X]]),$$

and this bound is sharp.

Theorem 2. *If G is a graph, then*

$$\delta(G)\alpha(G) \leq \Delta(G)\mu(G),$$

and this bound is sharp.

As can easily be seen, these two results generalize and in many cases improve on many of the known relationships between $\alpha(G)$ and $\mu(G)$. The remainder of our paper is structured as follows. In Section 2 we prove Theorem 1. In Section 3 we prove Theorem 2. In Section 4 we give constructions of infinite families of graphs which achieve equality in Theorem 2 and also discuss how Theorem 2 improves on several known upper bounds for $\alpha(G)$. Finally in Section 5, we give concluding remarks, suggestions for future work, and a new conjecture.

For notation and terminology not found here, we refer the reader to West [26]. We will also make use of the standard notation $[k] = \{1, \dots, k\}$.

2. PROOF OF THEOREM 1

In this section we prove Theorem 1. Before doing so, we will need the following two lemmas. The following lemma is widely known, and may be considered

“folklore” in the literature.

Lemma 3. *If A is an independent set and X is a maximum independent set, then there is a matching from $A \setminus (A \cap X)$ to $X \setminus A$ that saturates each vertex in $A \setminus (A \cap X)$ ¹.*

Using Lemma 3, we next prove a technical lemma that bounds the difference between the size of independent sets and the matchings numbers of their closed neighborhoods.

Lemma 4. *If A is an independent set and X is any intersection of maximum independent sets with $X \subseteq A$, then*

$$|A| - \mu(G[N_G[A]]) \leq |X| - \mu(G[N_G[X]]).$$

Proof. Let $X = X_1 \cap \cdots \cap X_k$, where X_i is a maximum independent set in G for each $i \in [k]$, and let $A \subseteq V(G)$ be any independent set satisfying $X \subseteq A$. For notational convenience, let $X_0 = A$ and $A_r = \bigcap_{i=0}^{r-1} X_i$. Note A_r is an independent set for all $r \in [k]$. By Lemma 3 there is a matching from $X_r \setminus A_r$ to $A_r \setminus (A_r \cap X_r)$ that saturates every vertex contained in $A_r \setminus (A_r \cap X_r)$. Let M_r denote one such matching for each $r \in [k]$, and consider M_1 and M_2 , and let $vw \in M_2$. Without loss of generality suppose $w \in X_2$. Then, by the construction of M_2 , v is necessarily contained in $X_0 \cap X_1$. Since M_1 matches vertices from $X_1 \setminus X_0$ to vertices in $X_0 \setminus (X_0 \cap X_1)$, and since X_0 and X_1 are independent sets, it must be the case that v is not the endpoint of any edge in M_1 . This same reasoning will hold for M_2 with respect to M_3 , and so on. Hence, edges in M_j and M_i will not share endpoints for any $i \neq j$ and $i, j \in [k]$. Furthermore, each edge in M_r contains at least one endpoint in A , again for each $r \in [k]$. Thus, $M = M_1, \dots, M_k$ is a matching in the induced subgraph $G[N_G[A]]$.

Thus far we have only saturated vertices in $A_k = X_0 \cap \cdots \cap X_{k-1}$. Let Q be a maximum matching in $G[N_G[X]]$. Next observe that Q is edge independent from the matching $M_1 \cup M_2 \cup \cdots \cup M_{k-1}$. Thus, $M = M_1 \cup M_2 \cup \cdots \cup M_k \cup Q$ is a matching in $G[N_G[A]]$. This implies the following inequality

$$\begin{aligned} \mu(G[N_G[A]]) &\geq |M| = \left| \bigcup_{i=1}^k M_i \right| + |Q| \\ &= \sum_{i=1}^k \left(|A_i \setminus (A_i \cap X_i)| \right) + \mu(G[N_G[X]]) \\ (2) \quad &= \sum_{i=1}^k \left(|A_i| - |A_{i+1}| \right) + \mu(G[N_G[X]]) \\ &= |A| - |X| + \mu(G[N_G[X]]), \end{aligned}$$

¹We acknowledge that $A \setminus (A \cap X)$ is equivalent to $A \setminus X$. However, we use $A \setminus (A \cap X)$ in place of $A \setminus X$ because this view of the set difference becomes useful in subsequent proofs.

where all the terms in the summation, except the first and last, cancel out because the summation in the inequality is a telescoping series. Rearranging the above inequality, we finish the proof of our lemma. ■

We now use Lemma 4 to prove Theorem 1, whose statement is recalled below.

Theorem 1. *If G is a graph and X is any intersection of maximum independent sets, then*

$$\alpha(G) \leq \mu(G) + |X| - \mu(G[N_G[X]]),$$

and this bound is sharp.

Proof. Let X be an intersection on maximum independent sets, one of which is the set A . By Lemma 4, we have

$$|A| - \mu(G[N_G[A]]) \leq |X| - \mu(G[N_G[X]]).$$

Since $N_G[A] = V(G)$ implies $\mu(G[N_G[A]]) = \mu(G)$, and since $|A| = \alpha(G)$, we obtain

$$\alpha(G) - \mu(G) \leq |X| - \mu(G[N_G[X]]).$$

Rearranging the above inequality proves the inequality posed in the theorem. To see this inequality is sharp, see Example 7. ■

Remark 5. Let Q be a set of maximum independent sets in a graph G , where $|Q| \geq 3$. In light of Theorem 1, it is natural to ask what number of elements in Q together form the optimal intersection with respect to the upper bound on $\alpha(G)$ given by the theorem? The answer comes from Lemma 4, X is the intersection of all elements in Q and A is the intersection of two elements in Q , then $|A| - \mu(G[N_G[A]]) \leq |X| - \mu(G[N_G[X]])$. Rearranging, we obtain

$$\alpha(G) \leq \mu(G) + |A| - \mu(G[N_G[A]]) \leq \mu(G) + |X| - \mu(G[N_G[X]]).$$

Thus, every collection of three or more elements in Q has a pair that yields a better bound on $\alpha(G)$.

As a consequence of Remark 1, we obtain the following corollary.

Corollary 6. *If G is a graph with no unique maximum independent set, and Q is the set of all maximum independent sets in G , then*

$$\alpha(G) \leq \mu(G) + \min \{|A \cap B| - \mu(G[N_G[A \cap B]]) : A, B \in Q\}.$$

The following example gives an infinite family of graphs satisfying Theorem 1 with equality. Moreover, it also shows a family of graphs where any intersection of maximum independent sets will satisfy Theorem 1 with equality.

Example 7. With this example, we establish the inequality of Theorem 1 being sharp, and in doing so, we also show the existence of graphs where any choice of intersecting maximum independent sets satisfies the inequality with equality. Let p , q , and r be non-negative integers, where p , q , and r are odd. Let $G(p, q, r)$ be the graph obtained by attaching each vertex of G_1 (the complete graph K_p with a pendant attached to each vertex) to each vertex of $G_2 = K_q$, and then attaching each vertex of the empty graph with order r , denoted G_3 , to every vertex of G_2 . For the graph $G(p, q, r)$, observe the following.

A. $\text{core}(G(p, q, r)) = V(G_3)$.

B. $\alpha(G) = p + r$.

C. $\mu(G) = \begin{cases} p + q, & \text{if } r \geq q, \\ p + \frac{r+q}{2}, & \text{if } r < q. \end{cases}$

D. If X is any intersection maximum independent sets in $G(p, q, r)$, then

$$|X| - \mu(G[N_G[X]]) = \begin{cases} r - q, & \text{if } r \geq q, \\ \frac{r-q}{2}, & \text{if } r < q. \end{cases}$$

With the above equations, if X is any intersection of maximum independent sets in $G(p, q, r)$, then

$$\alpha(G(p, q, r)) = \mu(G(p, q, r)) + |X| - \mu(G[N_{G(p,q,r)}[X]]).^2$$

The infinite family of graphs given in Example 7 provide examples where any choice of intersecting maximum independent sets will satisfy Theorem 1 with equality. The graph presented in Figure 1 provides an example where no choice of intersecting maximum independent sets will satisfy Theorem 1 with equality.

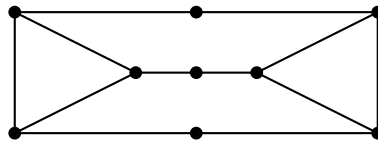


Figure 1. A graph G where no choice of intersecting maximum independent sets satisfies Theorem 1 with equality.

One interesting application of Theorem 1 can be seen by considering well-covered graphs, a heavily studied notion; see for example [10, 22]. A graph is *well-covered* if all maximal independent sets are also maximum. Observe that if

²Note: $|X| - \mu(G[N_{G(p,q,r)}[X]]) < 0$ whenever $r < q$.

G is an isolate-free and well-covered graph, then for every vertex $v \in V(G)$ there is a neighbor of v , say w , so that v cannot appear in any maximum independent set containing w . Since we may greedily construct a maximal independent set (which is also a maximum independent set in well-covered graphs), starting from either v or w , it follows that the intersection of all maximum independent sets in G is necessarily empty. Therefore, taking $X = \text{core}(G) = \emptyset$ in Theorem 1 implies the following corollary.

Corollary 8. *If G is an isolate-free and well-covered graph, then*

$$\alpha(G) \leq \mu(G).$$

Theorem 1 also generalizes and improves several known results. For example, recall $\alpha(G) \leq \mu(G)$, whenever G contains two disjoint maximum independent sets (Deniz *et al.* [8]). Since Theorem 1 implies $\alpha(G) \leq \mu(G)$ whenever any collection of maximum independent sets has an empty intersection, their result is a corollary of Theorem 1. Another example comes from considering the bound $\alpha(G) \leq \mu(G) + |\text{core}(G)| - 1$, whenever $\alpha(G) > \mu(G)$ (Boros *et al.* [2]). Taking $X = \text{core}(G)$ in the statement of Theorem 1, observe that if $\alpha(G) > \mu(G)$ and $\mu(G[N_G[\text{core}(G)]]) > 1$, then Theorem 1 improves upon this result. In particular, we make note of the following corollary.

Corollary 9. *If G is a graph, then*

$$\alpha(G) \leq \mu(G) + |\text{core}(G)| - \mu(G[N_G[\text{core}(G)]]).$$

3. PROOF OF THEOREM 2

In this section we prove Theorem 2. Before doing so we first prove a theorem and recall a lemma. The following result was motivated by a conjecture of the automated conjecturing program TxGRAFFITI, which in turn was motivated by GRAFFITI of Fajtlowicz [9], and later GRAFITTI.pc of DeLaVina [7]. The program TxGRAFFITI [6] was written by the second author, and generates possible graph inequalities on simple connected graphs. When asked to conjecture on the independence number, TxGRAFITTI conjectured $\alpha(G) \leq \mu(G)$ for all 3-regular and connected graphs. The following theorem confirms and generalizes this conjecture.

Theorem 10. *If G is a r -regular graph with $r > 0$, then*

$$\alpha(G) \leq \mu(G).$$

Proof. Let G be an r -regular graph with $r > 0$, $X \subseteq V(G)$ be a maximum independent set, and $Y = V(G) \setminus X$. By removing edges from G with both endpoints in Y , we next form a bipartite graph H with partite sets X and Y . Since those edges removed from G in order to form H were only edges with both endpoints in Y , any vertex chosen in X will have the same open neighborhood in H as it does in G . It follows that since G is r -regular and since X is an independent set, any vertex in X will have exactly r neighbors in Y , both in G and in H .

Let $S \subseteq X$ be chosen arbitrarily, and let $e(S, N_H(S))$ denote the number of edges from S to $N_H(S)$. Since each vertex in S has exactly r neighbors in Y , we observe that $e(S, N_H(S)) = r|S|$. However, since each vertex in $N_H(S)$ has at most r neighbors in X , we also have $e(S, N_H(S)) \leq r|N_H(S)|$. It follows that $r|S| \leq r|N_H(S)|$, and so, $|S| \leq |N_H(S)|$. By Hall's Theorem [14], there exists a matching M that can match X to a subset of Y . Since X is a maximum independent set and since M is also a matching in G , we conclude $\alpha(G) = |M| \leq \mu(G)$, proving the theorem. ■

A k -edge-coloring of G is an assignment of k colors to the edges of G so that no two edges with the same color share an endpoint. The minimum integer k so that G has a k -edge-coloring is the *edge chromatic number* of G , written $\chi'(G)$. By Vizing's Theorem, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$ for all graphs. Graphs satisfying $\Delta(G) = \chi'(G)$ are *class 1*, whereas graphs satisfying $\chi'(G) = \Delta(G) + 1$ are *class 2*. Let G_Δ denote the subgraph induced by the set of maximum degree vertices in G . With these definitions, we next recall a result due to Fournier [11].

Lemma 11. *If G is class 2, then G_Δ contains at least one cycle.*

As a consequence of Lemma 11, all class 2 graphs satisfy $|G_\Delta| \geq 3$ and $E(G_\Delta) \neq \emptyset$. With this observation, we are now ready to prove Theorem 2. Recall its statement.

Theorem 2. *If G is a graph, then*

$$\delta(G)\alpha(G) \leq \Delta(G)\mu(G),$$

and this inequality is sharp.

Proof. Clearly, if $\delta(G) = 0$, we are done. So we will assume $\delta(G) > 0$. Proceeding by way of contradiction, suppose the theorem is false. Among all counterexamples, let G be one with a minimum number of edges. By Theorem 10, any r -regular graph with $r > 0$ will satisfy the theorem, and so, the graph G must satisfy $\delta(G) < \Delta(G)$. Before proceeding, we remind the reader that all graphs are either class 1 or class 2.

If G is a class 1, then $\chi'(G) = \Delta(G)$. Since each color class in a $\chi'(G)$ -edge coloring forms a matching in G , and since every edge in G belongs to exactly one color class, it is clear that $m(G) \leq \chi'(G)\mu(G)$. Moreover, each vertex in any maximum independent set will have at least $\delta(G)$ edges incident with it, implying $\delta(G)\alpha(G) \leq m(G)$. It follows that $\delta(G)\alpha(G) \leq \Delta(G)\mu(G)$, which is impossible, because G is a counter-example. Thus, G is not class 1.

If G is class 2, then Lemma 11 implies G_Δ has a non-empty edge set. Let vw be one such edge and let $H = G - vw$. Clearly, $\alpha(G) \leq \alpha(H)$ and $\mu(H) \leq \mu(G)$. Since $\delta(G) < d_G(v) = d_G(w) = \Delta(G)$, it follows that $\delta(H) = \delta(G)$. Since G_Δ contains a cycle, it has at least 3 vertices, and so, $\Delta(H) = \Delta(G)$. Finally, G being a minimum counterexample implies $\delta(H)\alpha(H) \leq \Delta(H)\mu(H)$. It follows that $\delta(G)\alpha(G) \leq \Delta(G)\mu(G)$, which is again impossible, since G is a counter-example. Since G is neither class 1 nor class 2, yet all graphs are either class 1 or class 2, we reach a contradiction and the theorem is proven.

We shall give various constructions demonstrating sharpness in Theorem 2 in the discussion section below. ■

4. DISCUSSION OF THEOREM 2

In this section we show infinite families of graphs where Theorem 2 is sharp, as well as discuss improvements on known bounds for $\alpha(G)$.

4.1. Examples where equality holds for Theorem 2.

We now give constructions of bipartite graphs with $\delta(G)\alpha(G) = \Delta(G)\mu(G)$, and regular non-bipartite graphs with $\alpha(G) = \mu(G)$.

Construction 1. (δ, Δ) -bipartite graphs. Namely, the graph G with $V(G) = A \cup B$ where A and B are independent sets and all degrees in A equal $\Delta(G)$ and all degrees in B are equal $\delta(G)$. In these graphs $\alpha(G) = |B|$, $\mu(G) = |A|$, and $\delta(G)|B| = \Delta(G)|A|$. See Figure 2 for one such example.

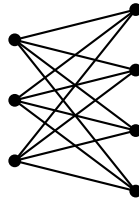


Figure 2. The (δ, Δ) -bipartite graph G with $\delta(G) = 3$, $\Delta(G) = 4$, and $\delta(G)\alpha(G) = \Delta(G)\mu(G)$.

Construction 2. Let $r \geq 4$ with $r \equiv 0 \pmod{2}$. Take $r/2$ copies of $K_{r,r}$, and for each $K_{r,r}$ delete a single edge. Next let $G(1, r)$ be the graph obtained by attaching each degree $r - 1$ vertex of each copy of $K_{r,r}$ to an isolated vertex v (see Figure 3 for an example with $r = 4$). This resulting graph is r -regular, non-bipartite, and satisfies $\alpha(G(1, r)) = \mu(G(1, r)) = r^2/2 = (n(G(1, r)) - 1)/2$.

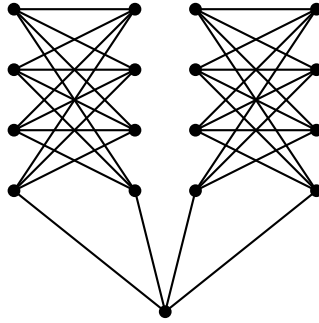


Figure 3. The non-bipartite regular graph $G(1, 4)$ with $\alpha(G(1, 4)) = \mu(G(1, 4))$.

Construction 3. Let $r \geq 3$. Consider r copies of $K_{r,r}$, say G_1, \dots, G_r . For each graph G_i , subdivide a single edge by a vertex g_i , for $i \in [r]$. Next take an empty graph E_{r-2} on $r - 2$ vertices and attach by an edge, each g_i for $i \in [r]$ to each vertex of E_{r-2} . Denote the resulting graph by $G(2, r)$. Note, $G(2, r)$ is r -regular and non-bipartite with $n(G(2, r)) = 2r^2 + 2r - 2$ and $\alpha(G(2, r)) = r^2 + r - 2 = (n(G(2, r)) - 2)/2 = n(G(2, r))/2 - 1 = \mu(G(2, r))$. As G has no perfect matching (by Tutte's Theorem since when we deleted B , $|B| = k - 2$ we are left with k odd blocks). See Figure 4 for an example of $G(2, r)$.

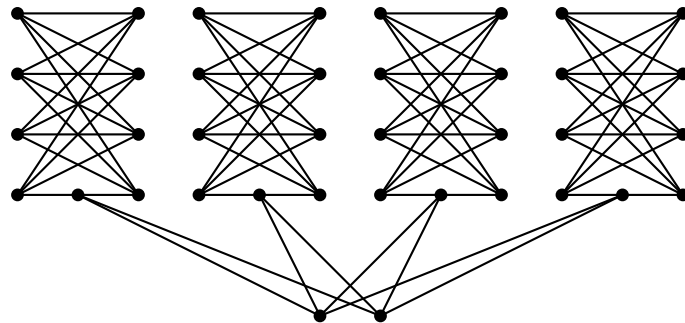


Figure 4. The non-bipartite regular graph $G(2, 4)$ with $\alpha(G(2, 4)) = \mu(G(2, 4))$.

Construction 4. Suppose $r \geq 3$ and let G be r -regular bipartite graph with $\alpha(G) = \mu(G)$ (e.g. $K_{r,r}$). Take r copies of G and subdivide by a vertex w_i ,

for $i = 1, \dots, r$ an edge on a cycle to get G^* which is non-bipartite. Next take one copy of $K_{2,r}$. The r vertices of $K_{2,r}$ have degree 2. Connect each w_i , for $i = 1, \dots, r$, to $r-2$ vertices of the r vertices of $K_{2,r}$ so that the induced subgraph on the w_i vertices and the r vertices of $K_{2,r}$ is $r-2$ regular forming $G(3, r)$; see Figure 5 for an example of $G(3, r)$. Now, $n(G(3, r)) = 2r^2 + 2r + 2$ (if $G = K_{r,r}$), $\alpha(G(3, r)) = r^2 + r = (n(G(3, r)) - 2)/2 = n(G(3, r))/2 - 1 = \mu(G(3, r))$. Since $G(3, r)$ cannot have a perfect matching, because of Tutte's Theorem and the fact the deleting the r vertices of $K_{2,r}$ leaves $r+2$ odd components.

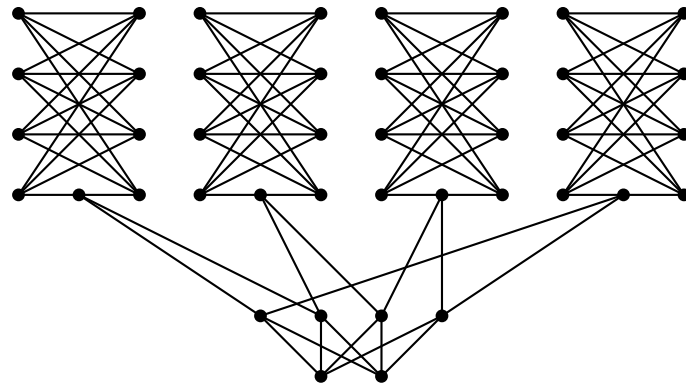


Figure 5. The non-bipartite regular graph $G(3, 4)$ with $\alpha(G(3, 4)) = \mu(G(3, 4))$.

4.2. Improvements on known bounds for $\alpha(G)$

If G is a graph with $\delta(G) \geq 1$, then Theorem 2 implies the following upper bound on $\alpha(G)$,

$$(3) \quad \alpha(G) \leq \frac{\Delta(G)}{\delta(G)} \mu(G).$$

This bound is interesting, as the righthand side of (3) is computable in polynomial time. Moreover, inequality (3) can also improve on known computationally efficient upper bounds for $\alpha(G)$ in some classes of graphs. For example, the *annihilation number* of G , written $a(G)$, is a degree sequence invariant for which $\alpha(G) \leq a(G)$ [21]. This bound improves on many known bounds, for example $\alpha(G) \leq a(G) \leq n(G) - \frac{m(G)}{\Delta(G)}$ (see [21]). However, $a(G) \geq \frac{n(G)}{2}$ for all graphs. Thus, for r -regular graphs with $r > 0$, inequality (3) gives the improvement

$$\alpha(G) \leq \mu(G) \leq \frac{n(G)}{2} \leq a(G) \leq n(G) - \frac{m(G)}{\Delta(G)},$$

and

$$\alpha(G) \leq \mu(G) \leq n(G) - \mu(G).$$

Further observe that for sufficiently large r -regular graphs with $r > 0$, inequality (3) can give dramatic improvements on the minimum degree bound $\alpha(G) \leq n(G) - \delta(G)$.

5. CONCLUDING REMARKS

In this paper we have proven two theorems relating $\alpha(G)$ and $\mu(G)$. These two theorems imply a myriad of interesting corollaries bounding $\alpha(G)$ from above; some of which we summarize in the following theorem.

Theorem 12. *Let G be a graph and $X \subseteq V(G)$ be any intersection of maximum independent sets.*

1. $\alpha(G) \leq \mu(G) + |X| - \mu(G[N_G[X]])$.
2. $\alpha(G) \leq \mu(G) + \text{core}(G) - \mu(G[N_G[\text{core}(G)]])$.
3. *If $\text{core}(G) = \emptyset$, then $\alpha(G) \leq \mu(G)$.*
4. *If X is isolate-free and well-covered, then $\alpha(G) \leq \mu(G)$.*
5. *If $\delta(G) \geq 1$, then $\alpha(G) \leq \frac{\Delta(G)}{\delta(G)}\mu(G)$.*
6. *If G is r -regular with $r > 0$, then $\alpha(G) \leq \mu(G)$.*

As mentioned before, many of the cases contained in Theorem 12 yield improvements on known upper bounds for $\alpha(G)$, most notably being the case of Theorem 12.6. Observing this, we believe the following problem merits further inspection.

Problem 13. Characterize $\alpha(G) = \mu(G)$ whenever G is 3-regular.

More generally, we also suggest the following problem.

Problem 14. Characterize all graphs G for which $\delta(G)\alpha(G) = \Delta(G)\mu(G)$.

Next we remark on the cardinality of minimum maximal matchings in G , written $\mu^*(G)$. Recall,

$$n(G) - 2\mu^*(G) \leq \alpha(G) \leq n(G) - \mu(G),$$

for any graph G . Thus, by Theorem 12 we obtain

$$(4) \quad n(G) - 2\mu^*(G) \leq \mu(G),$$

whenever G is r -regular with $r > 0$, or isolate-free and well-covered, or has an empty core. Rearranging (4), we obtain the inequality

$$\frac{\alpha(G)}{2} \leq \frac{n(G) - \mu(G)}{2} \leq \mu^*(G),$$

for all graphs satisfying one or more of the above mentioned properties. Hence, $\alpha(G) \leq 2\mu^*(G)$ for these families of graphs. We suggest that future work include studying relationships between independent sets and $\mu^*(G)$. More specifically, we suggest considering the following conjecture of TxGRAFFITI. Recall the *independent domination number* of G , written $i(G)$, is the minimum cardinality of a maximal independent set in G ; for an excellent survey on independent domination see the paper of Goddard and Henning [13].

Conjecture 15 (TxGraffiti 2019). *If G is an r -regular graph with $r > 0$, then*

$$i(G) \leq \mu^*(G).$$

If Conjecture 15 is true, we believe the following question merits further investigation.

Problem 16. Is it true that $\delta(G)i(G) \leq \Delta(G)\mu^*(G)$ for all graphs?

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