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TOTAL ROMAN {2}-DOMINATING FUNCTIONS IN GRAPHS

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Abstract

A Roman {2}-dominating function (R2F) is a function $f : V \to \{0, 1, 2\}$ with the property that for every vertex $v \in V$ with $f(v) = 0$ there is a neighbor u of v with $f(u) = 2$, or there are two neighbors x, y of v with $f(x) = f(y) = 1$. A total Roman {2}-dominating function (TR2DF) is an R2F f such that the set of vertices with $f(v) > 0$ induce a subgraph with no isolated vertices. The weight of a TR2DF is the sum of its function values over all vertices, and the minimum weight of a TR2DF of G is the total Roman {2}-domination number $\gamma_{tR2}(G)$. In this paper, we initiate the study of total Roman {2}-dominating functions, where properties are established. Moreover, we present various bounds on the total Roman {2}-domination number. We also show that the decision problem associated with $\gamma_{tR2}(G)$ is NP-complete for bipartite and chordal graphs. Moreover, we show that it is possible to compute this parameter in linear time for bounded clique-width graphs (including trees).

Keywords: Roman domination, Roman {2}-domination, total Roman {2}-domination.

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1. Introduction

In this paper, G is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by $n(G)$. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. A *leaf* of G is a vertex of degree one, while a *support vertex* of G is a vertex adjacent to a leaf. A support vertex is said to be *weak* (respectively, *strong*) if it is adjacent to exactly one leaf (respectively, at least two leaves).

Let P_n, C_n and K_n be the *path*, *cycle* and *complete graph* of order n and $K_{p,q}$ the complete bipartite graph with one partite set of cardinality p and the other of cardinality q. The *complement* of a graph G is denoted by $\overline{G}$. The *join* of two graphs G and H, denoted by $G \vee H$, is a graph obtained from G and H by joining each vertex of G to all vertices of H. A *tree* is an acyclic connected graph. A *double star* is a tree containing exactly two vertices that are not leaves. A double star with respectively p and q leaves attached at each support vertex is denoted by $S_{p,q}$. The *corona* of a graph H is the graph obtained from H by appending a vertex of degree 1 to each vertex of H. The *distance* $d_G(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u-v$ path in G. The *diameter* of G, denoted by $\mathrm{diam}(G)$, is the maximum value among distances between all pair of vertices of G.

A subset $S \subseteq V$ is a *dominating set* if every vertex in $V \setminus S$ has a neighbor in S, and S is a *total dominating set*, abbreviated TSD, if every vertex in V has a neighbor in S, that is, $N(v) \cap S \neq \emptyset$ for all $v \in V$. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set of G, and the *total domination number* $\gamma_t(G)$ is the minimum cardinality of a TDS of G.

For a graph G and a positive integer k, let $f : V(G) \to \{0, 1, 2, \ldots, k\}$ be a function, and let $(V_0, V_1, V_2, \ldots, V_k)$ be the ordered partition of $V = V(G)$ induced by f, where $V_i = \{v \in V : f(v) = i\}$ for $i \in \{0, 1, \ldots, k\}$. There is a 1-1 correspondence between the functions $f : V \to \{0, 1, 2, \ldots, k\}$ and the ordered partitions $(V_0, V_1, V_2, \ldots, V_k)$ of V, so we will write $f = (V_0, V_1, V_2, \ldots, V_k)$. The weight of f is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$.

A function $f : V(G) \to \{0, 1, 2\}$ is a *Roman dominating function*, abbreviated RDF, on G if every vertex $u \in V$ for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The *Roman domination number* $\gamma_R(G)$ is the minimum weight of a RDF on G. Roman domination was introduced by Cockayne *et al.* in [4] and was inspired by the work of ReVelle and Rosing [10], and Stewart [11].

The definition of Roman dominating functions was motivated by an article in *Scientific American* by Stewart entitled "Defend the Roman Empire" [11] and suggested even earlier by ReVelle [9]. Each vertex in our graph represents a location in the Roman Empire. A location (vertex v) is considered *unsecured* if no legions are stationed there (i.e., $f(v) = 0$) and *secured* otherwise (i.e., if $f(v) \in \{1, 2\}$). An unsecured location (vertex v) can be secured by sending a legion to v from a neighboring location (an adjacent vertex u). But Constantine the Great (Emperor of Rome) issued a decree in the 4th century A.D. for the defense of the regions, where a legion cannot be sent from a secured location having only one legion stationed there to an unsecured location, for otherwise it leaves that location unsecured. Thus, two legions must be stationed at a location ($f(v) = 2$) before one of the legions can be sent to a neighboring location. In this way, Emperor Constantine the Great can defend the Roman Empire. Since it is expensive to maintain a legion at a location, the Emperor would like to station as few legions as possible, while still defending the Roman Empire. A Roman dominating function of weight $\gamma_R(G)$ corresponds to such an optimal assignment of legions to locations.

In [2], Chellali *et al.* introduced the Roman {2}-domination (called in [7] and elsewhere Italian domination) defined as follows: a *Roman* {2}*-dominating function* is a function $f = (V_0, V_1, V_2)$ with the property that for every vertex $v \in V_0$ there is a vertex $u \in N(v)$, with $u \in V_2$ or there are two vertices $x, y \in N(v)$ with $x, y \in V_1$. The *Roman* {2}*-domination number* $\gamma_{\{R2\}}(G)$ is the minimum weight of a Roman {2}-dominating function on G.

There are many papers in the literature devoted to the study of Roman domination type problems and its variations. One of the questions that arise naturally when a Roman domination type problem is studied is to focus on its complexity and algorithmic aspects. In 2008, Liedloff *et al.* [8] showed, among other results, that it is possible to compute the Roman domination number of a graph with bounded cliquewidth in linear time. Clearly, this implies that there exists algorithms for computing the Roman domination number of trees in linear time. Chellali *et al.* [2] proved that Roman {2}-domination problem is NP-complete for bipartite graphs while Chen and Lu [3] recently showed it is NP-complete even when restricted to split graphs. Moreover, the authors [3] presented a linear time algorithm to obtain the Roman {2}-domination number of a block graph.

In this paper, we initiate the study of the total version of Roman {2}-dominating function. A *total Roman* {2}*-dominating function*, abbreviated TR2DF, is a Roman {2}-dominating function $f = (V_0, V_1, V_2)$ such that the subgraph induced by $V_1 \cup V_2$ has no isolated vertices. The *total Roman* {2}*-domination number* $\gamma_{tR2}(G)$ is the minimum weight of a TR2DF on G. A TR2DF on G with weight $\gamma_{tR2}(G)$ is called a $\gamma_{tR2}(G)$-function. Total Roman {2}-domination number is well-defined for all graphs G with no isolated vertices since assigning a 1 to every vertex of G provides a TR2DF of G. Hence for all graphs G of order n with $\delta(G) \geq 1$, $2 \leq \gamma_{tR2}(G) \leq n$. We present various bounds on the total Roman {2}-domination number and several properties are established. We show that the decision problem associated with $\gamma_{tR2}(G)$ is NP-complete for bipartite and chordal graphs. Moreover, we show that it is possible to compute this parameter in linear time for bounded clique-width graphs (including trees).

We note that throughout this paper, we only consider nontrivial connected graphs that we will call *ntc graphs*.

2. Preliminary Results

We begin by giving some properties of total Roman {2}-dominating functions. The following two facts lead to our first observation. Clearly assigning a 2 to every vertex in a minimum total dominating set of a ntc graph G and a 0 to the remaining vertices of G provides a TR2DF. Also, for every TR2DF $f = (V_0, V_1, V_2)$ the set $V_1 \cup V_2$ total dominates $V(G)$.

Observation 1. *For every ntc graph G,*

$$\gamma_t(G) \leq \gamma_{tR2}(G) \leq 2\gamma_t(G).$$

Note that the lower bound of Observation 1 is attained for $K_2 \vee \overline{K_{n-2}}$ while the upper bound is attained for the double star $S_{3,3}$.

It is well-known that $\gamma_t(G) \leq 2\gamma(G)$ for every ntc graph G. So by Observation 1, we will have $\gamma_{tR2}(G) \leq 4\gamma(G)$. Our next result improves this upper bound to $\gamma_{tR2}(G) \leq 3\gamma(G)$. We need the following result due to Bollobás and Cockayne [1]. If S is a set of vertices, then we say that a vertex v is a *private neighbor* of a vertex $u \in S$ (with respect to S) if $N[v] \cap S = \{u\}$. The *external private neighborhood* $\text{epn}(u, S)$ of u with respect to S consists of those private neighbors of u in $V \backslash S$. For a TR2DF $f = (V_0, V_1, V_2)$ of an ntc graph, let $V_{02} = \{w \in V_0 : N(w) \cap V_2 \neq \emptyset\}$ and $V_{01} = V_0 \backslash V_{02}$.

Theorem 2 (Bollobás and Cockayne [1])**.** *If G is a graph without isolated vertices, then G has a minimum dominating set D such that for all $d \in D$, there exists a neighbor $f(d) \in V \backslash D$ of d such that $f(d)$ is not a neighbor of any vertex $x \in D \backslash \{d\}$.*

Proposition 3. *For every ntc graph G, $\gamma_{tR2}(G) \leq 3\gamma(G)$.*

Proof. Let D be a minimum dominating set of G satisfying the property of Theorem 2. Since each vertex of D has an external private neighbor in $V \backslash D$, let W be a subset of $V \backslash D$ formed by the private neighbors chosen so that each vertex of D has exactly one external private neighbor in D. Clearly $|W| = |D|$. Now define the function f as follows: $f(x) = 2$ for all $x \in D$, $f(x) = 1$ for all $x \in W$, and $f(x) = 0$ otherwise. Clearly f is a TR2DF of G of weight $2|D| + |W| = 3\gamma(G)$, and thus $\gamma_{tR2}(G) \leq 3\gamma(G)$. ■

For the sharpness of the bound in Proposition 3, consider the tree T obtained from two stars $K_{1,4}$ by adding an edge between a leaf of one star to a leaf of the other star. The next observation is straightforward and is tight for double stars.

Observation 4. *For every ntc graph G, $\gamma_{R2}(G) \leq \gamma_{tR2}(G)$.*

Proposition 5. *Let G be an ntc graph. Then for every $\gamma_{tR2}(G)$-function $f = (V_0, V_1, V_2)$ such that $|V_2|$ is as small as possible, we have the following.*

(i) *Each vertex in V_2 has at least two private neighbors in V_0 with respect to V_2.*

(ii) $2|V_2| \leq |V_{02}|$.

Proof. (i) Suppose there exists a vertex $v \in V_2$ with at most one private neighbor in V_0 with respect to V_2. Then reassigning v and its private neighbor (if any) the value 1 instead of 2 and 0, respectively provides a $\gamma_{tR2}(G)$-function with less vertices assigned a 2, which contradicts the choice of f.

(ii) Follows from (i). ■

Proposition 6. *Let G be an ntc graph with maximum degree $\Delta \leq 2$. Then there exists a $\gamma_{tR2}(G)$-function $f = (V_0, V_1, V_2)$ such that $V_2 = \emptyset$.*

Proof. Among all $\gamma_{tR2}(G)$-functions, let $f = (V_0, V_1, V_2)$ be a one such that $|V_2|$ is as small as possible. If $V_2 \neq \emptyset$, then by Proposition 5, every vertex of V_2 has at least two private neighbors in V_0 with respect to V_2. But then since $\Delta \leq 2$, each vertex in V_2 would be isolated in $G[V_1 \cup V_2]$, a contradiction. Hence $V_2 = \emptyset$. ■

Recall that a subset S of V is a *double dominating set* of G if for every vertex $v \in V$, $|N[v] \cap S| \geq 2$, that is, v is in S and has at least one neighbor in S or v is in $V \backslash S$ and has at least two neighbors in S. The *double domination number* $\gamma_{\times 2}(G)$ is the minimum cardinality of a double dominating set of G. Double domination was introduced by Harary and Haynes [6] who proved that $\gamma_{\times 2}(P_n) = \lceil \frac{2n+2}{3} \rceil$ and $\gamma_{\times 2}(C_n) = \lceil \frac{2n}{3} \rceil$. The following result shows that the equality between $\gamma_{tR2}(G)$ and $\gamma_{\times 2}(G)$ occurs under certain conditions.

Proposition 7. *Let G be an ntc graph. If G has a $\gamma_{tR2}(G)$-function $f = (V_0, V_1, V_2)$ such that $V_2 = \emptyset$, then $\gamma_{tR2}(G) = \gamma_{\times 2}(G)$.*

Proof. If S is a double dominating set of G, then $(V\setminus S, S, \emptyset)$ is clearly a TR2DF on G, and thus $\gamma_{tR2}(G) \leq \gamma_{\times 2}(G)$. Now if $f = (V_0, V_1, V_2)$ is a $\gamma_{tR2}(G)$-function such that $V_2 = \emptyset$, then V_1 double dominates $V(G)$, and thus $\gamma_{\times 2}(G) \leq \gamma_{tR2}(G)$. Therefore $\gamma_{tR2}(G) = \gamma_{\times 2}(G)$. ■

The following results are immediate consequences of Propositions 6, 7 and the exact values of the double domination number of paths and cycles given above.

Proposition 8. *For* $n \geq 2$, $\gamma_{tR2}(P_n) = \lceil \frac{2n+2}{3} \rceil$.

Proposition 9. *For* $n \geq 3$, $\gamma_{tR2}(C_n) = \lceil \frac{2n}{3} \rceil$.

3. Complexity

Our aim in this section is to study the complexity of the following decision problem, to which we shall refer as TOTAL ROMAN {2}-DOMINATION.

TOTAL ROMAN {2}-DOMINATION
Instance: Graph $G = (V, E)$, positive integer $k \leq |V|$.
Question: Does G have a total Roman {2}-dominating function of weight at most k?

We show that this problem is NP-complete by reducing the well-known NP-complete problem, Exact-3-Cover (X3C), to TOTAL ROMAN {2}-DOMINATION.

EXACT 3-COVER (X3C)
Instance: A finite set X with $|X| = 3q$ and a collection C of 3-element subsets of X.
Question: Is there a subcollection C' of C such that every element of X appears in exactly one element of C'?

Theorem 10. *TOTAL ROMAN {2}-DOMINATION is NP-Complete for bipartite graphs.*

Proof. TOTAL ROMAN {2}-DOMINATION is a member of $\mathcal{NP}$, since we can check in polynomial time that a function $f : V \to \{0, 1, 2\}$ has weight at most k and is a total Roman {2}-dominating function. Now let us show how to transform any instance of X3C into an instance G of TOTAL ROMAN {2}-DOMINATION so that one of them has a solution if and only if the other one has a solution. Let $X = \{x_1, x_2, \ldots, x_{3q}\}$ and $C = \{C_1, C_2, \ldots, C_t\}$ be an arbitrary instance of X3C.

For each $x_i \in X$, we build a graph H_i obtained from a path $P_2 : x_i$-y_i and two stars $K_{1,3}$ centered at a_i and b_i, by adding edges $y_i a_i$ and $y_i b_i$. Hence, each H_i

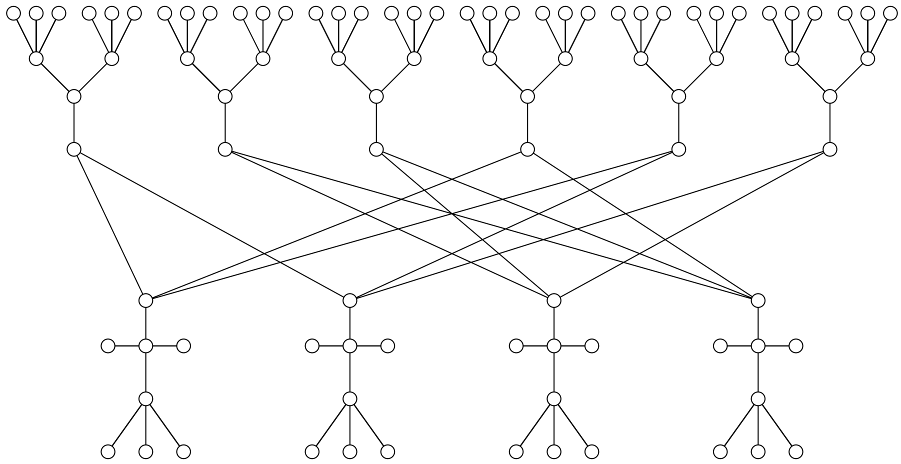

Figure 1. NP-Completeness for bipartite graphs.

has order 10. For each $C_j \in C$, we build a double star $S_{3,3}$ with support vertices u_j and v_j. Let c_j be a leaf of the double star $S_{3,3}$. Let $Y = \{c_1, c_2, \ldots, c_t\}$. Now to obtain a graph G, we add edges $c_j x_i$ if $x_i \in C_j$. Clearly G is a bipartite graph. Set $k = 4t + 16q$. Observe that for every total Roman $\{2\}$-dominating function f on G, each H_i has weight at least 5 and each double star $S_{3,3}$ has weight at least 4.

Suppose that the instance X, C of X3C has a solution C'. We construct a total Roman $\{2\}$-dominating function f on G of weight k. For each i, assign the value 2 to a_i, b_i; the value 1 to y_i and 0 to the remaining vertices of H_i. For every j, assign the value 2 to u_j and v_j, and 0 to each leaf. In addition, for every c_j, assign the value 1 if $C_j \in C'$ and the value 0 if $C_j \notin C'$. Note that since C' exists, its cardinality is precisely q, and so the number of c_j's with value 1 is q, having disjoint neighborhoods in $\{x_1, x_2, \ldots, x_{3q}\}$. Since C' is a solution for X3C, every vertex in X is adjacent to two vertices assigned a 1. Hence, it is straightforward to see that f is a total Roman $\{2\}$-dominating function with weight $f(V) = 4t + q + 15q = k$.

Conversely, suppose that G has a total Roman $\{2\}$-dominating function with weight at most k. Among all such functions, let $g = (V_0, V_1, V_2)$ be one such that $\{y_1, y_2, \ldots, y_{3q}\} \cap V_2$ is as small as possible. As observed above, since each H_i has weight at least 5, we may assume that $g(a_i) = g(b_i) = 2$ and $g(y_i) > 0$ so that vertices a_i, b_i are not isolated in the subgraph induced by $V_1 \cup V_2$. Hence each leaf neighbor of a_i or b_i is assigned a 0 under g. If $g(y_i) = 2$ for some i, then we must have $g(x_i) = 0$. In this case, reassigning a 1 to each of y_i and x_i instead of 2 and 0, respectively, provides a total Roman $\{2\}$-dominating function g' with less vertices y_i assigned a 2 than under g, contradicting our choice of g. Hence $g(y_i) = 1$ for every $i \in \{1, 2, \ldots, 3q\}$. On the other hand, the total weight of all double stars corresponding to elements of C is at least $4t$. In this case, we can assume that $g(u_j) = g(v_j) = 2$ and so each leaf neighbor of u_j or b_j is assigned a 0 under g. Note that each c_j can be assigned a 0 since $g(u_j) = 2$. Since $w(g) \leq 4t + 16q$ and the total weight assigned to vertices of $V(G)\backslash(X \cup Y)$ is $4t + 15q$, we have to assign to vertices of $(X \cup Y)$ weights whose total not exceeding q in order that each vertex $x_i \in X$ has either $g(x_i) > 0$ or has two neighbors in V_1. Since $|X| = 3q$, it is clear that this is only possible if there are q vertices of $\{c_1, c_2, \ldots, c_t\}$ that are assigned a 1. Since each c_j has a exactly three neighbors in $\{x_1, x_2, \ldots, x_{3q}\}$, we deduce that $C' = \{C_j : g(c_j) = 1\}$ is an exact cover for C. ■

The next result is obtained by using the same proof of Theorem 10 on the (same) graph G built for the transformation by adding all edges between the vertices labelled c_j in order that the resulting graph is chordal.

Theorem 11. *TOTAL ROMAN* $\{2\}$*-DOMINATION is NP-Complete for chordal graphs.*

In the rest of this section, we prove that the decision problem associated to $\gamma_{tR2}(G)$ can be solved in linear time for the class of graphs with bounded clique-width, which implies that it also can be computed in linear time for trees.

We make use of several useful objects and results, which are formally described in [5, 8], related to logic structures. Namely, in what follows, a k-expression on the vertices of a graph G with labels $\{1, 2, \ldots, k\}$ is an expression using the following operations.

$\bullet\, i(x)$ create a new vertex x with label i,

$G_1 \oplus G_2$ create a new graph which is the disjoint union of the graphs G_i,

$\eta_{ij}(G)$ add all edges in G joining vertices with label i with vertices with label j,

$\rho_{i\to j}(G)$ change the label of all vertices with label i into label j.

We call the *clique-width* of a graph G the minimum positive integer k that is needed to describe G by means of a k-expression. For example, the complete graph K_3 with set of vertices $\{a, b, c\}$ can be described by the following 2-expression.

$$\rho_{2\to 1}\left(\eta_{12}\left(\rho_{2\to 1}\left(\eta_{12}\left(\bullet 1(a) \oplus \bullet 2(b)\right)\right) \oplus \bullet 2(c)\right)\right).$$

In what follows, MSOL(τ_1) stands for the monadic second order logic with quantification over subsets of vertices. We denote by $G(\tau_1)$ the logic structure $< V(G), R >$ where R is a binary relation such that $R(u, v)$ holds if and only if uv is an edge in G. An optimization problem is a *LinEMSOL*(τ) *optimization problem* if it can be described in the following way (see [8] for more details, since this is a version of the definition given by [5] restricted to finite simple graphs),

$$Opt \left\{ \sum_{1 \le i \le l} a_i |X_i| : < G(\tau_1), X_1, \ldots, X_l > \vDash \theta(X_1, \ldots, X_l) \right\},$$

where θ is an MSOL(τ_1) formula that contains free set-variables $X_1, \ldots, X_l$, integers a_i and Opt is either min or max.

We use the following result on LinEMSOL optimization problems.

Theorem 12 (Courcelle *et al.* [5])**.** *Let $k \in \mathbb{N}$ and let $\mathcal{C}$ be a class of graphs of clique-width at most k. Then every LinEMSOL(τ_1) optimization problem on $\mathcal{C}$ can be solved in linear time if a k-expression of the graph is part of the input.*

We extend a result proved by Liedloff *et al.* (see Theorem 31 in [8]) regarding the complexity of the Roman domination decision problem to the corresponding decision problem for the total Roman {2}-domination number.

Theorem 13. *The total Roman {2}-domination problem is a LinEMSOL(τ_1) optimization problem.*

Proof. Let us show that the total Roman {2}-domination problem can be expressed as a LinEMSOL(τ_1) optimization problem. Let $f = (V_0, V_1, V_2)$ be a total Roman {2}-domination function in $G = (V, E)$ and let us define the free set-variables X_i such that $X_i(v) = 1$ whenever $v \in V_i$ and $X_i(v) = 0$, otherwise. For the sake of congruence with the logical system notation, we denote by $|X_i| = \sum_{v \in V} X_i(v)$, even when, clearly, is $|X_i| = |V_i|$. Observe that the total Roman {2}-domination decision problem corresponds to achieve the optimum for the following expression.

$$\min_{X_i} \{|X_1| + 2|X_2| : < G(\tau_1), X_0, X_1, X_2 > \vDash \theta(X_0, X_1, X_2)\},$$

where θ is the formula given by

$$\begin{aligned}\theta(X_0, X_1, X_2) &= (\forall v\, ((X_1(v) \vee X_2(v)) \to \exists u\, ((X_1(u) \vee X_2(u)) \wedge R(u, v)))) \\ &\quad \wedge (\forall v\, (X_1(v) \vee X_2(v) \vee \exists u\, (R(u, v) \wedge X_2(u)) \\ &\quad \vee\ \ \exists u, w\, (R(u, v) \wedge R(w, v) \wedge X_1(u) \wedge X_1(w)))).\end{aligned}$$

Clearly, θ is an MSOL(τ_1) formula that describes the total Roman {2}-domination problem. Namely, the formula has two main clauses. The first one requires that every vertex v with a positive label 1 or 2 must have a neighbor u with a positive label. The latter implies that the induced subgraph by the set of vertices $V_1 \cup V_2$ has no isolated vertices. The second clause of the formula assures that for any vertex v of the graph either the vertex itself has a positive label, or either it has a neighbor with a label 2, or either it has two different neighbors having label 1 each. Hence, when the formula θ is satisfied, the requirements of a total Roman {2}-domination assignment in G holds. ∎

As a consequence, we may derive the following corollary.

Corollary 14. *The decision problem associated to the total Roman $\{2\}$-domination problem can be solved in linear time on any graph G with clique-width bounded by a constant k, provided that either there exists a linear-time algorithm to construct a k-expression of G, or a k-expression of G is part of the input.*

Since any graph with bounded treewidth is also a bounded clique-width graph, and it is well-known that any tree graph has treewidth equal to 1, then we can deduce that the total Roman $\{2\}$-domination problem can be solved in linear time for the class of trees. Besides, there are several classes of graphs G with bounded clique-width $cw(G)$ like, for example, the cographs ($cw(G) \leq 2$) and the distance hereditary graphs ($cw(G) \leq 3$), for which it is also possible to solve the total Roman $\{2\}$-domination problem in linear time.

4. Graphs with Small or Large Total Roman $\{2\}$-domination

As mentioned in Section 1, for all ntc graphs G of order n, $2 \leq \gamma_{tR2}(G) \leq n$. In this section, we characterize all ntc graphs G such that $\gamma_{tR2}(G) \in \{2, 3, n\}$.

Proposition 15. *Let G be an ntc graph. For any graph H, we have $\gamma_{tR2}(K_2 \vee H) = 2$. Conversely, if $\gamma_{tR2}(G) = 2$, there is some graph H such that $G = K_2 \vee H$.*

Proof. If $G = K_2 \vee H$, then clearly $\gamma_{tR2}(G) = 2$. Conversely, assume that $\gamma_{tR2}(G) = 2$ and let $f = (V_0, V_1, V_2)$ be a $\gamma_{tR2}(G)$-function. By definition of TR2DF of G, we have $\gamma_{tR2}(G) = |V_1| + 2|V_2| = 2$. Since $G[V_1 \cup V_2]$ has no isolated vertex, we deduce that $|V_2| = 0$ and $|V_1| = 2$. Now let $V_1 = \{x, y\}$. Clearly, $xy \in E(G)$, because $G[V_1 \cup V_2]$ has no isolated vertex, and every vertex in $V(G) \backslash \{x, y\}$ is adjacent to both x and y. Thus $G \cong K_2 \vee H$, and H is any graph of order $n - 2$. ■

Proposition 16. *Let G be an ntc graph of order $n \geq 5$. Then $\gamma_{tR2}(G) = 3$ if and only if either G has exactly one vertex of degree $n - 1$ or $\Delta(G) \leq n - 2$ and G is obtained from two disjoint graphs G_1 and G_2 such that $G_1 \in \{P_3, C_3\}$ and G_2 is any graph of order $n - 3$ by adding edges between vertices of G_1 and G_2 in order that every vertex of G_2 has at least two neighbors in G_1.*

Proof. If $\Delta(G) = n - 1$ and G has exactly one vertex u of degree $n - 1$, then the function f defined on $V(G)$ by $f(u) = 2$, $f(v) = 1$ for some $v \in V(G) \backslash \{u\}$ and $f(w) = 0$ for all $w \in V \backslash \{u, v\}$ is a TR2DF and so $\gamma_{tR2}(G) \leq 3$. Since G has exactly one vertex of degree $n - 1$, we deduce from Proposition 15 that $\gamma_{tR2}(G) \geq 3$ and the equality follows.

Now assume that $\Delta(G) \leq n-2$ and G is obtained from two disjoint graphs G_1 and G_2 such that $G_1 \in \{P_3, C_3\}$ and G_2 is any graph of order $n-3$ by adding edges between vertices of G_1 and G_2 in order that every vertex of G_2 has at least two neighbors in G_1. Then the function f defined on $V(G)$ by $f(u) = 1$ for every vertex $u \in V(G_1)$ and $f(v) = 0$ for all $v \in V(G_2)$ is a TR2DF of G. Hence $\gamma_{tR2}(G) \leq 3$, and the equality follows as above from Proposition 15.

Conversely, assume that $\gamma_{tR2}(G) = 3$. Then G has at most one vertex of degree $n-1$, for otherwise $\gamma_{tR2}(G) = 2$ (by Proposition 15). Let $f = (V_0, V_1, V_2)$ be a $\gamma_{tR2}(G)$-function. Since $\gamma_{tR2}(G) = |V_1| + 2|V_2| = 3$, then it must be either $|V_1| = |V_2| = 1$ or either $|V_1| = 3$ and $|V_2| = 0$. If $|V_1| = |V_2| = 1$, with $V_2 = \{u\}$ and $V_1 = \{v\}$, then $uv \in E(G)$ and every vertex in $V \backslash \{x, y\}$ must be adjacent to u, because f is a TR2DF. So u is the unique vertex of degree $n-1$. Now assume that $V_2 = \emptyset$ and $|V_1| = 3$, where $V_1 = \{u, v, w\}$. Since $G[V_1]$ has no isolated vertices, $G[V_1] \in \{P_3, C_3\}$. Moreover, every vertex in $V_0 = V \backslash \{u, v, w\}$ must be adjacent to at least two vertices of V_1. Clearly, if $G_1 = G[V_1]$ and $G_2 = G[V_0]$, then G is an ntc graph as described in the statement. ■

Theorem 17. *Let G be an ntc graph of order n. Then $\gamma_{tR2}(G) = n$ if and only if $G \in \{K_2, K_{1,2}\}$ or every vertex of G is either a leaf or a weak support vertex.*

Proof. Assume that $\gamma_{tR2}(G) = n$. Clearly, if $n \in \{2, 3\}$, then $G \in \{K_2, K_{1,2}\}$. Hence assume that $n \geq 4$. Suppose first that G has a vertex w which is neither a leaf nor a support vertex. Define the function f by $f(w) = 0$ and $f(x) = 1$ for all $x \in V(G) \backslash \{w\}$. Clearly f is a TR2DF on G with weight $n-1$, a contradiction. Thus, each vertex of G is either a leaf or a support vertex. Now suppose that G has a strong support vertex, say u. Let u_1 and u_2 be two leaves adjacent to u. Define the function f by $f(u) = 2$, $f(u_1) = f(u_2) = 0$ and $f(x) = 1$ otherwise. Since $n \geq 4$, f is clearly a TR2DF on G of weight $n-1$, a contradiction too. Therefore, every vertex of G is either a leaf or a weak support vertex as desired.

The converse is obvious. ■

5. BOUNDS

We present in this section some bounds on the total Roman {2}-domination number of ntc graphs in terms of the order, maximum and minimum degrees.

Proposition 18. *Let G be an ntc graph of order n with girth $g \geq 6$ and minimum degree $\delta \geq 2$. Then $\gamma_{tR2}(G) \leq n + 2 - (\Delta + \delta)$.*

Proof. Let u be a vertex of G of maximum degree and let v be any neighbour of u. Define the function f on $V(G)$ by $f(u) = 1, f(v) = 1$, $f(w) = 0$ for all $w \in N(\{u, v\}) \backslash \{u, v\}$ and $f(w) = 1$ otherwise. Since since $\delta \geq 2$ and $g \geq 6$, set $A = V(G)\backslash N[\{u,v\}]$ is non-empty and no vertex of A has two neighbors $N[\{u,v\}]$. Hence f is well defined and is a TR2DF of weight $2+n-(\Delta+\deg_G(v))$, and thus

$$\gamma_{tR2}(G) \leq 2+n-(\Delta+\deg_G(v)) \leq n+2-(\Delta+\delta). \qquad \blacksquare$$

The sharpness of the previous bound can be seen by considering the cycles C_6 and C_7. Moreover, to see that the condition $\delta \geq 2$ is essential in the statement of Proposition 18, consider the star $K_{1,n-1}$ with $n \geq 3$, where $\gamma_{tR2}(K_{1,n-1}) = 3 > n+2-(\Delta+\delta) = 2$.

Corollary 19. *Let G be an ntc graph of order n with girth $g \geq 6$ and minimum degree $\delta \geq 2$ such that $\gamma_{tR2}(G) = n+2-(\Delta+\delta)$. Then for every vertex u of maximum degree we have that $d(v) = \delta(G)$ for all $v \in N(u)$.*

Proposition 20. *Let G be an ntc graph. Then*

$$\gamma_{tR2}(G) \geq \left\lceil \frac{2n}{\Delta+1} \right\rceil.$$

If $\gamma_{tR2}(G) = \frac{2n}{\Delta+1}$, then $V_2 = \emptyset$ for all $\gamma_{tR2}(G)$-function $f = (V_0, V_1, V_2)$.

Proof. Let $f = (V_0, V_1, V_2)$ be a $\gamma_{tR2}(G)$-function and let us denote by $V_{02} = \{w \in V_0 : N(w) \cap V_2 \neq \emptyset\}$ and by $V_{01} = V_0 \backslash V_{02}$. Thus $V(G) = V_{01} \cup V_{02} \cup V_1 \cup V_2$. Since any vertex $v \in V_2$ must have at least one neighbor in $V_1 \cup V_2$, we deduce that for each $v \in V_2$, $|N(v) \cap V_{02}| \leq \Delta - 1$ and thus $|V_{02}| \leq (\Delta-1)|V_2|$. Analogously, $2|V_{01}| \leq (\Delta-1)|V_1|$, because each vertex in V_{01} must have at least two neighbors in V_1. Hence

$$\begin{aligned} n &= |V_{01}| + |V_{02}| + |V_1| + |V_2| \leq \frac{\Delta-1}{2}|V_1| + (\Delta-1)|V_2| + |V_1| + |V_2| \\ &= \frac{\Delta+1}{2}|V_1| + \Delta|V_2| \leq \frac{\Delta+1}{2}\left(|V_1| + 2|V_2|\right) = \frac{\Delta+1}{2}\gamma_{tR2}(G), \end{aligned}$$

which leads to the desired result. If $\gamma_{tR2}(G) = \frac{2n}{\Delta+1}$, then all the previous inequalities become equalities and hence $|V_2| = 0$. $\blacksquare$

The sharpness of the bound in Proposition 20 can be shown for cycles.

6. Total Roman {2}-Domination of Trees

In this section, we present a lower and upper bounds on the total Roman {2}-domination number of trees. We start with a simple observation.

Observation 21. *Let G be a graph without isolated vertices and $v \in V(G)$ a support vertex of G.*

- *For any total Roman {2}-dominating function f of G, $f(v) \geq 1$.*
- *If v is a strong support vertex, then there exists a $\gamma_{tR2}(G)$-function f such that $f(v) = 2$.*

Theorem 22. *Let T be a tree of order $n \geq 2$ with $\ell(T)$ leaves. Then*

$$\gamma_{tR2}(T) \geq \left\lceil \frac{2(n-\ell(T)+3)}{3} \right\rceil.$$

This bound is sharp for paths, stars and double stars.

Proof. The proof is by induction on n. Clearly for all nontrivial trees of order $n \leq 4$ we have $\gamma_{tR2}(T) > \left\lceil \frac{2(n-\ell(T)+3)}{3} \right\rceil$. For the inductive hypothesis, let $n \geq 5$ and assume that for every tree of order at least 2 and less than n the result is true. Let T be a tree of order n. If $\operatorname{diam}(T) = 2$, then T is a star, which yields $\gamma_{tR2}(T) = 3 = \left\lceil \frac{2(n-n+1+3)}{3} \right\rceil$. If $\operatorname{diam}(T) = 3$, then T is a double star and we have $\gamma_{tR2}(T) = 4 = \left\lceil \frac{2(n-n+2+3)}{3} \right\rceil$. Henceforth we can assume $\operatorname{diam}(T) \geq 4$. Let f be a $\gamma_{tR2}(T)$-function.

If T has a strong support vertex u with at least two leaves, say u_1 and u_2, then let $T' = T - u_1$. By Observation 21, $f(u) \geq 1$ and we may assume without loss of generality that $f(u_2) \geq f(u_1)$. Now the function f, restricted to T' is a TR2DF of T' and we deduce from the inductive hypothesis that

$$\gamma_{tR2}(T) = \omega(f) \geq \gamma_{tR2}(T') \geq \left\lceil \frac{2((n-1)-(\ell(T)-1)+3)}{3} \right\rceil = \left\lceil \frac{2(n-\ell(T)+3)}{3} \right\rceil.$$

Thus in the sequel, we can assume that T has no strong support vertex. Let $v_1 v_2 \cdots v_k$ be a diametral path in T and root T in v_k. Since T has no strong support vertex, any child of v_3 is a leaf or a support vertex of degree 2. We consider the following cases.

Case 1. $\deg_T(v_3) \geq 3$. First suppose v_3 is a support vertex. By Observation 21, we may assume $f(v_2) = f(v_3) = 2$. Let $T' = T - v_1$ and define $h : V(T') \to \{0, 1, 2\}$ by $h(v_2) = 1$ and $h(x) = f(x)$ for $x \in V(T') \setminus \{v_2\}$. Clearly h is a TR2DF of T'. It follows from the induction hypothesis that

$$\begin{aligned} \gamma_{tR2}(T) &= \omega(f) = \omega(h) + 1 \geq \gamma_{tR2}(T') + 1 \\ &\geq \left\lceil \tfrac{2((n-1)-\ell(T)+3)}{3} \right\rceil + 1 > \left\lceil \tfrac{2(n-\ell(T)+3)}{3} \right\rceil, \end{aligned}$$

as desired. Now suppose v_3 is not a support vertex. Assume u_2 is a child of v_3 and u_1 is a leaf adjacent to u_2. Clearly $f(u_1) + f(u_2) \geq 2$ and $f(v_1) + f(v_2) \geq 2$. Assume without loss of generality that $f(v_2) \geq f(u_2)$. Let $T' = T - \{u_1, u_2\}$. If

$f(v_3) \geq 1$ or $\deg(v_3) \geq 4$, then clearly the function f restricted to T' is a TR2DF of T and we conclude from the inductive hypothesis that

$$\begin{aligned}\gamma_{tR2}(T) &= \omega(f) = \omega(f|_{T'}) + 2 \geq \gamma_{tR2}(T') + 2 \\ &\geq \left\lceil \tfrac{2((n-2)-(\ell(T)-1)+3)}{3} \right\rceil + 2 > \left\lceil \tfrac{2(n-\ell(T)+3)}{3} \right\rceil,\end{aligned}$$

as desired. Hence assume that $f(v_3) = 0$ and $\deg(v_3) = 3$. Let $T' = T - \{u_1, u_2, v_1\}$. Then the function $g : V(T') \to \{0, 1, 2\}$ defined by $g(v_3) = 1$ and $g(x) = f(x)$ for $x \in V(T')\backslash\{v_3\}$, is a TR2DF of T' of weight $\gamma_{tR2}(T) - 2$. By the inductive hypothesis we have

$$\begin{aligned}\gamma_{tR2}(T) &= \omega(f) = \omega(g) + 2 \geq \gamma_{tR2}(T') + 2 \\ &\geq \left\lceil \tfrac{2((n-3)-(\ell(T)-1)+3)}{3} \right\rceil + 2 > \left\lceil \tfrac{2(n-\ell(T)+3)}{3} \right\rceil.\end{aligned}$$

Case 2. $\deg_T(v_3) = 2$. As above we have $f(v_1) + f(v_2) \geq 2$. If $f(v_3) \geq 1$, then the function $g : V(T - v_1) \to \{0, 1, 2\}$ defined by $g(v_2) = 1$ and $g(x) = f(x)$ for $x \in V(T')\backslash\{v_2\}$, is a TR2DF of $T - v_1$ of weight $\gamma_{tR2}(T) - 1$ and by the inductive hypothesis we obtain

$$\begin{aligned}\gamma_{tR2}(T) &= \omega(f) = \omega(g) + 1 \geq \gamma_{tR2}(T - v_1) + 1 \\ &\geq \left\lceil \tfrac{2((n-1)-\ell(T)+3)}{3} \right\rceil + 1 > \left\lceil \tfrac{2(n-\ell(T)+3)}{3} \right\rceil.\end{aligned}$$

Hence let $f(v_3) = 0$. If $f(v_1) + f(v_2) \geq 3$, then reassigning v_1, v_2, v_3 the value 1 provides a $\gamma_{tR2}(T)$-function f' for which $f'(v_3) \geq 1$, and this situation was considered above. Therefore, we can assume that $f(v_1) + f(v_2) = 2$. More precisely, $f(v_1) = f(v_2) = 1$. It follows that $f(v_4) \geq 1$. Let $T' = T - \{v_1, v_2, v_3\}$. Clearly T' is nontrivial since $\mathrm{diam}(T) \geq 4$. Now if T' has order 2, then T is a path P_5 and $\gamma_{tR2}(P_5) = 4 \geq \left\lceil \frac{2(n-\ell(T)+3)}{3} \right\rceil$. Hence suppose that T' has order at least three. Note that $\ell(T) - 1 \leq \ell(T') \leq \ell(T)$. Also, the function f restricted to T' is a T2RDF of T' of weight $\omega(f) - 2$. We deduce from the inductive hypothesis on T' that

$$\begin{aligned}\gamma_{tR2}(T) &= \omega(f) = \omega(f|_{T'}) + 2 \geq \gamma_{tR2}(T') + 2 \\ &\geq \left\lceil \tfrac{2((n-3)-\ell(T)+3)}{3} \right\rceil + 2 \geq \left\lceil \tfrac{2(n-\ell(T)+3)}{3} \right\rceil,\end{aligned}$$

which competes the proof. ■

Lemma 23. *If T is a tree obtained from a path $v_1 v_2 \cdots v_k$ $(k \geq 4)$ by adding a pendant path $v_{k-1}w$, then $\gamma_{tR2}(T) < \frac{2(k+3)}{3}$.*

Proof. If $k \equiv 0 \pmod 3$, then define the function f by $f(v_{3i+1}) = f(v_{3i+2}) = 1$ for $0 \leq i \leq \frac{k}{3} - 2$, $f(v_{k-2}) = 1$, $f(v_{k-1}) = 2$ and $f(v) = 0$ for any remaining vertex v. If $k \equiv 1 \pmod 3$, then define the function f by $f(v_{3i+1}) = f(v_{3i+2}) = 1$ for $0 \leq i \leq \frac{k-1}{3} - 1$, $f(v_{k-1}) = 2$ and $f(v) = 0$ for any remaining vertex v. If $k \equiv 2 \pmod 3$, then define the function f by $f(v_{3i+1}) = f(v_{3i+2}) = 1$ for $0 \leq i \leq \frac{k-2}{3} - 1$, $f(v_{k-2}) = 1$, $f(v_{k-1}) = 2$ and $f(v) = 0$ for any remaining vertex v. Clearly f is an TR2DF of weight smaller than $\frac{2(k+3)}{3}$. ∎

Theorem 24. *For every tree T of order $n(T) \geq 4$ with $s(T)$ support vertices,*

$$\gamma_{tR2}(T) \leq \frac{3n(T) + 2s(T)}{4}$$

with equality if and only if T is the corona of a tree.

Proof. If T is the corona of a tree T', then $\gamma_{tR2}(T) = n(T) = \frac{3n(T)+2s(T)}{4}$. To prove that if T is a tree of order $n(T) \geq 4$ with $s(T)$ support vertices, then $\gamma_{tR2}(T) \leq \frac{3n(T)+2s(T)}{4}$ with equality only if T is the corona of a tree, we proceed by induction on the order $n(T)$. If $n(T) = 4$, then T is either a star $K_{1,3}$, where $\gamma_{tR2}(K_{1,3}) = 3 < \frac{3n(T)+2s(T)}{4}$ or a path P_4 where $\gamma_{tR2}(P_4) = 4 = \frac{3n(T)+2s(T)}{4}$ and P_4 is the corona of the path P_2. Let $n(T) \geq 5$ and assume that every T' of order $n(T') < n(T)$ with $s(T')$ support vertices satisfies $\gamma_{tR2}(T') \leq \frac{3n(T')+2s(T')}{4}$ with equality only if T' is the corona of a tree. Let T be a tree of order $n(T)$. If T is a star, then $\gamma_{tR2}(T) = 3 < \frac{3n(T)+2s(T)}{4}$. Likewise, if T is a double star, then $\gamma_{tR2}(T) = 4 < \frac{3n(T)+2s(T)}{4}$ (since $n(T) \geq 5$). Henceforth, we can assume that T has diameter at least 4. Denote by T_x the subtree induced by a vertex x and its descendants in the rooted tree T.

If T has a strong support vertex u with at least three leaves, then let T' be the tree obtained from T by removing a leaf neighbor w of u. Let f be a $\gamma_{tR2}(T')$-function f such that $f(u) = 2$, $f(v) \geq 1$ for some $v \in N_{T'}(u)$. Clearly, f can be extended to TR2D-function of T by assigning a 0 to w, and thus $\gamma_{tR2}(T) \leq \gamma_{tR2}(T')$. Now using the induction on T' and the fact that $n(T') = n(T) - 1$ and $s(T') = s(T)$, we obtain the desired result. Henceforth, we can assume that every support vertex of T is adjacent to at most two leaves.

Let $v_1 v_2 \cdots v_k$ be a diametral path in T such that $\deg_T(v_2)$ is as large as possible and root T at v_k. Clearly $\deg_T(v_2) \in \{2, 3\}$. We consider the following cases.

Case 1. $\deg_T(v_2) = 3$. We distinguish the following subcases.

Subcase 1.1. $\deg_T(v_3) \geq 3$. If v_3 is a support vertex or v_3 has a child with degree 3 other than v_2, then any $\gamma_{tR2}(T - T_{v_2})$-function can be extended to a TR2D-function of T by assigning 2 to v_2 and 0 to the leaf neighbors of v_2 and so $\gamma_{tR2}(T) \leq \gamma_{tR2}(T - T_{v_2}) + 2$. Since $T - T_{v_2}$ is a tree of order at least four, by induction on $T - T_{v_2}$ and using the facts $n(T - T_{v_2}) = n(T) - 3$ and $s(T - T_{v_2}) = s(T) - 1$, we obtain $\gamma_{tR2}(T) \leq \gamma_{tR2}(T') + 2 \leq \frac{3n(T')+2s(T')}{4} + 2 < \frac{3n(T)+2s(T)}{4}$. Hence assume that every child of v_3 except v_2 is of degree 2. Let w_2 be a child of v_3 besides v_2 and let w_1 be the leaf neighbor of w_2. Clearly any $\gamma_{tR2}(T - T_{w_2})$-function can be extended to a TR2D-function of T by assigning 1 to w_1, w_2 and so $\gamma_{tR2}(T) \leq \gamma_{tR2}(T - T_{w_2}) + 2$. Note that $T - T_{w_2}$ is a tree of order at least four with $n(T - T_{w_2}) = n(T) - 2$ and $s(T - T_{w_2}) = s(T) - 1$. Using the induction on $T - T_{w_2}$, we obtain $\gamma_{tR2}(T) \leq \gamma_{tR2}(T') + 2 < \frac{3n(T')+2s(T')}{4} + 2 < \frac{3n(T)+2s(T)}{4}$.

Subcase 1.2. $\deg_T(v_3) = 2$ and $\deg_T(v_4) \geq 3$. If $T - T_{v_3} = P_3$, then clearly $\gamma_{tR2}(T) = 5 < \frac{(3n(T)+2s(T))}{4}$. Hence assume that $T - T_{v_3}$ has order at least four. Clearly any $\gamma_{tR2}(T - T_{v_3})$-function can be extended to a TR2D-function of T by assigning a 2 to v_2, a 1 to v_1 and a 0 to the leaves of v_2. It follows from the induction hypothesis on $T - T_{v_3}$ and the facts $n(T - T_{v_3}) = n - 4$ and $s(T - T_{v_3}) = s(T) - 1$ that

$$\begin{aligned}\gamma_{tR2}(T) &\leq \gamma_{tR2}(T - T_{v_2}) + 3 \leq \frac{3n(T - T_{v_3}) + 2s(T - T_{v_3})}{4} + 3 \\ &\leq \frac{3(n(T) - 4) + 2(s(T) - 1)}{4} + 3 < \frac{(3n(T) + 2s(T))}{4}.\end{aligned}$$

Subcase 1.3. $\deg_T(v_3) = 2$ and $\deg_T(v_4) = 2$. First let $\deg_T(v_5) \geq 3$. Hence $T - T_{v_4}$ has order at least three. If $T - T_{v_4} = P_3$, then it is easy to see that $\gamma_{tR2}(T) = 6 < \frac{3n(T)+2s(T)}{4}$. Thus let $T - T_{v_4} \neq P_3$. Then any $\gamma_{tR2}(T - T_{v_4})$-function can be extended to a TR2D-function of T by assigning a 2 to v_2 and v_3, and a 0 to other vertices in T_{v_4}. Using the induction hypothesis on $T - T_{v_4}$ and the facts $n(T - T_{v_4}) = n(T) - 5$ and $s(T - T_{v_4}) = s(T) - 1$ we obtain

$$\gamma_{tR2}(T) \leq \frac{3n(T - T_{v_4}) + 2s(T - T_{v_4})}{4} + 4 < \frac{3n(T) + 2s(T)}{4}.$$

Assume now that $\deg_T(v_5) = 2$. If $\deg(v_i) \leq 2$ for each $i \geq 5$, then the result follows from Lemma 23. Hence let t be the smallest integer such that $\deg(v_t) \geq 3$ for some $t \geq 6$. Let $T' = T - T_{v_{t-1}}$. Note that T' has order at least three. Suppose that $n(T') = 3$, that is $T' = P_3$. Then any $\gamma_{tR2}(T_{v_{t-1}})$-function as defined in Lemma 23 can be extended to a TR2DF of T by assigning a 2 to v_t and a 0 to other vertices of T', and clearly we have $\gamma_{tR2}(T) < \frac{3n(T)+2s(T)}{4}$. Suppose now that $n(T') \geq 4$. If $t \equiv 1 \pmod 3$, then any $\gamma_{tR2}(T - T_{v_{t-1}})$-function can be extended to a TR2D-function of T by assigning a 2 to v_2, a 1 to v_3, v_{3i+2}, v_{3i+3} for $1 \leq i \leq \frac{t-1}{3} - 1$ and a 0 to the remaining vertices of $T_{v_{t-1}}$. Using the induction on $T - T_{v_{t-1}}$ and the fact $\frac{2(t-1)}{3}$ can be rewritten $\frac{3(t-1)}{4} - \frac{t-1}{12}$, we have

$$\begin{aligned}\gamma_{tR2}(T) &\le \gamma_{tR2}(T - T_{v_{t-1}}) + \frac{2(t-1)}{3} + 1\\ &\le \frac{3n(T - T_{v_4}) + 2s(T - T_{v_4})}{4} + \frac{3(t-1)}{4} - \frac{t-1}{12} + 1\\ &= \frac{3(n(T) - t) + 2(s(T) - 1)}{4} + \frac{3(t-1)}{4} - \frac{t-1}{12} + 1 < \frac{3n(T) + 2s(T)}{4}.\end{aligned}$$

Assume now that $t \equiv 2 \pmod 3$. Then any $\gamma_{tR2}(T - T_{v_{t-1}})$-function can be extended to a TR2D-function of T by assigning a 2 to v_2, a 1 to v_{3i}, v_{3i+1} for $1 \le i \le \frac{t-2}{3}$ and a 0 to the remaining vertices of $T_{v_{t-1}}$. By the induction hypothesis on $T - T_{v_{t-1}}$ we obtain

$$\begin{aligned}\gamma_{tR2}(T) &\le \gamma_{tR2}(T - T_{v_{t-1}}) + \frac{2(t-2)}{3} + 2\\ &\le \frac{3n(T - T_{v_4}) + 2s(T - T_{v_4})}{4} + \frac{3(t-2)}{4} - \frac{t-2}{12} + 2\\ &= \frac{3(n(T) - t) + 2(s(T) - 1)}{4} + \frac{3(t-2)}{4} - \frac{t-2}{12} + 2 < \frac{3n(T) + 2s(T)}{4}.\end{aligned}$$

Finally, assume that $t \equiv 0 \pmod 3$. Then any $\gamma_{tR2}(T - T_{v_{t-1}})$-function can be extended to a TR2D-function of T by assigning a 2 to v_2, a 1 to v_3, v_{3i+1}, v_{3i+2} for $1 \le i \le \frac{t}{3} - 1$ and a 0 to the remaining vertices of $T_{v_{t-1}}$. By the induction hypothesis on $T - T_{v_{t-1}}$ we have

$$\begin{aligned}\gamma_{tR2}(T) &\le \gamma_{tR2}(T - T_{v_{t-1}}) + \frac{2t}{3} + 1\\ &\le \frac{3(n(T) - t) + 2(s(T) - 1)}{4} + \frac{3t}{4} - \frac{t}{12} + 1\\ &= \frac{3n(T) + 2s(T)}{4} + \frac{6-t}{12} \le \frac{3n(T) + 2s(T)}{4}.\end{aligned}$$

If further $\gamma_{tR2}(T) = \frac{3n(T)+2s(T)}{4}$, then we have equality throughout the previous inequality chain. In particular, we have $t = 6$ and $\gamma_{tR2}(T - T_{v_{t-1}}) = \frac{3(n(T)-t)+2(s(T)-1)}{4}$. It follows from the induction on $T - T_{v_{t-1}}$ that $T - T_{v_{t-1}}$ is the corona of a tree and v_6 is support vertex (since $\deg_T(v_6) \ge 3$). It follows that for any $\gamma_{tR2}(T - T_{v_{t-1}})$-function g, $g(v_6) \ge 1$ and clearly g can be extended to a TR2D-function of T by assigning a 2 to v_2, a 1 to v_3, v_5 and a 0 to other vertices in T_{v_5}. By the induction hypothesis we obtain $\gamma_{tR2}(T) \le \gamma_{tR2}(T - T_{v_{t-1}}) + 4 < \frac{3n(T)+2s(T)}{4}$.

Case 2. $\deg_T(v_2) = 2$. By the choice of the diametral path, we deduce that every child of v_3 with depth one has degree two. Consider the following subcases.

Subcase 2.1. $\deg_T(v_3) \geq 3$. Suppose first that v_3 is a strong support vertex, and let u, w be two leaves of v_3. Let $T' = T - \{u, v_1, v_2\}$. Clearly T' is a tree of order $n(T') = n(T) - 3 \geq 4$ with $s(T') = s(T) - 1$ support vertices. Let g be a $\gamma_{tR2}(T')$-function. Then we extend g to a TR2D-function of T by assigning a 1 to v_1, v_2 and a 0 to u. In addition if $g(v_3) \neq 2$, then we reassign v_3 and w the values 2 and 0 instead of 1 to both. Now using the induction hypothesis on T', we get

$$\begin{aligned}\gamma_{tR2}(T) &\leq \gamma_{tR2}(T') + 2 \leq \frac{3n(T') + 2s(T')}{4} + 2 \\ &= \frac{3(n(T) - 3) + 2(s(T) - 1)}{4} + 2 < \frac{3n(T) + 2s(T)}{4}.\end{aligned}$$

Now, suppose that v_3 is not support vertex. Recall that every child of v_3 is a support vertex of degree two. Let $T' = T - T_{v_3}$. Clearly T_{v_3} has order $2\deg_T(v_3) - 1$ and T' has order $n(T') \geq 2$ (since $\mathrm{diam}(T) \geq 4$). If $n(T') = 2$, then $\gamma_{tR2}(T) = 2\deg_T(v_3) < \frac{3n(T)+2s(T)}{4}$, and if $n(T') = 3$, then $\gamma_{tR2}(T) = 2\deg_T(v_3) + 1 < \frac{3n(T)+2s(T)}{4}$. Hence we assume that $n(T') \geq 4$, and thus by induction on T', $\gamma_{tR2}(T') \leq \frac{3n(T')+2s(T')}{4}$. Since any $\gamma_{tR2}(T')$-function can be extended to a TR2D-function of T by assigning a 0 to v_3 and a 1 to each of the remaining vertices of T_{v_3}, $\gamma_{tR2}(T) \leq \gamma_{tR2}(T') + 2(\deg_T(v_3) - 1)$. Using the fact that $s(T') \leq s(T) - \deg_T(v_3) + 2$, we obtain

$$\begin{aligned}\gamma_{tR2}(T) &\leq \gamma_{tR2}(T') + 2(\deg_T(v_3) - 1) = \frac{3n(T') + 2s(T')}{4} + 2(\deg_T(v_3) - 1) \\ &\leq \frac{3(n(T) - 2\deg_T(v_3) + 1) + 2(s(T) - \deg_T(v_3) + 2)}{4} + 2(\deg_T(v_3) - 1) \\ &< \frac{3n(T) + 2s(T)}{4}.\end{aligned}$$

Next we can assume that v_3 is a support vertex with $\deg_T(v_3) = 3$. Let $T' = T - \{v_1, v_2\}$. As above we can easily see that

$$\gamma_{tR2}(T) \leq \gamma_{tR2}(T') + 2 \leq \frac{3n(T') + 2s(T')}{4} + 2 \leq \frac{3n(T) + 2s(T)}{4}.$$

If further $\gamma_{tR2}(T) = \frac{3n(T)+2s(T)}{4}$, then we have equality throughout the previous inequality chain. In particular, $\gamma_{tR2}(T - \{v_1, v_2\}) = \frac{3(n(T)-2)+2(s(T)-1)}{4}$. It follows from the induction on $T - \{v_1, v_2\}$ that $T - \{v_1, v_2\}$ is the corona of some tree, implying that T is the corona of a tree.

Subcase 2.2. $\deg_T(v_3) = 2$ and $\deg_T(v_4) \geq 3$. If $T' = T - T_{v_3} = P_3$, then clearly $\gamma_{tR2}(T) = 5 < \frac{3n(T)+2s(T)}{4}$. Hence assume that $T' \neq P_3$. If v_4 is support vertex or has a child with depth 1 and degree at least 3, then clearly there exists a $\gamma_{tR2}(T')$-function that assigns a non-zero positive value to v_4 and such a $\gamma_{tR2}(T')$-function can be extended to a TR2D-function of T by assigning a 1 to v_1, v_2 and a 0 to v_3. It follows from the induction hypothesis on T' that

$$\gamma_{tR2}(T) \leq \gamma_{tR2}(T') + 2 \leq \frac{3n(T') + 2s(T')}{4} + 2 < \frac{3n(T) + 2s(T)}{4}.$$

Now let v_4 have child w_2 with depth **1** and degree two, and let w_1 be the leaf neighbor of w_2. Let $T' = T - \{w_2, w_1\}$. Clearly, $\gamma_{tR2}(T) \leq \gamma_{tR2}(T') + 2$. By the inductive hypothesis on T' and since T' is not a corona, $\gamma_{tR2}(T') < \frac{3n(T')+2s(T')}{4}$. Using the facts that $n(T') = n(T) - 2$ and $s(T') = s(T) - 1$ we obtain

$$\gamma_{tR2}(T) \leq \gamma_{tR2}(T') + 2 < \frac{3n(T') + 2s(T')}{4} + 2 \leq \frac{3n(T) + 2s(T)}{4}.$$

Henceforth we assume that any child of v_4 is of depth 2. Thus T_{v_4} is a tree obtain from a star by subdividing every edge twice. Let $w_1^i w_2^i w_3^i v_4$ be paths in T where w_3^i is a child of v_4 for each $i \in \{1, 2, \ldots, t\}$ and $w_3^1 = v_3$. If $t \geq 3$, then any $\gamma_{tR2}(T - T_{v_4})$-function can be extended to a TR2D-function of T by assigning 1 to $v_1, v_2, v_3, v_4, w_2^i, w_1^i$ for $i \geq 2$. Now we deduce from the induction hypothesis on T' and the facts $n(T') = n(T) - 3t - 1$ and $s(T') \leq s(T) - t + 1$ that

$$\begin{aligned}\gamma_{tR2}(T) &\leq \gamma_{tR2}(T') + 2t + 2 \leq \frac{3(n(T) - 3t - 1) + 2(s(T) - t + 1)}{4} \\ &\quad + 2t + 2 < \frac{3n(T) + 2s(T)}{4}.\end{aligned}$$

Hence assume that $t = 2$. If $\deg(v_5) \geq 3$, then let $T' = T - T_{v_4}$. Then $\gamma_{tR2}(T) \leq \gamma_{tR2}(T - T_{v_4}) + 6$. By the induction hypothesis on T' and the facts $n(T - T_{v_4}) = n(T) - 7$ and $s(T - T_{v_4}) = s(T) - 2$ we obtain

$$\gamma_{tR2}(T) \leq \gamma_{tR2}(T') + 6 \leq \frac{3(n(T) - 7) + 2(s(T) - 2)}{4} + 6 < \frac{3n(T) + 2s(T)}{4}.$$

Thus let $\deg(v_5) = 2$ and let $T' = T - T_{v_5}$. Note that T' has order $n(T') \geq 2$. If $n(T') \in \{2, 3\}$, then one can check that $\gamma_{tR2}(T) < \frac{3n(T)+2s(T)}{4}$. Hence we assume that $n(T') \geq 4$. Then $\gamma_{tR2}(T) \leq \gamma_{tR2}(T') + 6$. It follows from the induction hypothesis on $T - T_{v_5}$ and the facts $n(T') = n(T) - 8$ and $s(T') \leq s(T) - 1$ that

$$\gamma_{tR2}(T) \leq \gamma_{tR2}(T') + 6 \leq \frac{3(n(T) - 8) + 2(s(T) - 1)}{4} + 6 < \frac{3n(T) + 2s(T)}{4}.$$

Subcase 2.3. $\deg_T(v_3) = \deg(v_4) = 2$. First let $\deg_T(v_5) \geq 3$. If $T' = T - T_{v_4} = P_3$, then it is easy to see that $\gamma_{tR2}(T) = 5 < \frac{3n(T)+2s(T)}{4}$. Hence assume that

$T' \neq P_3$. If v_5 is a support vertex, then v_5 is assigned a non-zero positive value under any $\gamma_{tR2}(T')$-set and thus one can easily see that $\gamma_{tR2}(T) \leq \gamma_{tR2}(T') + 3$. Using the induction hypothesis on T' and the facts $n(T') = n(T) - 4$ and $s(T') = s(T) - 1$ we obtain

$$\gamma_{tR2}(T) \leq \frac{3n(T') + 2s(T')}{4} + 3 < \frac{3n(T) + 2s(T)}{4}.$$

If v_5 has child w with depth one, then since there is a $\gamma_{tR2}(T - T_w)$-function that assigns a non-zero positive value to v_5, such a $\gamma_{tR2}(T - T_w)$-function can be extended to a TR2D-function of T by assigning a 2 to w and 0 to other vertices in T_w. By the inductive hypothesis on $T - T_w$ and since $T - T_w$ is not a corona, $\gamma_{tR2}(T - T_w) < \frac{3n(T-T_w)+2s(T-T_w)}{4}$. Moreover, we have $n(T - T_w) \leq n(T) - 2$ and $s(T - T_w) = n(T) - 1$, and thus

$$\begin{aligned}\gamma_{tR2}(T) &\leq \gamma_{tR2}(T - T_w) + 2 < \frac{3n(T - T_w) + 2s(T - T_w)}{4} + 2 \\ &\leq \frac{3(n(T) - 2) + 2(s(T) - 1)}{4} + 2 = \frac{3n(T) + 2s(T)}{4}.\end{aligned}$$

Suppose now that v_5 has child w with depth two. Let w have t_3 leaves, t_2 children with depth one and degree at least three and t_1 children with depth one and degree two. Let $T' = T - T_w$. Then any $\gamma_{tR2}(T')$-function can be extended to a TR2D-function of T by assigning a 2 to every child of w with depth one, $1 + t$ to w and 0 to other vertices in T_w, where $t = 0$ if $t_3 = 0$ and $t = 1$ if $t_3 \geq 1$. Clearly by the inductive hypothesis on T' and since T' is not a corona, $\gamma_{tR2}(T') < \frac{3n(T')+2s(T')}{4}$. Moreover, we know that $n(T') \leq n(T) - 3t_2 - 2t_1 - t_3 - 1$ and $s(T') = s(T) - t_1 - t_2 - t$. Now

- Assume that $t_2 \neq 0$ or $t_3 \neq 0$. Then we have

$$\begin{aligned}\gamma_{tR2}(T) &\leq \gamma_{tR2}(T') + 2t_2 + 2t_1 + 1 + t < \frac{3n(T') + 2s(T')}{4} + 2t_2 + 2t_1 + 1 + t \\ &\leq \frac{3(n(T) - 3t_2 - 2t_1 - t_3 - 1) + 2(s(T) - t_1 - t_2 - t)}{4} + 2t_2 + 2t_1 + 1 + t \\ &\leq \frac{3n(T) + 2s(T)}{4}.\end{aligned}$$

- Assume that $t_2 = 0$ and $t_3 = 0$. Thus $t_3 \geq 1$. Using the fact that there is a $\gamma_{tR2}(T')$-function that assigns a non-zero positive value to v_5, clearly then such a $\gamma_{tR2}(T')$-function can be extended to a TR2D-function of T by assigning a 0 to w and 1 to the remaining vertices of T_w. It follows that

$$\begin{aligned}\gamma_{tR2}(T) &\leq \gamma_{tR2}(T') + 2t_1 < \frac{3n(T') + 2s(T')}{4} + 2t_1 \\ &\leq \frac{3(n(T) - 2t_1 - 1) + 2(s(T) - t_1)}{4} + 2t_1 < \frac{3n(T) + 2s(T)}{4}.\end{aligned}$$

Assume that v_5 has child with depth three and let $w_1w_2w_3w_4v_5$ be a path in T where w_4 is a child of v_5 different from v_4. Considering the above cases and subcases we may assume that $\deg(w_i) = 2$ for $i \in \{1, 2, 3, 4\}$. Clearly $T - T_{w_4}$ has a $\gamma_{tR2}(T - T_{w_4})$-function f such that $f(v_5) \geq 1$, and f can be extended to a TR2D-function of T by assigning a 1 to w_3, w_2, w_1 and 0 to v_4. Using the induction hypothesis on $T - T_{w_4}$ we obtain

$$\gamma_{tR2}(T) \leq \gamma_{tR2}(T - T_{w_4}) + 3 < \frac{3n(T - T_{w_4}) + 2s(T - T_{w_4})}{4} + 3 < \frac{3n(T) + 2s(T)}{4}.$$

Finally, assume that $\deg_T(v_5) = 2$.

Let f be $\gamma_{tR2}(T - T_{v_3})$-function such that $f(v_4)$ is as large as possible. It is easy to see that $f(v_4) \geq 1$ and f can be extended to a TR2D-function of T by assigning a 1 to v_1, v_2 and a 0 to v_3. Using the induction hypothesis on $T - T_{v_3}$ we obtain

$$\gamma_{tR2}(T) \leq \gamma_{tR2}(T - T_{v_3}) + 2 \leq \frac{3n(T - T_{v_3}) + 2s(T)}{4} + 2 < \frac{3n(T) + 2s(T)}{4}. \blacksquare$$

We conclude this section with two open problems.

Problem 1. Is the problem of deciding whether $\gamma_{tR2}(G) = 3\gamma(G)$ for a given graph G NP-hard.

Problem 2. Characterize all graphs G such that $\gamma_{tR2}(G) = 3\gamma(G)$.

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