

SEPARATION OF CARTESIAN PRODUCTS OF GRAPHS INTO SEVERAL CONNECTED COMPONENTS BY THE REMOVAL OF VERTICES

TJAŠA PAJ ERKER AND SIMON ŠPACAPAN¹

University of Maribor
FME
Smetanova 17

e-mail: tjasa.paj@um.si
simon.spacapan@um.si

Abstract

A set $S \subseteq V(G)$ is a vertex k -cut in a graph $G = (V(G), E(G))$ if $G - S$ has at least k connected components. The k -connectivity of G , denoted as $\kappa_k(G)$, is the minimum cardinality of a vertex k -cut in G . We give several constructions of a set S such that $(G \square H) - S$ has at least three connected components. Then we prove that for any 2-connected graphs G and H , of order at least six, one of the defined sets S is a minimum vertex 3-cut in $G \square H$. This yields a formula for $\kappa_3(G \square H)$.

Keywords: k -connectivity, Cartesian product.

2010 Mathematics Subject Classification: 05C40, 05C76.

1. INTRODUCTION

Graph connectivity is one of the most fundamental concepts in graph theory. It has been studied for several classes of graphs, including graph products. The objective of this study is to express connectivity of the product in terms of connectivities and other invariants of factors.

The study of connectivity of graph products started with Sabidussi who proved in [7] that vertex connectivity of $G \square H$ is bounded by the sum of connectivities of G and H , expressed by this inequality

$$\kappa(G \square H) \geq \kappa(G) + \kappa(H).$$

¹The author is supported by research grants P1-0297, J1-9109 of Ministry of Education of Slovenia.

Eventually the vertex connectivity of Cartesian products was settled by the following equation

$$\kappa(G \square H) = \min\{\kappa(G)|V(H)|, \kappa(H)|V(G)|, \delta(G) + \delta(H)\},$$

which was first proved in [11]. An alternative proof of the above result was later given in [5, 12] and [13], where a generalization to digraphs is obtained. A characterization of super-connected Cartesian products is also given in [12] (a graph G is *super-connected* if every minimum separating set in G is the neighborhood of a vertex in G).

Since vertex connectivity $\kappa(G)$ does not provide any information about the number of connected components that may occur after removing $\kappa(G)$ vertices, another generalization of classical notion of connectivity was proposed in [2]. In this generalization we ask for the number of vertices that need to be deleted to obtain a graph with k connected components. A set $S \subseteq V(G)$ is a *vertex k -cut* in G if $G - S$ (the graph obtained from G by removing vertices in S) has at least k connected components. Note that a vertex k -cut in G exists if and only if $k \leq \alpha(G)$, where $\alpha(G)$ denotes the independence number of G . For $k \leq \alpha(G)$, the *k -connectivity*, denoted as $\kappa_k(G)$, is the minimum size of a vertex k -cut in G . For $k > \alpha(G)$ we define $\kappa_k(G)$ to be the minimum size of a set S such that $G - S$ has less than k vertices, that is $\kappa_k(G) = |V(G)| - k + 1$ (this part of the definition is only a formality, and is not of particular interest). A graph G is *(n, k) -connected* if $\kappa_k(G) \geq n$. The parameter $\kappa_k(G)$ is closely related to *toughness* of a graph $\tau(G)$, defined as

$$\tau(G) = \min \left\{ \frac{\kappa_k(G)}{k}; 2 \leq k \leq \alpha(G) \right\},$$

since knowing $\kappa_k(G)$ for all $k \leq \alpha(G)$ immediately gives $\tau(G)$. The toughness of a graph (and hence also the k -connectivity) plays the central role in many results and conjectures, such as the following famous conjecture of Chvátal (see [3] and [1]).

Conjecture 1. *There exists a $k \in \mathbb{R}$, such that every graph G with $\tau(G) > k$ is Hamiltonian.*

The k -connectivity of graphs was studied in articles [2, 4, 6, 8, 9, 10]. In [2] and [6] the authors give sufficient conditions for a graph to be (n, k) -connected. It is proved in [2] that every graph with sufficiently large minimum degree is (n, k) -connected. This result is improved in [6], where it is shown that a graph is (n, k) -connected if its vertices satisfy some special degree conditions. In [10] graphs satisfying $\kappa_k(G) = t$ are studied, and the maximum and minimum sizes of such graphs are determined, where the sizes are given in terms of n (the order of the graph), k and t .

The 3-connectivity of Cartesian products of graphs was addressed in [8] where several upper bounds for $\kappa_3(G \square H)$ are given. These bounds are obtained by constructions of various vertex 3-cuts in the product. Also, exact values of $\kappa_3(G \square H)$ for products of several classes of graphs are determined — for products of complete graphs, products of paths with cycles, and products of complete graphs with trees.

In this article we give a characterization of 3-connectivity of Cartesian products of graphs. First, in Section 3, we describe several types of vertex 3-cuts in the product (they are shown in Figure 2). Then, in Section 4, we prove Theorem 10, which asserts that a minimum vertex 3-cut in $G \square H$ can always be obtained by one of the constructions described in Section 3. This gives a formula for $\kappa_3(G \square H)$ for products of any graphs G and H , except in some special cases when one of the factors is a complete graph, or it has a cut-vertex.

2. NOTATION

For an $x \in V(G)$ the set

$$N_G(x) = \{y \in V(G); xy \in E(G)\}$$

is the *neighborhood* of x in G , and $N_G[x] = N(x) \cup \{x\}$ is the *closed neighbourhood* of x in G . We write $N(x)$ instead of $N_G(x)$, when G is clear from the context. For $X \subseteq V(G)$ we define the open neighbourhood of X as

$$N_G(X) = \bigcup_{x \in X} N_G(x) \setminus X.$$

A set $S \subseteq V(G)$ is a *separating set* or a *vertex cut* in G , if $G - S$ (the graph obtained from G by deleting all vertices in S) is not connected. For a graph G the *vertex connectivity* of G , denoted as $\kappa(G)$, is the minimum cardinality of a set $S \subseteq V(G)$ such that $G - S$ is not connected or has only one vertex. A separating set S is a *vertex 3-cut* in G if $G - S$ has at least 3 connected components. The *3-connectivity* of a graph G , denoted as $\kappa_3(G)$, is the size of a smallest vertex 3-cut in G , if a 3-cut in G exists, otherwise $\kappa_3(G) = |V(G)| - 2$. Note that $\kappa_3(G) \geq \kappa(G)$.

Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be graphs. The *Cartesian product* of G and H is the graph $G \square H$ defined by $V(G \square H) = V(G) \times V(H)$, where $(x_1, y_1)(x_2, y_2)$ is an edge in $G \square H$ if $x_1 = x_2$ and $y_1 y_2 \in E(H)$, or $x_1 x_2 \in E(G)$ and $y_1 = y_2$. For an $y \in V(H)$ the *G-layer* G_y is the set

$$G_y = \{(x, y); x \in V(G)\}.$$

Analogously, for an $x \in V(G)$ the H -layer H_x is the set

$$H_x = \{(x, y); y \in V(H)\}.$$

For $y_1, y_2 \in V(H)$ we say that G_{y_1} and G_{y_2} are *adjacent layers* if $y_1 y_2 \in E(H)$. We denote by $p_G : V(G \square H) \rightarrow V(G)$ the *projection* of $V(G \square H)$ to $V(G)$. The projection is given by $p_G(x, y) = x$. Similarly $p_H(x, y) = y$ is the projection of $V(G \square H)$ to $V(H)$. For $(x, y) \in V(G \square H)$ we define the G -neighbourhood and the H -neighbourhood of (x, y) in $G \square H$ as

$$N^G(x, y) = N(x) \times \{y\} \text{ and } N^H(x, y) = \{x\} \times N(y).$$

For $S \subseteq V(G \square H)$ let

$$N^G(S) = \bigcup_{(x,y) \in S} N^G(x, y) \setminus S \text{ and } N^H(S) = \bigcup_{(x,y) \in S} N^H(x, y) \setminus S.$$

Let S be a vertex 3-cut in $G \square H$, and let C'_1, \dots, C'_n be connected components of $(G \square H) - S$. We define

$$(1) \quad C_1 = C'_1, C_2 = C'_2 \text{ and } C_3 = \bigcup_{k=3}^n C'_k,$$

and we use this notation throughout the rest of this article. For an $x \in V(G)$ and $y \in V(H)$, we define

$$S_x = S \cap H_x \text{ and } S_y = S \cap G_y.$$

We say that layer G_y is of *type* $(1, 2, 3)$ if $C_i \cap G_y \neq \emptyset$ for all $i \in [3]$. We say, for example, that G_y is of type $(1, 3)$ if $C_1 \cap G_y \neq \emptyset, C_3 \cap G_y \neq \emptyset$ and $C_2 \cap G_y = \emptyset$. Similar terminology is used for all types of layers, where the type of the layer is determined by components C_i that intersect this layer. Layers G_{y_1} and G_{y_2} are of *different types*, if the type of G_{y_1} is not equal to the type of G_{y_2} .

G_{y_2}	C_1	S_{y_2}		C_2	C_3
G_{y_1}	C_1		S_{y_1}	C_2	C_3

Figure 1. Compatible layers G_{y_1} and G_{y_2} .

We say that layers G_{y_1} and G_{y_2} are *compatible* if

$$p_G(C_i \cap G_{y_1}) \cap p_G(C_j \cap G_{y_2}) = \emptyset$$

for all $i, j \in [3], i \neq j$ (see Figure 1). Note that any two adjacent layers in $(G \square H) - S$ are compatible. If G_{y_1} is compatible with G_{y_2} we write $G_{y_1} \sim G_{y_2}$.

3. CONSTRUCTIONS OF VERTEX 3-CUTS

There are several natural ways to remove some vertices of the product $G \square H$ to get (at least) three connected components. We shall describe them in this section.

If $S \subseteq V(G)$ is a vertex 3-cut with $|S| = \kappa_3(G)$, then $S \times V(H)$ is a vertex 3-cut in $G \square H$. Since $|S \times V(H)| = \kappa_3(G) |V(H)|$ we find that $\kappa_3(G \square H) \leq \kappa_3(G) |V(H)|$, if G has a vertex 3-cut. If there is no vertex 3-cut in G , then $\kappa_3(G) = |V(G)| - 2$. If also $|V(G)| \geq 6$ and $V(H) \geq 3$, then let

$$C = \{(x_1, y_1), (x_2, y_1), (x_3, y_2), (x_4, y_2)\} \cup (\{x_5, x_6\} \times (V(H) \setminus \{y_1, y_2\})),$$

where $x_1, \dots, x_6 \in V(G)$, and $y_1, y_2 \in V(H)$ are arbitrary vertices. Since the graph induced by C has at least three connected components, we find that $S = V(G \square H) - C$ is a vertex 3-cut in $G \square H$. Clearly, $|S| = \kappa_3(G) |V(H)|$. Hence, if G and H both have at least 6 vertices, then $\kappa_3(G \square H) \leq \kappa_3(G) |V(H)|$.

Analogously we argue that $\kappa_3(G \square H) \leq \kappa_3(H) |V(G)|$ if G and H have at least 6 vertices. A vertex 3-cut of the form $S \times V(H)$ or $V(G) \times S$ is called a *type 1*, respectively, *type 2* vertex 3-cut (see Figure 2).

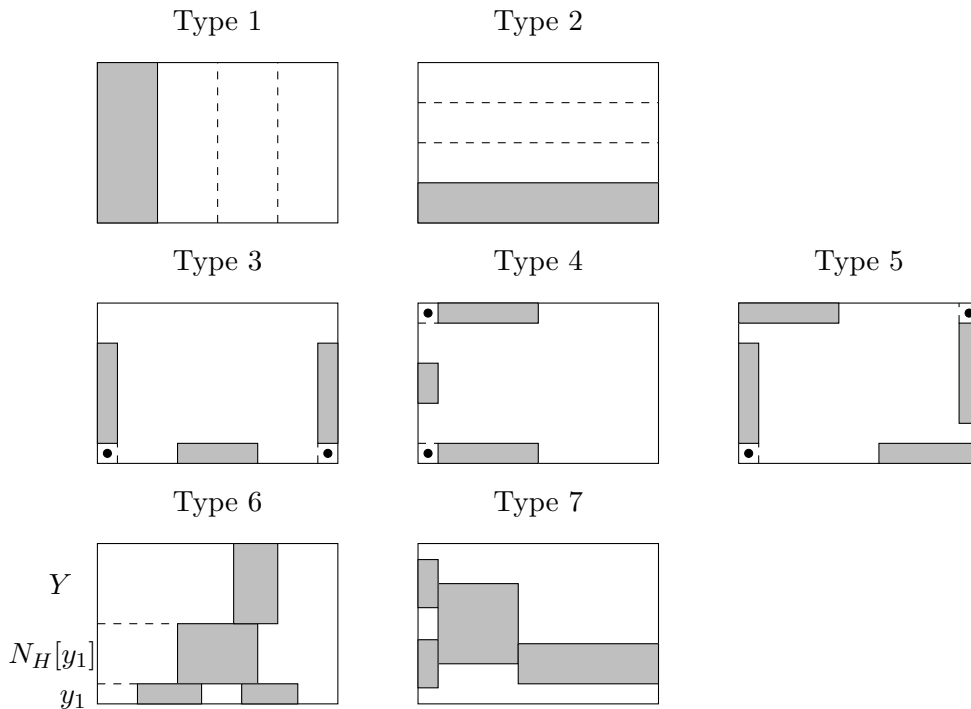


Figure 2. Graph G is the horizontal axis, and H is the vertical axis. In each type the vertex 3-cut S is denoted by dark color. Dots denote isolated vertices of $(G \square H) - S$.

When we isolate two nonadjacent vertices of $G \square H$ we obtain a graph with three connected components, as noted in the remark below.

Remark 2. Let G and H be graphs such that $|V(G)|, |V(H)| \geq 3$, and let $(x_1, y_1), (x_2, y_2) \in V(G \square H)$. If $x_1 x_2 \notin E(G)$ and $y_1 = y_2$, or $y_1 y_2 \notin E(G)$ and $x_1 = x_2$, or $x_1 \neq x_2$ and $y_1 \neq y_2$, then

$$S = N_{G \square H}(x_1, y_1) \cup N_{G \square H}(x_2, y_2)$$

is a vertex 3-cut in $G \square H$.

We now discuss the minimum size of $N_{G \square H}(x_1, y_1) \cup N_{G \square H}(x_2, y_2)$ and describe it in terms of invariants of factors. For a graph G which is not complete let

$$\delta_2(G) = \min \{|N(x_1) \cup N(x_2)|; x_1, x_2 \in V(G), x_1 x_2 \notin E(G)\}.$$

Observe that $\delta_2(G)$ is the minimum number of vertices whose removal isolates two vertices of G , and note that $\kappa_3(G) \leq \delta_2(G) \leq |V(G)| - 2$. This can also be applied to products, hence $\kappa_3(G \square H) \leq \delta_2(G \square H)$.

If we choose x_1 and x_2 in remark 2 so that $|N_G(x_1) \cup N_G(x_2)| = \delta_2(G)$, and y_1, y_2 so that $y_1 = y_2$ and $\deg(y_1) = \delta(H)$, then we find that $|S| = \delta_2(G) + 2\delta(H)$. Hence, $\kappa_3(G \square H) \leq \delta_2(G \square H) \leq \delta_2(G) + 2\delta(H)$. Analogously we have $\kappa_3(G \square H) \leq \delta_2(H) + 2\delta(G)$. It follows that

$$\kappa_3(G \square H) \leq \delta_2(G \square H) \leq \min\{\delta_2(G) + 2\delta(H), \delta_2(H) + 2\delta(G)\},$$

if G and H are not complete graphs.

A vertex 3-cut that isolates two vertices of $G \square H$ is called a *type 3* or *type 4* or *type 5* vertex 3-cut (see Figure 2 where isolated vertices are denoted by black dots).

The 5 types of vertex 3-cuts we described so far are in one way or another obtained from a vertex 3-cut of a factor. It turns out that for some pairs of graphs G and H , no minimum vertex 3-cut in $G \square H$ is obtained from a factor, or at least its size cannot be straightforwardly described in terms of invariants of factors. These types of vertex 3-cuts are type 6 and type 7 vertex 3-cuts, described below. Let $\omega(G, H)$ be the minimum size of a vertex 3-cut S of the form

$$(2) \quad S = (S_1 \times \{y_1\}) \cup (S_2 \times N_H(y_1)) \cup (S_3 \times Y),$$

where $S_1, S_2, S_3 \subseteq V(G)$, $y_1 \in V(H)$, and $Y \subseteq \overline{N_H[y_1]}$ (see type 6 in Figure 2). If for nonadjacent vertices x_1 and x_2 in G we choose $S_1 = V(G) \setminus \{x_1, x_2\}$ and $S_2 = S_3 = \{x_1, x_2\}$, we get a vertex 3-cut of the form (2). So it follows, that $\omega(G, H)$ is well defined, if G is not complete. We define

$$D(G, H) = \min\{\omega(G, H), \omega(H, G)\}.$$

It follows straightforward from the definition of $D(G, H)$ that $\kappa_3(G \square H) \leq D(G, H)$, if G and H are not complete. We will call a vertex 3-cut of the form (2) a vertex 3-cut of *type 6*. Note that a special case of a vertex 3-cut of the form (2) is the case when $S_2 = S_3$.

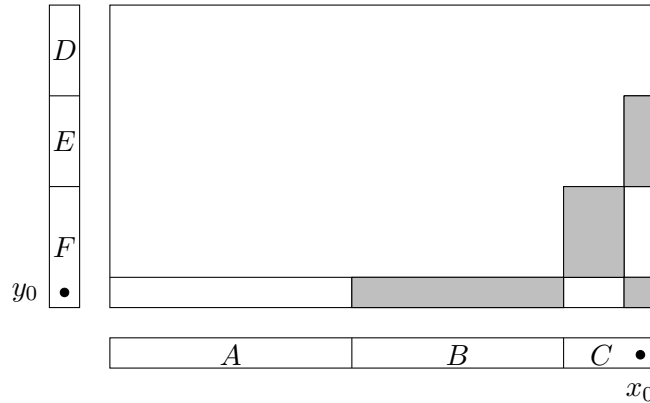


Figure 3. An example of a minimum vertex 3-cut of type 6.

We give an example of a minimum vertex 3-cut S of type 6, where sets S_2 and S_3 are not equal. Let $A = K_{1000}$, $B = K_{100}$, $C = K_3$, and $x_0 \in C$. Let G be the graph with vertex set $A \cup B \cup C$, such that every vertex in B is adjacent to every vertex in $A \cup C \setminus \{x_0\}$. Furthermore, let $D = K_{1000}$, $E = K_{100}$, $F = K_6$, and $y_0 \in F$. Let H be the graph with vertex set $D \cup E \cup F$, such that every vertex in E is adjacent to every vertex in $D \cup F \setminus \{y_0\}$ (see Figure 3). The set

$$S = ((B \cup \{x_0\}) \times \{y_0\}) \cup (C \setminus \{x_0\} \times F \setminus \{y_0\}) \cup (\{x_0\} \times E)$$

is a vertex 3-cut of type 6. It is a minimum vertex 3-cut in $G \square H$, and $|S| = 211$ (note that $\delta_2(G \square H) > 211$, $\kappa_3(G)|V(H)| > 211$, and $\kappa_3(H)|V(G)| > 211$).

4. THE PROOF OF OPTIMALITY

We prove that one of the 7 types of vertex 3-cuts described in the previous section is a minimum vertex 3-cut in $G \square H$, provided that G and H are 2-connected graphs on at least 6 vertices, and that G and H are not complete graphs. So assume from now on that both G and H are 2-connected of order at least 6.

Lemma 3. *Let $G = (V(G), E(G))$ be a 2-connected graph. If $X \cup Y$ is a partition of $V(G)$ such that $|X| \geq 2$ and $|Y| \geq 2$, then there exist edges x_1y_1 and x_2y_2 such that $x_1, x_2 \in X$, $y_1, y_2 \in Y$ and $x_1 \neq x_2$, $y_1 \neq y_2$.*

Proof. Suppose on the contrary. Then all edges with one endvertex in X and the other in Y are incident to a single vertex $u \in V(G)$. It follows that u is a cutvertex in G , a contradiction. ■

Lemma 4. *Let S be a vertex 3-cut in $G \square H$ and let G_{y_1} and G_{y_2} be adjacent layers of different types (there exists $i \in [3]$, such that $C_i \cap G_{y_1} \neq \emptyset$ and $C_i \cap G_{y_2} = \emptyset$). For sets $S'_{y_1} = N^G(C_i \cap G_{y_1})$, and $S'_{y_2} = p_G(C_i \cap G_{y_1}) \times \{y_2\}$ and any $x \in p_G(C_i \cap G_{y_1})$ we have*

$$|S'_{y_1}| + |S'_{y_2}| \geq \deg_G(x) + 1, \text{ and } p_G(S'_{y_1}) \cap p_G(S'_{y_2}) = \emptyset,$$

and $S'_{y_i} \subseteq S_{y_i}$, for $i = 1, 2$.

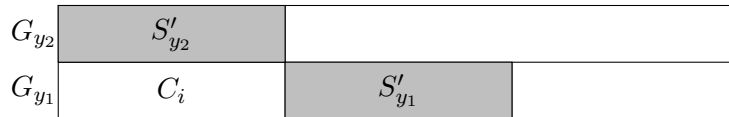


Figure 4. Adjacent layers in Lemma 4.

Proof. Let $x \in p_G(C_i \cap G_{y_1})$. Since all G -neighbors of (x, y_1) are contained in $G_{y_1} \cap C_i$ or S'_{y_1} we find that $|S'_{y_1}| + |S'_{y_2}| \geq \deg_G(x) + 1$. Clearly, $p_G(S'_{y_1}) \cap p_G(S'_{y_2}) = \emptyset$ follows from the definition of sets S'_{y_1} and S'_{y_2} . ■

Corollary 5. *Let S be a vertex 3-cut in $G \square H$. If G_{y_1} and G_{y_2} are adjacent layers such that $C_i \cap G_{y_1} \neq \emptyset$ and $C_i \cap G_{y_2} = \emptyset$, then for sets $S'_{y_1} = N^G(C_i \cap G_{y_1})$, and $S'_{y_2} = p_G(C_i \cap G_{y_1}) \times \{y_2\}$ we have*

$$|S'_{y_1}| + |S'_{y_2}| \geq \delta(G) + 1 \text{ and } p_G(S'_{y_1}) \cap p_G(S'_{y_2}) = \emptyset,$$

and $S'_{y_i} \subseteq S_{y_i}$, for $i = 1, 2$.

Lemma 6. *Let S be a vertex 3-cut in $G \square H$. If G_{y_1} and G_{y_2} are adjacent layers such that $(C_i \cup C_j) \cap G_{y_1} \neq \emptyset$, and $C_i \cap G_{y_2} = C_j \cap G_{y_2} = \emptyset$ for some $i, j \in [3]$, $i \neq j$, then there exist sets $S'_{y_1} \subseteq S_{y_1}$ and $S'_{y_2} \subseteq S_{y_2}$ such that*

$$|S'_{y_1}| + |S'_{y_2}| \geq \delta_2(G) + 2, \text{ and } p_G(S'_{y_1}) \cap p_G(S'_{y_2}) = \emptyset.$$

Proof. Let $x_1 \in p_G(C_i \cap G_{y_1})$ and $x_2 \in p_G(C_j \cap G_{y_1})$, and note that x_1 and x_2 are not adjacent in G . Let us define $S'_{y_1} = N^G(\{(x_1, y_1), (x_2, y_1)\}) \setminus (C_i \cup C_j)$ and $S'_{y_2} = p_G((C_i \cup C_j) \cap G_{y_1}) \times \{y_2\}$ (see Figure 5). Clearly, $|S'_{y_1}| + |S'_{y_2}| \geq |N_G(\{x_1, x_2\})| + 2 \geq \delta_2(G) + 2$, and $p_G(S'_{y_1}) \cap p_G(S'_{y_2}) = \emptyset$. Since $C_i \cap G_{y_2} = C_j \cap G_{y_2} = \emptyset$ we have $S'_{y_2} \subseteq S_{y_2}$, and $S'_{y_1} \subseteq S_{y_1}$ follows from the definition of S'_{y_1} . ■

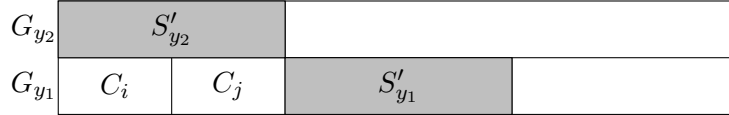


Figure 5. Adjacent layers in Lemma 6.

Lemma 7. *Let S be a vertex 3-cut in $G \square H$ and suppose that there is a $y \in V(H)$ such that $G_y \subseteq C_1$. If $x_1x_2 \notin E(G)$ and $(x_1, y_1) \notin C_1$, $(x_2, y_2) \notin C_1$ for some (possibly equal) $y_1, y_2 \in V(H)$, then there exists a set $S' \subseteq S$ such that $|S'| \geq \delta_2(G) + 2$ and $|S' \cap H_x| \leq 1$ for every $x \in V(G)$.*

Proof. Assume that $(x_1, y_1) \in C_i$ and $(x_2, y_2) \in C_j$ where $i, j \neq 1$ (here we allow the possibility that $i = j$, as well as the possibility $y_1 = y_2$). Let $A_1, A_2 \subseteq V(G)$ be the following sets

$$A_1 = p_G(G_{y_1} \cap C_i) \quad \text{and} \quad A_2 = p_G(G_{y_2} \cap C_j).$$

Since $G_y \subseteq C_1$ for some $y \in V(H)$ and H is connected, we find that for every $x \in A_1 \cup A_2$ the layer H_x contains at least one vertex of S . We construct S' so that for every $x \in A_1 \cup A_2$ we choose a vertex of $S \cap H_x$ and give it in S' . If $x \in N_G(A_1 \cup A_2)$, then $(x, y_1) \in S$ or $(x, y_2) \in S$, so for every $x \in N(A_1 \cup A_2)$ we choose either (x, y_1) or (x, y_2) and give it in S' . Since $|A_1 \cup A_2| + |N_G(A_1 \cup A_2)| \geq \delta_2(G) + 2$, this construction of S' has all properties claimed in the lemma. ■

Lemma 8. *Let S be a vertex 3-cut in $G \square H$ and $|V(H)| \geq 4$, $\kappa(H) \geq 2$. If there are at least three different types of G -layers in $(G \square H) - S$, then there exist edges $u_1v_1, u_2v_2 \in E(H)$, such that u_1, v_1, u_2, v_2 are pairwise distinct, and G_{u_i} is of different type than G_{v_i} for $i = 1, 2$.*

Proof. Observe the graph H' obtained from H by deleting all edges uv such that G_u and G_v are of equal type. If H' has at least two nontrivial connected components, then we choose two edges in different connected components, and the endvertices of these two edges are the desired vertices u_1, v_1, u_2, v_2 . Otherwise H' has exactly one nontrivial connected component. If this component contains a path on four vertices, we may choose first two vertices as u_1 and v_1 , and the second two as u_2 and v_2 . It remains that H' is a disjoint union of a star and some isolated vertices, or H' is a subgraph of a triangle and some isolated vertices. In the former case the center of the star of H' is a cut-vertex in H (when we remove the center of the star, there are no edges that connect layers of different types), and in the latter case one of the three vertices of the triangle is a cut-vertex in H (because $|V(H)| \geq 4$ and there are three types of G -layers), a contradiction. ■

Lemma 9. *Suppose that G_{y_1} and G_{y_n} are non-compatible layers, and let y_1, y_2, \dots, y_n be a path in H such that G_{y_1} and G_{y_n} are compatible with G_{y_k} for every $k \in [n], k \notin \{1, n\}$. Then for every $k \in [n], k \notin \{1, n\}$, we have*

$$|S_{y_1} \cup S_{y_k} \cup S_{y_n}| \geq \delta(G) + 1.$$

Proof. Since G_{y_1} and G_{y_n} are non-compatible there is an $x \in V(G)$ such that $(x, y_1) \in C_i$ and $(x, y_n) \in C_j$ for some $i, j \in [3], i \neq j$. Observe that for every $x' \in N[x]$ we have either $(x', y_1) \in C_i$ and $(x', y_n) \in C_j$, or at least one of (x', y_1) and (x', y_n) is in S . In the first case $(x', y_k) \in S$, because both G_{y_1} and G_{y_n} are compatible with G_{y_k} for every $k \in [n], k \neq 1, n$. ■

Theorem 10 given below is the main result of this article. Note that in the previous section we exhibited an example of a product $G \square H$, such that $D(G, H)$ is strictly smaller than the other three terms $\kappa_3(G)|V(H)|, \kappa_3(H)|V(G)|$ and $\delta_2(G \square H)$. We also give examples of products, such that each of the other three terms is the smallest of the four terms. Clearly, $\delta_2(C_m \square C_n) = 6$ for $m, n \geq 4$, and in this case $\delta_2(G \square H)$ is the smallest term. To find an example, where $\kappa_3(G)|V(H)|$ is the smallest term, let H be the graph obtained from K_{10} by deleting an edge. Moreover let G be obtained from three copies of K_{10} and two additional vertices x and y , by identifying two vertices in each copy with x and y . In this case we have $\kappa_3(G)|V(H)| = 20$, and all other three terms are strictly greater than 20. It follows that all four terms that appear in the theorem below are needed.

Theorem 10. *Let G and H be 2-connected graphs of order at least six. If G and H are not complete graphs, then*

$$\kappa_3(G \square H) = \min \{ \kappa_3(G) |V(H)|, \kappa_3(H) |V(G)|, \delta_2(G \square H), D(G, H) \}.$$

Proof. It follows from the discussion in Section 3 that

$$\kappa_3(G \square H) \leq \min \{ \kappa_3(G) |V(H)|, \kappa_3(H) |V(G)|, \delta_2(G \square H), D(G, H) \}.$$

To prove \geq inequality let S be a vertex 3-cut in $G \square H$, and C_1, C_2 and C_3 (unions of) connected components of $(G \square H) - S$ defined by (1) in Section 2. We call C_1, C_2 and C_3 components (although C_3 might be a union of several connected components).

If $G_y \cap C_i \neq \emptyset$ for all $i \in [3]$ and for all $y \in V(H)$, we get $|S| \geq \kappa_3(G) |V(H)|$. Similarly we get $|S| \geq \kappa_3(H) |V(G)|$, if $H_x \cap C_i \neq \emptyset$ for all $i \in [3]$ and for all $x \in V(G)$. So we can assume that there is a $y \in V(H)$ so that G_y is not of type $(1, 2, 3)$ and that there is an $x \in V(G)$ so that H_x is not of type $(1, 2, 3)$.

Case A. There is exactly one G-layer which is not of type (1, 2, 3) (or there is exactly one H-layer which is not of type (1, 2, 3), in which case the proof is analogous).

Let G_{y_0} be G-layer that is not of type (1, 2, 3), and let G_{y_1} and G_{y_2} be adjacent to G_{y_0} (recall that H is 2-connected and so $\delta(H) \geq 2$). We have $|S_{y_1}| > |S_{y_0}|$ for otherwise $|S| \geq \kappa_3(G) |V(H)|$. Similarly, $|S_{y_2}| > |S_{y_0}|$. By Corollary 5 we get $|S_{y_0}| + |S_{y_1}| \geq \delta(G) + 1$ and $|S_{y_0}| + |S_{y_2}| \geq \delta(G) + 1$. Thus $2|S_{y_0}| + |S_{y_1}| + |S_{y_2}| \geq 2\delta(G) + 2$. So we find that $2|S_{y_1}| + 2|S_{y_2}| \geq |S_{y_1}| + |S_{y_2}| + |S_{y_0}| + 1 + |S_{y_0}| + 1 \geq 2\delta(G) + 4$, and therefore $|S_{y_1}| + |S_{y_2}| \geq \delta(G) + 2$.

Now let $y_m \in V(H)$, $y_m \neq y_0$ be such that $|S_{y_m}| = \min\{|S_y|; y \neq y_0\}$. If $|S_{y_m}| \leq \frac{1}{2} |S_{y_1}|$, then

$$|S| = |S_{y_0}| + |S_{y_1}| + \sum_{y \neq y_0, y_1} |S_y| \geq |S_{y_m}| \cdot |V(H)| \geq \kappa_3(G) |V(H)|.$$

Therefore $|S_{y_m}| > \frac{1}{2} |S_{y_1}|$ and similarly $|S_{y_m}| > \frac{1}{2} |S_{y_2}|$. So we have

$$4|S_{y_m}| \geq |S_{y_1}| + |S_{y_2}| + 2 \geq \delta(G) + 4.$$

Since G is 2-connected, every G-layer contains at least two vertices of S . Note that this is also true for G_{y_0} , because either $G_{y_0} - S_{y_0}$ is not connected, or G_{y_0} is a type (1) layer (or type (2) or type (3)), which is adjacent to a layer of type (1, 2, 3), and every layer of type (1) adjacent to a layer of type (1, 2, 3) contains at least two vertices of S . If $\kappa_3(G) = 2$, then $|S| \geq \kappa_3(G) |V(H)|$. So we may assume that $\kappa_3(G) \geq 3$. If $|V(H)| \geq 7$, then

$$\begin{aligned} |S| &\geq |S_{y_0}| + |S_{y_1}| + |S_{y_2}| + 4|S_{y_m}| + \kappa_3(G) (|V(H)| - 7) \\ &\geq 2 + (\delta(G) + 2) + (\delta(G) + 4) + 3(|V(H)| - 7) \\ &= 2\delta(G) + |V(H)| - 2 + 2|V(H)| - 11 \\ &\geq 2\delta(G) + \delta_2(H) + 2|V(H)| - 11 \\ &\geq 2\delta(G) + \delta_2(H). \end{aligned}$$

If $|V(H)| = 6$ we have $\delta_2(H) \leq 4$ and therefore

$$|S| \geq |S_{y_0}| + |S_{y_1}| + 4|S_{y_m}| \geq \delta(G) + 1 + \delta(G) + 4 \geq 2\delta(G) + \delta_2(H).$$

Case B. There is more than one G-layer which is not of type (1, 2, 3), and more than one H-layer which is not of type (1, 2, 3), and suppose that there exist vertices $y \in V(H)$ and $x \in V(G)$ such that $S_y = \emptyset$ and $S_x = \emptyset$.

Without loss of generality we can assume $G_y \subseteq C_1$ and $H_x \subseteq C_1$. Observe that in this case there are at least two G-layers of type (1), because all layers adjacent to G_y are type (1) layers, and H is connected.

We claim that there are at least two G -layers that are not of type (1). If there would be only one G -layer which is not of type (1), then this layer is of type (1, 2, 3). Since this layer is adjacent to a layer of type (1), we find that these two adjacent layers contain at least $\delta_2(G) + 2$ vertices of S (see Lemma 6). Moreover, $G \square H$ has exactly three types of H -layers, these are types (1), (1, 2) and (1, 3). It follows from Lemma 8 and Corollary 5 that we have at least $2\delta(H) + 2$ vertices of S in four H -layers. Since we count at most four vertices in the intersection of these six layers (two G -layers and four H -layers), we have $|S| \geq \delta_2(G) + 2 + 2\delta(H) + 2 - 4 = \delta_2(G) + 2\delta(H)$. This proves that there are at least two G -layers that are not of type (1). Since H is 2-connected, there are at least two pairs of adjacent G -layers (with no layer in both pairs simultaneously), such that one of them is of type (1), and the other is not of type (1), see Lemma 3. Analogous claim is true for H -layers.

Let $A = \{x \in V(G); H_x \cap (C_2 \cup C_3) \neq \emptyset\}$ and $B = \{y \in V(H); G_y \cap (C_2 \cup C_3) \neq \emptyset\}$. We claim that A and B induce a complete graph. To prove it, assume on the contrary, that $u_1, u_2 \in A$ and $u_1u_2 \notin E(G)$. According to Lemma 7 there is a set $S' \subseteq S$, such that $|S'| \geq \delta_2(G) + 2$ and $|S' \cap H_x| \leq 1$ for every $x \in V(G)$. By Lemma 3 there are edges x_1x_2 and x_3x_4 , with no common endvertex, such that $x_1, x_3 \in A$ and $x_2, x_4 \notin A$, and therefore we find, by an application of Corollary 5, that

$$\left| S \cap \bigcup_{i=1}^4 H_{x_i} \right| \geq 2\delta(H) + 2.$$

Define

$$S'' = S \cap \bigcup_{i=1}^4 H_{x_i}$$

and note that $|S' \cap S''| \leq 4$. It follows that $|S| \geq |S'| + |S''| - 4 \geq \delta_2(G) + 2\delta(H)$. This proves that A (and similarly also B) induces a complete graph.

Let $A_i = \{x \in V(G); H_x \cap C_i \neq \emptyset\}$ and $B_i = \{y \in V(H); G_y \cap C_i \neq \emptyset\}$ for $i = 2, 3$. Since A induces a complete graph $B_2 \cap B_3 = \emptyset$ and analogously $A_2 \cap A_3 = \emptyset$, because B induces a complete graph. Note also that $A_2 \times B_3 \subseteq S$ and $A_3 \times B_2 \subseteq S$. Define $a_i = |A_i|, b_i = |B_i|$ for $i = 2, 3$.

Assume that $a_i, b_i \geq 2$ for $i = 2, 3$.

Suppose first that $|S \cap ((A_2 \times B_2) \cup (A_3 \times B_3))| \geq 2$. Fix any $(x_2, y_2) \in (A_2 \times B_2) \cap C_2$ and $(x_3, y_3) \in (A_3 \times B_3) \cap C_3$ and observe that

$$|S| \geq a_2b_3 + a_3b_2 + 2 + |N_{G \square H}(x_2, y_2) \cap \overline{(A \times B)}| + |N_{G \square H}(x_3, y_3) \cap \overline{(A \times B)}|.$$

Since

$$\begin{aligned} |N_{G \square H}(x_2, y_2) \cup N_{G \square H}(x_3, y_3)| &= 2(a_2 + a_3 + b_2 + b_3 - 2) - 2 \\ &+ |N_{G \square H}(x_2, y_2) \cap \overline{(A \times B)}| + |N_{G \square H}(x_3, y_3) \cap \overline{(A \times B)}| \end{aligned}$$

in order to prove that $|S| \geq |N_{G \square H}(x_2, y_2) \cup N_{G \square H}(x_3, y_3)|$ we need to prove that

$$a_2b_3 + a_3b_2 + 2 \geq 2(a_2 + a_3 + b_2 + b_3) - 6.$$

Since $a_2, b_3 \geq 2$ we have $a_2b_3 + 4 \geq 2a_2 + 2b_3$ and similarly $a_3b_2 + 4 \geq 2a_3 + 2b_2$, which proves the above inequality and so $|S| \geq |N_{G \square H}(x_2, y_2) \cup N_{G \square H}(x_3, y_3)| \geq \delta_2(G \square H)$.

Suppose next that $|S \cap ((A_2 \times B_2) \cup (A_3 \times B_3))| \leq 1$. In this case let $v_2 \in B_2$ and $v_3 \in B_3$ be such that $A_2 \times \{v_2\} \subseteq C_2$ and $A_3 \times \{v_3\} \subseteq C_3$. Let $(x_2, y_2) \in C_2$ be any vertex with $y_2 \neq v_2$ and let $(x_3, y_3) \in C_3$ be any vertex with $y_3 \neq v_3$. Recall that there are edges u_1u_2 and u_3u_4 , with no common endvertex, such that $u_1, u_3 \in A$ and $u_2, u_4 \notin A$ and so if, for example, $u_1 \in A_2$ and $u_3 \in A_3$ then $(u_2, v_2), (u_4, v_3) \in S$ (in any case we get two vertices in S , say if $u_1, u_3 \in A_2$ the conclusion is similar). It follows that

$$|S| \geq |N_{G \square H}(x_2, y_2) \cap \overline{(A \times B)}| + |N_{G \square H}(x_3, y_3) \cap \overline{(A \times B)}| + a_2b_3 + a_3b_2 + 2$$

where the $+2$ is because of the vertices $(u_2, v_2), (u_4, v_3) \in S$. The rest of the proof is the same as above and so we have $|S| \geq \delta_2(G \square H)$.

Assume that not $a_i, b_i \geq 2$ for $i = 2, 3$.

We may assume, without loss of generality, that $a_2 = 1$. By the definition of B this implies that $A_2 \times B_2 \subseteq C_2$. Let $A_2 = \{x_0\}$ and let $(x_0, y_0) \in A_2 \times B_2$ and $(x_1, y_1) \in (A_3 \times B_3) \cap C_3$ be arbitrary vertices. Since G is 2-connected there is at least one edge with one endvertex in A_3 and the other in \bar{A} (for otherwise x_0 is a cut-vertex in G). It follows that for every $y \in B_3$ we have $(A_3 \times \{y\}) \cap S \neq \emptyset$ or there is a $x' \notin A$ such that $(x', y) \in S$. Note also that $(N(x_0) \times B_2) \cup (\{x_0\} \times B_3) \subseteq S$ and $(N(x_0, y_0) \cup N(x_1, y_1)) \cap \overline{(A \times B)} \subseteq S$, and therefore

$$(3) \quad |S| \geq \deg(x_0)|B_2| + |N_H(y_0) \setminus B| + 2|B_3| + |N_H(y_1) \setminus B| + |N_G(x_1) \setminus A| - 1$$

where -1 comes from the fact that the vertex (x', y_1) might be a vertex of $N_G(x_1) \setminus A$ so we might have counted it twice. We also note that inequality (3) becomes a strict inequality if $(A_3 \times B_3) \cap S \neq \emptyset$, for in this case we may choose $y_1 \in B_3$ such that $(x_1, y_1) \in C_3$ and $(A_3 \times \{y_1\}) \cap S \neq \emptyset$ which leads to a strict inequality in the estimate of $|S|$ above (double counting of (x', y_1) cannot happen).

We claim that $|S| \geq \delta_2(G \square H)$ unless $b_2 = 1$ or $b_3 = 1$. To prove this, suppose that $b_2 \geq 2$ and $b_3 \geq 2$. Observe that

$$\begin{aligned} |N(x_0, y_0) \cup N(x_1, y_1)| &= |N(x_0, y_0)| + |N(x_1, y_1)| - 2 \\ &= 2(a_2 + a_3 + b_2 + b_3 - 2) + |(N(x_0, y_0) \cup N(x_1, y_1)) \cap \overline{(A \times B)}| - 2 \\ &= 2(a_2 + a_3 + b_2 + b_3 - 2) + (\deg(x_0) - a_3) + |N_H(y_0) \setminus B| + |N_H(y_1) \setminus B| \\ &\quad + |N_G(x_1) \setminus A| - 2 \end{aligned}$$

and therefore, when combining inequality (3) with the above equality we find that $|S| \geq |N(x_0, y_0) \cup N(x_1, y_1)|$ holds whenever

$$(\deg(x_0) - 2)b_2 - 1 \geq 2a_3 + \deg(x_0) - a_3 - 4.$$

which (taking into account $b_2 \geq 2$) reduces to

$$\deg(x_0) - 1 \geq a_3$$

which is correct unless $\deg(x_0) = a_3$ and (3) is an equality rather than a strict inequality. To finish the proof of the claim we will prove that if $\deg(x_0) = a_3$ and $b_3 \geq 2$ then (3) becomes a strict inequality. To prove this recall that if (3) is an equality then $A_3 \times B_3 \subseteq C_3$ as observed above (so we may assume this, otherwise we are done). Since $b_3 \geq 2$ there is $y_2 \in B_3, y_2 \neq y_1$ such that $|(S \cap \bar{A}) \times \{y_2\}| \geq 2$, because x_0 is not adjacent to any vertex in \bar{A} and so there are at least two vertices in \bar{A} adjacent to a vertex in A_3 (because G is 2-connected). This two additional vertices make (3) a strict inequality. This completes the proof of the claim.

It remains to prove the theorem in case $b_2 = 1$ or $b_3 = 1$ (under the assumption $a_2 = 1$). If $a_2 = 1$ and $b_3 = 1$, then $|S| < \delta_2(G \square H)$ only if there is an equality in (3) and $\deg(x_0) = a_3$.

Now, if this is the case, then

$$|S| = \deg(x_0)|B_2| + |N_H(y_0) \setminus B| + |B_3| + |N_H(y_1) \setminus B| + |N_G(x_1) \setminus A|.$$

Since (3) is an equality, $A_3 \times \{y_1\} \subseteq C_3$ (as noted above). Since $a_3 = \deg(x_0) \geq 2$ this implies that $|N_H(y_1) \setminus B| = 0$, for otherwise $A_3 \times (N_H(y_1) \setminus B) \subseteq S$, and then (3) becomes a strict inequality. It follows that

$$S = (A_3 \times B_2) \cup (\{x_0\} \times N_H(y_0) \setminus B) \cup (A_2 \times B_3) \cup (N_H(x_1) \setminus A \times \{y_1\}).$$

But in this case S is a type 6 vertex 3-cut and so $|S| \geq D(G, H)$. The case $a_2 = 1$ and $b_2 = 1$ leads to $|S| \geq \delta_2(G \square H)$ with an easy proof, which is left to the reader.

Case C. There is more than one G -layer which is not of type (1, 2, 3), and more than one H -layer which is not of type (1, 2, 3), and suppose that for all $y \in V(H)$ we have $S_y \neq \emptyset$, or for all $x \in V(G)$ we have $S_x \neq \emptyset$.

Without loss of generality we can assume that for all $y \in V(H)$, $S_y \neq \emptyset$. Suppose that there exist nonadjacent edges $v_1v'_1 \in E(H)$ and $v_2v'_2 \in E(H)$, such that G_{v_1} and $G_{v'_1}$ are of different types, and G_{v_2} and $G_{v'_2}$ are of different types. Since $S_y \neq \emptyset$ for every $y \neq v_1, v'_1, v_2, v'_2$ and since $|S_{v_i} \cup S_{v'_i}| \geq \delta(G) + 1$ for $i = 1, 2$ (see Corollary 5), we find that $|S| \geq 2(\delta(G) + 1) + |V(H)| - 4 \geq 2\delta(G) + \delta_2(H)$. It follows from Lemma 8 that there are exactly two different types of G -layers, for otherwise (if there are three types) two such edges $v_1v'_1$ and $v_2v'_2$ would exist. Moreover, if there are at least two G -layers of each type, we find again that there

are edges $v_1v'_1$ and $v_2v'_2$ with endvertices corresponding to layers of different types (see Lemma 3). Therefore we can assume that one G -layer is of different type than all the others (which are all of the same type). Denote by G_{y_0} the G -layer which is of different type than all the other G -layers.

If all these G -layers, except G_{y_0} , are of type (1) (if they are of type (2) or (3), then the arguments that follow are analogous), then G_{y_0} is of type (2, 3) or of type (1, 2, 3). For every $(x, y_0) \in C_2 \cup C_3$ we have $\{x\} \times N_H(y_0) \subseteq S$ and hence $|S| \geq \delta_2(G) + 2\delta(H)$. Therefore G_{y_0} is not of type (1) or (2) or (3).

We claim that for every $y \in N_H(y_0)$ and $y' \notin N_H(y_0)$ the layers G_y and $G_{y'}$ are compatible. Suppose this is not true. Then y and y' are not adjacent. There is a path between y and y' that avoids y_0 (recall that H is 2-connected). Moreover, if $\deg_H(y_0) = 2$, then there is a path from y to y' that intersects $N[y_0]$ only in y , again because H is 2-connected. We may assume that G_y and $G_{y'}$ are compatible with all G -layers $G_{y''}$ where y'' is an internal vertex of this path (for otherwise we may redefine y and y' to be two vertices on this path that fulfil this condition). It follows from Lemma 9 that $|S_y \cup S_{y'} \cup S_{y''}| \geq \delta(G) + 1$, where y'' is a vertex between y and y' on this path. If $\deg_H(y_0) \geq 3$, then there is a neighbour y_1 of y_0 , such that $y_1 \neq y, y''$, and then $|S_{y_0} \cup S_{y_1}| \geq \delta(G) + 1$ (see Corollary 5). If $\deg_H(y_0) = 2$, also in this case there is a neighbour y_1 of y_0 , such that $y_1 \neq y, y''$, leading to the same conclusion $|S_{y_0} \cup S_{y_1}| \geq \delta(G) + 1$. Since G is 2-connected, and no G -layer is of type (1) or (2) or (3), we find that for all $y \neq y_0$, $|S_y| \geq 2$. All together we find that

$$|S| \geq |S_y \cup S_{y''} \cup S_{y'}| + |S_{y_0} \cup S_{y_1}| + 2(|V(H)| - 5) \geq 2\delta(G) + |V(H)| - 2,$$

because $|V(H)| \geq 6$. We conclude that $|S| \geq 2\delta(G) + \delta_2(H)$, and so the claim is proved.

Let $y_1 \in N_H(y_0)$ be such that $|S_{y_1}| \leq |S_y|$ for all $y \in N_H(y_0)$, and let $y_2 \notin N_H[y_0]$ be such that $|S_{y_2}| \leq |S_y|$ for all $y \notin N_H[y_0]$. We construct the set

$$S' = S_{y_0} \cup (S_{y_1} \times N_H(y_0)) \cup (S_{y_2} \times \overline{N_H[y_0]}).$$

Since $G_{y_0} \sim G_{y_1}$ and $G_{y_1} \sim G_{y_2}$ we find that S' is a vertex 3-cut in $G \square H$, moreover $|S'| \leq |S|$. Observe that S' is a type 6 vertex 3-cut and therefore $S' \geq D(G, H)$, which concludes the proof of the theorem. ■

REFERENCES

- [1] D. Bauer, H. Broersma and E. Schmeichel, *Toughness in graphs—A survey*, Graphs Combin. **22** (2006) 1–35.
<https://doi.org/10.1007/s00373-006-0649-0>
- [2] G. Chartrand, S.F. Kapoor, L. Lesniak and D.R. Lick, *Generalized connectivity in graphs*, Bull. Bombay Math. Colloq. **2** (1984) 1–6.

- [3] V. Chvátal, *Tough graphs and Hamiltonian circuits*, Discrete Math. **5** (1973) 215–228.
[https://doi.org/10.1016/0012-365X\(73\)90138-6](https://doi.org/10.1016/0012-365X(73)90138-6)
- [4] D.P. Day, O.R. Oellermann and H.C. Swart, *The ℓ -connectivity function of trees and complete multipartite graphs*, J. Combin. Math. Combin. Comput. **10** (1991) 183–192.
- [5] J. Govorčin, and R. Škrekovski, *On the connectivity of Cartesian product of graphs*, Ars Math. Contemp. **7** (2014) 293–297.
<https://doi.org/10.26493/1855-3974.313.e10>
- [6] O.R. Oellermann, *On the ℓ -connectivity of a graph*, Graphs Combin. **3** (1987) 285–291.
<https://doi.org/10.1007/BF01788551>
- [7] G. Sabidussi, *Graphs with given group and given graph-theoretical properties*, Canad. J. Math. **9** (1957) 515–525.
<https://doi.org/10.4153/CJM-1957-060-7>
- [8] Y. Sun and X. Li, *On the difference of two generalized connectivities of a graph*, J. Comb. Optim. **33** (2017) 283–291.
<https://doi.org/10.1007/s10878-015-9956-9>
- [9] Y. Sun, F. Li and Z. Jin, *On two generalized connectivities of graphs*, Discuss. Math. Graph Theory **38** (2018) 245–261.
<https://doi.org/10.7151/dmgt.1987>
- [10] Y. Sun, *On the maximum and minimum sizes of a graph with given k -connectivity*, Discuss. Math. Graph Theory **37** (2017) 623–632.
<https://doi.org/10.7151/dmgt.1941>
- [11] S. Špacapan, *Connectivity of Cartesian products of graphs*, Appl. Math. Lett. **21** (2008) 682–685.
<https://doi.org/10.1016/j.aml.2007.06.010>
- [12] J.M. Xu and C. Yang, *Connectivity and super-connectivity of Cartesian product graphs*, Ars Combin. **95** (2010) 235–245.
- [13] C. Yang and J.M. Xu, *Reliability of interconnection networks modeled by Cartesian product digraphs*, Networks **52** (2008) 202–205.
<https://doi.org/10.1002/net.20231>

Received 17 October 2019

Revised 12 March 2020

Accepted 13 March 2020