

## THE ROMAN DOMATIC PROBLEM IN GRAPHS AND DIGRAPHS: A SURVEY

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### Abstract

In this paper, we survey results on the Roman domatic number and its variants in both graphs and digraphs. This fifth survey completes our works on Roman domination and its variations published in two book chapters and two other surveys.

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## 1. ROMAN DOMATIC NUMBER AND ITS VARIANTS IN GRAPHS

## 1.1. Introduction

For notation and graph theory terminology, we in general follow Haynes, Hedetniemi and Slater [21,22]. Specifically, let  $G$  be a graph with vertex set  $V(G) = V$  and edge set  $E(G) = E$ . The integers  $n = n(G) = |V(G)|$  and  $m = m(G) = |E(G)|$  are the *order* and the *size* of the graph  $G$ , respectively. The *open neighborhood* of vertex  $v$  is  $N_G(v) = N(v) = \{u \in V(G) \mid uv \in E(G)\}$ , and the *closed neighborhood* of  $v$  is  $N_G[v] = N[v] = N(v) \cup \{v\}$ . The *degree* of a vertex  $v$  is  $d_G(v) = d(v) = |N(v)|$ . The *minimum* and *maximum degree* of a graph  $G$  are denoted by  $\delta(G) = \delta$  and  $\Delta(G) = \Delta$ , respectively. If  $X \subseteq V(G)$ , then  $G[X]$  is the subgraph of  $G$  induced by  $X$ . For a set  $X \subseteq V(G)$ , its *open neighborhood* is the set  $N_G(X) = N(X) = \bigcup_{v \in X} N(v)$ , and its *closed neighborhood* is the set  $N_G[X] = N[X] = N(X) \cup X$ . The *complement* of a graph  $G$  is denoted by  $\overline{G}$ . Let  $K_n$  be the complete graph of order  $n$  and  $K_{p,q}$  be the complete bipartite graph with the bipartition  $X$  and  $Y$  such that  $|X| = p$  and  $|Y| = q$ . We write  $C_n$  and  $P_n$  for the *cycle* and *path* of order  $n$ , respectively.

A subset  $S$  of vertices of  $G$  is a *dominating set* if  $N[S] = V(G)$ . The *domination number*  $\gamma(G)$  is the minimum cardinality of a dominating set of  $G$ . A *domatic partition* is a partition of  $V(G)$  into dominating sets, and the *domatic number*  $d(G)$  is the largest number of sets in a domatic partition. The domatic number was introduced by Cockayne and Hedetniemi [18]. In their paper they showed that  $\gamma(G) \cdot d(G) \leq n(G)$ . For more information on the domatic number and their variants, we refer the reader to the survey article of Zelinka [57].

Inspired by the strategies for defending the Roman Empire presented by Steward [37] and ReVelle and Rosing [28], Cockayne, Dreyer Jr., S.M. Hedetniemi and S.T. Hedetniemi [11] defined in 2004 a *Roman dominating function* (RDF) on a graph  $G$  as a function  $f : V(G) \rightarrow \{0, 1, 2\}$  satisfying the condition that every vertex  $u$  for which  $f(u) = 0$  is adjacent to at least one vertex  $v$  for which  $f(v) = 2$ . The *weight* of an RDF  $f$  on a graph  $G$  is defined by  $\omega(f) = \sum_{v \in V(G)} f(v)$ . The *Roman domination number*, denoted  $\gamma_R(G)$ , is the minimum weight of an RDF in  $G$ . An RDF of weight  $\gamma_R(G)$  is called a  $\gamma_R(G)$ -function.

The concept of the domatic number is in a certain sense dual to the domination number. Rall has defined a variant of the domatic number of  $G$ , namely the fractional domatic number of  $G$ , using functions. (This was mentioned by Slater and Trees in [36].) Analogous to the fractional domatic number we may define the Roman domatic number of a graph.

We emphasize that all graphs considered in this section are undirected. Through the next subsections, we will present the Roman domatic number and its variants. It should be noted that the results on Roman domination and its variations have recently been collected in book chapters and surveys. For more

details, we refer the reader to [13–16].

## 1.2. Roman domatic number

A set  $\{f_1, f_2, \dots, f_d\}$  of distinct Roman dominating functions on  $G$  such that  $\sum_{i=1}^d f_i(v) \leq 2$  for each  $v \in V(G)$ , is called by Sheikholeslami and Volkmann in [29] a *Roman dominating family* (of functions) on  $G$ . The maximum number of functions in a Roman dominating family (RD family) on  $G$  is the *Roman domatic number* of  $G$ , denoted by  $d_R(G)$ . The Roman domatic number is well-defined and  $d_R(G) \geq 1$  for all graphs  $G$ , since the set consisting of any RDF forms an RD family on  $G$ . It is easy to see that  $d_R(G) = 1$  if and only if  $G$  has no edges. In addition, Sheikholeslami and Volkmann [29] showed that  $d_R(K_n) = n$  for the complete graph and  $d_R(T) = 2$  for each tree  $T$  with at least two vertices. If  $G$  is a cactus graph, then  $d_R(G) \leq 3$ . This bound is sharp, since we have  $d_R(C_n) = 3$  if  $n \equiv 0 \pmod{3}$  and  $d_R(C_n) = 2$  if  $n \equiv 1, 2 \pmod{3}$ . An analogue to the result  $\gamma(G) \cdot d(G) \leq n(G)$  also holds.

**Theorem 1** [29]. *If  $G$  is a graph of order  $n$ , then  $\gamma_R(G) \cdot d_R(G) \leq 2n$ . Moreover, if  $\gamma_R(G) \cdot d_R(G) = 2n$ , then for each RD family  $\{f_1, f_2, \dots, f_d\}$  on  $G$  with  $d = d_R(G)$ , each function  $f_i$  is a  $\gamma_R(G)$ -function and  $\sum_{i=1}^d f_i(v) = 2$  for all  $v \in V(G)$ .*

As applications of Theorem 1, we obtain the following results.

**Corollary 2** [29]. *If  $G$  is a graph of order  $n$ , then  $d(G) \leq d_R(G) \leq n$ .*

**Proposition 3** [29]. *Let  $G$  be a graph of order  $n \geq 2$ . Then  $\gamma_R(G) = n$  and  $d_R(G) = 2$  if and only if  $\Delta(G) = 1$ .*

**Proposition 4** [29]. *If  $G$  is a graph of order  $n \geq 2$ , then  $d_R(G) = n$  if and only if  $G = K_n$ .*

**Theorem 5** [29]. *If  $G$  is a graph of order  $n \geq 2$ , then  $\gamma_R(G) + d_R(G) \leq n + 2$ , with equality if and only if  $\Delta(G) = 1$  or  $G = K_n$ .*

Next we present a sharp upper bound on the Roman domatic number in terms of minimum degree.

**Theorem 6** [29]. *If  $G$  is a graph, then  $d_R(G) \leq \delta(G) + 2$ .*

The authors of [29] gave the following example to illustrate the sharpness of Theorem 6. Let  $G_i$  be the copy of  $K_{k+3}$  with vertex set  $V(G_i) = \{v_1^i, v_2^i, \dots, v_{k+3}^i\}$  for  $1 \leq i \leq k$ , and let  $G$  be the graph obtained from  $\bigcup_{i=1}^k G_i$  by adding a new vertex  $v$  attached to each  $v_1^i$ . Then  $\delta(G) = k$  and  $d_R(G) = k + 2$ .

In [29], the authors presented the bound  $\gamma_R(G) \geq \lceil \frac{2n}{\Delta+1} \rceil + \epsilon$  for graphs  $G$  of order  $n$  and maximum degree  $\Delta \geq 1$ , with  $\epsilon = 0$  when  $n \equiv 0, 1 \pmod{\Delta+1}$  and

$\epsilon = 1$  when  $n \not\equiv 0, 1 \pmod{\Delta + 1}$ . The following example shows that this bound is incorrect. Let  $H$  be the graph obtained from a path  $v_1 v_2 \dots v_{12}$  by adding a pendant edge at the vertices  $v_2, v_5$  and  $v_8$ . Then  $\gamma_R(H) = 8$ , however, it follows from the bound above that  $\gamma_R(H) \geq 9$ .

For  $\Delta \geq 1$ , one can find in [17] the inequality  $\gamma_R(G) \geq \lceil \frac{2n}{\Delta+1} \rceil$ . Combining this bound with Theorem 1, we obtain the following corollary immediately.

**Corollary 7.** *If  $G$  is a graph with maximum degree  $\Delta \geq 1$ , then  $d_R(G) \leq \Delta + 1$ .*

Corollary 7 leads to  $d_R(G) \leq r + 1$  for  $r$ -regular graphs. Again using the inequality  $\gamma_R(G) \geq \lceil \frac{2n}{\Delta+1} \rceil$ , this bound can be improved slightly.

**Theorem 8** [13]. *If  $G$  is an  $r$ -regular graph of order  $n$ , then  $d_R(G) \leq r + \epsilon$  with  $\epsilon = 1$  when  $n \equiv 0, \frac{r+1}{2} \pmod{r+1}$  and  $\epsilon = 0$  when  $n \not\equiv 0, \frac{r+1}{2} \pmod{r+1}$ .*

As applications of Theorems 6 and 8, we obtain the following Nordhaus-Gaddum type results.

**Theorem 9** [29]. *If  $G$  is a graph of order  $n$ , then  $d_R(G) + d_R(\overline{G}) \leq n + 2$ .*

**Theorem 10** [13]. *If  $G$  is an  $r$ -regular graph of order  $n$ , then  $d_R(G) + d_R(\overline{G}) \leq n + 1$ , and if  $d_R(G) + d_R(\overline{G}) = n + 1$ , then  $n \equiv 0, \frac{r+1}{2} \pmod{r+1}$  and  $n \equiv 0, \frac{n-r}{2} \pmod{n-r}$ .*

If  $G$  is isomorphic to the complete graph, then we observe that  $d_R(G) = n$  and  $d_R(\overline{G}) = 1$  and therefore  $d_R(G) + d_R(\overline{G}) = n + 1$ , and thus Theorem 10 is sharp. Tan, Liang, Wang and Zhou [38] proved the following useful theorem from which several (old and new) results follow easily.

**Theorem 11** [38]. *Let  $G$  be an arbitrary graph. Construct  $H$  by adding a new vertex attached to each vertex of  $G$ . Then  $d_R(H) = d_R(G) + 1$ .*

For  $n \geq 3$ , the fan  $F_n$  (the wheel  $W_n$ ) is the graph obtained from a path  $P_n$  (a cycle  $C_n$ ), by adding a new vertex attached to each vertex of  $P_n$  (of  $C_n$ ). Using Theorem 11 and the Roman domatic number of paths and cycles, given above, the next results in [38] are immediate. For  $n \geq 3$ ,  $d_R(F_n) = 3$  and  $d_R(W_n) = 4$  if  $n \equiv 0 \pmod{3}$  and  $d_R(W_n) = 3$  if  $n \equiv 1, 2 \pmod{3}$ . Furthermore they proved  $d_R(K_{m,n}) = \max\{2, \min\{m, n\}\}$ . In addition, Tian *et al.* [38] showed that the decision problem corresponding to the problem of computing  $d_R(G)$  is NP-complete even when restricted to bipartite graphs. They also proved that there is a  $(\ln n + O(\ln \ln n))$ -approximation algorithm for Roman Domatic Number, where  $n$  is the order of the input graph.

### 1.3. Roman $k$ -domatic and Roman $(k, k)$ -domatic numbers

If  $k \geq 1$  is an integer, then Kammerling and Volkmann [24] defined a *Roman  $k$ -dominating function* (RkDF) on a graph  $G$  as a function  $f : V(G) \rightarrow \{0, 1, 2\}$  satisfying the condition that every vertex  $u$  for which  $f(u) = 0$  is adjacent to at least  $k$  vertices  $v$  for which  $f(v) = 2$ . The *weight* of an RkDF  $f$  on a graph  $G$  is defined by  $\omega(f) = \sum_{v \in V(G)} f(v)$ . The *Roman  $k$ -domination number*, denoted  $\gamma_{kR}(G)$ , is the minimum weight of an RkDF in  $G$ . An RkDF of weight  $\gamma_{kR}(G)$  is called a  $\gamma_{kR}(G)$ -function.

Sheikholeslami and Volkmann [30] introduced the Roman  $k$ -domatic number of a graph. A set  $\{f_1, f_2, \dots, f_d\}$  of distinct Roman  $k$ -dominating functions on  $G$  with the property that  $\sum_{i=1}^d f_i(v) \leq 2$  for each  $v \in V(G)$ , is called a *Roman  $k$ -dominating family* (of functions) on  $G$ . The maximum number of functions in a Roman  $k$ -dominating family (RkD family) on  $G$  is the *Roman  $k$ -domatic number* of  $G$ , denoted by  $d_{kR}(G)$ . The Roman  $k$ -domatic number is well-defined and  $d_{kR}(G) \geq 1$  for all graphs  $G$ , since the set consisting of any RkDF forms an RkD family on  $G$ . Sheikholeslami and Volkmann [30] showed that  $d_{kR}(G) = 1$  if and only if  $G$  has no connected bipartite subgraph with minimum degree at least  $k$ , and they proved that  $d_{kR}(K_n) = 1$  if  $n < k$  and  $d_{kR}(K_n) = \lfloor \frac{n}{k} \rfloor$  if  $n \geq k$ . The following extension of Theorem 1 holds.

**Theorem 12** [30]. *If  $G$  is a graph of order  $n$ , then  $\gamma_{kR}(G) \cdot d_{kR}(G) \leq 2n$ . Moreover, if  $\gamma_{kR}(G) \cdot d_{kR}(G) = 2n$ , then for each RkD family  $\{f_1, f_2, \dots, f_d\}$  on  $G$  with  $d = d_{kR}(G)$ , each function  $f_i$  is a  $\gamma_{kR}(G)$ -function and  $\sum_{i=1}^d f_i(v) = 2$  for all  $v \in V(G)$ .*

The next results are applications of Theorem 12.

**Proposition 13** [30]. *Let  $G$  be a graph of order  $n \geq 2$ . Then  $d_{kR}(G) = n$  if and only if  $k = 1$  and  $G$  is the complete graph on  $n$  vertices.*

**Theorem 14** [30]. *If  $G$  is a graph of order  $n \geq 2$ , then  $\gamma_{kR}(G) + d_{kR}(G) \leq n + 2$ , with equality if and only if  $k = 1$  and  $G$  is a complete graph, or  $k \geq 2$  and  $G$  contains a bipartite subgraph  $H$  with bipartition  $X, Y$  such that  $|X| = |Y| \geq k$  and  $d_H(v) \geq k$  for each  $v \in X \cup Y$  and  $G$  has no bipartite subgraph  $H'$  with bipartition  $A, B$  such that  $|A| > |B| \geq k$  and  $d_{H'}(v) \geq k$  for each  $v \in A$ .*

**Proposition 15** [30]. *If  $k \geq 1$  and  $p \geq 1$  are integers, then  $d_{kR}(K_{p,p}) = 1$  if  $p < k$ ,  $d_{kR}(K_{p,p}) = 2$  if  $k \leq p < 3k$  and  $d_{kR}(K_{p,p}) = \lfloor \frac{p}{k} \rfloor$  if  $p \geq 3k$ .*

Theorem 6 is the special case  $k = 1$  of the next upper bound of the Roman  $k$ -domatic number.

**Theorem 16** [30]. *For every graph  $G$ , we have  $d_{kR}(G) \leq \left\lfloor \frac{\delta(G)}{k} \right\rfloor + 2$ .*

Examples in [30] show that the bound in Theorem 16 is sharp. For  $\delta(G)$ -regular graphs the slightly better bound  $d_{kR}(G) \leq \left\lfloor \frac{\delta(G)}{k} \right\rfloor + 1$  holds. As an application of Theorems 12 and 16, we obtain the following Nordhaus-Gaddum type result, of which Theorem 9 is a special case.

**Theorem 17** [30]. *If  $G$  is a graph of order  $n$ , then  $d_{kR}(G) + d_{kR}(\overline{G}) \leq \frac{n-1}{k} + 3$ , with equality only for graphs  $G$  with  $\Delta(G) - \delta(G) = k$ .*

Kazemi, Sheikholeslami and Volkmann [27] introduced the Roman  $(k, k)$ -domatic number of a graph. A set  $\{f_1, f_2, \dots, f_d\}$  of distinct Roman  $k$ -dominating functions on  $G$  with the property that  $\sum_{i=1}^d f_i(v) \leq 2k$  for each  $v \in V(G)$ , is called a *Roman  $(k, k)$ -dominating family* (of functions) on  $G$ . The maximum number of functions in a Roman  $(k, k)$ -dominating family ( $R(k, k)D$  family) on  $G$  is the *Roman  $(k, k)$ -domatic number* of  $G$ , denoted by  $d_{(k,k)R}(G)$ . The Roman  $(k, k)$ -domatic number is well-defined and  $d_{(k,k)R}(G) \geq 1$  for all graphs  $G$ , since the set consisting of any  $RkDF$  forms an  $R(k, k)D$  family on  $G$ , and if  $k \geq 2$ , then  $d_{(k,k)R}(G) \geq 2$ , since the functions  $f_i : V(G) \rightarrow \{0, 1, 2\}$  defined by  $f_i(v) = i$  for each  $v \in V(G)$  and for  $i = 1, 2$  forms an  $R(k, k)D$  family on  $G$  of size 2. It is easy to see that  $d_{(k,k)R}(G) = 1$  if and only if  $k = 1$  and  $G$  is empty, and if  $k \geq 2$ , then  $d_{(k,k)R}(G) = 2$  if and only if  $G$  is trivial. For large  $k$ , we have the following nice result.

**Theorem 18** [27]. *Let  $G$  be a graph of order  $n$ . If  $k \geq 3 \cdot 2^{n-2}$ , then  $d_{(k,k)R}(G) = 2^n$ .*

The next extension of Theorem 1 also holds.

**Theorem 19** [27]. *If  $G$  is a graph of order  $n$ , then  $\gamma_{kR}(G) \cdot d_{(k,k)R}(G) \leq 2kn$ . Moreover, if  $\gamma_{kR}(G) \cdot d_{(k,k)R}(G) = 2kn$ , then for each  $R(k, k)D$  family  $\{f_1, f_2, \dots, f_d\}$  on  $G$  with  $d = d_{(k,k)R}(G)$ , each function  $f_i$  is a  $\gamma_{kR}(G)$ -function and  $\sum_{i=1}^d f_i(v) = 2k$  for all  $v \in V(G)$ .*

The next results are applications of Theorem 19.

**Theorem 20** [27]. *If  $G$  is a graph, then  $d_{(k,k)R}(G) \leq \max\{\Delta(G), k-1\} + k$ .*

**Theorem 21** [27]. *If  $G$  is a graph of order  $n \geq 2$ , then*

$$\gamma_{kR}(G) + d_{(k,k)R}(G) \leq n + 2k,$$

*with equality if and only if  $\gamma_{kR}(G) = 2n$  and  $d_{(k,k)R}(G) = 2k$ , or  $\gamma_{kR}(G) = 2k$  and  $d_{(k,k)R}(G) = n$ .*

**Theorem 22** [27]. *We have  $d_{(k,k)R}(K_n) = n$  if  $n \geq 2k$ ,  $d_{(k,k)R}(K_n) \leq 2k-1$  if  $n \leq 2k-1$  and  $d_{(k,k)R}(K_n) = 2k-1$  if  $k \geq 2$  and  $2k-2 \leq n \leq 2k-1$ .*

In [27], one can find the Roman  $(k, k)$ -domatic numbers for some special cases of complete bipartite graphs. The next result is an extension of Theorem 6.

**Theorem 23** [27]. *For every graph  $G$ , we have  $d_{(k,k)R}(G) \leq \delta(G) + 2k$ .*

Examples in [27] demonstrate that the bound in Theorem 23 is sharp. For regular graphs, the following improvement of Theorem 23 holds.

**Theorem 24** [27]. *If  $G$  is an  $r$ -regular graph, then  $d_{(k,k)R}(G) \leq \max\{2k - 1, r + k\} \leq r + 2k - 1$ .*

As an application of Theorems 23 and 24, we obtain the following Nordhaus-Gaddum type result, which extends Theorem 9.

**Theorem 25** [27]. *If  $G$  is a graph of order  $n$ , then  $d_{(k,k)R}(G) + d_{(k,k)R}(\overline{G}) \leq n + 4k - 2$ , with equality only for graphs  $G$  with  $\Delta(G) - \delta(G) = 1$ .*

#### 1.4. Distance Roman domatic number

For two vertices  $x$  and  $y$  of a graph  $G$ , let  $d_G(x, y) = d(x, y)$  be the distance between  $x$  and  $y$ . The  $k$ -th power  $G^k$  of a graph  $G$  is the graph with vertex set  $V(G)$ , where two different vertices  $x$  and  $y$  are adjacent if and only if  $d_G(x, y) \leq k$ . If  $k \geq 1$  is an integer, then Aram, Norouzian, Sheikholeslami and Volkmann [7] defined a  $k$ -distance Roman dominating function ( $k$ DRDF) on a graph  $G$  as a function  $f : V(G) \rightarrow \{0, 1, 2\}$  satisfying the condition that every vertex  $u$  for which  $f(u) = 0$ , there is a vertex  $v$  for which  $f(v) = 2$  and  $d(u, v) \leq k$ . The weight of an  $k$ DRDF  $f$  on a graph  $G$  is defined by  $\omega(f) = \sum_{v \in V(G)} f(v)$  and the  $k$ -distance Roman domination number, denoted  $\gamma_R^k(G)$ , is the minimum weight of an  $k$ DRDF in  $G$ .

Following Aram, Sheikholeslami and Volkmann [8], a set  $\{f_1, f_2, \dots, f_d\}$  of distinct  $k$ -distance Roman dominating functions on  $G$  with the property that  $\sum_{i=1}^d f_i(v) \leq 2$  for each  $v \in V(G)$ , is called a  $k$ -distance Roman dominating family (of functions) on  $G$ . The maximum number of functions in a  $k$ -distance Roman dominating family ( $k$ DRD family) on  $G$  is the  $k$ -distance Roman domatic number of  $G$ , denoted by  $d_R^k(G)$ . The  $k$ -distance Roman domatic number is well-defined and  $d_R^k(G) \geq 1$  for all graphs  $G$ , since the set consisting of any  $k$ DRDF forms an  $k$ DRD family on  $G$ . In the case  $k = 1$ , we write  $d_R(G)$  instead of  $d_R^1(G)$ . Obviously,  $d_R^k(G) = d_R(G^k)$ . Aram, Sheikholeslami and Volkmann [8] showed that  $d_R^k(G) = 1$  if and only if  $G$  is empty,  $d_R^k(K_n) = n$ . Moreover, if the diameter  $\text{diam}(G) \leq k$ , then  $d_R^k(G) = d_R(K_n)$ . The following upper bounds were obtained in [8].

**Theorem 26** [8]. *If  $G$  is a graph of order  $n$ , then  $\gamma_R^k(G) \cdot d_R^k(G) \leq 2n$ .*

Since  $\gamma_R^k(G) \geq 2$  for each graph  $G$  of order  $n \geq 2$ , Theorem 26 implies that  $d_R^k(G) \leq n$ . Therefore  $d_R^k(G) + d_R^k(\overline{G}) \leq 2n$  for each graph of order  $n \geq 2$ . If  $G^k$  and  $\overline{G}^k$  are complete graphs, then we observe that  $d_R^k(G) + d_R^k(\overline{G}) = 2n$ , and thus the presented Nordhaus-Gaddum inequality is sharp.

**Theorem 27** [8]. *For every graph  $G$ , we have  $d_R^k(G) \leq \delta(G^k) + 2$ .*

Using a result of Zelinka [55] on the  $k$ -distance domatic number, the next result is easy to prove.

**Corollary 28** [8]. *If  $G$  is a connected graph of order  $n$ , then  $d_R^k(G) \geq \min\{n, k+1\}$ .*

As applications of Corollary 28, we arrive at the following observations.

**Proposition 29** [8]. *If  $n \geq k+2$ , then  $d_R^k(P_n) = k+1$ .*

**Proposition 30** [8]. *Let  $G$  be a connected graph of order  $n \geq 2$ . Then  $\gamma_R^k(G) = n$  and  $d_R^k(G) = 2$  if and only if  $G = K_2$ .*

**Proposition 31** [8]. *If  $G$  is a connected graph of order  $n \geq 2$ , then  $d_R^k(G) = n$  if and only if  $G^k$  is the complete graph.*

Combining Theorem 26 and Propositions 30 and 31, we obtain the next result.

**Theorem 32** [8]. *If  $G$  is a connected graph of order  $n \geq 2$ , then*

$$\gamma_R^k(G) + d_R^k(G) \leq n + 2,$$

*with equality if and only if  $G^k = K_n$ .*

### 1.5. Italian domatic number

In 2016, Chellali, Haynes, Hedetniemi and McRae [12] defined the following variant of Roman dominating function. An *Italian dominating function* (IDF) (or *Roman  $\{2\}$ -dominating function* as called in [12]) on a graph  $G$  is a function  $f : V(G) \rightarrow \{0, 1, 2\}$  having the property that  $\sum_{x \in N(u)} f(x) \geq 2$  if  $f(u) = 0$ . The *Italian domination number*  $\gamma_I(G)$  (or *Roman  $\{2\}$ -domination number*) equals the minimum weight of an IDF on  $G$ , and an IDF of  $G$  with weight  $\gamma_I(G)$  is called a  $\gamma_I(G)$ -function.

Following Volkmann [50], a set  $\{f_1, f_2, \dots, f_d\}$  of distinct Italian dominating functions on  $G$  with the property that  $\sum_{i=1}^d f_i(v) \leq 2$  for each  $v \in V(G)$ , is called an *Italian dominating family* (of functions) on  $G$ . The maximum number of functions in an Italian dominating family (ID family) on  $G$  is the *Italian domatic number* of  $G$ , denoted by  $d_I(G)$ . The Italian domatic number is well-defined and



$d_I(G) \geq 1$  for all graphs  $G$ , since the set consisting of any IDF forms an Italian dominating family on  $G$ . We note that  $\gamma_I(G) \leq \gamma_R(G)$  and  $d_I(G) \geq d_R(G)$ . Therefore, every lower bound of  $d_R(G)$  is also a lower bound of  $d_I(G)$ . The following upper bounds were proved in [50].

**Theorem 33** [50]. *If  $G$  is a graph of order  $n$ , then  $\gamma_I(G) \cdot d_I(G) \leq 2n$ . Moreover, if  $\gamma_I(G) \cdot d_I(G) = 2n$ , then for each Italian dominating family  $\{f_1, f_2, \dots, f_d\}$  on  $G$  with  $d = d_I(G)$ , each function  $f_i$  is a  $\gamma_I(G)$ -function and  $\sum_{i=1}^d f_i(v) = 2$  for all  $v \in V(G)$ .*

Theorem 33 implies that  $d_I(G) \leq n$  for each graph  $G$  of order  $n \geq 2$ . Furthermore, it is proved in [50] that  $d_I(G) = n$  if and only if  $G$  is isomorphic to the complete graph  $K_n$ , and  $d_I(G) = 1$  if and only if  $G$  is empty. Using Theorem 33, one can prove the following result analogously to Theorem 5.

**Theorem 34** [50]. *If  $G$  is a graph of order  $n \geq 2$ , then  $\gamma_I(G) + d_I(G) \leq n + 2$ , with equality if and only if  $\Delta(G) = 1$  or  $G = K_n$ .*

**Theorem 35** [50]. *For every graph  $G$ ,  $d_I(G) \leq \delta(G) + 2$ .*

Applying Theorems 33 and 35, it is shown in [50] that  $d_I(C_n) = 3$  for  $n \geq 3$  and  $d_I(P_n) = 3$  for  $n \geq 6$ .

Volkman [50] also determines the Italian domatic number of different complete  $r$ -partite graphs for  $r \geq 2$ . Using the bound  $\gamma_I(G) \geq \lceil (2n(G))/(\Delta(G) + 2) \rceil$  (see [12]) and Theorem 35, one can prove the following Nordhaus-Gaddum type inequality.

**Theorem 36** [50]. *If  $G$  is a graph of order  $n$ , then  $d_I(G) + d_I(\overline{G}) \leq n + 2$ , except when  $G$  is 4-regular of order 9, 7-regular of order 18 or 16-regular of order 45.*

However, we think that Theorem 36 holds for all graphs.

**Conjecture 37** [50]. *If  $G$  is a graph of order  $n$ , then  $d_I(G) + d_I(\overline{G}) \leq n + 2$ .*

**Conjecture 38** [50]. *If  $G$  is a  $\delta$ -regular graph, then  $d_I(G) \leq \delta + 1$ .*

Conjecture 37 would be a consequence of Conjecture 38. The next conjecture would be another consequence of Conjecture 38.

**Conjecture 39** [50]. *If  $G$  is an regular graph of order  $n$ , then  $d_I(G) + d_I(\overline{G}) \leq n + 1$ .*

The complete graph,  $C_3, C_4$  and  $C_5$  show that the bound stated in Conjecture 39 would be sharp.

### 1.6. Double Roman domatic number

A *double Roman dominating function* (DRD function) on a graph  $G$  is defined by Beeler, Haynes and Hedetniemi in [9] as a function  $f : V(G) \rightarrow \{0, 1, 2, 3\}$  having the property that if  $f(v) = 0$ , then the vertex  $v$  must have at least two neighbors assigned 2 under  $f$  or one neighbor  $w$  with  $f(w) = 3$ , and if  $f(v) = 1$ , then the vertex  $v$  must have at least one neighbor  $u$  with  $f(u) \geq 2$ . The *double Roman domination number*  $\gamma_{dR}(G)$  equals the minimum weight of a double Roman dominating function on  $G$ , and a double Roman dominating function of  $G$  with weight  $\gamma_{dR}(G)$  is called a  $\gamma_{dR}(G)$ -function.

In [47], a set  $\{f_1, f_2, \dots, f_d\}$  of distinct double Roman dominating functions on  $G$  with the property that  $\sum_{i=1}^d f_i(v) \leq 3$  for each  $v \in V(G)$ , is called a *double Roman dominating family* (of functions) on  $G$ . The maximum number of functions in a double Roman dominating family (DRD family) on  $G$  is the *double Roman domatic number* of  $G$ , denoted by  $d_{dR}(G)$ . The double Roman domatic number is well-defined and  $d_{dR}(G) \geq 1$  for all graphs  $G$ , since the set consisting of any DRD function forms a DRD family on  $G$ . If  $G$  has no isolated vertices, then we even have  $d_{dR}(G) \geq 2$ . We start with the following basic property.

**Theorem 40** [47]. *If  $G$  is a graph of order  $n$ , then  $\gamma_{dR}(G) \cdot d_{dR}(G) \leq 3n$ . Moreover, if  $\gamma_{dR}(G) \cdot d_{dR}(G) = 3n$ , then for each DRD family  $\{f_1, f_2, \dots, f_d\}$  on  $G$  with  $d = d_{dR}(G)$ , each function  $f_i$  is a  $\gamma_{dR}(G)$ -function and  $\sum_{i=1}^d f_i(v) = 3$  for all  $v \in V(G)$ .*

Since  $\gamma_{dR}(G) \geq 3$  for each graph  $G$  of order  $n \geq 2$ , Theorem 40 implies that  $d_{dR}(G) \leq n$ . In [18], the authors note that  $d(K_n) = n$ . Using the simple observation  $d(G) \leq d_{dR}(G)$ , we obtain  $d_{dR}(K_n) = n$ .

**Theorem 41** [47]. *If  $G$  is a graph, then  $\gamma_{dR}(G) \leq \delta(G) + 1$ .*

Using Theorem 41, we observe that if  $G$  is a graph of order  $n \geq 2$ , then  $d_{dR}(G) = n$  if and only if  $G = K_n$ . As a further application of Theorem 41, we note that  $d_{dR}(C_n) = 3$  if  $n \equiv 0 \pmod{3}$  and  $d_{dR}(C_n) = 2$  if  $n \equiv 1, 2 \pmod{3}$ .

**Theorem 42** [48]. *Let  $G = K_{n_1, n_2, \dots, n_r}$  be the complete  $r$ -partite graph with  $r \geq 2$  and  $n_1 = n_2 = \dots = n_r = q \geq 2$ . Then  $d_{dR}(G) = \lfloor \frac{rq}{2} \rfloor = \lfloor \frac{n(G)}{2} \rfloor$ .*

**Theorem 43** [47]. *Let  $G$  be a graph of order  $n \geq 2$ . If  $\Delta(G) \leq n - 2$ , then  $\gamma_{dR}(G) \leq n/2$ .*

As an application of Theorems 41 and 43, we arrive at the following Nordhaus-Gaddum type bound.

**Theorem 44** [47]. *If  $G$  is a graph of order  $n$ , then  $d_{dR}(G) + d_{dR}(\overline{G}) \leq n + 1$ , with equality if and only if  $G = K_n$  or  $\overline{G} = K_n$ .*

For a large family of graphs, the following improvement of Theorem 44 holds.

**Theorem 45** [48]. *Let  $G$  be a graph of order  $n$  such that  $\delta(G), \delta(\overline{G}) \geq 1$ . If  $n$  is odd, or if  $n$  is even and  $\min\{\delta(G), \delta(\overline{G})\} \leq \frac{n}{2} - 2$ , then  $d_{dR}(G) + d_{dR}(\overline{G}) \leq n - 1$ .*

If  $G = K_{p,p}$  for  $p \geq 2$ , then  $d_{dR}(G) + d_{dR}(\overline{G}) = 2p = n(G)$ . Thus Theorem 45 does not hold for  $n$  even and  $\delta(\overline{G}) = \frac{n}{2} - 1$  in general. In [9], the authors proved the sharp bound  $\gamma_{dR}(G) \leq \frac{5n}{4}$  for all connected graphs of order  $n \geq 3$ . Combining this bound with Theorem 40, we obtain the following sharp result.

**Theorem 46** [47]. *If  $G$  is a connected graph of order  $n \geq 3$ , then*

$$d_{dR}(G) + \gamma_{dR}(G) \leq \frac{5n}{4} + 2,$$

*except when  $G = K_3$ , in which case  $d_{dR}(K_3) + \gamma_{dR}(K_3) = 6$ .*

Restricted to bipartite graphs  $G$  with  $\delta(G) \geq 2$ , Volkmann [47] presents the improvement  $d_{dR}(G) + \gamma_{dR}(G) \leq n(G) + 3$  of Theorem 46.

### 1.7. Total Roman domatic number

In 2013, Liu and Chang [26] introduced the concept of total Roman domination as follows. A *total Roman dominating function* of a graph  $G$  with no isolated vertex, abbreviated TRD-function, is a Roman dominating function on  $G$  with the additional property that the subgraph of  $G$  induced by the set of all vertices of positive weight has no isolated vertices. The *total Roman domination number*  $\gamma_{tR}(G)$  is the minimum weight of a TRD-function on  $G$ . A TRD-function with weight  $\gamma_{tR}(G)$  in  $G$  is called a  $\gamma_{tR}(G)$ -function.

Following Amjadi, Nazari-Moghaddam and Sheikholeslami [5], a set  $\{f_1, f_2, \dots, f_d\}$  of distinct total Roman dominating functions on  $G$  with the property that  $\sum_{i=1}^d f_i(v) \leq 2$  for each  $v \in V(G)$ , is called a *total Roman dominating family* (of functions) on  $G$ . The maximum number of functions in a total Roman dominating family (TRD family) on  $G$  is called the *total Roman domatic number* of  $G$ , denoted by  $d_{tR}(G)$ . The total Roman domatic number is well-defined and  $d_{tR}(G) \geq 1$  for all graphs  $G$  since the set consisting of the constant function 1 forms a TRD family on  $G$ . It is shown that if  $G$  is a connected graph of order  $n \geq 2$ , then  $d_{tR}(G) = 1$  if and only if any edge of  $G$  is a pendant edge or is adjacent to a pendant edge.

The following upper bounds hold.

**Theorem 47** [5]. *For every graph  $G$  with  $\delta(G) \geq 1$ ,*

$$d_{tR}(G) \leq \delta(G) + 1.$$

Moreover, if  $d_{tR}(G) = \delta(G) + 1$ , then for each TRD family  $\{f_1, f_2, \dots, f_d\}$  of  $G$  with  $d = d_{tR}(G)$ , and for all vertices  $v$  of degree  $\delta(G)$ ,  $\sum_{u \in N[v]} f_i(u) = 2$  for each  $i \in \{1, 2, \dots, d\}$ .

Examples in [5] show that the bound in Theorem 47 is sharp. For  $\delta$ -regular graphs the aforementioned bound can be slightly improved.

**Theorem 48** [5]. *If  $G$  is a connected  $\delta$ -regular graph of order  $n$  with minimum degree  $\delta \geq 3$ , then*

$$d_{tR}(G) \leq \delta - 1 + \epsilon$$

with  $\epsilon = 1$  when  $n \equiv 0 \pmod{\delta}$  or  $\delta$  is even and  $n \equiv \frac{\delta}{2} \pmod{\delta}$ , and  $\epsilon = 0$  otherwise.

As an application of Theorems 47 and 48, the following Nordhaus-Gaddum type result is derived.

**Theorem 49** [5]. *For every graph  $G$  of order  $n \geq 5$  with  $\delta(G) \geq 1$  and  $\delta(\overline{G}) \geq 1$ ,*

$$(1) \quad d_{tR}(G) + d_{tR}(\overline{G}) \leq n.$$

**Theorem 50** [5]. *Let  $G$  be a graph of order  $n$  with  $\delta(G) \geq 1$ . Then*

$$\gamma_{tR}(G) \cdot d_{tR}(G) \leq 2n.$$

Moreover, if  $\gamma_{tR}(G) \cdot d_{tR}(G) = 2n$ , then for each TRD family  $\{f_1, f_2, \dots, f_d\}$  on  $G$  with  $d = d_{tR}(G)$ , each function  $f_i$  is a  $\gamma_{tR}(G)$ -function and  $\sum_{i=1}^d f_i(v) = 2$  for all  $v \in V$ .

Using Theorem 50 and the facts  $\gamma_{tR}(G) \geq \lceil \frac{2n}{\Delta} \rceil$  (see [2]) and  $\gamma_R(P_n) = \gamma_R(C_n) = \lceil \frac{2n}{3} \rceil$  for  $n \geq 3$  the next results are easy to prove.

**Corollary 51** [5]. *For any graph  $G$  of order  $n \geq 2$  and maximum degree  $\Delta$ ,  $d_{tR}(G) \leq \Delta$ .*

**Corollary 52** [5]. *For paths and cycles,  $d_{tR}(C_n) = 2$  when  $n \geq 3$ , and  $d_{tR}(P_n) = 2$  when  $n \geq 6$ .*

The following upper bound on the sum of total Roman domination and total Roman domatic numbers was obtained in [5].

**Theorem 53** [5]. *If  $G$  is a connected graph of order  $n \geq 5$ , then  $\gamma_{tR}(G) + d_{tR}(G) \leq n + 2$ .*

All graphs attaining the bound in Theorem 53 were characterized in [5].

### 1.8. Signed (total) Roman domatic number

A *signed (total) Roman dominating function* (SRDF, STRDF) on a graph  $G$  is defined in [1] ([40]) as a function  $f : V(G) \rightarrow \{-1, 1, 2\}$  such that  $\sum_{x \in N[v]} f(x) \geq 1$  ( $\sum_{x \in N(v)} f(x) \geq 1$ ) for each  $v \in V(G)$ , and such that every vertex  $u \in V(G)$  with  $f(u) = -1$  is adjacent to at least one vertex  $w$  for which  $f(w) = 2$ . The *signed (total) Roman domination number*  $\gamma_{sR}(G)$  ( $\gamma_{stR}(G)$ ) equals the minimum weight of an SRDF (STRDF) on  $G$ , and a signed (total) Roman dominating function of  $G$  with weight  $\gamma_{sR}(G)$  ( $\gamma_{stR}(G)$ ) is called a  $\gamma_{sR}(G)$ -function ( $\gamma_{stR}(G)$ -function).

A set  $\{f_1, f_2, \dots, f_d\}$  of distinct signed (total) Roman dominating functions on  $G$  with the property that  $\sum_{i=1}^d f_i(v) \leq 1$  for each  $v \in V(G)$ , is called in [32] ([41]) a *signed (total) Roman dominating family* (of functions) on  $G$ . The maximum number of functions in a signed (total) Roman dominating family (SRD family, STRD family) on  $G$  is the *signed (total) Roman domatic number* of  $G$ , denoted by  $d_{sR}(G)$  ( $d_{stR}(G)$ ). The signed (total) Roman domatic number is well-defined and  $d_{sR}(G) \geq 1$  for all graphs  $G$  ( $d_{stR}(G) \geq 1$  for all graphs  $G$  with  $\delta(G) \geq 1$ ), since the set consisting of any SDRF (STRDF) forms an SRD (STRD) family on  $G$ . We start with some basic properties of  $d_{sR}(G)$  and  $d_{stR}(G)$ .

**Theorem 54** [32, 41]. (i) If  $G$  is a graph of order  $n$ , then  $\gamma_{sR}(G) \cdot d_{sR}(G) \leq n$ . Moreover, if  $\gamma_{sR}(G) \cdot d_{sR}(G) = n$ , then for each SRD family  $\{f_1, f_2, \dots, f_d\}$  on  $G$  with  $d = d_{sR}(G)$ , each function  $f_i$  is a  $\gamma_{sR}(G)$ -function and  $\sum_{i=1}^d f_i(v) = 1$  for all  $v \in V(G)$ .

(ii) If  $G$  is a graph of order  $n$  with  $\delta(G) \geq 1$ , then  $\gamma_{stR}(G) \cdot d_{stR}(G) \leq n$ . Moreover, if  $\gamma_{stR}(G) \cdot d_{stR}(G) = n$ , then for each STRD family  $\{f_1, f_2, \dots, f_d\}$  on  $G$  with  $d = d_{stR}(G)$ , each function  $f_i$  is a  $\gamma_{stR}(G)$ -function and  $\sum_{i=1}^d f_i(v) = 1$  for all  $v \in V(G)$ .

**Theorem 55** [32, 41]. (i) For every graph  $G$ , we have  $d_{sR}(G) \leq \delta(G) + 1$ . Moreover, if  $d_{sR}(G) = \delta(G) + 1$ , then for each SRD family  $\{f_1, f_2, \dots, f_d\}$  on  $G$  with  $d = d_{sR}(G)$  and each vertex  $v$  of minimum degree,  $\sum_{x \in N[v]} f_i(x) = 1$  for each function  $f_i$  and  $\sum_{i=1}^d f_i(x) = 1$  for all  $x \in N[v]$ .

(ii) For every graph  $G$  with  $\delta(G) \geq 1$ , we have  $d_{stR}(G) \leq \delta(G)$ . Moreover, if  $d_{stR}(G) = \delta(G)$ , then for each STRD family  $\{f_1, f_2, \dots, f_d\}$  on  $G$  with  $d = d_{stR}(G)$  and each vertex  $v$  of minimum degree,  $\sum_{x \in N(v)} f_i(x) = 1$  for each function  $f_i$  and  $\sum_{i=1}^d f_i(x) = 1$  for all  $x \in N(v)$ .

As applications of Theorems 54 and 55, we obtain  $d_{sR}(K_{1,n}) = 1$  for  $n \geq 2$ ,  $d_{sR}(C_n) = d_{sR}(P_n) = 1$  for  $n \geq 3$ ,  $d_{sR}(K_n) = n$ , unless  $n = 3$  in which case  $d_{sR}(K_3) = 1$ ,  $d_{stR}(K_{p,p}) = p$  for  $p \geq 4$  and  $d_{stR}(K_{9k+6}) = 3k + 2$  for each integer  $k \geq 0$ . The next results are also consequences of Theorems 54 and 55.

**Theorem 56** [41]. *Let  $G$  be a graph of order  $n \geq 3$  with  $\delta(G) \geq 1$ . Then  $d_{stR}(G) \leq n - 2$ , with equality if and only if  $G$  is isomorphic to  $K_3, P_3$  or  $C_4$ .*

If  $G$  is a  $\delta$ -regular graph of order  $n$  with  $\delta \geq 1$ , then  $\gamma_{stR}(G) \geq \lceil n/\delta \rceil \geq 2$  (see [40]) and thus Theorem 54 leads to the improvement  $d_{stR}(G) \leq n/2$  of Theorem 56 in this special case.

**Theorem 57** [32]. *If  $G$  is a graph of order  $n$ , then  $d_{sR}(G) + \gamma_{sR}(G) \leq n + 1$ , with equality if and only if  $n \neq 3$  and  $G = K_n$  or  $G = \overline{K_n}$ .*

**Theorem 58** [41]. *Let  $G$  be a graph of order  $n$  with  $\delta(G) \geq 1$ . Then  $d_{stR}(G) + \gamma_{stR}(G) \leq n + 1$ , with equality if and only if the components of  $G$  are  $K_2, K_3, P_3$  or  $C_6$ .*

If  $\delta(G) \geq 5$ , then we even have  $d_{stR}(G) + \gamma_{stR}(G) \leq n - 2$ , unless  $n = 6$ , in which case  $G = K_6$  with  $d_{stR}(G) + \gamma_{stR}(G) = 5 = n - 1$ .

**Theorem 59** [32]. *If  $G$  is a graph of order  $n$ , then  $d_{sR}(G) + d_{sR}(\overline{G}) \leq n + 1$ , with equality if and only if  $n \neq 3$  and  $G = K_n$  or  $\overline{G} = K_n$ .*

**Theorem 60** [41]. *If  $G$  is a graph of order  $n$  with  $\delta(G), \delta(\overline{G}) \geq 1$ , then  $d_{stR}(G) + d_{stR}(\overline{G}) \leq n - 1$ , with equality if and only if  $G = C_4$  or  $\overline{G} = C_4$ .*

### 1.9. Signed (total) Roman $k$ -domatic number

Let  $k \geq 1$  be an integer. A *signed (total) Roman  $k$ -dominating function* (SR $k$ DF, STR $k$ DF) on a graph  $G$  is defined by Henning and Volkmann in [23] (Volkmann [46]) as a function  $f : V(G) \rightarrow \{-1, 1, 2\}$  such that  $\sum_{x \in N[v]} f(x) \geq k$  ( $\sum_{x \in N(v)} f(x) \geq k$ ) for each  $v \in V(G)$ , and such that every vertex  $u \in V(G)$  with  $f(u) = -1$  is adjacent to at least one vertex  $w$  for which  $f(w) = 2$ . The *signed (total) Roman  $k$ -domination number*  $\gamma_{sR}^k(G)$  ( $\gamma_{stR}^k(G)$ ) equals the minimum weight of an SR $k$ DF (STR $k$ DF) on  $G$ , and a signed (total) Roman  $k$ -dominating function of  $G$  with weight  $\gamma_{sR}^k(G)$  ( $\gamma_{stR}^k(G)$ ) is called a  $\gamma_{sR}^k(G)$ -function ( $\gamma_{stR}^k(G)$ -function).

A set  $\{f_1, f_2, \dots, f_d\}$  of distinct signed (total) Roman  $k$ -dominating functions on  $G$  with the property that  $\sum_{i=1}^d f_i(v) \leq k$  for each  $v \in V(G)$ , is called in [39] ([45]) a *signed (total) Roman  $k$ -dominating family* (of functions) on  $G$ . The maximum number of functions in a signed (total) Roman  $k$ -dominating family (SR $k$ D family, STR $k$ D family) on  $G$  is the *signed (total) Roman  $k$ -domatic number* of  $G$ , denoted by  $d_{sR}^k(G)$  ( $d_{stR}^k(G)$ ). The signed (total) Roman  $k$ -domatic number is well-defined and  $d_{sR}^k(G) \geq 1$  ( $d_{stR}^k(G) \geq 1$ ) for all graphs  $G$  with  $\delta(G) \geq k - 1$  ( $\delta(G) \geq k$ ). Analogous to Theorems 54 and 55, we have the following results.

**Theorem 61** [39]. *If  $G$  is a graph of order  $n$  with  $\delta(G) \geq k - 1$ , then  $\gamma_{sR}^k(G) \cdot d_{sR}^k(G) \leq kn$ .*

**Theorem 62** [45]. *If  $G$  is a graph of order  $n$  with  $\delta(G) \geq k$ , then  $\gamma_{stR}^k(G) \cdot d_{stR}^k(G) \leq kn$ .*

**Theorem 63** [39]. *If  $G$  is a graph with  $\delta(G) \geq k - 1$ , then  $d_{sR}^k(G) \leq \delta(G) + 1$ .*

**Theorem 64** [45]. *If  $G$  is a graph with  $\delta(G) \geq k$ , then  $d_{stR}^k(G) \leq \delta(G)$ .*

Using Theorem 63, we obtain the signed Roman  $k$ -domatic number of the complete graph for  $k \geq 2$ .

**Proposition 65** [39]. *If  $K_n$  is the complete graph of order  $n \geq k \geq 2$ , then  $d_{sR}^k(K_n) = n$ , unless  $n = k = 2$  in which case  $d_{sR}^2(K_2) = 1$ .*

Since  $\gamma_{sR}^k(K_n) = k$  for  $n \geq k \geq 2$  (see [23]), Proposition 65 demonstrates that Theorems 61 and 63 are both sharp. Using Theorem 64, we obtain  $d_{stR}^k(K_{k,k}) = k$  for  $k \geq 3$  and  $d_{stR}^k(K_{p,p}) = p$  for integers  $p \geq k + 1 \geq 2$ , with exception of the case  $k = 1$  and  $p = 3$ , in which case  $d_{stR}^1(K_{3,3}) = 1$ . These examples demonstrate that Theorem 64 is sharp.

As an extension of the bounds in Theorems 57 and 58, we have the following results.

**Theorem 66** [39, 45]. *If  $G$  is a graph of order  $n$  with  $\delta(G) \geq k - 1$ , then  $d_{sR}^k(G) + \gamma_{sR}^k(G) \leq n + k$ . If  $G$  is a graph of order  $n$  with  $\delta(G) \geq k$ , then  $d_{stR}^k(G) + \gamma_{tsR}^k(G) \leq n + k$ .*

**Theorem 67** [39]. *If  $G$  is a graph of order  $n$  such that  $\delta(G), \delta(\overline{G}) \geq k - 1$ , then  $d_{sR}^k(G) + d_{sR}^k(\overline{G}) \leq n + 1$ . Furthermore, if  $d_{sR}^k(G) + d_{sR}^k(\overline{G}) = n + 1$ , then  $G$  is regular.*

For  $k = 2$ , the following improvement of Theorem 67 was given in [39].

**Theorem 68** [39]. *If  $G$  is a graph of order  $n$  such that  $\delta(G), \delta(\overline{G}) \geq 1$ , then  $d_{sR}^2(G) + d_{sR}^2(\overline{G}) \leq n$ .*

**Theorem 69** [39]. *Let  $k \geq 3$  be an integer. Then there is only a finite number of graphs  $G$  with  $\delta(G), \delta(\overline{G}) \geq k - 1$  such that  $d_{sR}^k(G) + d_{sR}^k(\overline{G}) = n(G) + 1$ .*

In connection with Theorems 68 and 69, we have the following conjecture.

**Conjecture 70** [39]. *Let  $k \geq 3$  be an integer. If  $G$  is a graph of order  $n$  such that  $\delta(G), \delta(\overline{G}) \geq k - 1$ , then  $d_{sR}^k(G) + d_{sR}^k(\overline{G}) \leq n$ .*

If  $k \geq 4$  is an even integer, then  $d_{sR}^k(K_{k,k}) = k$  (see [39]) and  $d_{sR}^k(K_k) = k$  (see Proposition 65 and so  $d_{sR}^k(K_{k,k}) + d_{sR}^k(\overline{K_{k,k}}) = 2k = n(K_{k,k})$ . Thus Conjecture 70 would be tight, at least for  $k \geq 4$  even.

**Theorem 71** [45]. *If  $G$  is a graph of order  $n$  such that  $\delta(G), \delta(\overline{G}) \geq k$ , then  $d_{stR}^k(G) + d_{stR}^k(\overline{G}) \leq n - 1$ . Furthermore, if  $d_{stR}^k(G) + d_{stR}^k(\overline{G}) = n - 1$ , then  $G$  is regular.*

As a supplement to Theorem 60, we have the following Nordhaus-Gaddum type result.

**Theorem 72** [45]. *Let  $k \geq 2$  be an integer. Then there are only finitely many graphs  $G$  with  $\delta(G), \delta(\overline{G}) \geq k$  such that  $d_{stR}^k(G) + d_{stR}^k(\overline{G}) = n(G) - 1$ .*

We finish this section with the following conjecture.

**Conjecture 73** [45]. *Let  $k \geq 2$  be an integer. If  $G$  is a graph of order  $n$  such that  $\delta(G), \delta(\overline{G}) \geq k$ , then  $d_{stR}^k(G) + d_{stR}^k(\overline{G}) \leq n - 2$ .*

If  $n \geq 5$  is an integer, then  $d_{stR}^{n-2}(K_{n,n}) = n$  (see above) and  $d_{stR}^{n-2}(\overline{K_{n,n}}) = n - 2$  (see [45]) and so  $d_{stR}^{n-2}(K_{n,n}) + d_{stR}^{n-2}(\overline{K_{n,n}}) = 2n - 2 = n(K_{n,n}) - 2$ . Thus Conjecture 73 would be tight, at least for  $k \geq 3$ .

## 2. ROMAN DOMATIC NUMBER AND ITS VARIANTS IN DIGRAPHS

### 2.1. Introduction

Let  $D$  be a finite digraph with neither loops nor multiple arcs (but pairs of opposite arcs are allowed) with vertex set  $V(D) = V$  and arc set  $A(D) = A$ . The integers  $n = n(D) = |V(D)|$  and  $m = m(D) = |A(D)|$  are the *order* and the *size* of the digraph  $D$ , respectively. For two different vertices  $u$  and  $v$ , we use  $uv$  to denote the arc with tail  $u$  and head  $v$ , and we also call  $v$  an *out-neighbor* of  $u$  and  $u$  an *in-neighbor* of  $v$ . For  $v \in V(D)$ , the *out-neighborhood* and *in-neighborhood* of  $v$ , denoted by  $N_D^+(v) = N^+(v)$  and  $N_D^-(v) = N^-(v)$ , are the sets of out-neighbors and in-neighbors of  $v$ , respectively. The *closed out-neighborhood* and *closed in-neighborhood* of a vertex  $v \in V(D)$  are the sets  $N_D^+[v] = N^+[v] = N^+(v) \cup \{v\}$  and  $N_D^-[v] = N^-[v] = N^-(v) \cup \{v\}$ , respectively. The *out-degree* and *in-degree* of a vertex  $v$  are defined by  $d_D^+(v) = d^+(v) = |N^+(v)|$  and  $d_D^-(v) = d^-(v) = |N^-(v)|$ . The *maximum out-degree*, *maximum in-degree*, *minimum out-degree* and *minimum in-degree* of a digraph  $D$  are denoted by  $\Delta^+(D) = \Delta^+$ ,  $\Delta^-(D) = \Delta^-$ ,  $\delta^+(D) = \delta^+$  and  $\delta^-(D) = \delta^-$ , respectively. A digraph  $D$  is  *$r$ -out-regular* when  $\Delta^+(D) = \delta^+(D) = r$  and  *$r$ -in-regular* when  $\Delta^-(D) = \delta^-(D) = r$ . If  $D$  is  $r$ -out-regular and  $r$ -in-regular, then  $D$  is called  *$r$ -regular*. The *complement*  $\overline{D}$  of a digraph  $D$  is the digraph with vertex set  $V(D)$  such that for any two distinct vertices  $u, v$  the arc  $uv$  belongs to  $\overline{D}$  if and only if  $uv$  does not belong to  $D$ . The *underlying graph*  $G(D)$  of a digraph  $D$  is the graph obtained from  $D$  by replacing each arc  $uv$  or symmetric pairs  $uv, vu$  of arcs by



the edge  $uv$ . A digraph  $D$  is *connected* if its underlying graph is connected. If  $X$  is a nonempty subset of the vertex set  $V(D)$  of a digraph  $D$ , then  $D[X]$  is the subdigraph of  $D$  induced by  $X$ . A digraph  $D$  is *bipartite* if its underlying graph is bipartite. Let  $K_n^*$  be the complete digraph of order  $n$  and  $K_{p,q}^*$  be the complete bipartite digraph with partite sets  $X$  and  $Y$ , where  $|X| = p$  and  $|Y| = q$ . The *associated digraph*  $D(G)$  of a graph  $G$  is the digraph obtained from  $G$  when each edge  $e$  of  $G$  is replaced by two oppositely oriented arcs with the same ends as  $e$ . Since  $N_{D(G)}^-[v] = N_G[v]$  for each vertex  $v \in V(G) = V(D(G))$ , the following useful observation is valid.

**Observation 74.** *Let  $D(G)$  be the associated digraph of the graph  $G$ . If  $\mu(G)$  is a graph parameter and  $\mu(D(G))$  the corresponding digraph parameter, then mostly  $\mu(D(G)) = \mu(G)$ .*

Similar to Section 1, the next subsections will be devoted to presenting the Roman domatic number and its variants in digraphs, some of which have recently been introduced.

## 2.2. Roman domatic number

Kamaraj and Jakkamal [25] defined a *Roman dominating function* (RDF) on a digraph  $D$  as a function  $f : V(D) \rightarrow \{0, 1, 2\}$  satisfying the condition that every vertex  $u$  with  $f(u) = 0$  has an in-neighbor  $v$  with  $f(v) = 2$ . The *weight* of an RDF  $f$  on a digraph  $D$  is defined by  $\omega(f) = \sum_{v \in V(D)} f(v)$ . The *Roman domination number* of a digraph  $D$ , denoted  $\gamma_R(D)$ , is the minimum weight of an RDF on  $D$ . An RDF of weight  $\gamma_R(D)$  is called a  $\gamma_R(D)$ -function. Roman domination on digraphs has been studied by several authors, as for example [20, 31].

A set  $\{f_1, f_2, \dots, f_d\}$  of distinct Roman dominating functions on  $D$  such that  $\sum_{i=1}^d f_i(v) \leq 2$  for each  $v \in V(G)$ , is called by Xie, Hao and Wei [54] a *Roman dominating family* (of functions) on  $D$ . The maximum number of functions in a Roman dominating family (RD family) on  $D$  is the *Roman domatic number* of  $D$ , denoted by  $d_R(D)$ . The Roman domatic number is well-defined and  $d_R(D) \geq 1$  for all digraphs  $D$ , since the set consisting of any RDF forms an RD family on  $D$ . The first result that we present characterizes the digraphs  $D$  with  $\gamma_R(D) = 1$ .

**Theorem 75** [54]. *If  $D$  is a digraph, then  $d_R(D) = 1$  if and only if  $D$  has no directed cycle of even length.*

An analogue to Theorem 1 also holds.

**Theorem 76** [54]. *If  $D$  is a digraph of order  $n$ , then  $\gamma_R(D) \cdot d_R(D) \leq 2n$ . Moreover, if  $\gamma_R(D) \cdot d_R(D) = 2n$ , then for each RD family  $\{f_1, f_2, \dots, f_d\}$  on  $D$  with  $d = d_R(D)$ , each function  $f_i$  is a  $\gamma_R(D)$ -function and  $\sum_{i=1}^d f_i(v) = 2$  for all  $v \in V(D)$ .*

As an application of Theorem 76, we obtain the following result.

**Theorem 77** [54]. *If  $D$  is a digraph of order  $n \geq 2$ , then  $d_R(D) \leq n$ , with equality if and only if  $D = K_n^*$ .*

Using the statement  $\gamma_R(D) < n$  if and only if  $\Delta^+(D) \geq 2$  (see [31]), and Theorems 75, 76, and 77, the next sharp bound on the sum  $\gamma_R(D) + d_R(D)$  is obtained.

**Theorem 78** [54]. *Let  $D$  be a digraph of order  $n \geq 2$ . Then  $\gamma_R(D) + d_R(D) \leq n + 2$ , with equality if and only if  $D = K_n^*$  or  $\Delta^+(D) = 1$  and  $D$  has a directed cycle of even length.*

If  $C_n$  is a directed cycle of length  $n$ , then  $\gamma_R(C_n) = n$  (see [31]). Using this observation as well as Theorems 75 and 76 the authors of [54] derived that  $d_R(C_n) = 1$  if  $n$  is odd and  $d_R(C_n) = 2$  if  $n$  is even. The next bounds are corresponding results to Theorems 6 and 8.

**Theorem 79** [54]. *If  $D$  is a digraph, then  $d_R(D) \leq \delta^-(D) + 2$ .*

The reader can find an example in [54] that illustrates the sharpness of Theorem 79.

**Theorem 80** [54]. *If  $D$  is a  $k$ -out-regular digraph of order  $n$ , where  $n = p(k + 1) + r$  with integers  $p \geq 1$  and  $0 \leq r \leq k$ , then  $d_R(D) \leq k + \epsilon$  with  $\epsilon = 1$  when  $k = 0$  or  $r = 0$  or  $2r = k + 1$  and  $\epsilon = 0$  otherwise.*

Using Theorems 79 and 80, the following Nordhaus-Gaddum type result is obtained.

**Theorem 81** [54]. *If  $D$  is a digraph of order  $n \geq 2$ , then  $d_R(D) + d_R(\overline{D}) \leq n + \epsilon$ , where  $\epsilon = 1$  when  $D$  is out-regular,  $\epsilon = 2$  when  $D$  is not in-regular and  $\epsilon = 3$  otherwise.*

The following extension of Theorem 79 can be found in a note of Volkmann and Meierling [53].

**Theorem 82** [53]. *If  $D$  is a digraph, then  $d_R(D) \leq \delta^-(D) + 2$ . Moreover, if  $d_R(D) = \delta^-(D) + 2$ , then the set of vertices of minimum in-degree is an independent set.*

Theorem 82 leads to the following improvement of Theorem 81.

**Theorem 83** [53]. *If  $D$  is a digraph of order  $n \geq 2$ , then  $d_R(D) + d_R(\overline{D}) \leq n + 1$ .*

If  $D$  is isomorphic to the complete digraph  $K_n^*$ , then  $d_R(D) = n$  and  $d_R(\overline{D}) = 1$  and therefore  $d_R(D) + d_R(\overline{D}) = n + 1$ . This example demonstrates that Theorem 83 is sharp. Using Observation 74 and Theorems 76, 78 or 79 we obtain Theorems 1, 5 or 6, respectively. Combining Observation 74 and Theorem 83, we arrive at the following improvement of Theorems 9 and 10.

**Corollary 84** [53]. *If  $G$  is a graph of order  $n \geq 2$ , then  $d_R(G) + d_R(\overline{G}) \leq n + 1$ .*

### 2.3. Italian domatic number in digraphs

Volkman [51] defined the following variant of Roman dominating functions in digraphs. An *Italian dominating function* (IDF) (or *Roman  $\{2\}$ -dominating function*) on a digraph  $D$  is a function  $f : V(D) \rightarrow \{0, 1, 2\}$  such that every vertex  $v \in V(D)$  with  $f(v) = 0$  has at least two in-neighbors assigned 1 under  $f$  or one in-neighbor  $w$  with  $f(w) = 2$ . The *Italian domination number*  $\gamma_I(D)$  (or *Roman  $\{2\}$ -domination number*) equals the minimum weight of an Italian dominating function on  $D$ , and an Italian dominating function of  $D$  with weight  $\gamma_I(D)$  is called a  $\gamma_I(D)$ -function.

Following Volkman [49], a set  $\{f_1, f_2, \dots, f_d\}$  of distinct Italian dominating functions on  $D$  with the property that  $\sum_{i=1}^d f_i(v) \leq 2$  for each  $v \in V(D)$ , is called an *Italian dominating family* (of functions) on  $D$ . The maximum number of functions in an Italian dominating family (ID family) on  $D$  is the *Italian domatic number* of  $D$ , denoted by  $d_I(D)$ . The Italian domatic number is well-defined and  $d_I(D) \geq 1$  for all digraphs  $D$ , since the set consisting of any IDF forms an Italian dominating family on  $D$ . We note that  $\gamma_I(D) \leq \gamma_R(D)$  and  $d_I(D) \geq d_R(D)$ . Therefore, every lower bound of  $d_R(D)$  is also a lower bound of  $d_I(D)$ . The following upper bounds are valid.

**Theorem 85** [49]. *If  $D$  is a digraph of order  $n$ , then  $\gamma_I(D) \cdot d_I(D) \leq 2n$ . Moreover, if  $\gamma_I(D) \cdot d_I(D) = 2n$ , then for each Italian dominating family  $\{f_1, f_2, \dots, f_d\}$  on  $D$  with  $d = d_I(D)$ , each function  $f_i$  is a  $\gamma_I(D)$ -function and  $\sum_{i=1}^d f_i(v) = 2$  for all  $v \in V(D)$ .*

Since  $\gamma_I(D) \geq 2$  for each digraph  $D$  of order  $n \geq 2$ , Theorem 85 immediately implies that  $d_R(D) \leq d_I(D) \leq n$ . In addition, it is shown in [49] that  $d_I(D) = n$  if and only if  $D = K_n^*$ . The next result is an analogue to Theorem 75.

**Theorem 86** [49]. *If  $D$  is a digraph, then  $d_I(D) = 1$  if and only if  $\Delta^-(D) \leq 1$  and  $D$  has no directed cycle of even length.*

If  $D$  is a directed path  $P_n$  or a directed cycle  $C_n$  of order  $n$ , then  $\gamma_I(D) = n$  (see [51]). Using this observation and Theorems 85 and 86, we easily see that  $d_I(P_n) = 1$  and  $d_I(C_n) = 1$  if  $n$  is odd and  $d_I(C_n) = 2$  if  $n$  is even.

The upper bound on the product  $\gamma_I(D) \cdot d_I(D)$  leads to an upper bound on the sum of these terms.

**Theorem 87** [49]. *If  $D$  is a digraph of order  $n \geq 2$ , then  $\gamma_I(D) + d_I(D) \leq n + 2$ . Moreover, equality holds if and only if  $\Delta^+(D) = \Delta^-(D) = 1$  and  $D$  has a directed cycle of even length or  $D = K_n^*$ .*

**Theorem 88** [49]. *If  $D$  is a digraph, then  $d_I(D) \leq \delta^-(D) + 2$ .*

Since  $d_R(D) \leq d_I(D)$ , Theorem 88 yields  $d_R(D) \leq \delta^-(D) + 2$  (see Theorem 79). Theorem 88 easily leads to the following Nordhaus-Gaddum type result.

**Theorem 89** [49]. *If  $D$  is a digraph of order  $n$ , then  $d_I(D) + d_I(\overline{D}) \leq n + 3$ . If  $d_I(D) + d_I(\overline{D}) = n + 3$ , then  $D$  is in-regular.*

We can improve Theorem 88 as well as Theorem 89 for many regular digraphs.

**Theorem 90** [49]. *Let  $D$  be a  $\delta$ -regular digraph of order  $n$  with  $\delta \geq 1$ , and let  $n = p(\delta + 2) + r$  with integers  $p \geq 0$  and  $0 \leq r \leq \delta + 1$ . If  $1 \leq r < (\delta + 2)/2$  or  $(\delta + 2)/2 < r \leq \delta + 1$ , then  $d_I(D) \leq \delta + 1$ .*

**Theorem 91** [49]. *If  $D$  is a  $\delta$ -regular digraph of order  $n$ , then  $d_I(D) + d_I(\overline{D}) \leq n + 2$ , except when  $D$  is 4-regular of order 9, 7-regular of order 18 or 16-regular of order 45.*

We conclude this section by mentioning some conjectures suggested by the presented research.

**Conjecture 92** [49]. *If  $D$  is a  $\delta$ -regular digraph, then  $d_I(D) \leq \delta + 1$ .*

The next conjecture would be a consequence of Conjecture 92.

**Conjecture 93** [49]. *If  $D$  is a  $\delta$ -regular digraph of order  $n$ , then  $d_I(D) + d_I(\overline{D}) \leq n + 1$ .*

Conjecture 93 is true for  $\delta = 0$  and for  $\delta = 1$ . If  $n = p_1(\delta + 2) + r_1$  with  $r_1 \neq 0, (\delta + 2)/2$  and  $n = p_2(\delta + 2) + r_2$  with  $r_2 \neq 0, (\delta + 2)/2$ , then Theorem 90 shows that Conjecture 93 is also true. We even think that the bound in Conjecture 93 holds for all digraphs.

**Conjecture 94** [49]. *If  $D$  is a digraph of order  $n$ , then  $d_I(D) + d_I(\overline{D}) \leq n + 1$ .*

## 2.4. Double Roman domatic number in digraphs

Inspired by an idea in [9], Hao, Chen and Volkmann [19] defined the *double Roman dominating function* (DRD function) on a digraph  $D$  as a function  $f : V(D) \rightarrow \{0, 1, 2, 3\}$  having the property that if  $f(v) = 0$ , then the vertex  $v$  must have at least two in-neighbors assigned 2 under  $f$  or one in-neighbor  $w$  with  $f(w) = 3$ , and if  $f(v) = 1$ , then the vertex  $v$  must have at least one in-neighbor  $u$  with  $f(u) \geq 2$ . The *double Roman domination number*  $\gamma_{dR}(D)$  equals the minimum weight of a DRD function on  $D$ , and a DRD function of  $D$  with weight  $\gamma_{dR}(D)$  is called a  $\gamma_{dR}(D)$ -function.

In [52], a set  $\{f_1, f_2, \dots, f_d\}$  of distinct double Roman dominating functions on  $D$  with the property that  $\sum_{i=1}^d f_i(v) \leq 3$  for each  $v \in V(D)$ , is called a *double Roman dominating family* (of functions) on  $D$ . The maximum number

of functions in a double Roman dominating family (DRD family) on  $D$  is the *double Roman domatic number* of  $D$ , denoted by  $d_{dR}(D)$ . The double Roman domatic number is well-defined and  $d_{dR}(D) \geq 1$  for all digraphs  $D$ , since the set consisting of any DRD function forms a DRD family on  $D$ . We start with the following basic properties.

**Theorem 95** [52]. *If  $D$  is a digraph of order  $n$ , then  $\gamma_{dR}(D) \cdot d_{dR}(D) \leq 3n$ .*

**Theorem 96** [52]. *If  $D$  is a digraph, then  $\gamma_{dR}(D) \leq \delta^-(D) + 1$ .*

Examples in [52] show that Theorems 95 and 96 are sharp. As a consequence of the last theorem we see that  $d_{dR}(D) \leq n$ . Following an idea of Zelinka [56], it is easy to see that  $d_{dR}(D) \geq \lfloor n/(n - \delta^-(D)) \rfloor$ . Using this and Theorem 96, it is shown in [52] that  $d_{dR}(D) = n$  if and only if  $D$  is isomorphic to the complete digraph. In addition, it is not hard to verify that  $d_{dR}(D) \geq 2$  for each bipartite digraph  $D$  with  $\delta^-(D) \geq 1$ . Consequently,  $d_{dR}(C_n) = 2$  for each oriented cycle  $C_n$  of even length. Next we present an upper bound on the double Roman domatic number in terms of the maximum out-degree.

**Theorem 97** [52]. *Let  $D$  be a digraph of order  $n \geq 2$ , and let  $k$  be an integer with  $2 \leq k \leq n$ . If  $\Delta^+(D) \leq (n - k)/(k - 1)$ , then  $d_{dR}(D) \leq n/k$ .*

Using Theorem 97 for  $k = (n + 1)/2$ , we deduce that  $d_{dR}(C_n) = 1$  for each oriented cycle  $C_n$  of odd length. As an application of Theorems 96 and 97, we arrive at the following Nordhaus-Gaddum type bounds.

**Theorem 98** [52]. *Let  $D$  be a digraph of order  $n \geq 3$ . Then  $d_{dR}(D) + d_{dR}(\overline{D}) \leq n + 1$ . If  $d_{dR}(D) + d_{dR}(\overline{D}) = n + 1$ , then  $D$  is in-regular. If  $\Delta^+(D) \leq n - 2$  and  $\Delta^+(\overline{D}) \leq n - 2$ , then  $d_{dR}(D) + d_{dR}(\overline{D}) \leq n$ , and if  $n$  is odd, then  $d_{dR}(D) + d_{dR}(\overline{D}) \leq n - 1$ .*

Finally, we present upper and lower bounds on the sum  $\gamma_{dR}(D) + d_{dR}(D)$ . Using the bound  $\gamma_{dR}(D) \leq 2n - 2$  for each connected digraph of order  $n \geq 4$  (see [19]) and Theorem 95, we obtain the following upper bound.

**Theorem 99** [52]. *If  $D$  is a connected digraph of order  $n \geq 5$ , then  $\gamma_{dR}(D) + d_{dR}(D) \leq 2n - 1$ .*

Since  $\gamma_{dR}(C_4) + d_{dR}(C_4) = 8$ ,  $\gamma_{dR}(C_3) + d_{dR}(C_3) = 6$  and  $\gamma_{dR}(C_2) + d_{dR}(C_2) = 5$ , we observe that Theorem 99 does not hold for  $2 \leq n \leq 4$ . In addition, let  $H$  be the digraph of order  $n \geq 5$  with vertex set  $\{v_1, v_2, \dots, v_n\}$  and arc set  $\{v_2v_1, v_3v_1, \dots, v_nv_1\}$ . Then  $\gamma_{dR}(H) = 2(n - 1)$  and  $d_{dR}(H) = 1$  and thus  $\gamma_{dR}(H) + d_{dR}(H) = 2n - 1$ . This example shows that Theorem 99 is sharp. In [19], it is shown that  $\gamma_{dR}(D) = 3$  if and only if  $\Delta^+(D) = n - 1$ . As an application of this observation one can prove a sharp lower bound on  $\gamma_{dR}(D) + d_{dR}(D)$ .

**Theorem 100** [52]. *If  $D$  is a digraph of order  $n \geq 2$ , then  $\gamma_{dR}(D) + d_{dR}(D) \geq 4$ , with equality if and only if  $D$  contains a vertex  $v$  with  $d_D^+(v) = n - 1$  and  $d_D^-(v) = 0$ .*

## 2.5. Signed (total) Roman domatic number in digraphs

A *signed (total) Roman dominating function* (SRDF, STRDF) on a digraph  $D$  is defined in [33] ([44]) as a function  $f : V(D) \rightarrow \{-1, 1, 2\}$  such that  $\sum_{x \in N^-[v]} f(x) \geq 1$  ( $\sum_{x \in N^-(v)} f(x) \geq 1$ ) for each  $v \in V(D)$ , and such that every vertex  $u \in V(D)$  with  $f(u) = -1$  has an in-neighbor  $w$  for which  $f(w) = 2$ . The *signed (total) Roman domination number*  $\gamma_{sR}(D)$  ( $\gamma_{stR}(D)$ ) equals the minimum weight of an SRDF (STRDF) on  $D$ , and a signed (total) Roman dominating function of  $D$  with weight  $\gamma_{sR}(D)$  ( $\gamma_{stR}(D)$ ) is called a  $\gamma_{sR}(D)$ -function ( $\gamma_{stR}(D)$ -function).

A set  $\{f_1, f_2, \dots, f_d\}$  of distinct signed (total) Roman dominating functions on  $D$  with the property that  $\sum_{i=1}^d f_i(v) \leq 1$  for each  $v \in V(D)$ , is called in [34] ([3]) a *signed (total) Roman dominating family* (of functions) on  $D$ . The maximum number of functions in a signed (total) Roman dominating family (SRD (STRD) family) on  $D$  is the *signed (total) Roman domatic number* of  $D$ , denoted by  $d_{sR}(D)$  ( $d_{stR}(D)$ ). The signed (total) Roman domatic number is well-defined and  $d_{sR}(D) \geq 1$  ( $d_{stR}(D) \geq 1$ ) for all digraphs  $D$  (with  $\delta^-(D) \geq 1$ ), since the set consisting of the SDRF (DTRDF) with constant value 1 forms an SRD (STRDF) family on  $D$ . Using Observation 74 and the fact that  $d_{sR}(K_3) = 1$  and  $d_{sR}(K_n) = n$  ( $n \neq 3$ ), (see Subsection 1.8), we deduce that  $d_{sR}(K_3^*) = 1$  and  $d_{sR}(K_n^*) = n$  for  $n \neq 3$ .

We start with some basic properties.

**Theorem 101** [3, 34]. (i) *If  $D$  is a digraph of order  $n$ , then  $\gamma_{sR}(D) \cdot d_{sR}(D) \leq n$ . Moreover, if  $\gamma_{sR}(D) \cdot d_{sR}(D) = n$ , then for each SRD family  $\{f_1, f_2, \dots, f_d\}$  on  $D$  with  $d = d_{sR}(D)$ , each function  $f_i$  is a  $\gamma_{sR}(D)$ -function and  $\sum_{i=1}^d f_i(v) = 1$  for all  $v \in V(D)$ .*

(ii) *If  $D$  is a digraph of order  $n$ , then  $\gamma_{stR}(D) \cdot d_{stR}(D) \leq n$ . Moreover, if  $\gamma_{stR}(D) \cdot d_{stR}(D) = n$ , then for each STRD family  $\{f_1, f_2, \dots, f_d\}$  on  $D$  with  $d = d_{stR}(D)$ , each function  $f_i$  is a  $\gamma_{stR}(D)$ -function and  $\sum_{i=1}^d f_i(v) = 1$  for all  $v \in V(D)$ .*

**Theorem 102** [3, 34]. (i) *For every digraph  $D$ , we have  $d_{sR}(D) \leq \delta^-(D) + 1$ . Moreover, if  $d_{sR}(D) = \delta^-(D) + 1$ , then for each SRD family  $\{f_1, f_2, \dots, f_d\}$  on  $D$  with  $d = d_{sR}(D)$  and each vertex  $v$  of minimum in-degree,  $\sum_{x \in N^-[v]} f_i(x) = 1$  for each function  $f_i$  and  $\sum_{i=1}^d f_i(x) = 1$  for all  $x \in N^-[v]$ .*

(ii) *For every digraph  $D$  with  $\delta^-(D) \geq 1$ ,  $d_{stR}(D) \leq \delta^-(D)$ . Moreover, if equality holds, then for each STRD family  $\{f_1, f_2, \dots, f_d\}$  on  $D$  with  $d = d_{stR}(D)$*

and each vertex  $v$  of minimum in-degree,  $\sum_{u \in N^-(v)} f_i(u) = 1$  for each function  $f_i$  and  $\sum_{i=1}^d f_i(u) = 1$  for all  $u \in N^-(v)$ .

As applications of these results, we obtain  $d_{sR}(K_{1,n}^*) = 1$  ( $d_{sR}(K_{1,n}^*) = 1$ ) for  $n \geq 2$ ,  $d_{sR}(P_n) = 1$  for an oriented path  $P_n$ ,  $d_{sR}(C_n) = 1$  for an oriented cycle  $C_n$  of odd length and  $d_{sR}(C_n) = 2$  for an oriented cycle  $C_n$  of even length. In addition, if  $p \geq 4$  is an even integer, then  $d_{sR}(K_{p,p}^*) = \frac{p}{2}$  when  $p \neq 6$ . Since  $\gamma_{sR}(K_{p,p}^*) = 4$  for  $p \geq 3$  (see [33]), the complete bipartite digraph  $K_{p,p}^*$  shows for even  $p \neq 6$  that Theorem 101 (i) is sharp. The next two results are also consequences of Theorems 101 and 102.

**Theorem 103** [3, 34]. (i) If  $D$  is a digraph of order  $n$ , then  $\gamma_{sR}(D) + d_{sR}(D) \leq n + 1$ , with equality if and only if  $D = K_n^*$  ( $n \neq 3$ ) or  $D$  consists of the disjoint union of isolated vertices and oriented triangles.

(ii) If  $D$  is a digraph of order  $n \geq 1$  with  $\delta^-(D) \geq 1$ , then  $\gamma_{stR}(D) + d_{stR}(D) \leq n + 1$ . Moreover, if  $\delta^-(D) \geq 5$ , then  $\gamma_{stR}(D) + d_{stR}(D) \leq n - 2$  unless  $n = 6$ , in which case  $\gamma_{stR}(D) + d_{stR}(D) = n - 1$ .

**Theorem 104** [3, 34]. (i) If  $D$  is a digraph of order  $n$ , then  $d_{sR}(D) + d_{sR}(\overline{D}) \leq n + 1$ . Furthermore, if  $d_{sR}(D) + d_{sR}(\overline{D}) = n + 1$ , then  $D$  is in-regular.

(ii) Let  $D$  be a digraph of order  $n$  such that  $\delta^-(D), \delta^-(\overline{D}) \geq 1$ . Then  $d_{stR}(D) + d_{stR}(\overline{D}) \leq n - 1$ . Furthermore, if  $d_{stR}(D) + d_{stR}(\overline{D}) = n - 1$ , then  $D$  is in-regular.

Using Observation 74, Theorems 101, 102, 103 or 104, we obtain the next known results (see Subsection 1.8).

**Corollary 105** [34]. If  $G$  is a graph of order  $n$ , then  $d_{sR}(G) \cdot \gamma_{sR}(G) \leq n$ ,  $d_{sR}(G) \leq \delta(G) + 1$ ,  $d_{sR}(G) + \gamma_{sR}(G) \leq n + 1$  and  $d_{sR}(G) + d_{sR}(\overline{G}) \leq n + 1$ .

For some out-regular digraphs one can improve the upper bound given in Theorem 102.

**Theorem 106** [34]. Let  $D$  be an  $r$ -out-regular digraph of order  $n$  with  $r \geq 1$ . If  $n \not\equiv 0 \pmod{r+1}$ , then  $d_{sR}(D) \leq r$ .

As an application of Theorem 106, we present the following supplement to Theorem 104 for regular digraphs.

**Theorem 107** [34]. Let  $D$  be a  $\delta$ -regular digraph of order  $n$ . Then  $d_{sR}(D) + d_{sR}(\overline{D}) = n + 1$  if and only if  $D = K_n^*$  or  $\overline{D} = K_n^*$  and  $n \neq 3$ .

For tournaments of odd order, the following improvement of Theorem 104 is valid.

**Theorem 108** [3, 34]. If  $T$  is a tournament of odd order  $n \geq 3$ , then  $d_{sR}(T) + d_{sR}(\overline{T}) \leq n - 1$  ( $d_{stR}(T) + d_{stR}(\overline{T}) \leq n - 3$ ).

## 2.6. Twin signed (total) Roman domatic number in digraphs

If  $D$  is a digraph, then let  $D^-$  be the digraph obtained by reversing all arcs of  $D$ . A signed (total) Roman dominating function  $f$  on a digraph  $D$  is defined by Bodaghli, Sheikholeslami and Volkmann in [10] (Amjadi and Soroudi [6]) as a *twin signed Roman dominating function* (TSRDF, TSTRDF) if it is also a signed (total) Roman dominating function of  $D^{-1}$ , i.e.,  $f(N^+[v]) \geq 1$  ( $f(N^+(v)) \geq 1$ ) for each  $v \in V(D)$  and every vertex  $u$  for which  $f(u) = -1$  has an out-neighbor  $w$  with  $f(w) = 2$ . The *twin signed (total) Roman domination number*  $\gamma_{sR}^*(D)$  ( $\gamma_{stR}^*(D)$ ) equals the minimum weight of a TSRDF (TSTRDF) on  $D$ , and a twin signed total Roman dominating function of  $D$  with weight  $\gamma_{sR}^*(D)$  ( $\gamma_{stR}^*(D)$ ) is called a  $\gamma_{sR}^*(D)$ -function ( $\gamma_{stR}^*(D)$ -function). Since every TSRDF of  $D$  is an SRDF on both  $D$  and  $D^{-1}$ , and since the constant function 1 is a TSRDF of  $D$ , we have  $\max\{\gamma_{sR}(D), \gamma_{sR}(D^{-1})\} \leq \gamma_{sR}^*(D) \leq n$ . Likewise, for any digraph  $D$  with  $\min\{\delta^-(D), \delta^+(D)\} \geq 1$  we have  $\max\{\gamma_{stR}(D), \gamma_{stR}(D^{-1})\} \leq \gamma_{stR}^*(D) \leq n$ .

A set  $\{f_1, f_2, \dots, f_d\}$  of distinct twin signed total Roman dominating functions on  $D$  with the property that  $\sum_{i=1}^d f_i(v) \leq 1$  for each  $v \in V(D)$ , is called in [35] ([4]) a *twin signed total Roman dominating family* (of functions) on  $D$ . The maximum number of functions in a twin signed total Roman dominating family (TSRD (TSTRD) family) on  $D$  is the *twin signed total Roman domatic number* of  $D$ , denoted by  $d_{sR}^*(D)$  ( $d_{stR}^*(D)$ ). The twin signed total Roman domatic number is well-defined and  $d_{sR}^*(D) \geq 1$  ( $d_{stR}^*(D) \geq 1$ ) for all digraphs  $D$ , since the set consisting of the TSRDF (TSTRDF) with constant value 1 forms a TSRD (TSTRD) family on  $D$ . Since every TSRD (TSTRD) family of  $D$  is an SRD (STRD) family on both  $D$  and  $D^{-1}$ , we have  $d_{sR}^*(D) \leq \min\{d_{sR}(D), d_{sR}(D^{-1})\}$  ( $d_{stR}^*(D) \leq \min\{d_{stR}(D), d_{stR}(D^{-1})\}$ ). Combining this inequality with Theorem 102, we arrive at the following result.

**Proposition 109** [6, 35]. (i) If  $D$  is a digraph, then  $d_{sR}^*(D) \leq \min\{\delta^-(D), \delta^+(D)\} + 1$ .

(ii) If  $D$  is a digraph with  $\min\{\delta^-(D), \delta^+(D)\} \geq 1$ , then  $d_{stR}^*(D) \leq \min\{\delta^-(D), \delta^+(D)\}$ .

We observed that  $d_{sR}^*(D) \leq d_{sR}(D)$  and  $d_{stR}^*(D) \leq d_{stR}(D)$ . However, Sheikholeslami and Volkmann [35] gave examples which show that the difference  $d_{sR}(D) - d_{sR}^*(D)$  can be arbitrarily large. Modifying their example, Amjadi and Soroudi showed that the difference  $d_{stR}(D) - d_{stR}^*(D)$  can be arbitrarily large. Using Observation 74 and the fact that  $d_{sR}(K_3) = 1$  and  $d_{sR}(K_n) = n$  ( $n \neq 3$ ) (see Subsection 1.8), we deduce that  $d_{sR}^*(K_3) = 1$  and  $d_{sR}^*(K_n) = n$  for  $n \neq 3$ .

**Theorem 110** [6, 35]. (i) If  $D$  is a digraph of order  $n$ , then  $\gamma_{sR}^*(D) \cdot d_{sR}^*(D) \leq n$ . Moreover, if  $\gamma_{sR}^*(D) \cdot d_{sR}^*(D) = n$ , then for each TSRD family  $\{f_1, f_2, \dots, f_d\}$  on  $D$  with  $d = d_{sR}^*(D)$ , each function  $f_i$  is a  $\gamma_{sR}^*(D)$ -function and  $\sum_{i=1}^d f_i(v) = 1$  for all  $v \in V(D)$ .



(ii) If  $D$  is a digraph of order  $n$ , then  $\gamma_{stR}^*(D) \cdot d_{stR}^*(D) \leq n$ . Moreover, if  $\gamma_{stR}^*(D) \cdot d_{stR}^*(D) = n$ , then for each TSTRD family  $\{f_1, f_2, \dots, f_d\}$  on  $D$  with  $d = d_{stR}^*(D)$ , each function  $f_i$  is a  $\gamma_{stR}^*(D)$ -function and  $\sum_{i=1}^d f_i(v) = 1$  for each  $v \in V(D)$ .

The complete digraph  $K_n^*$  with  $n \geq 4$  and the complete bipartite digraph  $K_{p,p}^*$  with  $p \geq 4$  even and  $p \neq 6$  satisfy equality in Theorem 110(i). If  $C_n$  is an oriented cycle, then  $\gamma_{sR}^*(C_n) = n/2$  when  $n$  is even and  $\gamma_{sR}^*(C_n) = (n+3)/2$  when  $n$  is odd (see [10]). Using this, Proposition 109 and Theorem 110(i), it was shown in [35] that  $d_{sR}^*(C_n) = 2$  when  $n$  is even and  $d_{sR}^*(C_n) = 1$  when  $n$  is odd. Thus the oriented cycle  $C_n$  is another example which fulfill Theorem 110(i) with equality when  $n$  is even. As further applications of Proposition 109 and Theorem 110, we obtain a sharp bound on the sum  $\gamma_{sR}^*(D) + d_{sR}^*(D)$  and a Nordhaus-Gaddum type inequality.

**Theorem 111** [6, 35]. (i) If  $D$  is a digraph of order  $n$ , then  $\gamma_{sR}^*(D) + d_{sR}^*(D) \leq n+1$ , with equality if and only if  $D = K_n^*$  ( $n \neq 3$ ) or  $\gamma_{sR}^*(D) = n$  and  $d_{sR}^*(D) = 1$ .

(ii) If  $D$  is a digraph of order  $n$  with  $\min\{\delta^-(D), \delta^+(D)\} \geq 1$ , then  $\gamma_{stR}^*(D) + d_{stR}^*(D) \leq n+1$ .

If  $H$  is the disjoint union of oriented triangles, then we see that  $\gamma_{sR}^*(H) = n(H)$  and  $d_{sR}^*(H) = 1$ . Thus  $\gamma_{sR}^*(D) = n(D)$  and  $d_{sR}^*(D) = 1$  is possible in Theorem 111(i).

**Theorem 112** [6, 35]. (i) For every digraph  $D$  of order  $n$ , we have  $d_{sR}^*(D) + d_{sR}^*(\overline{D}) \leq n+1$ , with equality if and only if  $D = K_n^*$  or  $\overline{D} = K_n^*$  and ( $n \neq 3$ ).

(ii) Let  $D$  be a digraph of order  $n$  such that  $\min\{\delta(D), \delta(\overline{D})\} \geq 1$ . Then  $d_{stR}^*(D) + d_{stR}^*(\overline{D}) \leq n-1$ . Furthermore, if  $d_{stR}^*(D) + d_{stR}^*(\overline{D}) = n-1$ , then  $D$  is in-regular.

## 2.7. Signed Roman $k$ -domatic number in digraphs

If  $k \geq 1$  is an integer, then the *signed Roman  $k$ -dominating function* (SRkDF) on a digraph  $D$  is defined in [42] as a function  $f : V(D) \rightarrow \{-1, 1, 2\}$  such that  $\sum_{x \in N^-[v]} f(x) \geq k$  for each  $v \in V(D)$ , and such that every vertex  $u \in V(D)$  with  $f(u) = -1$  has an in-neighbor  $w$  for which  $f(w) = 2$ . The *signed Roman  $k$ -domination number*  $\gamma_{sR}^k(D)$  equals the minimum weight of an SRkDF on  $D$ , and a signed Roman  $k$ -dominating function of  $D$  with weight  $\gamma_{sR}^k(D)$  is called a  $\gamma_{sR}^k(D)$ -function. If  $k = 1$ , then write  $\gamma_{sR}^1(D) = \gamma_{sR}(D)$ , as in Subsection 2.5.

A set  $\{f_1, f_2, \dots, f_d\}$  of distinct signed Roman  $k$ -dominating functions on  $D$  with the property that  $\sum_{i=1}^d f_i(v) \leq k$  for each  $v \in V(D)$ , is called in [43] a *signed Roman  $k$ -dominating family* (of functions) on  $D$ . The maximum number of functions in a signed Roman  $k$ -dominating family (SRkD family) on  $D$  is the

signed Roman  $k$ -domatic number of  $D$ , denoted by  $d_{sR}^k(D)$ . The special case  $k = 1$  was introduced and studied in Subsection 2.5.

The signed Roman  $k$ -domatic number exists when  $\delta^-(D) \geq \frac{k}{2} - 1$ . However, for investigations of the signed Roman  $k$ -domatic number it is reasonable to claim that  $\delta^-(D) \geq k - 1$ . Thus we assume throughout this subsection that  $\delta^-(D) \geq k - 1$ . The signed Roman  $k$ -domatic number is well-defined and  $d_{sR}^k(D) \geq 1$  for all digraphs  $D$ , since the set consisting of the SRkDF with constant value 1 forms an SRkD family on  $D$ . Using Observation 74, Proposition 65 and the corresponding results in Subsection 1.8, we obtain the signed Roman  $k$ -domatic number of the complete digraph.

**Corollary 113** [43]. *If  $n \geq k \geq 1$ , then  $d_{sR}^k(K_n^*) = n$  unless  $k = 1$  and  $n = 3$  in which case  $d_{sR}(K_3^*) = 1$  and unless  $n = k = 2$  in which case  $d_{sR}^2(K_2^*) = 1$ .*

The following extensions of Theorems 101 and 102 hold.

**Theorem 114** [43]. *If  $D$  is a digraph of order  $n$ , then  $\gamma_{sR}^k(D) \cdot d_{sR}^k(D) \leq kn$ . Moreover, if  $\gamma_{sR}^k(D) \cdot d_{sR}^k(D) = kn$ , then for each SRkD family  $\{f_1, f_2, \dots, f_d\}$  on  $D$  with  $d = d_{sR}^k(D)$ , each function  $f_i$  is a  $\gamma_{sR}^k(D)$ -function and  $\sum_{i=1}^d f_i(v) = k$  for all  $v \in V(D)$ .*

**Theorem 115** [43]. *For every digraph  $D$ , we have  $d_{sR}^k(D) \leq \delta^-(D) + 1$ . Moreover, if  $d_{sR}^k(D) = \delta^-(D) + 1$ , then for each SRkD family  $\{f_1, f_2, \dots, f_d\}$  on  $D$  with  $d = d_{sR}^k(D)$  and each vertex  $v$  of minimum in-degree,  $\sum_{x \in N^-[v]} f_i(x) = k$  for each function  $f_i$  and  $\sum_{i=1}^d f_i(x) = k$  for all  $x \in N^-[v]$ .*

Different examples in [43] show that Theorems 114 and 115 are sharp. Using Theorem 114, we obtain a sharp upper bound on the sum  $\gamma_{sR}^k(D) + d_{sR}^k(D)$ .

**Theorem 116** [43]. *If  $D$  is a digraph of order  $n$ , then  $\gamma_{sR}^k(D) + d_{sR}^k(D) \leq n + k$ . If  $\gamma_{sR}^k(D) + d_{sR}^k(D) = n + k$ , then*

- (a)  $\gamma_{sR}^k(D) = k$  and  $d_{sR}^k(D) = n$  (in this case  $D = K_n^*$  unless  $k = 1$  and  $n = 3$  or  $n = k = 2$ ) or
- (b)  $\gamma_{sR}^k(D) = n$  and  $d_{sR}^k(D) = k$  (in this case  $D$  is the disjoint union of isolated vertices and oriented triangles when  $k = 1$ ,  $k \neq 2$  and  $k - 1 \leq \delta^-(D) \leq k$  when  $k \geq 3$ ).

For some out-regular digraphs one can improve the upper bound given in Theorem 115.

**Theorem 117** [43]. *Let  $D$  be a  $\delta$ -out-regular digraph of order  $n$  with  $\delta \geq k - 1$  such that  $n = p(\delta + 1) + r$  with integers  $p \geq 1$  and  $1 \leq r \leq \delta$  and  $kr = t(\delta + 1) + s$  with integers  $t \geq 0$  and  $1 \leq s \leq \delta$ . Then  $d_{sR}^k(D) \leq \delta$ .*

A digraph without directed cycles of length 2 is called an *oriented graph*. As an application of Theorems 115 and 117, we obtain the following corollaries.

**Corollary 118** [43]. *If  $D$  is an oriented graph of order  $n$  such that  $\delta^-(D), \delta^-(D^{-1}) \geq k$ , then  $d_{sR}^k(D) + d_{sR}^k(D^{-1}) \leq n$ .*

**Corollary 119** [43]. *If  $T$  is a  $\delta$ -regular tournament of order  $n$  such that  $\delta^-(T) \geq k$ , then  $d_{sR}^k(T) + d_{sR}^k(T^{-1}) \leq n$ .*

Theorem 115 easily leads to the following Nordhaus-Gaddum type inequality.

**Theorem 120** [43]. *If  $D$  is a digraph of order  $n$  such that  $\delta^-(D), \delta^-(\overline{D}) \geq k-1$ , then  $d_{sR}^k(D) + d_{sR}^k(\overline{D}) \leq n+1$ . Furthermore, if  $d_{sR}^k(D) + d_{sR}^k(\overline{D}) = n+1$ , then  $D$  is in-regular.*

Using Theorems 115 and 117, one can improve Theorem 120 for tournaments of odd order.

**Theorem 121** [43]. *If  $T$  is a tournament of odd order  $n \geq 3$  such that  $\delta^-(T), \delta^-(\overline{T}) \geq k$ , then  $d_{sR}^k(T) + d_{sR}^k(\overline{T}) \leq n-1$ .*

As a supplement to Theorem 107, we present the next result for  $k \geq 2$ .

**Theorem 122** [43]. *Let  $k \geq 2$  be an integer, and let  $D$  be a  $\delta$ -regular digraph such that  $\delta \geq k-1$  and  $\overline{\delta} = \delta^-(\overline{D}) \geq k-1$ . Then there are only a finite number of digraphs  $D$  such that  $d_{sR}^k(D) + d_{sR}^k(\overline{D}) = n(D) + 1$ .*

In connection with Theorem 122, we state the following conjecture.

**Conjecture 123** [43]. *Let  $k \geq 2$  be an integer. If  $D$  is a  $\delta$ -regular digraph of order  $n$  such that  $\delta, \overline{\delta} \geq k-1$ , then  $d_{sR}^k(D) + d_{sR}^k(\overline{D}) \leq n$ .*

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