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SUM-LIST COLOURING OF UNIONS OF A HYPERCYCLE AND A PATH WITH AT MOST TWO VERTICES IN COMMON

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Abstract

Given a hypergraph $\mathcal{H}$ and a function $f : V(\mathcal{H}) \longrightarrow \mathbb{N}$, we say that $\mathcal{H}$ is f-choosable if there is a proper vertex colouring ϕ of $\mathcal{H}$ such that $\phi(v) \in L(v)$ for all $v \in V(\mathcal{H})$, where $L : V(\mathcal{H}) \longrightarrow 2^{\mathbb{N}}$ is any assignment of $f(v)$ colours to a vertex v. The sum choice number $\mathcal{H}i_{sc}(\mathcal{H})$ of $\mathcal{H}$ is defined to be the minimum of $\sum_{v\in V(\mathcal{H})} f(v)$ over all functions f such that $\mathcal{H}$ is f-choosable. For an arbitrary hypergraph $\mathcal{H}$ the inequality $\chi_{sc}(\mathcal{H}) \leq |V(\mathcal{H})| + |\mathcal{E}(\mathcal{H})|$ holds, and hypergraphs that attain this upper bound are called *sc*-greedy. In this paper we characterize *sc*-greedy hypergraphs that are unions of a hypercycle and a hyperpath having at most two vertices in common. Consequently, we characterize the hypergraphs of this type that are forbidden for the class of *sc*-greedy hypergraphs.

Keywords: hypergraphs, sum-list colouring, induced hereditary classes, forbidden hypergraphs.

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1. Introduction

A hypergraph $\mathcal{H}$ consists of a non-empty finite set $V(\mathcal{H})$ of *vertices* and a finite set $\mathcal{E}(\mathcal{H})$ of *edges*, each of which is a subset of $V(\mathcal{H})$ having at least two elements. We allow the existence of edges that are subsets of others, in particular, multiple edges. Let $k \in \mathbb{N}$. An edge of a hypergraph that contains exactly k vertices is called a *k-edge*. A hypergraph is *linear* if every two of its edges have at most one vertex in common. A linear hypergraph having only 2-edges is called a *graph*. We follow the notation and terminology of [1, 4] concerning hypergraphs and graphs.

Let $\mathcal{H}$ be a hypergraph. A *proper colouring of* $\mathcal{H}$ is a mapping $\phi : V(\mathcal{H}) \longrightarrow \mathbb{N}$ such that every edge $E \in \mathcal{E}(\mathcal{H})$ contains at least two different vertices v_1, v_2 satisfying $\phi(v_1) \neq \phi(v_2)$ that is, there are no monochromatic edges. Given a mapping $L : V(\mathcal{H}) \longrightarrow 2^{\mathbb{N}}$ we call a mapping $\phi : V(\mathcal{H}) \longrightarrow \mathbb{N}$ *an* L*-colouring of* $\mathcal{H}$ if for every vertex $v \in V(\mathcal{H})$ it holds that $\phi(v) \in L(v)$. Let $f : V(\mathcal{H}) \longrightarrow \mathbb{N}$. To emphasize the meaning of f in the notions that we introduce, we call f a *size function*. By size(f) we mean $\sum_{v \in V(\mathcal{H})} f(v)$. A mapping $L : V(\mathcal{H}) \longrightarrow 2^{\mathbb{N}}$ such that $|L(v)| = f(v)$ for every vertex v in $V(\mathcal{H})$ is called an f*-assignment* for $\mathcal{H}$. The hypergraph $\mathcal{H}$ is f*-choosable* if for each f-assignment L for $\mathcal{H}$ there is a proper L-colouring of $\mathcal{H}$. Thus, $\mathcal{H}$ is f-choosable if $\mathcal{H}$ is properly L-colourable for each f-assignment L for $\mathcal{H}$. If $\mathcal{H}$ is f-choosable, then f is called a *choice* function for $\mathcal{H}$. The *sum-choice-number* $\chi_{sc}(\mathcal{H})$ of $\mathcal{H}$ is the minimum of size(f) taken over all choice functions f for $\mathcal{H}$. If f is a choice function for $\mathcal{H}$ and $\chi_{sc}(\mathcal{H}) = \text{size}(f)$, then we say that f *realizes* $\chi_{sc}(\mathcal{H})$.

The concept of sum-list colouring, and consequently, the notion of the sum-choice-number of a graph has been introduced by Isaak in [9]. Since then many results on the sum-list colourability appeared. Among others Heinold [8] investigated the sum-list colourability of θ-graphs. Brause *et al.* [3] determined upper bounds on the sum-choice-numbers of all generalized θ-graphs $\theta_{k_1,k_2,\ldots,k_r}$, where $k_i \geq 2$ for every i, and next they characterized all generalized θ-graphs G that attain the upper bound $|V(G)| + |E(G)|$. Graphs that meet this bound, named *sc-greedy*, gain a lot of attention in the literature. It is known that paths, cycles, trees, complete graphs, and all graphs on at most four vertices are *sc*-greedy. All *sc*-greedy graphs on five vertices were determined by Lastrina [16] and on six vertices by Kemnitz *et al.* [13]. Moreover, all *sc*-greedy complete multipartite graphs [12], wheels [14], and broken wheels [8] were determined. Isaak [10] proved that block graphs are *sc*-greedy. Kemnitz *et al.* [13] considered *sc*-greediness of the Cartesian product, the direct product, the strong product, and the lexicographic product of two graphs.

In [5, 6, 11, 15] some generalization of the sum-list colourability concept has been investigated. In this variant the graphs induced by the vertices of the same colour belong to some specific fixed class of graphs (not necessarily to the class of edgeless graphs). For example, in [5, 6] the authors considered sum-list colouring in which colour classes induce acyclic graphs.

The idea of sum-list colouring for hypergraphs has been introduced in [7], where it also has been shown that for each hypergraph $\mathcal{H}$ the inequality $\chi_{sc}(\mathcal{H}) \leq |V(\mathcal{H})| + |\mathcal{E}(\mathcal{H})|$ holds. It leads to the generalization of the notion of *sc*-greediness from graphs to hypergraphs. Namely, a hypergraph $\mathcal{H}$ is called *sc-greedy* if $\chi_{sc}(\mathcal{H}) = |V(\mathcal{H})| + |\mathcal{E}(\mathcal{H})|$. Let $\mathcal{H}_1 \cup \mathcal{H}_2$ denote the *union* of hypergraphs $\mathcal{H}_1$, $\mathcal{H}_2$, i.e., $V(\mathcal{H}_1 \cup \mathcal{H}_2) = V(\mathcal{H}_1) \cup V(\mathcal{H}_2)$, $\mathcal{E}(\mathcal{H}_1 \cup \mathcal{H}_2) = \mathcal{E}(\mathcal{H}_1) \cup \mathcal{E}(\mathcal{H}_2)$. By the definition of the sum-choice-number of a hypergraph it immediately follows that if $\mathcal{H}_1, \mathcal{H}_2$ are vertex disjoint hypergraphs, then $\chi_{sc}(\mathcal{H}_1 \cup \mathcal{H}_2) = \chi_{sc}(\mathcal{H}_1) + \chi_{sc}(\mathcal{H}_2)$. Also, based on the result from [2], it has been shown in [7] that if $\mathcal{H}_1$, $\mathcal{H}_2$ are two hypergraphs that have exactly one vertex in common, then $\chi_{sc}(\mathcal{H}_1 \cup \mathcal{H}_2) = \chi_{sc}(\mathcal{H}_1) + \chi_{sc}(\mathcal{H}_2) - 1$. Thus if two hypergraphs have at most one vertex in common and both are *sc*-greedy, then their union is also *sc*-greedy. This leads to the following problem.

Problem 1. Let $\mathcal{H}_1$, $\mathcal{H}_2$ be two *sc*-greedy hypergraphs that have exactly two vertices in common. Under which conditions is $\mathcal{H}_1 \cup \mathcal{H}_2$ *sc*-greedy?

It is known (see [7]) that each hypercycle and each hyperpath is *sc*-greedy (the precise definitions of these notions are given in Section 3). In this paper we focus our attention on the solution of Problem 1 in the case when $\mathcal{H}_1$ is a hypercycle and $\mathcal{H}_2$ is a hyperpath. The results on θ-graphs obtained in [8] and their extensions concerning θ-hypergraphs obtained in [7] can be viewed as a part of the solution of this special subproblem of Problem 1. The main results of this paper, together with the just mentioned results on θ-hypergraphs, determine the sum-choice-number of all hypergraphs that can be obtained as a union of a hyperpath and a hypercycle having exactly two vertices in common. Consequently, they completely characterize *sc*-greedy hypergraphs of this type and hypergraphs of this type that are forbidden for the family of *sc*-greedy hypergraphs.

This paper is organized as follows. In Section 2 we give some basic definitions and basic results concerning sum-list colourability of hypergraphs. In Section 3 we formulate some necessary and sufficient conditions to be *sc*-greedy for a hypergraph that is a union of a hypercycle and a hyperpath having exactly two vertices in common. In Section 4 we propose examples which show how to apply results for hypergraphs to different variants of sum-list colouring of graphs. Section 5 discusses other possible applications of the results.

2. Preliminaries

Let $\mathcal{H}$ be a hypergraph. By a subhypergraph of $\mathcal{H}$ we mean an arbitrary hypergraph $\mathcal{H}'$ such that $V(\mathcal{H}') \subseteq V(\mathcal{H})$ and $\mathcal{E}(\mathcal{H}') \subseteq \mathcal{E}(\mathcal{H})$.

Observation 1 [7]**.** *If f is a choice function for a hypergraph $\mathcal{H}$ and $\mathcal{H}'$ is a subhypergraph of $\mathcal{H}$, then $f|_{V(\mathcal{H}')}$ is a choice function for $\mathcal{H}'$.*

Suppose that f is a choice function for a hypergraph $\mathcal{H}$ and there is $u \in V(\mathcal{H})$ such that $f(u) \geq \deg_{\mathcal{H}}(u) + 2$. Using Observation 1 it was shown in [7] that also f' is a choice function for $\mathcal{H}$, where $f'(u) = f(u) - 1$ and $f'(v) = f(v)$ for the remaining vertices. Hence we have the following fact.

Observation 2 [7]**.** *If f realizes $\chi_{sc}(\mathcal{H})$, then $f(v) \leq \deg_{\mathcal{H}}(v)+1$ for every vertex v of a hypergraph $\mathcal{H}$.*

Let $\mathcal{H}$ be a hypergraph and $V' \subseteq V(\mathcal{H})$ and $\mathcal{E}' \subseteq \mathcal{E}(\mathcal{H})$. A subhypergraph of $\mathcal{H}$ induced by V', denoted by $\mathcal{H}[V']$, has the vertex set V' and the edge set $\{E \in \mathcal{E}(\mathcal{H}) : E \subseteq V'\}$. We use notation $\mathcal{H} - V'$ instead of $\mathcal{H}[V(\mathcal{H}) \setminus V']$ and even $\mathcal{H} - v$ instead of $\mathcal{H} - \{v\}$. A subhypergraph of $\mathcal{H}$ induced by $\mathcal{E}'$, denoted by $\mathcal{H}[\mathcal{E}']$, has a vertex set $\bigcup_{E\in\mathcal{E}'} E$ and an edge set $\mathcal{E}'$. If we say that $\mathcal{H}'$ is an induced subhypergraph of $\mathcal{H}$, then we mean that $\mathcal{H}'$ is induced in $\mathcal{H}$ by some set of vertices.

Given the hypergraph $\mathcal{H}$ and $v \in V(\mathcal{H})$, by $\mathcal{H}(v)$ we denote a hypergraph induced in $\mathcal{H}$ by the set of all edges containing the vertex v. The *degree* of v in $\mathcal{H}$, denoted by $\deg_{\mathcal{H}}(v)$, is defined as the number of edges of $\mathcal{H}(v)$.

As we mentioned earlier, for each hypergraph $\mathcal{H}$ we have $\chi_{sc}(\mathcal{H}) \leq |V(\mathcal{H})| + |\mathcal{E}(\mathcal{H})|$. Recall that hypergraphs that achieve this upper bound are called *sc*-greedy. By Γ_{sc} we denote the family of all *sc*-greedy hypergraphs, and let $GB(\mathcal{H}) = |V(\mathcal{H})| + |\mathcal{E}(\mathcal{H})|$. In [7] the following property of Γ_{sc} was established.

Observation 3 [7]**.** *Each hypergraph in Γ_{sc} is linear.*

Furthermore, a large family of linear *sc*-greedy hypergraphs was determined in [7].

Definition (Hypertree)**.** The 1-vertex hypergraph is a *hypertree* without edges. Next, a hypergraph that has exactly one edge consisting of all its vertices is a *hypertree* with one edge. A *hypertree* with m edges ($m \geq 2$) is an arbitrary union of two hypertrees having exactly one vertex in common and m edges in total.

Definition (Hyperpath, hypercycle)**.** A hypertree is a *hyperpath* if there is an ordering (called *canonical*) of its vertex set such that each edge consists of some number of consecutive vertices (with respect to this ordering). A union of two hyperpaths having exactly two vertices in common is called a *hypercycle*, provided that two common vertices are the first and last vertices in some canonical orderings of both these hyperpaths. The length of a hyperpath (hypercycle) is the number of its edges.

Theorem 4 [7]**.** *Γ_{sc} contains all hypertrees and all hypercycles.*

Given a family $\mathcal{G}$ of hypergraphs, let $\overline{\mathcal{G}}$ be the recursively defined family of hypergraphs such that $\mathcal{G} \subseteq \overline{\mathcal{G}}$, and for any two hypergraphs $\mathcal{H}_1$, $\mathcal{H}_2$ in $\overline{\mathcal{G}}$ that have at most one vertex in common, the hypergraph $\mathcal{H}_1 \cup \mathcal{H}_2$ is also in $\overline{\mathcal{G}}$. Recall that the union of two *sc*-greedy hypergraphs having at most one vertex in common is also *sc*-greedy. Hence, using the mentioned observation, denoting by $\mathcal{G}^*$ the family of all hypercycles and all hyperpaths and applying Theorem 4 we obtain the following facts.

Corollary 5. *If $\mathcal{G} \subseteq \Gamma_{sc}$, then $\overline{\mathcal{G}} \subseteq \Gamma_{sc}$ and, consequently, $\overline{\mathcal{G}^*} \subseteq \Gamma_{sc}$.*

Definition (Hypercycle with handle)**.** A union of a hypercycle and a hyperpath having exactly two vertices in common is a *hypercycle with handle* provided that it is linear and two common vertices are the first and last vertices in some canonical ordering of the hyperpath.

In the next section we discuss when the union of a hypercycle and a hyperpath having exactly two vertices in common is *sc*-greedy. In fact we determine all hypergraphs of this type in Γ_{sc} even though we consider only hypercycles with handle. Indeed, only linear union of a hypercycle and a hyperpath having exactly two vertices in common can be *sc*-greedy by Observation 3. Let $\mathcal{G}^{**}$ be the family of all hypercycles with handle. Clearly, a linear union of a hypercycle and a hyperpath having exactly two vertices in common is always an element of $\overline{\mathcal{G}^{**}}$. Thus the results for a hypercycle with handle will imply the results for a linear union of a hypercycle and a hyperpath having exactly two vertices in common

By a *class of hypergraphs* we mean a family of hypergraphs that is closed under isomorphism. A class of hypergraphs is *induced hereditary* if it is closed with respect to taking induced subhypergraphs. Each induced hereditary class $\mathcal{R}$ of hypergraphs can be uniquely characterized by the family $\mathcal{C}(\mathcal{R})$ of forbidden hypergraphs (for $\mathcal{R}$), where

$$\mathcal{C}(\mathcal{R}) = \{\mathcal{H} : \mathcal{H} \notin \mathcal{R} \text{ and } \mathcal{H}' \in \mathcal{R} \text{ for each proper induced subhypergraph } \mathcal{H}' \text{of } \mathcal{H}\}.$$

It was observed in [7] that Γ_{sc} is an induced hereditary class of hypergraphs. As we mentioned previously, this paper deals with hypercycles with handle that are in the class Γ_{sc}. The construction of hypercycles with handle implies that if such a hypergraph is not in Γ_{sc}, then it must be in $\mathcal{C}(\Gamma_{sc})$. Thus our consideration also deals with hypergraphs with handle in $\mathcal{C}(\Gamma_{sc})$.

Note that we must be very careful in our investigation, since unexpectedly, Γ_{sc} is not closed with respect to taking subhypergraphs. For example, it was shown in [16] that $K_{2,3}$ is not *sc*-greedy, unlike to a graph obtained from $K_{2,3}$ by the addition of an edge joining two vertices of degree three.

For forthcoming proofs we need the following lemma that has been proved in [7].

Lemma 6. *Let $\mathcal{H}$ be a linear hypergraph in $\mathcal{C}(\Gamma_{sc})$ and $v \in V(\mathcal{H})$. If f is a function that realizes $\chi_{sc}(\mathcal{H})$, then*

(i) *$f(v) \leq \deg_{\mathcal{H}}(v)$, and*

(ii) *$\deg_{\mathcal{H}}(v) \geq 2$ implies $f(v) \geq 2$ provided that each edge of $\mathcal{H}(v)$ contains at most two vertices having degree greater than one in $\mathcal{H}$.*

3. Hypercycles with Handle

In this section we discuss when a hypercycle with handle belongs to Γ_{sc}. The case when a hypercycle with handle has two different vertices of degree three has been considered in [7]. Such hypergraphs are called θ-hypergraphs. To be more formal, let $k_1, k_2, k_3 \in \mathbb{N}$. By $\theta^h_{k_1,k_2,k_3}$ we denote the hypergraph consisting of two vertices of degree three connected by three internally disjoint hyperpaths of lengths k_1, k_2, k_3. In what follows, we use the notion of a hyperpath of $\theta^h_{k_1,k_2,k_3}$ of length k_i, $i \in \{1, 2, 3\}$, meaning the hyperpath of length k_i used in the definition of $\theta^h_{k_1,k_2,k_3}$. By a θ-hypergraph we mean an arbitrary hypergraph $\theta^h_{k_1,k_2,k_3}$.

Theorem 7 [7]**.** *Let $k_1, k_2, k_3 \in \mathbb{N}$. A hypergraph $\theta^h_{k_1,k_2,k_3}$ is sc-greedy if and only if one of the hyperpaths of $\theta^h_{k_1,k_2,k_3}$, say the hyperpath of length k_2, has only 2-edges and, under this assumption, one of the following conditions holds:*

(i) *$k_1 + k_2$ and $k_2 + k_3$ are odd numbers and at least one of the inequalities $k_1 \geq 2$, $k_3 \geq 2$ holds, or*

(ii) *$k_1 + k_2$ is an odd number and $k_2 + k_3$ is an even number and $k_3 \geq 3$, or*

(iii) *$k_1 + k_2$ is an even number and $k_2 + k_3$ is an even number and $k_1 \geq 3$ and $k_3 \geq 3$.*

In the next part of this section we prove Theorem 11 concerning the case when all vertices of a hypercycle with handle have degrees less than three and next Theorem 16 concerning the case when one vertex of such a hypergraph has degree three and the remaining ones have degrees less than three. Since the degrees of all vertices of a hypercycle with handle are not greater than three and such a hypergraph can have at most two vertices of degree three, our consideration covers all possibilities, giving a characterization of *sc*-greedy hypercycles with handle.

Lemma 8. *If $\mathcal{H}$ is a linear hypercycle with handle, then $\chi_{sc}(\mathcal{H}) \geq GB(\mathcal{H}) - 1$.*

Proof. Let E be an edge of $\mathcal{H}$ and $\mathcal{H}'$ be a hypergraph obtained from $\mathcal{H}$ by the removal of the edge E. The assumptions on $\mathcal{H}$ imply that $\mathcal{H}' \in \overline{\mathcal{G}^*}$. Hence, we have $\mathcal{H}' \in \Gamma_{sc}$ by Corollary 5. Thus $\chi_{sc}(\mathcal{H}') = GB(\mathcal{H}') = GB(\mathcal{H}) - 1$. Observation 1 implies $\chi_{sc}(\mathcal{H}) \geq \chi_{sc}(\mathcal{H}') \geq GB(\mathcal{H}) - 1$. ■

Lemma 9. *Let $\mathcal{P}$ be a hyperpath of length p that has at least two vertices and let $v_1, \ldots, v_n$ be a canonical ordering of $V(\mathcal{P})$. Next, let f_1, f_2 be size functions for $\mathcal{P}$ defined by $f_1(v_i) = f_2(v_i) = \deg_{\mathcal{P}}(v_i)$ for $i \in \{2, \ldots, n-1\}$ and $f_1(v_n) = f_2(v_n) = 2$ and $f_1(v_1) = 1$ and $f_2(v_1) = 2$. If a, b are different positive integers, then*

(i) *there exists an f_1-assignment L for $\mathcal{P}$ such that for every proper L-colouring ϕ of $\mathcal{P}$ it holds that $\phi(v_1) = a$ and $\phi(v_n) = b$, and*

(ii) *if $p \geq 2$, then there exists an f_1-assignment L for $\mathcal{P}$ such that for every proper L-colouring ϕ of $\mathcal{P}$ it holds that $\phi(v_1) = \phi(v_n) = a$, and*

(iii) *there exists an f_2-assignment L for $\mathcal{P}$ such that for every proper L-colouring ϕ of $\mathcal{P}$ it holds that $\phi(v_1) = a$ or $\phi(v_n) = b$, and*

(iv) *there exists an f_2-assignment L for $\mathcal{P}$ such that for every proper L-colouring ϕ of $\mathcal{P}$ it holds that $\phi(v_1) = a$ or $\phi(v_n) = a$.*

Proof. Let $\{v_{i_j} : j \in \{1, \ldots, p-1\}\}$ be the set of vertices of degree two in $\mathcal{P}$ with $i_j < i_l$ for $j < l$ and let $v_{i_p} = v_n$.

We define L that confirms the statements (i) and (ii) in the following way:

(a) if either (p is odd and we claim that $\phi(v_1) = a$ and $\phi(v_n) = b$) or (p is even and we claim $\phi(v_1) = \phi(v_n) = a$), then
- $L(v_s) = \{a\}$ for $s \in \{1, \ldots, i_1 - 1\}$, and
- $L(v_{i_j}) = \{a, b\}$ for $j \in \{1, \ldots, p\}$, and
- $L(v_s) = \{b\}$ for $s \in \{i_j + 1, \ldots, i_{j+1} - 1\}$ and odd j in $\{1, \ldots, p-1\}$, and
- $L(v_s) = \{a\}$ for $s \in \{i_j + 1, \ldots, i_{j+1} - 1\}$ and even j in $\{1, \ldots, p-1\}$,

(b) if either (p is even and we claim that $\phi(v_1) = a$ and $\phi(v_n) = b$) or (p is odd, $p \geq 3$, and we claim that $\phi(v_1) = \phi(v_n) = a$), then
- $L(v_{i_1}) = \{a, c\}$, $L(v_{i_2}) = \{b, c\}$, $L(v_{i_j}) = \{a, b\}$ for $j \in \{3, \ldots, p\}$, and
- $L(v_s) = \{c\}$ for $s \in \{i_1 + 1, \ldots, i_2 - 1\}$, and
- $L(v_s) = \{a\}$ for $s \in \{1, \ldots, i_1 - 1\}$ and also for $s \in \{i_j + 1, \ldots, i_{j+1} - 1\}$ and odd j in $\{3, \ldots, p-1\}$, and
- $L(v_s) = \{b\}$ for $s \in \{i_j + 1, \ldots, i_{j+1} - 1\}$ and even j in $\{1, \ldots, p-1\}$.

Next we define L that confirms the statements (iii) and (iv) in the following way:

(c) if either (p is odd and we claim that $\phi(v_1) = a$ or $\phi(v_n) = a$) or (p is even and we claim that $\phi(v_1) = a$ or $\phi(v_n) = b$), then
- $L(v_1) = \{a, b\} = L(v_{i_j})$ for $j \in \{1, \ldots, p\}$, and
- $L(v_s) = \{b\}$ for $s \in \{2, \ldots, i_1 - 1\}$ and also for $s \in \{i_j + 1, \ldots, i_{j+1} - 1\}$ and even j in $\{1, \ldots, p-1\}$, and
- $L(v_s) = \{a\}$ for $s \in \{i_j + 1, \ldots, i_{j+1} - 1\}$ and odd j in $\{1, \ldots, p-1\}$,

(d) if p is odd and we claim that $\phi(v_1) = a$ or $\phi(v_n) = b$, then
- $L(v_{i_j}) = \{b, c\}$ for $j \in \{1, \ldots, p\}$, and
- $L(v_1) = \{a, c\}$, and
- $L(v_s) = \{c\}$ for $s \in \{2, \ldots, i_1 - 1\}$ and also for $s \in \{i_j + 1, \ldots, i_{j+1} - 1\}$ and even j in $\{1, \ldots, p-1\}$, and
- $L(v_s) = \{b\}$ for $s \in \{i_j + 1, \ldots, i_{j+1} - 1\}$ and odd j in $\{1, \ldots, p-1\}$,

(e) if p is even and we claim that $\phi(v_1) = a$ or $\phi(v_n) = a$, then

- $L(v_1) = \{a, b\}$, $L(v_{i_1}) = \{b, c\}$, $L(v_{i_j}) = \{a, c\}$ for $j \in \{2, \ldots, p\}$, and
- $L(v_s) = \{b\}$ for $s \in \{2, \ldots, i_1 - 1\}$, and
- $L(v_s) = \{c\}$ for $s \in \{i_j + 1, \ldots, i_{j+1} - 1\}$ and odd j in $\{1, \ldots, p-1\}$, and
- $L(v_s) = \{a\}$ for $s \in \{i_j + 1, \ldots, i_{j+1} - 1\}$ and even j in $\{1, \ldots, p-1\}$. ■

Lemma 10. *Let $\mathcal{H}$ be a hypercycle with handle and let the degrees of all vertices of $\mathcal{H}$ be less than three. If $\mathcal{H}$ satisfies at least one of the conditions*

(i) *$\mathcal{H}$ has no hypercycle of length three, or*

(ii) *$\mathcal{H}$ has an edge with four vertices of degree two,*

then $\mathcal{H}$ is sc-greedy.

Proof. Assume, by contradiction, that $\mathcal{H}$ has no hypercycle of length 3 or $\mathcal{H}$ has an edge with four vertices of degree two and additionally $\mathcal{H} \notin \Gamma_{sc}$. Since each proper induced subhypergraph of $\mathcal{H}$ is in $\overline{\mathcal{G}^*}$, we have $\mathcal{H} \in \mathcal{C}(\Gamma_{sc})$ by Corollary 5.

Case 1. First suppose that $\mathcal{H}$ has no edge with four vertices of degree two. Therefore, the handle connects vertices in different edges. $\mathcal{H}$ has no hypercycles of length 3. This implies that $\mathcal{H}$ has two edges E_1, E_2, $E_1 \neq E_2$, each with three vertices of degree two. We consider two subcases.

Subcase 1.1. $E_1 \cap E_2 = \emptyset$. From the construction of $\mathcal{H}$ we can assume that $V(\mathcal{H}) = E_1 \cup E_2 \cup \bigcup_{i=1}^{3} V(\mathcal{P}_i)$ and $\mathcal{E}(\mathcal{H}) = \bigcup_{i=1}^{3} \mathcal{E}(\mathcal{P}_i) \cup \{E_1, E_2\}$, where $\mathcal{P}_1$, $\mathcal{P}_2, \mathcal{P}_3$ are three disjoint hyperpaths, each on at least two vertices. Moreover, for $i \in \{1, 2, 3\}$ there is a canonical ordering $v_1^i, \ldots, v_{n_i}^i$ of $V(\mathcal{P}_i)$ such that $\{v_1^1, v_1^2, v_1^3\} \subseteq E_1$, $\{v_{n_1}^1, v_{n_2}^2, v_{n_3}^3\} \subseteq E_2$ and E_1, E_2 do not contain any other vertex of any of $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ (see Figure 1).

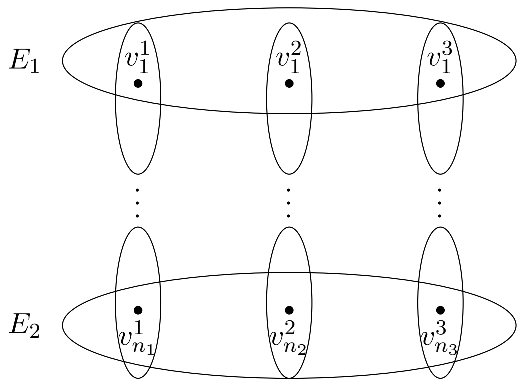


Figure 1. Illustration to the proof of Lemma 10, Subcase 1.1.

Let f be a function that realizes $\chi_{sc}(\mathcal{H})$. Since $\mathcal{H} \in \mathcal{C}(\Gamma_{sc})$ and every vertex of $\mathcal{H}$ is of degree at most two, Lemma 6(i) implies that $f(v) \leq 2$ for $v \in V(\mathcal{H})$. Let $V' = \{v_1^1, v_1^2, v_1^3, v_{n_1}^1, v_{n_2}^2, v_{n_3}^3\}$. Furthermore, Lemma 6(ii) implies that $f(v) = \deg_{\mathcal{H}}(v)$ for $v \in V(\mathcal{H}) \setminus V'$. Note that the number of vertices of degree two in $\mathcal{H}$ is equal to $|\mathcal{E}(\mathcal{H})| + 1$. Since $\mathcal{H}$ is not *sc*-greedy, we have $\chi_{sc}(\mathcal{H}) = GB(\mathcal{H}) - 1$ by Lemma 8. Thus the values of f for two vertices in V' are equal to one. Note that it cannot happen that for both end-vertices of a hyperpath $\mathcal{P}_i$ the size function f has values one, since in this case $\text{size}(f|_{V(\mathcal{P}_i)}) = GB(\mathcal{P}_i) - 1$ and $\mathcal{P}_i$ is not $f|_{V(\mathcal{P}_i)}$-choosable (recall that every hyperpath is *sc*-greedy) and hence $\mathcal{H}$ is not f-choosable. Thus, without loss of generality, either $f(v_1^1) = f(v_1^2) = 1$ or $f(v_1^1) = f(v_{n_2}^2) = 1$.

By Lemma 9(i), there is an $f|_{V(\mathcal{P}_1)}$-assignment L_1 such that for every proper L_1-colouring ϕ of $\mathcal{P}_1$ it holds that $\phi(v_1^1) = a$ and $\phi(v_{n_1}^1) = b$, and similarly, there is an $f|_{V(\mathcal{P}_2)}$-assignment L_2 such that for every proper L_2-colouring ϕ of $\mathcal{P}_2$ it holds that $\phi(v_1^2) = a$ and $\phi(v_{n_2}^2) = b$. Furthermore, Lemma 9(iii) implies that there is an $f|_{V(\mathcal{P}_3)}$-assignment L_3 such that for every proper L_3-colouring ϕ of $\mathcal{P}_3$ it holds that $\phi(v_1^3) = a$ or $\phi(v_{n_3}^3) = b$. Let L be an f-assignment for $\mathcal{H}$ defined in the following way.

$$L(v) = \begin{cases} L_1(v), & \text{if } v \in V(\mathcal{P}_1), \\ L_2(v), & \text{if } v \in V(\mathcal{P}_2), \\ L_3(v), & \text{if } v \in V(\mathcal{P}_3), \\ \{a\}, & \text{if } v \in E_1 \setminus \{v_1^1, v_1^2, v_1^3\}, \\ \{b\}, & \text{if } v \in E_2 \setminus \{v_{n_1}^1, v_{n_2}^2, v_{n_3}^3\}. \end{cases}$$

Thus, in every L-colouring of $\mathcal{H}$ such that vertices of $\mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$ are properly coloured, the vertices in $\{v_1^1, v_1^2, v_1^3\}$ are coloured with a or the vertices in $\{v_{n_1}^1, v_{n_2}^2, v_{n_3}^3\}$ are coloured with b. Hence, in every proper L-colouring, E_1 is monochromatic or E_2 is monochromatic. This implies that f is not a choice function for $\mathcal{H}$, a contradiction.

Subcase 1.2. $|E_1 \cap E_2| = 1$. In this case we can assume that $V(\mathcal{H}) = E_1 \cup E_2 \cup \bigcup_{i=1}^{2} V(\mathcal{P}_i)$ and $\mathcal{E}(\mathcal{H}) = \bigcup_{i=1}^{2} \mathcal{E}(\mathcal{P}_i) \cup \{E_1, E_2\}$, where $\mathcal{P}_1, \mathcal{P}_2$ are two disjoint hyperpaths. Moreover, for $i \in \{1, 2\}$, there is a canonical ordering of $V(\mathcal{P}_i)$, say $v_1^i, \ldots, v_{n_i}^i$, such that $\{v_1^1, v_1^2\} \subseteq E_1$, $\{v_{n_1}^1, v_{n_2}^2\} \subseteq E_2$ and E_1, E_2 do not contain any other vertex of any of $\mathcal{P}_1, \mathcal{P}_2$ (see Figure 2). Observe that our assumption that $\mathcal{H}$ has no edge with four vertices of degree two implies that $\mathcal{H}$ has no hypercycle of length 3. Thus, the lengths of both $\mathcal{P}_1$, $\mathcal{P}_2$ are at least two. Moreover, we have a common vertex of E_1, E_2, say u, such that $u \notin V(\mathcal{P}_1) \cup V(\mathcal{P}_2)$.

Let f be a function that realizes $\chi_{sc}(\mathcal{H})$. Since $\mathcal{H} \in \mathcal{C}(\Gamma_{sc})$ and every vertex of $\mathcal{H}$ is of degree at most two, Lemma 6(i) implies that $f(v) \leq 2$ for $v \in V(\mathcal{H})$. Let

$V'' = \{v_1^1, v_1^2, v_{n_1}^1, v_{n_2}^2, u\}$. Furthermore, Lemma 6(ii) implies that $f(v) = \deg_{\mathcal{H}}(v)$ for $v \in V(\mathcal{H}) \setminus V''$. Since $\chi_{sc}(\mathcal{H}) = GB(\mathcal{H}) - 1$ by Lemma 8, the values of f for two vertices in V'' are equal to one. Similarly as in Subcase 1.1, we can observe that it cannot happen that for both end-vertices of a hyperpath $\mathcal{P}_i$ the size function f has values one. Thus, without loss of generality, either $f(v_1^1) = f(v_1^2) = 1$ or $f(v_1^1) = f(v_{n_2}^2) = 1$ or $f(u) = f(v_1^1) = 1$.

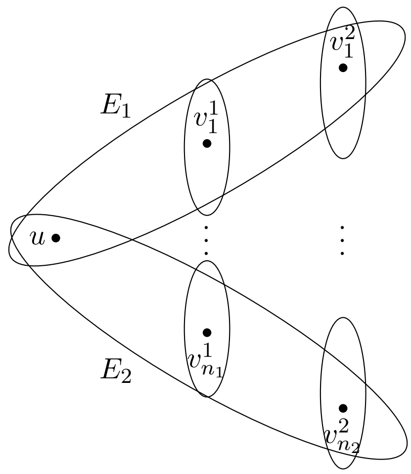


Figure 2. Illustration to the proof of Lemma 10, Subcase 1.2.

Suppose first that either $f(v_1^1) = f(v_1^2) = 1$ or $f(v_1^1) = f(v_{n_2}^2) = 1$. By Lemma 9(i), there is an $f|_{V(\mathcal{P}_1)}$-assignment L_1 such that for every proper L_1-colouring ϕ of $\mathcal{P}_1$ it holds that $\phi(v_1^1) = a$ and $\phi(v_{n_1}^1) = b$. Similarly, there is an $f|_{V(\mathcal{P}_2)}$-assignment L_2 such that for every proper L_2-colouring ϕ of $\mathcal{P}_2$ it holds that $\phi(v_1^2) = a$ and $\phi(v_{n_2}^2) = b$. Let L be an f-assignment for $\mathcal{H}$ defined in the following way.

$$L(v) = \begin{cases} L_1(v), & \text{if } v \in V(\mathcal{P}_1), \\ L_2(v), & \text{if } v \in V(\mathcal{P}_2), \\ \{a,b\}, & \text{if } v = u, \\ \{a\}, & \text{if } v \in E_1 \setminus \{v_1^1, v_1^2, u\}, \\ \{b\}, & \text{if } v \in E_2 \setminus \{v_{n_1}^1, v_{n_1}^2, u\}. \end{cases}$$

Thus, in every L-colouring of $\mathcal{H}$ such that vertices of $\mathcal{P}_1 \cup \mathcal{P}_2$ are properly coloured, vertices in $\{v_1^1, v_1^2\}$ are coloured with a and vertices in $\{v_{n_1}^1, v_{n_2}^2\}$ are coloured with b. Hence either E_1 is monochromatic or E_2 is monochromatic, which contradicts the fact that $\mathcal{H}$ is f-choosable.

Suppose now that $f(u) = f(v_1^1) = 1$. Since both $\mathcal{P}_1, \mathcal{P}_2$ have lengths at least two, we may apply Lemma 9(ii) for the path $\mathcal{P}_1$ and Lemma 9(iv) for the path $\mathcal{P}_2$. Let L_1 be an $f|_{V(\mathcal{P}_1)}$-assignment for $\mathcal{P}_1$ such that for every proper L_1-colouring ϕ of $\mathcal{P}_1$ it holds that $\phi(v_1^1) = a$ and $\phi(v_{n_1}^1) = a$. Let L_2 be an $f|_{V(\mathcal{P}_2)}$-assignment such that for every proper L_2-colouring ϕ of $\mathcal{P}_2$ it holds that $\phi(v_1^2) = a$ or $\phi(v_{n_2}^2) = a$. Let L be an f-assignment for $\mathcal{H}$ defined in the following way.

$$L(v) = \begin{cases} L_1(v), & \text{if } v \in V(\mathcal{P}_1), \\ L_2(v), & \text{if } v \in V(\mathcal{P}_2), \\ \{a\}, & \text{if } v \in (E_1 \cup E_2) \setminus \{v_1^1, v_1^2, v_{n_1}^1, v_{n_2}^2\}. \end{cases}$$

Thus, in every L-colouring of $\mathcal{H}$ such that vertices of $\mathcal{P}_1 \cup \mathcal{P}_2$ are properly coloured, all vertices in E_1 are coloured with a or all vertices in E_2 are coloured with a. It contradicts f-choosability of $\mathcal{H}$.

Case 2. Now suppose that $\mathcal{H}$ has an edge E' with four vertices of degree two. In this case we can assume that $V(\mathcal{H}) = E' \cup \bigcup_{i=1}^{2} V(\mathcal{P}_i)$ and $\mathcal{E}(\mathcal{H}) = \bigcup_{i=1}^{2} \mathcal{E}(\mathcal{P}_i) \cup \{E'\}$, where $\mathcal{P}_1, \mathcal{P}_2$ are two disjoint hyperpaths. For $i \in \{1, 2\}$ let $v_1^i, \ldots, v_{n_i}^i$ be a canonical ordering of $V(\mathcal{P}_i)$, such that $\{v_1^1, v_1^2, v_{n_1}^1, v_{n_2}^2\} \subseteq E'$ and E' does not contain any other vertex of any of $\mathcal{P}_1, \mathcal{P}_2$ (see Figure 3). Since $\mathcal{H}$ is linear, each of $\mathcal{P}_1$, $\mathcal{P}_2$ has at least two edges.

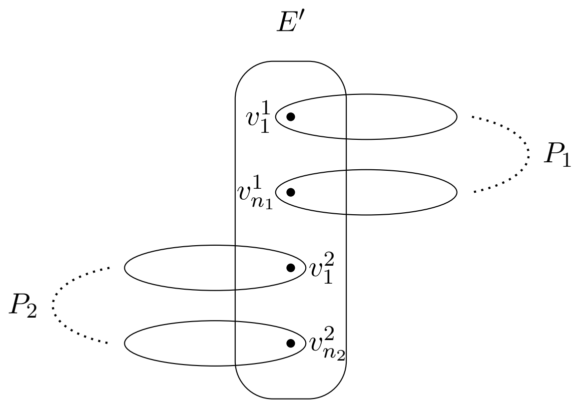


Figure 3. Illustration to the proof of Lemma 10, Case 2.

Let f be a function that realizes $\chi_{sc}(\mathcal{H})$ and let $V''' = \{v_1^1, v_1^2, v_{n_1}^1, v_{n_2}^2\}$. Since $\mathcal{H} \in \mathcal{C}(\Gamma_{sc})$, similarly as before, Lemma 6 implies that $f(v) = \deg_{\mathcal{H}}(v)$ for $v \in V(\mathcal{H}) \setminus V'''$ and $f(v) \leq \deg_{\mathcal{H}}(v)$ for $v \in V'''$. Furthermore, the values of f for two vertices in V''' not on the same path $\mathcal{P}_i$ are equal to one.

By Lemma 9(ii), there is an $f|_{V(\mathcal{P}_1)}$-assignment L_1 such that for every proper L_1-colouring ϕ of $\mathcal{P}_1$ it holds that $\phi(v_1^1) = a$ and $\phi(v_{n_1}^1) = a$ and there is an $f|_{V(\mathcal{P}_2)}$-assignment L_2 such that for every proper L_2-colouring ϕ of $\mathcal{P}_2$ it holds that $\phi(v_1^2) = a$ and $\phi(v_{n_2}^2) = a$. Let L be an f-assignment for $\mathcal{H}$ defined in the following way.

$$L(v) = \begin{cases} L_1(v), & \text{if } v \in V(\mathcal{P}_1), \\ L_2(v), & \text{if } v \in V(\mathcal{P}_2), \\ \{a\}, & \text{if } v \in E' \setminus V'''. \end{cases}$$

Thus, in every L-colouring of $\mathcal{H}$ such that vertices of $\mathcal{P}_1 \cup \mathcal{P}_2$ are properly coloured, vertices in V''' and thus all vertices in E' are coloured with a. This implies that $\mathcal{H}$ is not f-choosable, a contradiction. ■

Theorem 11. *Let $\mathcal{H}$ be a hypercycle with handle and let the degrees of all vertices of $\mathcal{H}$ be less than three. The hypergraph $\mathcal{H}$ is sc-greedy if and only if at least one of the following conditions holds:*

(i) *$\mathcal{H}$ has no hypercycle of length three, or*

(ii) *$\mathcal{H}$ has an edge with four vertices of degree two.*

Proof. Applying Lemma 10, it is enough to prove that if $\mathcal{H}$ belongs to Γ_{sc}, then it has no hypercycle of length 3 or has an edge with four vertices of degree two. Assume that $\mathcal{H} \in \Gamma_{sc}$. The construction of $\mathcal{H}$ and the assumptions on $\mathcal{H}$ apply that $\mathcal{H}$ has either exactly one edge with four vertices of degree two or two edges each with three vertices of degree two. Thus assume, by contradiction, that $\mathcal{H}$ contains a hypercycle of length 3, say $\mathcal{H}'$, and has two edges, say E_1, E_2, each with three vertices of degree two. Since $\mathcal{H}$ contains $\mathcal{H}'$, E_1 and E_2 are adjacent and belong to this hypercycle. Let $E_1 \cap E_2 = \{u\}$ and $v_1^1, v_1^2 \in E_1, v_2^1, v_n^2 \in E_2$ be vertices of degree two. Let E_3 be the third edge of $\mathcal{H}'$. Assume that $v_1^1, v_2^1 \in E_3$. Let $\mathcal{P}$ be a hyperpath that joins v_1^2 and v_n^2 in the subhypergraph induced by $\mathcal{E}(\mathcal{H}) \setminus \{E_1, E_2, E_3\}$ (see Figure 4).

We define the size function f for $\mathcal{H}$ so that $f(v) = \deg_{\mathcal{H}}(v)$ for $v \notin \{v_1^1, u\}$ and $f(u) = f(v_1^1) = 1$. Observe that $size(f) = GB(\mathcal{H}) - 1$. We will prove that f is a choice function for $\mathcal{H}$, which contradicts that $\mathcal{H}$ is *sc*-greedy.

Suppose that L is an arbitrary f-assignment for $\mathcal{H}$. We construct a proper L-colouring ϕ assigning first unique colours from the lists to the vertices v for which $f(v) = 1$.

If $L(u) = L(v_1^1)$, then we extend ϕ assigning to v_1^2 a colour from $L(v_1^2) \setminus L(u)$. Next we colour the vertices of the hyperpath $\mathcal{P}$, according to its arbitrary canonical ordering, starting with v_1^2 and finishing with v_n^2. In each step we choose the colour from the list that guarantees ϕ to be proper on $V(\mathcal{P})$. Finally, we assign to v_2^1 a colour from $L(v_2^1) \setminus L(v_1^1)$ that always exists.

If $L(u) \neq L(v_1^1)$, then we first assign to v_2^1 a colour from $L(v_2^1) \setminus L(v_1^1)$ and next we assign the colour from $L(v_n^2) \setminus L(u)$ to v_n^2. We continue colouring of all vertices of $\mathcal{P}$ according to its arbitrary canonical ordering that starts with v_n^2 and finishes with v_1^2. We choose in each step the colour from the list that yields the proper colouring of $\mathcal{P}$. ■

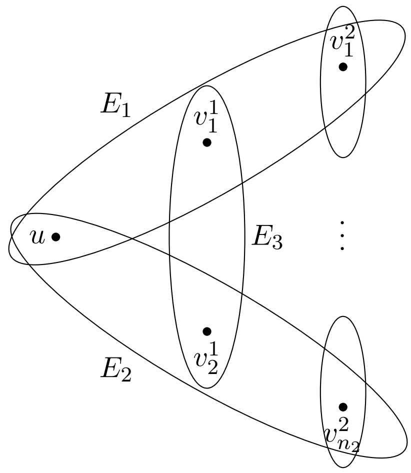


Figure 4. Illustration to the proof of Theorem 11.

Lemma 12. *Let $\mathcal{P}$ be a hyperpath with at least two edges, $v_1, \ldots, v_n$ be a canonical ordering of $V(\mathcal{P})$ and let q, $1 < q < n$, be such that $\deg_{\mathcal{P}}(v_q) = 2$. Next, let f be a size function for $\mathcal{P}$ defined by $f(v_i) = \deg_{\mathcal{P}}(v_i)$ for $i \neq q$ and $f(v_q) = 3$ and let $\mathcal{P}_1$, $\mathcal{P}_2$ be hyperpaths induced in $\mathcal{P}$ by the vertices $v_1, \ldots, v_q$ and by the vertices $v_q, \ldots, v_n$, respectively.*

(i) *If the length of at least one of $\mathcal{P}_1$, $\mathcal{P}_2$ is at least two and if a, b are different positive integers, then there exists an f-assignment L for $\mathcal{P}$ such that for every proper L-colouring ϕ of $\mathcal{P}$ it holds that $\phi(v_1) = \phi(v_n) = a$ and $\phi(v_q) = b$.*

(ii) *If the lengths of both $\mathcal{P}_1$, $\mathcal{P}_2$ are at least two and if a is a positive integer, then there exists an f-assignment L for $\mathcal{P}$ such that for every proper L-colouring ϕ of $\mathcal{P}$ it holds that $\phi(v_1) = \phi(v_n) = \phi(v_q) = a$.*

Proof. Let $\{v_{i_j} : j \in \{1, \ldots, r\}\}$ be the set of vertices of degree two in $\mathcal{P}$ with $i_j < i_l$ for $j < l$. Note that $r \geq 2$ holds, by the assumptions, and $q = i_t$ for some $t \in \{1, \ldots, r\}$. If $t > 1$, then we denote by $\mathcal{P}_1'$ the hyperpath induced by the vertices $v_1, \ldots, v_{i_{t-1}}$. Next, if $t < r$, then we denote by $\mathcal{P}_2'$ the hyperpath induced by the vertices $v_{i_{t+1}}, \ldots, v_n$. We take into account a canonical ordering $\pi_1 = v_1, \ldots, v_{i_{t-1}}$ of $\mathcal{P}_1'$ and a canonical ordering $\pi_2 = v_n, v_{n-1}, \ldots, v_{i_{t+1}}$ of $\mathcal{P}_2'$. Now we apply Lemma 9(i) to $\mathcal{P}_1'$ and π_1 to obtain an $f|_{V(\mathcal{P}_1')}$-assignment L_1 such that for every proper L_1-colouring ϕ of $\mathcal{P}_1'$ it holds that $\phi(v_1) = a$ and $\phi(v_{i_{t-1}}) = c$, $c \neq a$, $c \neq b$. Similarly, we apply Lemma 9(i) to $\mathcal{P}_2'$ and π_2 to obtain an $f|_{V(\mathcal{P}_2')}$-assignment L_2 such that for every proper L_2-colouring ϕ of $\mathcal{P}_2'$ it holds that $\phi(v_n) = a$ and $\phi(v_{i_{t+1}}) = d$, $d \neq a$, $d \neq b$, $d \neq c$. We define L in each of Cases (i), (ii) separately, but using in both the assignments L_1, L_2.

Case (i). The definition of L that confirms (i). Note that the assumptions on the lengths of $\mathcal{P}_1$, $\mathcal{P}_2$ force that at least one of the conditions $t > 1$, $t < r$ holds.

$$L(v) = \begin{cases} \{a\}, & \text{if } v \in \{v_1, \ldots v_{q-1}\} & \text{and } t = 1, \\ L_1(v), & \text{if } v \in V(\mathcal{P}_1') & \text{and } t > 1, \\ \{c\}, & \text{if } v \in \big\{v_{i_{t-1}+1}, \ldots, v_{i_t-1} = v_{q-1}\big\} & \text{and } t > 1, \\ \{a\}, & \text{if } v \in \big\{v_{q+1} = v_{i_t+1}, \ldots, v_n\big\} & \text{and } t = r, \\ L_2(v), & \text{if } v \in V(\mathcal{P}_2') & \text{and } t < r, \\ \{d\}, & \text{if } v \in \big\{v_{q+1}, \ldots v_{i_{t+1}-1}\big\} & \text{and } t < r, \\ \{a, b, d\}, & \text{if } v = v_q & \text{and } t = 1, \\ \{a, b, c\}, & \text{if } v = v_q & \text{and } t = r, \\ \{b, c, d\}, & \text{if } v = v_q & \text{and } 1 < t < r. \end{cases}$$

Case (ii). The definition of L that confirms (ii). Note that the assumptions on the lengths of $\mathcal{P}_1$, $\mathcal{P}_2$ force that both of the conditions $t > 1$, $t < r$ hold.

$$L(v) = \begin{cases} L_1(v), & \text{if } v \in V(\mathcal{P}_1'), \\ \{c\}, & \text{if } v \in \big\{v_{i_{t-1}+1}, \ldots, v_{i_t-1} = v_{q-1}\big\}, \\ L_2(v), & \text{if } v \in V(\mathcal{P}_2'), \\ \{d\}, & \text{if } v \in \big\{v_{q+1}, \ldots v_{i_{t+1}-1}\big\}, \\ \{a, c, d\}, & \text{if } v = v_q\,. \end{cases}$$

It is easy to check that L defined in each of the cases confirms the statement of the lemma. ■

Lemma 13. *Let $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ be hyperpaths such that $V(\mathcal{P}_k) \cap V(\mathcal{P}_j) = \{w\}$, for every two indices $k, j \in \{1, 2, 3\}$. Next, for each $i \in \{1, 2, 3\}$ let $\mathcal{P}_i$ have at least two vertices and $w = v_1^i, \ldots, v_{n_i}^i$ be a canonical ordering of $V(\mathcal{P}_i)$. Assume that $\mathcal{H} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$ and f is a size function for $\mathcal{H}$ defined by $f(v) = \deg_{\mathcal{H}}(v)$ for $v \notin \big\{v_{n_1}^1, v_{n_2}^2, w\big\}$ and $f\big(v_{n_1}^1\big) = f\big(v_{n_2}^2\big) = f(w) = 2$. If a, b are different positive integers, then there exists an f-assignment L for $\mathcal{H}$ such that for every proper L-colouring ϕ of $\mathcal{H}$ it holds that $\phi\big(v_{n_1}^1\big) = \phi\big(v_{n_2}^2\big) = \phi\big(v_{n_3}^3\big) = a$ and $\phi(w) = b$.*

Proof. Let $f_i : V(\mathcal{P}_i) \longrightarrow \mathbb{N}$, $i \in \{1, 2, 3\}$ with $f_i(v) = f(v)$ for $v \in V(\mathcal{P}_i) \setminus \{w\}$ and let $f_1(w) = f_2(w) = 1$, $f_3(w) = 2$. We define L that confirms the assertion using canonical orderings $v_{n_3}^3, \ldots, v_1^3 = w$ of $V(\mathcal{P}_3)$ and $w = v_1^i, \ldots, v_{n_i}^i$ of $V(\mathcal{P}_i)$ for $i \in \{1, 2\}$. So, there exists an f_3-assignment L_3 for $\mathcal{P}_3$ such that for every proper L_3-colouring ϕ it holds that $\phi\big(v_{n_3}^3\big) = a$ and $\phi(w) = b$, by Lemma 9(i). The same lemma guarantees two additional facts. There exists an f_1-assignment L_1 for $\mathcal{P}_1$ such that for every proper L_1-colouring ϕ it holds that $\phi(w) = b$ and $\phi\big(v_{n_1}^1\big) = a$ and there exists an f_2-assignment L_2 for $\mathcal{P}_2$ such that for every proper

L_2-colouring ϕ it holds that $\phi(w) = b$ and $\phi\big(v^2_{n_2}\big) = a$. We define an f-assignment L for $\mathcal{H}$ in the following way.

$$L(v) = \begin{cases} L_3(v), & \text{if } v \in V(\mathcal{P}_3), \\ L_2(v), & \text{if } v \in V(\mathcal{P}_2) \setminus \{w\}, \\ L_1(v), & \text{if } v \in V(\mathcal{P}_1) \setminus \{w\}. \end{cases}$$

It is easy to check that L confirms the statement of the lemma. ■

Lemma 14. *Let $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ be hyperpaths, each on at least two vertices, such that $V\big(\mathcal{P}_k\big) \cap V\big(\mathcal{P}_j\big) = \{w\}$, for every two indices $k, j \in \{1, 2, 3\}$. Assume that $w = v^i_1, \ldots, v^i_{n_i}$ is a canonical ordering of $V\big(\mathcal{P}_i\big)$ for $i \in \{1, 2, 3\}$ and also that the length of $\mathcal{P}_1 \cup \mathcal{P}_2$ is at least three. Next, let $\mathcal{H} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$ and let f be a size function for $\mathcal{H}$ defined by $f(v) = \deg_{\mathcal{H}}(v)$ for $v \notin \big\{v^3_{n_3}\big\}$ and $f\big(v^3_{n_3}\big) = 2$. If a is a positive integer, then there exists an f-assignment L for $\mathcal{H}$ such that for every proper L-colouring ϕ of $\mathcal{H}$ it holds that $\phi\big(v^1_{n_1}\big) = \phi\big(v^2_{n_2}\big) = \phi\big(v^3_{n_3}\big) = a$.*

Proof. Let $\mathcal{H}' = \mathcal{P}_1 \cup \mathcal{P}_2$ and let f' be a size function for $\mathcal{H}'$, and f'' be a size function for $\mathcal{P}_3$ defined by $f'(v) = f|_{V(\mathcal{H}')}(v)$ for all v, $f''(v) = f|_{V(\mathcal{P}_3)}(v)$ for $v \neq w$ and $f''(w) = 1$. We define L that confirms the assertion using canonical orderings $v^1_{n_1}, \ldots, v^1_1 = w = v^2_1, \ldots, v^2_{n_2}$ of $V\big(\mathcal{H}'\big)$ and $v^3_1, \ldots, v^3_{n_3}$ of $V(\mathcal{P}_3)$. Since the length of the hyperpath $\mathcal{H}'$ is at least three, there exists an f'-assignment L' for $\mathcal{H}'$ such that for every proper L'-colouring ϕ it holds that $\phi\big(v^1_{n_1}\big) = \phi\big(v^2_{n_2}\big) = a$ and $\phi(w) = b$ by Lemma 12(i). Lemma 9(i) guarantees that there exists an f''-assignment L'' for $\mathcal{P}_3$ such that for every proper L''-colouring ϕ it holds that $\phi\big(v^3_1\big) = b$ and $\phi\big(v^3_{n_3}\big) = a$. We define an f-assignment L for $\mathcal{H}$ in the following way.

$$L(v) = \begin{cases} L'(v), & \text{if } v \in V(\mathcal{H}'), \\ L''(v), & \text{if } v \in V(\mathcal{P}_3) \setminus \{w\}. \end{cases}$$

It is easy to check that L confirms the statement of the lemma. ■

Lemma 15. *Let $\mathcal{H}$ be a hypercycle with handle that has one vertex of degree three and the remaining vertices of degrees less than three. If $\mathcal{H}$ satisfies at least one of the conditions*

(i) *$\mathcal{H}$ has no hypercycle of length three, or*

(ii) *the vertex of degree three is in an edge with three vertices of degree at least two,*

then $\mathcal{H}$ is sc-greedy.

Proof. Let w be a vertex of degree three in $\mathcal{H}$ and E' be the edge that contains three vertices of degree at least two. Assume, by contradiction, that $\mathcal{H}$ has no hypercycle of length 3 or $w \in E'$ and additionally $\mathcal{H}$ is not sc-greedy. Since each proper induced subhypergraph of $\mathcal{H}$ is in $\overline{\mathcal{G}^*}$, we have $\mathcal{H} \in \mathcal{C}(\Gamma_{sc})$ by Corollary 5. We consider two cases.

Case 1. $w \notin E'$. This assumption implies that $\mathcal{H}$ does not contain a hypercycle of length three. From the construction of $\mathcal{H}$ we can assume that $V(\mathcal{H}) = E' \cup \bigcup_{i=1}^{3} V(\mathcal{P}_i)$ and $\mathcal{E}(\mathcal{H}) = \bigcup_{i=1}^{3} \mathcal{E}(\mathcal{P}_i) \cup \{E'\}$, where $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ are three hyperpaths, each on at lest two vertices, each with some canonical ordering $v_1^i, \ldots, v_{n_i}^i$ of $V(\mathcal{P}_i)$, where $w = v_1^1 = v_1^2 = v_1^3$ and $\{v_{n_1}^1, v_{n_2}^2, v_{n_3}^3\} \subseteq E'$ and E' does not contain any other vertex of any $\mathcal{P}_i$ (see Figure 5). Observe that at most one hyperpath has length one, since otherwise the hypercycle of length three appears.

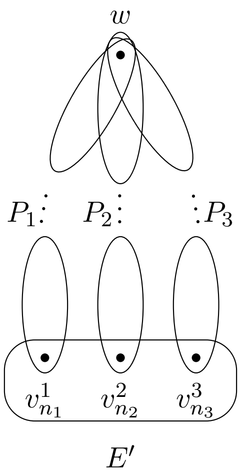


Figure 5. Illustration to the proof of Lemma 15, Case 1.

Let f be a function that realizes $\chi_{sc}(\mathcal{H})$ and let $V' = \{v_{n_1}^1, v_{n_2}^2, v_{n_3}^3, w\}$. Since $\mathcal{H} \in \mathcal{C}(\Gamma_{sc})$, Lemma 6 implies $f(v) = \deg_{\mathcal{H}}(v)$ for $v \in V(\mathcal{H}) \setminus V'$. Moreover, by Lemma 6(i), $f(v) \leq \deg_{\mathcal{H}}(v)$ for $v \in V'$, and by Lemma 6(ii), $f(w) \geq 2$.

By Lemma 8, $\chi_{sc}(\mathcal{H}) = GB(\mathcal{H}) - 1$, so either ($f(w) = 2$ and the values of f for one vertex in $V' \setminus \{w\}$ is equal to one) or ($f(w) = 3$ and the values of f for two vertices in $V' \setminus \{w\}$ are equal to one). Without loss of generality, we can consider two subcases.

Subcase 1.1. $f(w) = 2$, $f(v_{n_1}^1) = f(v_{n_2}^2) = 2$, $f(v_{n_3}^3) = 1$. Let $\mathcal{H}' = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$. From Lemma 13 there is an $f|_{V(\mathcal{H}')}$-assignment L' for $\mathcal{H}'$ such that for every proper L'-colouring ϕ we have $\phi(v_{n_1}^1) = \phi(v_{n_2}^2) = \phi(v_{n_3}^3) = a$. Let L be an f-assignment for $\mathcal{H}$ defined in the following way.

$$L(v) = \begin{cases} L'(v), & \text{if } v \in V(\mathcal{H}'), \\ \{a\}, & \text{otherwise.} \end{cases}$$

Thus, in every L-colouring of $\mathcal{H}$ such that vertices of $\mathcal{H}'$ are properly coloured, vertices in $\{v^1_{n_1}, v^2_{n_2}, v^3_{n_3}\}$ are coloured with a. Since a is the only colour on the lists of remaining vertices of E', the edge E' is monochromatic. This implies that $\mathcal{H}$ is not f-choosable, a contradiction.

Subcase 1.2. $f(w) = 3$, $f(v^1_{n_1}) = f(v^2_{n_2}) = 1$, $f(v^3_{n_3}) = 2$. Let $\mathcal{H}' = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$. Recall that by the assumptions, $\mathcal{H}$ does not contain a hypercycle of length three, and so, at most one of $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ has length one. Thus $\mathcal{H}$ satisfies the assumptions of Lemma 14, and so, there is an $f|_{V(\mathcal{H}')}$-assignment L' for $\mathcal{H}'$ such that for every proper L'-colouring ϕ we have $\phi(v^1_{n_1}) = \phi(v^2_{n_2}) = \phi(v^3_{n_3}) = a$. We construct an f-assignment L for $\mathcal{H}$ in the following way.

$$L(v) = \begin{cases} L'(v), & \text{if } v \in V(\mathcal{H}'), \\ \{a\}, & \text{otherwise.} \end{cases}$$

Such L-assignment confirms that $\mathcal{H}$ is not f-choosable, a contradiction.

Case 2. $w \in E'$. In this case we can assume that $V(\mathcal{H}) = E' \cup \bigcup_{i=1}^{2} V(\mathcal{P}_i)$ and $\mathcal{E}(\mathcal{H}) = \bigcup_{i=1}^{2} \mathcal{E}(\mathcal{P}_i) \cup \{E'\}$, where $\mathcal{P}_1, \mathcal{P}_2$ are two hyperpaths. Each $\mathcal{P}_i$ has a canonical ordering $v^i_1, \ldots, v^i_{n_i}$ of $V(\mathcal{P}_i)$, where $v^1_1 = v^2_1 = w$, $\{w, v^1_{n_1}, v^2_{n_2}\} \subseteq E'$ and E' does not contain any other vertex of any of $\mathcal{P}_1, \mathcal{P}_2$ (see Figure 6). Observe that both hyperpaths have lengths at least two, since $\mathcal{H}$ is linear.

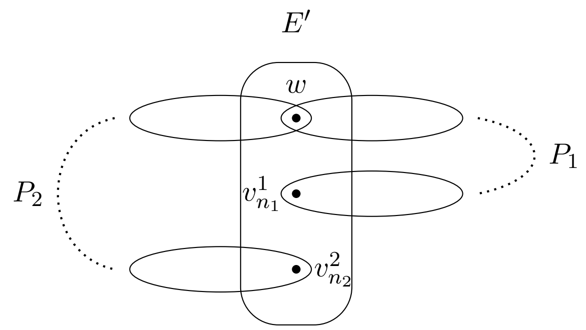


Figure 6. Illustration to the proof of Lemma 15, Case 2.

Let f be a function that realizes $\chi_{sc}(\mathcal{H})$ and let $V'' = \{w, v^1_{n_1}, v^2_{n_2}\}$. Since $\mathcal{H} \in \mathcal{C}(\Gamma_{sc})$, Lemma 6 implies $f(v) = \deg_{\mathcal{H}}(v)$ for $v \in V(\mathcal{H}) \setminus V''$. Moreover, by Lemma 6(i), $f(v) \leq \deg_{\mathcal{H}}(v)$ for $v \in V''$, and by Lemma 6(ii), $f(w) \geq 2$. Similarly as in Case 1, we can observe that either ($f(w) = 2$ and the value of f for one vertex in $V'' \setminus \{w\}$ is equal to one) or ($f(w) = 3$ and the value of f for two vertices in $V'' \setminus \{w\}$ are equal to one). Without loss of generality, we can consider two subcases.

Subcase 2.1. $f(w) = 2$, $f(v^1_{n_1}) = 1$, $f(v^2_{n_2}) = 2$. Let $f_1 : V(\mathcal{P}_1) \longrightarrow \mathbb{N}$, $f_2 : V(\mathcal{P}_2) \longrightarrow \mathbb{N}$ be defined by $f_1(v) = f(v)$ for all v, $f_2(v) = f(v)$ for $v \neq w$ and

$f_2(w) = 1$. From Lemma 9(ii), there is an f_1-assignment L_1 such that for every proper L_1-colouring ϕ we have $\phi\big(v^1_{n_1}\big) = a$ and $\phi(w) = a$. Similarly, there is an f_2-assignment L_2 such that for every proper L_2-colouring ϕ we have $\phi\big(v^2_{n_2}\big) = a$ and $\phi(w) = a$. Let L be an f-assignment for $\mathcal{H}$ defined in the following way.

$$L(v) = \begin{cases} L_1(v), & \text{if } v \in V(\mathcal{P}_1), \\ L_2(v), & \text{if } v \in V(\mathcal{P}_2) \setminus \{w\}, \\ \{a\}, & \text{if } v \in E' \setminus V''. \end{cases}$$

Thus, in every L-colouring of $\mathcal{H}$ such that vertices of $\mathcal{P}_1 \cup \mathcal{P}_2$ are properly coloured, vertices in $\big\{v^1_{n_1}, v^2_{n_2}, w\big\}$ are coloured with a. Since a is the only colour on the lists of the remaining vertices of E', the edge E' is monochromatic. This implies that $\mathcal{H}$ is not f-choosable, a contradiction.

Subcase 2.2. $f(w) = 3$, $f\big(v^1_{n_1}\big) = f\big(v^2_{n_2}\big) = 1$. The hyperpaths $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ satisfy assumptions of Lemma 12(ii) with $v_q = w$. Thus, there is an $f|_{V(\mathcal{P})}$-assignment L' such that for every proper L'-colouring ϕ we have $\phi\big(v^1_{n_1}\big) = a$, $\phi\big(v^2_{n_2}\big) = a$ and $\phi(w) = a$. Let L be an f-assignment for $\mathcal{H}$ defined in the following way.

$$L(v) = \begin{cases} L'(v), & \text{if } v \in V(\mathcal{P}), \\ \{a\}, & \text{otherwise.} \end{cases}$$

Thus $\mathcal{H}$ is not f-choosable, a contradiction. ■

Theorem 16. *Let $\mathcal{H}$ be a hypercycle with handle that has one vertex of degree three and remaining vertices of degrees less than three. The hypergraph $\mathcal{H}$ is sc-greedy if and only if at least one of the following conditions holds:*

(i) *$\mathcal{H}$ has no hypercycle of length three, or*

(ii) *the vertex of degree three is in an edge with three vertices of degree at least two.*

Proof. Let w be a vertex of degree three in $\mathcal{H}$ and E' be the edge that contains three vertices of degree at least two (such an edge always exists by the construction of $\mathcal{H}$). In the light of Lemma 15, we only have to prove that if $\mathcal{H} \in \Gamma_{sc}$, then it has no hypercycle of length 3 or $w \in E'$. Suppose that $\mathcal{H} \in \Gamma_{sc}$ and assume, by contradiction, that $\mathcal{H}$ has a hypercycle of length 3 and $w \notin E'$. The construction of $\mathcal{H}$ implies that E' is in the hypercycle of length three. Let E_1 and E_2 be remaining edges of the hypercycle of length three. By the assumption $w \notin E'$ we obtain $E_1 \cap E_2 = \{w\}$. Let $E_1 \cap E' = v_1$ and $E_2 \cap E' = v_2$. Let $\mathcal{P}_3$ be a hyperpath with a canonical ordering $v^3_1, v^3_2, \ldots, v^3_{n_3}$, where $v^3_1 = w$ and $v^3_{n_3} \in E'$ (see Figure 7).

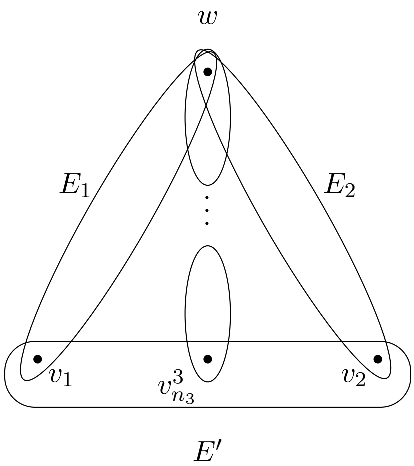


Figure 7. Illustration to the proof of Theorem 16.

We define the size function f for $\mathcal{H}$ so that $f(v) = \deg_{\mathcal{H}}(v)$ for $v \notin \{v_1, v_2\}$ and $f(v_1) = f(v_2) = 1$. Observe that $\text{size}(f) = GB(\mathcal{H}) - 1$. We will prove that $\mathcal{H}$ is f-choosable, which contradicts that $\mathcal{H}$ belongs to Γ_{sc}.

Suppose that L is an arbitrary f-assignment for $\mathcal{H}$. We construct a proper L-colouring ϕ first assigning unique colours to the vertices v for which $f(v) = 1$.

Suppose that $\phi(v_1) \neq \phi(v_2)$. In this case we colour w using the colour from $L(w) \setminus \{\phi(v_1), \phi(v_2)\}$. Next we colour the vertices of $\mathcal{P}_3$, according to its canonical ordering, putting in each step the colour from the list that guarantees ϕ to be proper on $\mathcal{P}_3$ and consequently on $\mathcal{H}$.

Now suppose that $\phi(v_1) = \phi(v_2)$. We colour the vertex $v^3_{n_3}$ using a colour from $L(v^3_{n_3}) \setminus \{\phi(v_1)\}$. Next we colour the vertices of $\mathcal{P}_3$, according to its new canonical ordering $v^3_{n_3}, \ldots, v^3_1 = w$, putting in each step the colour from the list that guarantees ϕ to be proper on $\mathcal{P}_3$. Finally, we assign to w the colour from its list that is different from $\phi(v_1), \phi(v_2)$ and $\phi(v^3_2)$. ■

4. Application

In this section we show how to apply the results from the previous section to different variants of sum-list colouring of graphs.

For each graph G and each induced hereditary class $\mathcal{R}$ of graphs one can construct a special hypergraph, say $\mathcal{H}(G, \mathcal{R})$, whose set of vertices is $V(G)$ and whose set of edges is $\{W : W \subseteq V(G) \text{ and } G[W] \in \mathcal{C}(\mathcal{R})\}$.

The problem of determining $\mathcal{C}(\mathcal{R})$ for a given induced hereditary class of graphs $\mathcal{R}$, is very difficult (see for example the class of k-colourable graphs, where $k \geq 3$). On the other hand, for some classes of graphs this problem is solved.

For example, if $\mathcal{R}$ is a class of 2-colourable graphs, then $\mathcal{C}(\mathcal{R})$ consists of all odd cycles, if $\mathcal{R}$ is a class of all forests, then $\mathcal{C}(\mathcal{R})$ consists of all cycles. Also, it is easy to see that if $\mathcal{R}$ is a class of subcubic graphs (graphs G satisfying $\Delta(G) \leq 3$), then $\mathcal{C}(\mathcal{R})$ consists only of 5-vertex graphs, each of which contains a vertex of degree 4, including $K_{1,4}$.

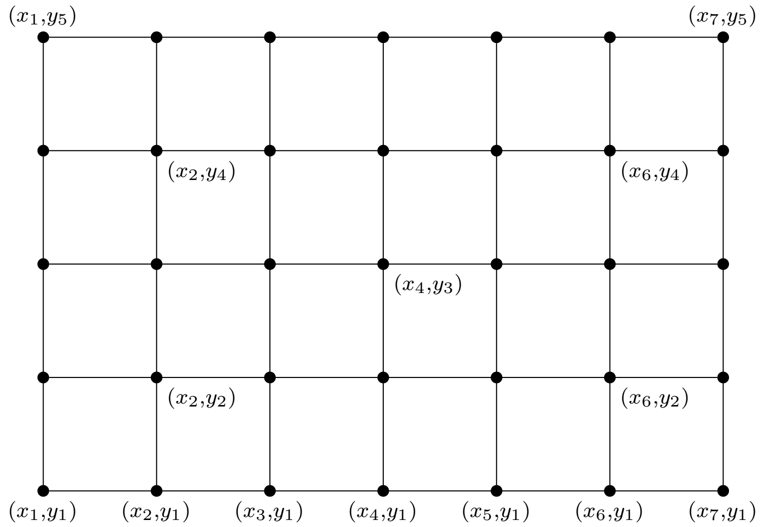
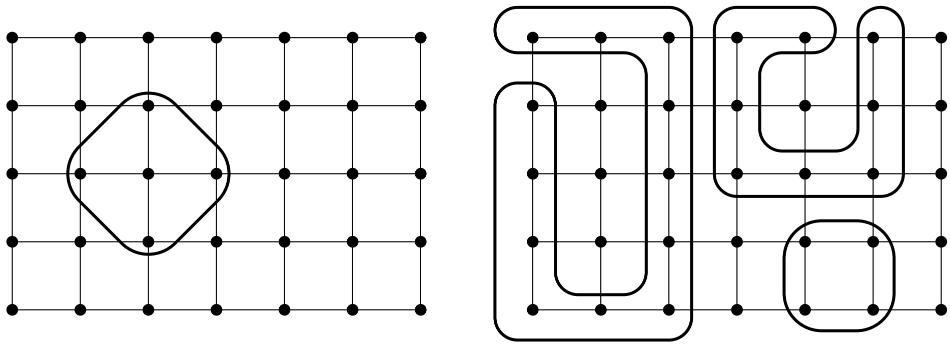


Figure 8. Grid.

Let $\mathcal{R}_1$, $\mathcal{R}_2$ be the classes of subcubic graphs and of forests, respectively. In Figure 8 we present a graph G that is a grid $P_5 \square P_7$ (the Cartesian product of two paths P_5, P_7) with natural labels of vertices. We use these labels in each case when we refer to the vertex (x_i, y_j) of $P_5 \square P_7$. In Figure 9 we show the origin of the creation of some edges of $\mathcal{H}(P_5 \square P_7, \mathcal{R}_1)$ and of $\mathcal{H}(P_5 \square P_7, \mathcal{R}_2)$, respectively. As we mentioned before $K_{1,4} \in \mathcal{C}(\mathcal{R}_1)$. Hence $\{N[(x_i, y_j)] : i \in \{2, \ldots, 6\}, j \in \{2, 3, 4\}\}$ is a subset of the edge set of $\mathcal{H}(P_5 \square P_7, \mathcal{R}_1)$. Moreover, no other subset of $V(P_5 \square P_7)$ is an edge of the hypergraph $\mathcal{H}(P_5 \square P_7, \mathcal{R}_1)$ since no such a subset induces a graph with the vertex of degree 4. Thus $\mathcal{H}(P_5 \square P_7, \mathcal{R}_1)$ has exactly 15 edges, all of them are of the same type, and one of them is presented in Figure 9(a). Figure 9(b) shows the graph $P_5 \square P_7$ and exemplary edges of the hypergraph $\mathcal{H}(P_5 \square P_7, \mathcal{R}_2)$. Recall that a set of vertices of $P_5 \square P_7$ creates the edge of $\mathcal{H}(P_5 \square P_7, \mathcal{R}_2)$ if and only if it induces a cycle in $P_5 \square P_7$.

A colouring of a graph G with respect to a class $\mathcal{R}$ of graphs is an assignment of colours to the vertices of G so that each colour class induces in G a graph in $\mathcal{R}$. Note that, immediately by definitions, an assignment of colours to vertices of G is a colouring of G with respect to an induced hereditary class $\mathcal{R}$ of graphs if and only if it is a proper colouring of $\mathcal{H}(G, \mathcal{R})$.



(a) (b)

Figure 9. Exemplary edges of hypergraphs $\mathcal{H}(P_5\Box P_7, \mathcal{R}_1)$, $\mathcal{H}(P_5\Box P_7, \mathcal{R}_2)$.

The problem of sum-list colouring of graphs with respect to induced hereditary class $\mathcal{R}$ of graphs was introduced in 2016 (see [6]) and it was also investigated in [5, 11, 15]. Given a function $f : V(G) \longrightarrow \mathbb{N}$ and an induced hereditary class $\mathcal{R}$ of graphs, a graph G is $(f, \mathcal{R})$-choosable if for every f-assignment L there is an L-colouring of G with respect to $\mathcal{R}$. The $\mathcal{R}$-sum-choice-number $\chi_{sc}^{\mathcal{R}}(G)$ of G is defined to be the minimum of $\sum_{v\in V(G)} f(v)$ taken over all f such that G is $(f, \mathcal{R})$-choosable.

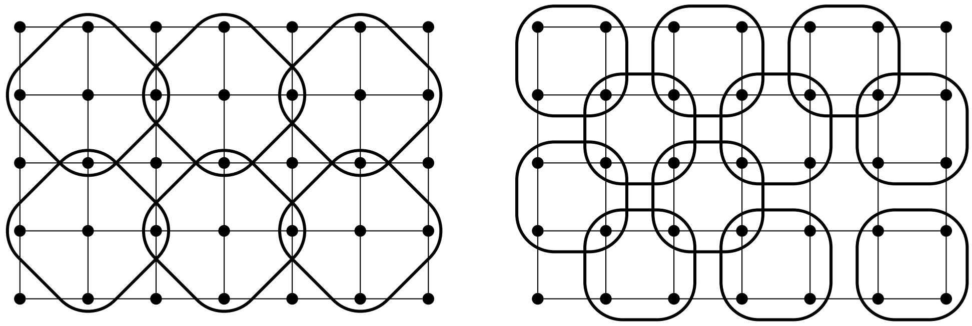
Observation 17. *If G is a graph and $\mathcal{R}$ is an induced hereditary class of graphs, then $\chi_{sc}^{\mathcal{R}}(G) = \chi_{sc}(\mathcal{H}(G, \mathcal{R}))$.*

Using Observations 1 and 17, one can note that for $i \in \{1, 2\}$ the inequality $\chi_{sc}^{\mathcal{R}_i}(G) \geq \chi_{sc}(\mathcal{H}^i)$ is satisfied for every subhypergraph $\mathcal{H}^i$ of $\mathcal{H}(G, \mathcal{R}_i)$. We present, in Figures 10(a) or Figure 10(b), respectively, the fixed hypergraphs $\mathcal{H}^1$, $\mathcal{H}^2$ that we use in the forthcoming analyze in this field.

Observe that $\mathcal{H}^1$ is the union of 12 disjoint hypergraphs K_1 and the hypergraph on 23 vertices that is a hypercycle of length 4 with handle of length two. Moreover, the degrees of all vertices of this 23-vertex hypergraph are less than three. Thus, $\mathcal{H}^1$ is *sc*-greedy, by Theorem 11 and Corollary 5. Consequently, $\chi_{sc}(\mathcal{H}^1) = 12 + 23 + 6 = 41$.

Also, note that $\mathcal{H}^2$ is a hypergraph that is the union of the following disjoint hypergraphs: two hypergraphs K_1, one hypertree on 4 vertices with one edge, and some hypergraph on 29 vertices. From Theorem 11, this 29-vertex hypergraph belongs to $\overline{\mathcal{G}'}$, where $\mathcal{G}'$ is the class of hypercycles with handle being *sc*-greedy. Consequently, by Corollary 5, $\chi_{sc}(\mathcal{H}^2) = 2 + 4 + 1 + 29 + 10 = 46$. In the light of the previous consideration we obtain the following facts.

Observation 18. $\chi_{sc}^{\mathcal{R}_1}(P_5\Box P_7) \geq 41$ *and* $\chi_{sc}^{\mathcal{R}_2}(P_5\Box P_7) \geq 46$.



(a) (b)

Figure 10. The subhypergraph $\mathcal{H}^1$ of $\mathcal{H}(G, \mathcal{R}_1)$ and the subhypergraph $\mathcal{H}^2$ of $\mathcal{H}(G, \mathcal{R}_2)$.

To find upper bounds on $\chi_{sc}^{\mathcal{R}_i}(P_5 \Box P_7)$ for $i \in \{1,2\}$ we use the following result that was proved in [7].

The *β-degree* of a vertex v in a hypergraph $\mathcal{H}$, denoted by $\deg_{\mathcal{H}}^{\beta}(v)$, is the largest number of edges of a linear subhypergraph of $\mathcal{H}(v)$.

Theorem 19 [7]**.** *If $\mathcal{H}$ is a hypergraph and $v_1, \ldots, v_n$ is an arbitrary ordering of $V(\mathcal{H})$, then*

$$\chi_{sc}(\mathcal{H}) \leq \sum_{i=1}^{n} \deg_{\mathcal{H}_i}^{\beta}(v_i) + n,$$

where $\mathcal{H}_i = \mathcal{H}[\{v_1, \ldots, v_i\}]$.

First we find the upper bound on $\chi_{sc}^{\mathcal{R}_1}(P_5 \Box P_7)$. Let V^1 be a set defined as $V(P_5 \Box P_7) \setminus \{(x_2, y_2), (x_2, y_4), (x_4, y_3), (x_6, y_2), (x_6, y_4)\}$. We associate labels $v_1, \ldots, v_{30}$ to vertices in V^1 in an arbitrary way. Next we put $v_{31} = (x_2, y_2)$, $v_{32} = (x_2, y_4)$, $v_{33} = (x_6, y_2)$, $v_{34} = (x_6, y_4)$ and $v_{35} = (x_4, y_3)$. Now we apply Theorem 19 to the hypergraph $\mathcal{H}^1$ (see Figure 10) and the ordering $v_1, \ldots, v_{35}$ of its vertex set. In Figure 11(a) we present a graph G^1 induced in $P_5 \Box P_7$ by V^1. Since G^1 is subcubic, the hypergraph induced in $\mathcal{H}^1$ by V^1 is edgeless and consequently $\deg_{\mathcal{H}_i^1}^{\beta}(v_i) = 0$ for each $i \in \{1, \ldots, 30\}$ (recall that $\mathcal{H}_i^1 = \mathcal{H}^1[\{v_1, \ldots, v_i\}]$). Next we observe that $\deg_{\mathcal{H}_i^1}^{\beta}(v_i) = 1$ for $i \in \{31, \ldots, 34\}$ and finally $\deg_{\mathcal{H}_{35}^1}^{\beta}(v_{35}) = \deg_{\mathcal{H}^1}^{\beta}(v_{35}) = 2$. It implies $\chi_{sc}(\mathcal{H}^1) \leq 4 + 2 + 35 = 41$, by Theorem 19.

Similar investigation allows us to find an upper bound on $\chi_{sc}^{\mathcal{R}_2}(P_5 \Box P_7)$. In this case we use a completely different ordering of the set $V(P_5 \Box P_7)$. Namely, let $V^2 = V(P_5 \Box P_7) \setminus \{(x_i, y_j) : i \in \{2, \ldots, 7\};\ j \in \{2, 4\}$. We associate labels $v'_1, \ldots, v'_{23}$ to vertices in V^2 in an arbitrary way. Next we put $v'_{22+p} = (x_p, y_2)$ for

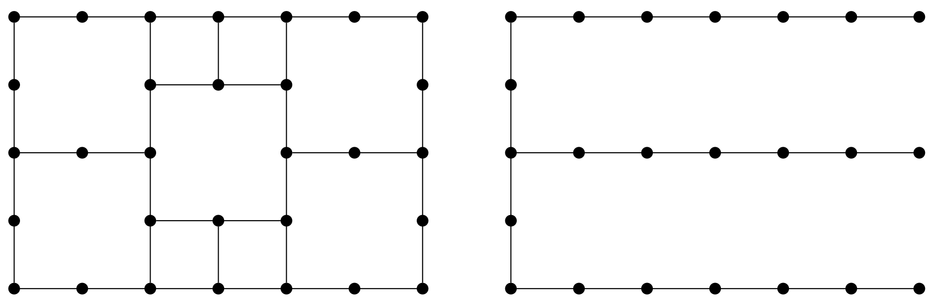


Figure 11. The subcubic subgraph G^1 of $P_5\Box P_7$ and the acyclic subgraph G^2 of $P_5\Box P_7$.

$p \in \{2,\dots,7\}$ and $v'_{28+p} = (x_p, y_4)$ for $p \in \{2,\dots,7\}$. Now we apply Theorem 19 to the hypergraph $\mathcal{H}^2$ and the ordering $v'_1,\dots,v'_{35}$ of its vertex set. In Figure 11(b) we present a graph G^2 induced in $P_5\Box P_7$ by V^2. Since G^2 is a forest, the hypergraph induced in $\mathcal{H}^2$ by V^2 is edgeless and consequently $\deg^{\beta}_{\mathcal{H}^2_i}(v'_i) = 0$ for each $i \in \{1,\dots,23\}$. Next we observe that $\deg^{\beta}_{\mathcal{H}^2_i}(v'_i) = 1$ for $i \in \{24,\dots,35\}$. This implies that $\chi_{sc}(\mathcal{H}^2) \le 12 + 35 = 47$.

We conclude this section with the result based on Observation 18 and the collection of all facts that we have made.

Observation 20. $\chi^{\mathcal{R}_1}_{sc}(P_5\Box P_7) = 41$ *and* $46 \le \chi^{\mathcal{R}_2}_{sc}(P_5\Box P_7) \le 47$.

Finally, it has to be mentioned that the parameter $\chi^{\mathcal{R}_1}_{sc}(G)$ have been not present in the literature so far, but $\chi^{\mathcal{R}_2}_{sc}(G)$ was considered in the paper [5], where among others the estimation $45 \le \chi^{\mathcal{R}_2}_{sc}(P_5\Box P_7) \le 47$ was proved. It means that Observation 20 improves the known fact concerning this special graph.

5. CONCLUDING REMARKS

Clearly, if a hypergraph $\mathcal{H}$ contains an induced subhypergraph that is not *sc*-greedy, then $\mathcal{H}$ is not *sc*-greedy, since Γ_{sc} is an induced hereditary class. Recalling that each *sc*-greedy hypergraph has to be linear, to show that $\mathcal{H}$ is not *sc*-greedy we have to find an induced subhypergraph of $\mathcal{H}$ that either is non-liner or it is linear but not *sc*-greedy. Since all hyperpaths and hypercycles are *sc*-greedy, in the sense of the structure, the smallest hypergraphs that are linear and need not be *sc*-greedy are hypercycles with handle. This fact shows the power of the results presented in the paper.

We would like to highlight that Theorems 7, 11, 16 completely characterize hypercycles with handle in Γ_{sc}, and consequently completely characterize hypercycles with handle in $\mathcal{C}(\Gamma_{sc})$. It leads to the characterization of unions of a hypercycle and a hyperpath having at most two vertices in common in both classes Γ_{sc} and $\mathcal{C}(\Gamma_{sc})$.

Additionally, a choice function for a hypergraph is also a choice function for its subhypergraph (up to domains). Thus, finding a subhypergraph $\mathcal{H}'$ of a given hypergraph $\mathcal{H}$ with relatively large $\chi_{sc}(\mathcal{H}')$ yields a relatively good lower bound on $\chi_{sc}(\mathcal{H})$. Since *sc*-greedy hypergraphs have the largest possible sum-choice-numbers (they achieve the upper bound on this number) the knowledge about elements of the class Γ_{sc} seems to be interesting in this context.

Finally, we would like to mention that Section 4 shows that the results proved in the paper can be applied to different variants of sum-list colouring of graphs.

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