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GALLAI-RAMSEY NUMBERS FOR RAINBOW S_3^+ AND MONOCHROMATIC PATHS[1]

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Abstract

Motivated by Ramsey theory and other rainbow-coloring-related problems, we consider edge-colorings of complete graphs without rainbow copy of some fixed subgraphs. Given two graphs G and H, the k-colored Gallai-Ramsey number $gr_k(G : H)$ is defined to be the minimum positive integer n such that every k-coloring of the complete graph on n vertices contains either a rainbow copy of G or a monochromatic copy of H. Let S_3^+ be the graph on four vertices consisting of a triangle with a pendant edge. In this paper, we prove that $gr_k(S_3^+ : P_5) = k+4$ $(k \geq 5)$, $gr_k(S_3^+ : mP_2) = (m-1)k+m+1$ $(k \geq 1)$, $gr_k(S_3^+ : P_3 \cup P_2) = k+4$ $(k \geq 5)$ and $gr_k(S_3^+ : 2P_3) = k+5$ $(k \geq 1)$.

Keywords: Gallai-Ramsey number, rainbow coloring, monochromatic paths.

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1. Introduction

In this paper, we only consider edge-colorings of finite simple graphs. For an integer $k \geq 1$, let $c : E(G) \to [k]$ be a k-coloring of a graph G, where $[k] := \{1, 2, \ldots, k\}$. A coloring of a graph is called *rainbow* if no two edges have the

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same color, and a coloring is called *monochromatic* if all edges are colored the same. In 1967, Gallai [12] first investigated the structures of rainbow triangle-free (i.e., there is no rainbow K_3) colorings of complete graphs and proved the following celebrated result. In the following statement, a coloring of a complete graph G is said to be *Gallai colored* if G is rainbow triangle-free in honor of Gallai's work.

Theorem 1 [12, 16]**.** *In any Gallai colored complete graph G, $V(G)$ can be partitioned into nonempty sets $H_1, H_2, \ldots, H_l$ with $l \geq 2$ such that there are at most two colors between the parts, and there is only one color on the edges between every pair of parts.*

In response to Theorem 1, Fujita and Magnant [9] considered the structures of rainbow S_3^+-free colorings, where S_3^+ is the graph on four vertices consisting of a triangle with a pendant edge, in which the vertex with degree three is said to be a center of S_3^+.

Theorem 2 [9]**.** *In any rainbow S_3^+-free coloring of a complete graph G, one of the following holds:*

(1) *There are three (differently colored) monochromatic spanning trees, and moreover, $V(G)$ can be partitioned into nonempty sets $H_1, H_2, \ldots, H_l$ with $l \geq 6$ such that there are exactly three colors on edges between parts, and there is only one color on the edges between every pair of parts; or*

(2) *$V(G)$ can be partitioned into nonempty sets $H_1, H_2, \ldots, H_l$ with $l \geq 2$ such that there are at most two colors on the edges between the parts.*

Remark 3. In Theorem 2(1), let K be a 3-coloring of K_l obtained by taking one vertex from each part in the partition of $V(G)$. Since there are three (differently colored) monochromatic spanning trees in G, there are three (differently colored) monochromatic spanning trees in K, say T_1, T_2 and T_3. Since $\binom{l}{2} = |E(K)| \geq \sum_{i=1}^{3} |E(T_i)| = 3(l-1)$, we have $l \geq 6$.

For more results about rainbow triangle-free colorings and rainbow S_3^+-free colorings of complete graphs, see [5, 7, 15, 16] and [9, 18], respectively.

Given a graph H, the *Ramsey number* $r_k(H)$ is the minimum positive integer n such that every k-coloring of K_n contains a monochromatic copy of H. Note that a k-coloring means an edge-coloring using at most k colors. Given two graphs G and H, the *k-colored Gallai-Ramsey number* $gr_k(G : H)$ is defined to be the minimum positive integer n such that every k-coloring of the complete graph on n vertices contains either a rainbow copy of G or a monochromatic copy of H. Note that for any graph H, we have $gr_k(G : H) \leq r_k(H)$ clearly.

In [7], Fox *et al.* posed the following conjecture about $gr_k(K_3 : H)$ where H is a complete graph.

Conjecture 4 [7]**.** *For integers $k \geq 1$ and $t \geq 3$,*

$$gr_k(K_3 : K_t) = \begin{cases} (r_2(K_t) - 1)^{k/2} + 1, & \text{if } k \text{ is even}, \\ (t-1)(r_2(K_t) - 1)^{(k-1)/2} + 1, & \text{if } k \text{ is odd}. \end{cases}$$

Conjecture 4 has been studied by Chung and Graham [3] for $t = 3$, Liu *et al.* [19] for $t = 4$, Magnant and Schiermeyer [20] for $t = 5$. There are also many results for rainbow triangle and monochromatic cycles or paths (see [1, 5, 8, 14, 17] and two surveys [10, 11]). However, there are not much known about Gallai-Ramsey numbers for other rainbow subgraphs.

In this paper, we consider $gr_k(G : H)$ for rainbow S_3^+ and monochromatic paths since very few results are known for the case where $G = S_3^+$ and finding this number for a path is a fundamental work. For any graph H, since any rainbow K_3-free coloring certainly contains no rainbow S_3^+, we have $gr_k(S_3^+ : H) \geq gr_k(K_3 : H)$. And since every 3-coloring contains no rainbow S_3^+, we have $gr_k(S_3^+ : H) = r_k(H)$ for $1 \leq k \leq 3$, so we will generally suppose $k \geq 4$.

The first result of Gallai-Ramsey numbers for paths was given by Faudree *et al.* [5].

Theorem 5 [5]**.** *For integers $k \geq 1$ and $4 \leq n \leq 6$, $gr_k(K_3 : P_n) = \left\lfloor \frac{n-2}{2} \right\rfloor k + \left\lceil \frac{n}{2} \right\rceil + 1$.*

Faudree *et al.* [5] provided a lower bound and Hall *et al.* [17] provided an upper bound for $gr_k(K_3 : P_n)$, respectively.

Theorem 6 [5, 17]**.** *For integers $k \geq 1$ and $n \geq 3$,*

$$\left\lfloor \frac{n-2}{2} \right\rfloor k + \left\lceil \frac{n}{2} \right\rceil + 1 \leq gr_k(K_3 : P_n) \leq \left\lfloor \frac{n-2}{2} \right\rfloor k + 3 \left\lfloor \frac{n}{2} \right\rfloor.$$

In [14], the exact values of $gr_k(K_3 : P_7)$ for all integers $k \geq 1$ were determined.

Theorem 7 [14]**.** *For any integer $k \geq 1$, $gr_k(K_3 : P_7) = 2k + 5$.*

In [9], Fujita and Magnant considered the Gallai-Ramsey numbers for rainbow S_3^+.

Theorem 8 [9]**.** *For any integer $k \geq 4$, $gr_k(S_3^+ : P_4) = k + 3$.*

In this paper, we continue to study the Gallai-Ramsey numbers for rainbow S_3^+ and monochromatic paths. We prove the following main result.

Theorem 9. *For any integer $k \geq 5$, $gr_k(S_3^+ : P_5) = k + 4$.*

Moreover, for two disjoint graphs G and H, let $G \cup H$ denote the union of G and H, and nG denote the union of n disjoint copies of G. In [14], Gregory obtained the following result.

Theorem 10 [14]**.** *For any integer* $k \geq 1$, $gr_k(K_3 : 2P_3) = k + 5$.

In this work, we prove the following result concerning monochromatic $2P_3$.

Theorem 11. *For any integer* $k \geq 1$, $gr_k(S_3^+ : 2P_3) = k + 5$.

Furthermore, by Theorem 8, we can deduce that $gr_k(S_3^+ : 2P_2) = k + 3$ for $k \geq 4$. In fact, for a matching mP_2 with m edges, we can show that the Gallai-Ramsey number $gr_k(S_3^+ : mP_2)$ is exactly the same as the Ramsey number $r_k(mP_2)$ (proven in [4]) since $gr_k(S_3^+ : mP_2) \leq r_k(mP_2)$ and the sharpness example for $r_k(mP_2)$ (see [4]) contains no rainbow S_3^+. Therefore, we have the following result.

Theorem 12. *For integers* $m \geq 1$ *and* $k \geq 1$, $gr_k(S_3^+ : mP_2) = (m-1)k+m+1$.

Finally, since $P_3 \cup P_2$ is a subgraph of P_5, we have $gr_k(S_3^+ : P_3 \cup P_2) \leq gr_k(S_3^+ : P_5)$. From the proof of Theorem 9 (see Section 3), we can show that $gr_k(S_3^+ : P_3 \cup P_2) \geq gr_k(S_3^+ : P_5)$ since the sharpness example for $gr_k(S_3^+ : P_5)$ contains no monochromatic $P_3 \cup P_2$. Therefore, we have the following immediate corollary.

Corollary 13. *For any integer* $k \geq 5$, $gr_k(S_3^+ : P_3 \cup P_2) = k + 4$.

The remainder of this paper is organized as follows. In Section 2, we provide several terminologies and lemmas which will be used in the proofs of our main results. In Section 3, we give the proof of Theorem 9. In Section 4, we give the proof of Theorem 11.

2. Preliminaries

We first state some known classical Ramsey numbers which will be used in the proofs of Theorems 9 and 11.

Theorem 14 [13, 22]**.** $r_2(P_5) = 6$, $r_3(P_5) = 9$.

Theorem 15 [2, 21]**.** $r_2(2P_3) = 7$, $r_3(2P_3) = 8$.

We will commonly use the following definition $L(n, k)$ in our construction of sharpness examples. For $k \leq n$, $L(n, k)$ is a k-coloring of K_n with vertex set $\{u_1, u_2, \ldots, u_n\}$ such that for all $1 \leq i \leq k$ and $i < j \leq n$, the edge u_iu_j has color i, and all the remaining edges have color k. It is easy to check that $L(n, k)$ contains no rainbow copy of K_3.

Moreover, we give some terminologies and lemmas which will be used later. Given a colored graph G and an edge uv, let $c(uv)$ denote the color used on uv and $C(G)$ be the set of colors used in G. For two disjoint vertex sets U,

$V \subseteq V(G)$, let $E(U, V)$ denote the set of edges between U and V, and $C(U, V)$ denote the set of colors used on the edges in $E(U, V)$. In the case that all the edges in $E(U, V)$ have a single color, we will use $c(U, V)$ to denote this color. If $|U| = 1$, say $U = \{u\}$, then we simply write $E(u, V)$, $C(u, V)$ and $c(u, V)$ for $E(\{u\}, V)$, $C(\{u\}, V)$ and $c(\{u\}, V)$, respectively. For $U \subseteq V(G)$, let $G[U]$ denote the subgraph of G induced by U, and $C(U)$ denote the set of colors used on the edges of $G[U]$.

Lemma 16. *Let G be a rainbow S_3^+-free coloring of K_n ($n \geq 4$) and c_1, c_2 be two distinct colors. Then the following holds:*

(1) *Let v_1v_2 and v_3v_4 be two non-adjacent edges with $c(v_1v_2) = c_1$ and $c(v_3v_4) = c_2$. If $C(\{v_1, v_2\}, \{v_3, v_4\}) \cap \{c_1, c_2\} = \emptyset$, then $|C(\{v_1, v_2\}, \{v_3, v_4\})| = 1$.*

(2) *Let v_1v_2 and v_2v_3 be two adjacent edges with $c(v_1v_2) = c_1$ and $c(v_2v_3) = c_2$. For any $v \in V(G) \setminus \{v_1, v_2, v_3\}$, if $C(v, \{v_1, v_2, v_3\}) \cap \{c_1, c_2\} = \emptyset$, then $|C(v, \{v_1, v_2, v_3\})| = 1$.*

Proof. For (1), without loss of generality, let $c(v_1v_3) = c_3$, where $c_3 \notin \{c_1, c_2\}$. Since G is rainbow S_3^+-free, we have $c(v_1v_4) = c(v_2v_3) = c_3$, and then $c(v_2v_4) = c_3$. Hence, (1) is proved. For (2), without loss of generality, let $c(vv_2) = c_3$. Since G is rainbow S_3^+-free, we have $c(vv_1) = c(vv_3) = c_3$. Hence, (2) is proved. ■

Lemma 17. *Let G be a rainbow S_3^+-free coloring of K_8 using exactly four colors. If $V(G)$ can be partitioned into two subsets U and V with $|U| = |V| = 4$ such that $|C(U, V)| \leq 2$, then G contains a monochromatic P_5.*

Proof. Suppose G contains no monochromatic P_5. Let $U = \{u_1, u_2, u_3, u_4\}$, $V = \{v_1, v_2, v_3, v_4\}$, $C(U, V) \subseteq \{1, 2\}$, $c(e_1) = 3$ and $c(e_2) = 4$. If e_1 and e_2 are two adjacent edges, say $e_1 = u_1u_2$ and $e_2 = u_2u_3$, then by Lemma 16(2) and the pigeonhole principle we may assume that $c(\{v_1, v_2\}, \{u_1, u_2, u_3\}) = 1$, resulting in a monochromatic P_5, a contradiction. If e_1 and e_2 are two non-adjacent edges in a same part, say $e_1 = u_1u_2$ and $e_2 = u_3u_4$, then by the above discussion and Lemma 16(1) we may assume that $c(\{u_1, u_2\}, \{u_3, u_4\}) = 1$. In order to avoid a P_5 in color 1, we have $c(U, V) = 2$, resulting in a monochromatic P_5 in color 2, a contradiction. If $e_1 \in E(G[U])$ and $e_2 \in E(G[V])$, say $e_1 = u_1u_2$ and $e_2 = v_1v_2$, then by Lemma 16(1) we may assume that $c(\{u_1, u_2\}, \{v_1, v_2\}) = 1$. Then $c(\{u_3, u_4\}, \{v_1, v_2\}) = 2$ for avoiding a monochromatic P_5 in color 1. In order to avoid a rainbow S_3^+ and a monochromatic P_5, we have that $c(u_2u_3) \notin \{1, 2, 3, 4\}$, a contradiction. ■

Lemma 18 [6]. *In any 2-coloring of $K_{3,5}$, there is a monochromatic P_5.*

Lemma 19 [21]. *In any 2-coloring of $K_{4,4}$, there is a monochromatic $2P_3$.*

Lemma 20. *In any* 2*-coloring of* $K_{3,5}$*, there is a monochromatic* $2P_3$*. Moreover, there exists a* 2*-coloring of* $K_{3,4}$ *without monochromatic* $2P_3$*.*

Proof. Let G be a 2-coloring of $K_{3,5}$ with bipartition (U,V), where $U=\{u_1,u_2,u_3\}$ and $V=\{v_1,v_2,v_3,v_4,v_5\}$. Suppose G contains no monochromatic $2P_3$. By the pigeonhole principle, we may assume that $c(u_1,\{v_1,v_2,v_3\})=1$. If $1\notin C(\{v_4,v_5\},\{u_2,u_3\})$, i.e., $c(\{v_4,v_5\},\{u_2,u_3\})=2$, then we consider $C(v_1,\{u_2,u_3\})$. Since G is monochromatic $2P_3$-free, at most one of v_1u_2 and v_1u_3 is colored by 1. Thus we may assume that $c(v_1u_2)=2$. Then $c(v_3u_2)\neq 2$, $c(v_3u_3)\neq 2$, i.e., $c(v_3,\{u_2,u_3\})=1$, which implies that $\{v_1u_1v_2,u_2v_3u_3\}$ forms a monochromatic $2P_3$, a contradiction. Thus $1\in C(\{v_4,v_5\},\{u_2,u_3\})$, say $c(v_4u_2)=1$. Then $c(v_4u_3)=c(u_2,\{v_1,v_2,v_3,v_5\})=2$, and hence $2\notin C(u_3,\{v_1,v_2,v_3,v_5\})$, which implies that $\{v_1u_1v_2,v_3u_3v_5\}$ forms a monochromatic $2P_3$ in color 1, a contradiction.

For the moreover part, consider a 2-coloring G' of $K_{3,4}$ with bipartition (U',V'), where $U'=\{u'_1,u'_2,u'_3\}$ and $V'=\{v'_1,v'_2,v'_3,v'_4\}$. We color the edges such that $c(\{v'_1,v'_2\},U')=1$ and $c(\{v'_3,v'_4\},U')=2$. Then G' contains no monochromatic $2P_3$. ■

3. Proof of Theorem 9

We will deduce Theorem 9 from the following statement.

Theorem 21. *For any integer* $k\geq 4$*, there is a monochromatic* P_5 *in every rainbow* S_3^+*-free coloring* (*using all* k *colors*) *of* K_{k+4}*.*

***Proof of Theorem* 9 (*assuming Theorem* 21).** For the lower bound, consider $L(k+3,k)$, which contains no rainbow S_3^+ and no monochromatic P_5. For the upper bound, let G be a k-coloring of K_{k+4} ($k\geq 5$) without rainbow S_3^+. Suppose that G contains no monochromatic P_5. We say a coloring is bad if it contains neither a rainbow S_3^+ nor a monochromatic P_5. Among all bad k-colorings of K_{k+4}, we choose G (using exactly k' colors, $k'\leq k$) such that k is minimum. If $k'\leq 3$, then since $|V(G)|=k+4\geq 9$, there is a monochromatic P_5 by Theorem 14. If $k'=4$, then since $|V(G)|=k+4\geq 9$, there exists a set V of eight vertices such that $G[V]$ contains all the four colors (since we can choose four edges with distinct colors which incident to at most eight vertices), so there is a monochromatic P_5 by Theorem 21. If $k'=k$, then there is a monochromatic P_5 by Theorem 21. Thus $5\leq k'\leq k-1$. For any $V'\subseteq V(G)$ with $|V'|=k'+4$, we have $|C(V')|\leq k'$, i.e., $G[V']$ is a k'-coloring of $K_{k'+4}$. Since k is minimum, there is a rainbow S_3^+ or a monochromatic P_5 in $G[V']$, a contradiction. ■

***Proof of Theorem* 21.** For a contradiction, suppose that G is a rainbow S_3^+-free coloring (using all k colors) of K_{k+4} without monochromatic P_5 and $V(G)=$

$\{v_1, v_2, \ldots, v_{k+4}\}$. By Theorem 2, we divide the proof into two cases.

Case 1. Theorem 2(1) holds. In this case, we may assume that the three colors used between the parts are 1, 2 and 3. Let $H_1, H_2, \ldots, H_l$ be the parts with $|H_1| \leq |H_2| \leq \cdots \leq |H_l|$ and $l \geq 6$. We first claim that $|H_l| = 2$. In fact, since there are $k \geq 4$ colors used in G, we have $|H_l| \geq 2$. Suppose $|H_l| \geq 3$. Since $l \geq 6$, there are at least two parts with a single color to H_l. Thus there is a monochromatic $K_{2,3}$, which contains a P_5, a contradiction.

Without loss of generality, we may assume that $H_l = \{v_1, v_2\}$ and $c(v_1v_2) = 4$. Since $|V(G) \setminus H_l| = k + 2 \geq 6$, we may further assume that $c(H_l, \{v_3, v_4\}) = 1$ and $c(H_l, \{v_5, v_6\}) = 2$. Note that v_3 and v_5 must be contained in different parts since $c(H_l, v_3) \neq c(H_l, v_5)$. Thus $c(v_3v_5) \in \{1, 2, 3\}$. If $c(v_3v_5) \in \{1, 2\}$, then there is a monochromatic P_5, and if $c(v_3v_5) = 3$, then there is a rainbow S_3^+, a contradiction.

Case 2. Theorem 2(2) holds. In this case, note that there exists a bipartition of $V(G)$ with at most two colors, say colors 1 and 2, on the edges between parts. We choose such a bipartition (H_1, H_2) with $|H_1| \leq |H_2|$ and $|H_1|$ is minimum. Since $k + 4 \geq 8$ and by Lemmas 17 and 18, we have $|H_1| \leq 2$. If $|H_1| = 2$, say $H_1 = \{v_1, v_2\}$, then we have $c(v_1v_2) \notin \{1, 2\}$ by the choice of (H_1, H_2). So we may assume that $c(v_1v_2) = 3$ and $c(v_3v_4) = 4$. By Lemma 16(1), we may let $c(\{v_1, v_2\}, \{v_3, v_4\}) = 1$. Since G contains no monochromatic P_5, we have $1 \notin C(\{v_1, v_2\}, \{v_5, v_6, \ldots, v_{k+4}\})$. Thus $c(\{v_1, v_2\}, \{v_5, v_6, \ldots, v_{k+4}\}) = 2$, also resulting in a monochromatic P_5. Thus we have $|H_1| = 1$ and $|H_2| = k + 3$. Let $H_1 = \{v_1\}$ and $H_2 = \{v_2, v_3, \ldots, v_{k+4}\}$.

Claim 22. *There is no rainbow triangle $\mathcal{C}$ satisfying $|C(\mathcal{C}) \cap \{1, 2\}| = 1$ in $G[H_2]$.*

Proof. For a contradiction, suppose that $c(v_2v_3) = 3$, $c(v_3v_4) = 4$ and $c(v_2v_4) = 1$. Since G contains no rainbow S_3^+, we have $c(v_1, \{v_2, v_3, v_4\}) = 1$. In order to avoid a rainbow S_3^+ and a monochromatic P_5, we have $C(\{v_2, v_3, v_4\}, \{v_5, v_6, \ldots, v_{k+4}\}) \subseteq \{3, 4\}$. If $c(v_1v_i) = 2$ for some $i \in \{5, 6, \ldots, k + 4\}$, then $c(v_3v_i) \notin \{1, 2, \ldots, k\}$, a contradiction. Thus $c(H_1, H_2) = 1$. Since all the k colors are used in G, we may assume that $c(v_5v_6) = 2$ and without loss of generality let $c(v_3v_5) = 3$. Then $c(v_3v_6) = 3$. If $c(v_4v_5) = 3$, then $c(v_4v_6) \notin \{1, 2, \ldots, k\}$. Thus $c(v_4v_5) = 4$, so $c(v_2v_5) = c(v_4v_6) = 4$. But then $c(v_2v_6) \notin \{1, 2, \ldots, k\}$. □

Claim 23. *There is no rainbow triangle $\mathcal{C}$ satisfying $|C(\mathcal{C}) \cap \{1, 2\}| = 2$ in $G[H_2]$. Moreover, there is no rainbow triangle in $G[H_2]$.*

Proof. For a contradiction, suppose that $c(v_2v_3) = 1$, $c(v_3v_4) = 2$ and $c(v_2v_4) = 3$. In order to avoid a rainbow S_3^+, we have $C(\{v_2, v_3, v_4\}, V(G) \setminus \{v_2, v_3, v_4\}) \subseteq \{1, 2, 3\}$. Without loss of generality, let $c(v_1v_3) = 1$. Since all the k (≥ 4) colors are used in G, we may assume that $c(v_5v_6) = 4$. If $3 \notin C(\{v_2, v_4\}, \{v_5, v_6\})$, then

$c(\{v_2, v_4\}, \{v_5, v_6\}) = 1$ or 2 by Lemma 16(1), resulting in a monochromatic P_5. Thus $3 \in C(\{v_2, v_4\}, \{v_5, v_6\})$.

First, suppose $c(v_4v_5) = 3$. In order to avoid a rainbow S_3^+ and by Claim 22, we have $c(v_4v_6) = 3$. If $c(v_3v_5) = 3$, then $c(v_3v_6) \notin \{1, 2, \ldots, k\}$. Thus $c(v_3v_5) = 2$, and by symmetry $c(v_3v_6) = 2$. In order to avoid a monochromatic P_5 and by Lemma 16(2), we have $c(v_1, \{v_4, v_5, v_6\}) = 1$. For avoiding a rainbow S_3^+, we have $C(v_2, \{v_5, v_6\}) \subseteq \{1, 2\}$, and for avoiding a monochromatic P_5, we have $1 \notin C(v_2, \{v_5, v_6\})$, so $c(v_2, \{v_5, v_6\}) = 2$. But then $v_4v_3v_5v_2v_6$ is a P_5 in color 2. Therefore, $c(v_4v_5) \neq 3$, and by symmetry $c(v_4v_6) \neq 3$.

Finally, we may assume that $c(v_2v_5) = 3$, so $c(v_2v_6) = 3$ by Claim 22 and for avoiding a rainbow S_3^+. Note that we have $C(v_3, \{v_5, v_6\}) \subseteq \{1, 3\}$ and $c(v_3v_5) = c(v_3v_6)$. In order to avoid a monochromatic P_5, we have a $c(v_3v_5) = c(v_3v_6) = 1$. Then we have $C(v_4, \{v_5, v_6\}) \subseteq \{1, 2\}$ and $c(v_4v_5) = c(v_4v_6)$, so $c(v_4v_5) = c(v_4v_6) = 2$ for avoiding a monochromatic P_5. Furthermore, we have $C(v_1, \{v_5, v_6\}) \subseteq \{1, 2\}$ and $c(v_1v_5) = c(v_1v_6)$. But then there is a monochromatic P_5, a contradiction.

Moreover, if there is a rainbow triangle $\mathcal{C}'$ in $G[H_2]$, then $C(\mathcal{C}') \subseteq \{3, 4, \ldots, k\}$ from the above argument and by Claim 22, so there is a rainbow S_3^+ in G, a contradiction. □

Since G is monochromatic P_5-free, there exists a rainbow triangle in G by Theorem 5. By Claim 23, there is no rainbow triangle in $G[H_2]$, so we may assume that $c(v_1v_2) = 1$, $c(v_1v_3) = 2$ and $c(v_2v_3) = 3$. Since all the k (≥ 4) colors are used in G, we may assume that $c(v_4v_5) = 4$. Note that $C(\{v_2, v_3\}, \{v_4, v_5, \ldots, v_{k+4}\}) \subseteq \{1, 2, 3\}$. If $3 \notin C(\{v_2, v_3\}, \{v_4, v_5\})$, then $c(\{v_2, v_3\}, \{v_4, v_5\}) = 1$ or 2 by Lemma 16(1), resulting in a monochromatic P_5. Thus $3 \in C(\{v_2, v_3\}, \{v_4, v_5\})$, say $c(v_3v_4) = 3$. Since there is no rainbow triangle in $G[H_2]$, we have $c(v_3v_5) = 3$. Since $c(v_1v_3) = 2$, we have $c(v_1, \{v_4, v_5\}) = 2$ by Lemma 16(2). Note that $c(v_2v_4) = c(v_2v_5)$ for avoiding a rainbow triangle in $G[H_2]$. In order to avoid a rainbow S_3^+ and a monochromatic P_5, we have $c(v_2v_4) = c(v_2v_5) = 1$.

We first claim that $c(v_1, \{v_6, v_7, \ldots, v_{k+4}\}) = 2$. In fact, if $c(v_1v_i) = 1$ for some $i \geq 6$, then $c(v_i, \{v_4, v_5\}) = 2$, resulting in a P_5 in color 2. We next claim that $c(v_2, \{v_6, v_7, \ldots, v_{k+4}\}) = 1$. In fact, if $c(v_2v_i) = 2$ or 3 for some $i \geq 6$, then $c(v_i, \{v_4, v_5\}) = 1$ for avoiding a rainbow S_3^+ and a monochromatic P_5, resulting in a P_5 in color 1. Finally, we claim that $c(v_3, \{v_6, v_7, \ldots, v_{k+4}\}) = 3$. Note that $C(v_3, \{v_6, v_7, \ldots, v_{k+4}\}) \subseteq \{1, 3\}$ for avoiding a rainbow triangle in $G[H_2]$. If $c(v_3v_i) = 1$ for some $i \geq 6$, then we have $c(\{v_4, v_5\}, v_i) \subseteq \{1, 3\}$ and $c(v_4v_i) = c(v_5v_i)$, resulting in a monochromatic P_5, a contradiction.

Let $A_1 = \{v_2\}$, $A_2 = \{v_1\}$, $A_3 = \{v_3\}$ and $A_4 = \{v_4, v_5, \ldots, v_{k+4}\}$. Then A_1, A_2, A_3 and A_4 form a partition of $V(G)$ with $c(A_i, A_4) = i$ for $i \in \{1, 2, 3\}$. If $C(A_4) \cap \{1, 2, 3\} = \emptyset$, then $G[A_4]$ is a $(k-3)$-coloring of K_{k+1}. By Claim 23, there is no rainbow triangle in $G[A_4]$. Thus there is monochromatic P_5 by

Theorem 5, a contradiction. Therefore, $C(A_4) \cap \{1,2,3\} \neq \emptyset$, say $c(uv) = 1$ for some $u, v \in A_4$. Note that for every $i \in \{1,2,3\}$, there exists at most one edge with color i in $G[A_4]$ for avoiding a monochromatic P_5. Moreover, since there is no rainbow triangle in $G[A_4]$, we have $c(uw) = c(vw)$ for every vertex $w \in A_4 \setminus \{u,v\}$. Thus, $C(\{u,v\}, A_4 \setminus \{u,v\}) \subseteq \{4,\ldots,k\}$, which implies that $|C(\{u,v\}, A_4 \setminus \{u,v\})| \leq k-3$. If $k=4$, then the edges between $\{u,v\}$ and $A_4 \setminus \{u,v\}$ form a $K_{2,3}$ (containing a P_5) in color 4, a contradiction. If $k \geq 5$, then in order to avoid a monochromatic $K_{2,3}$ and since $|A_4 \setminus \{u,v\}| = k-1$, there exist four vertices $x, y, z, w \in A_4 \setminus \{u,v\}$ such that $c(\{u,v\},\{x,y\}) = c_1$ and $c(\{u,v\},\{z,w\}) = c_2$, where $4 \leq c_1 < c_2 \leq k$. If $c(xz) \in \{c_1, c_2\}$, then there is a monochromatic P_5, and if $c(xz) \notin \{c_1, c_2\}$, then there is a rainbow S_3^+. This contradiction completes the proof of Theorem 21. ■

4. PROOF OF THEOREM 11

***Proof of Theorem* 11.** For the lower bound, consider $L(k+4,k)$, which contains no rainbow S_3^+ and no monochromatic $2P_3$. Thus we have $gr_k(S_3^+ : 2P_3) \geq k+5$.

For the upper bound, the case $k=1$ is trivial and the case $2 \leq k \leq 3$ is precisely Theorem 15, so we may assume that $k \geq 4$ in the following. Suppose that G is a k-coloring of K_{k+5} containing no rainbow S_3^+ and no monochromatic $2P_3$ with $V(G) = \{v_1, v_2, \ldots, v_{k+5}\}$. We choose G such that k is minimum. Let k' $(\leq k)$ be the number of colors used in G. If $k' < k$, then for any $V' \subseteq V(G)$ with $|V'| = k'+5$, we have $|C(V')| \leq k'$, i.e., $G[V']$ is a k'-coloring of $K_{k'+5}$. Since k is minimum, there is a rainbow S_3^+ or a monochromatic $2P_3$ in $G[V']$, a contradiction. Therefore, all the k colors are used in G. By Theorem 2, we divide the rest of the proof into two cases.

Case 1. Theorem 2(1) holds. Let $H_1, H_2, \ldots, H_l$ be the parts with $1 \leq |H_1| \leq |H_2| \leq \cdots \leq |H_l|$ and $l \geq 6$. Let 1, 2 and 3 be the three colors used between the parts. Since all the k (≥ 4) colors are used in G, we have $|H_l| \geq 2$. Since $l \geq 6$, there are at least two parts with a single color to H_l. Thus we have $|H_l| \leq 3$ for avoiding a monochromatic $2P_3$.

If $|H_l| = 3$, say $H_l = \{v_1, v_2, v_3\}$, then since $|V(G)| = k+5 \geq 9$ and for avoiding a monochromatic $2P_3$, we may assume that $c(\{v_4, v_5\}, H_l) = 1$, $c(\{v_6, v_7\}, H_l) = 2$ and $c(\{v_8, v_9\}, H_l) = 3$. In order to avoid a rainbow S_3^+, $C(\{v_4, v_5\}, \{v_6, v_7\}) \subseteq \{1,2,3\}$, and to avoid a monochromatic $2P_3$, $1, 2 \notin C(\{v_4, v_5\}, \{v_6, v_7\})$, so $c(\{v_4, v_5\}, \{v_6, v_7\}) = 3$. But then $\{v_6v_4v_7, v_8v_1v_9\}$ forms a $2P_3$ in color 3. Thus $|H_l| = 2$, say $H_l = \{v_1, v_2\}$. Since there are k (≥ 4) colors in total, we may assume that $c(v_1v_2) = 4$. Note that there are at least $\lceil \frac{k+3}{3} \rceil \geq 3$ vertices in $V(G) \setminus H_l$ with a single color to H_l, say $c(H_l, \{v_3, v_4, v_5\}) = 1$. Then

$C(H_l, \{v_6, v_7, \ldots, v_{k+5}\}) = \{2,3\}$. Without loss of generality, let $c(H_l, \{v_6, v_7\}) = 2$. Note that $C(\{v_3, v_4, v_5\}, \{v_6, v_7\}) \subseteq \{1,2,3\}$. If $1 \in C(\{v_3, v_4, v_5\}, \{v_6, v_7\})$, then there is a monochromatic $2P_3$, and if $3 \in C(\{v_3, v_4, v_5\}, \{v_6, v_7\})$, then there is rainbow S_3^+. Thus $c(\{v_3, v_4, v_5\}, \{v_6, v_7\}) = 2$, resulting in a monochromatic $2P_3$, a contradiction.

Case 2. Theorem 2(2) holds. In this case, there exists a bipartition of $V(G)$ with at most two colors, say colors 1 and 2, on the edges between parts. We choose such a bipartition (H_1, H_2) with $|H_1| \leq |H_2|$ and $|H_1|$ is minimum. Since $|V(G)| = k + 5 \geq 9$, we have $|H_1| \leq 2$ by Lemmas 19 and 20.

If $|H_1| = 2$, say $H_1 = \{v_1, v_2\}$, then we have $c(v_1v_2) \notin \{1,2\}$ by the choice of (H_1, H_2). Thus we may assume that $c(v_1v_2) = 3$ and $c(v_3v_4) = 4$. By Lemma 16(1), let $c(\{v_1, v_2\}, \{v_3, v_4\}) = 1$. Since G has no monochromatic $2P_3$, there is at most one edge using color 1 between v_1 (respectively, v_2) and $\{v_5, v_6, \ldots, v_{k+5}\}$, i.e., there are at least three vertices in $\{v_5, v_6, \ldots, v_{k+5}\}$, say v_5, v_6 and v_7, with $c(\{v_1, v_2\}, \{v_5, v_6, v_7\}) = 2$. Then for avoiding a monochromatic $2P_3$ in color 2, we have $2 \notin C(v_1, \{v_8, v_9, \ldots, v_{k+5}\})$, i.e., $c(v_1, \{v_8, v_9, \ldots, v_{k+5}\}) = 1$. Now $\{v_3v_2v_4, v_8v_1v_9\}$ forms a $2P_3$ in color 1. Thus $|H_1| = 1$. Let $H_1 = \{v_1\}$ and $H_2 = \{v_2, v_3, \ldots, v_{k+5}\}$. Since G is monochromatic $2P_3$-free, there exists a rainbow triangle $\mathcal{C}$ in G by Theorem 10. We may consider three different types of $\mathcal{C}$: (Type 1) $v_1 \notin V(\mathcal{C})$, $|C(\mathcal{C}) \cap \{1,2\}| = 1$; (Type 2) $v_1 \notin V(\mathcal{C})$, $|C(\mathcal{C}) \cap \{1,2\}| = 2$; (Type 3) $v_1 \in V(\mathcal{C})$.

Claim 24. *There is no triangle of Type* 1 *in* G.

Proof. For a contradiction, suppose $c(v_2v_3) = 1$, $c(v_3v_4) = 3$ and $c(v_2v_4) = 4$. In order to avoid a rainbow S_3^+, we have $c(v_1, \{v_2, v_3, v_4\}) = 1$. If $2 \in C(H_1, H_2)$, say $c(v_1v_5) = 2$, then we have $c(v_4v_5) = 1$ and $c(v_1, \{v_6, v_7, \ldots, v_{k+5}\}) = 1$ (otherwise if $c(v_1v_i) = 2$ for some $i \geq 6$, then $c(v_4v_i) = 1$, resulting in a monochromatic $2P_3$ in color 1). So $c(v_2v_5) = 4$. But then $c(v_3v_5) \notin \{1, 2, \ldots, k\}$. Therefore, $c(H_1, H_2) = 1$. Since all the k colors are used in G, we may assume that $c(v_5v_6) = 2$. In order to avoid a monochromatic $2P_3$ and a rainbow S_3^+, we have $1, 2 \notin C(\{v_2, v_3\}, \{v_5, v_6\})$, so by Lemma 16(1) we have $c(\{v_2, v_3\}, \{v_5, v_6\}) = 3$ or 4, say 3. For avoiding a monochromatic $2P_3$, $1, 3 \notin C(\{v_2, v_3\}, \{v_7, v_8, \ldots, v_{k+5}\})$. Thus we have $c(\{v_2, v_3\}, \{v_7, v_8, \ldots, v_{k+5}\}) = 4$. Then $\{v_4v_2v_7, v_8v_3v_9\}$ forms a $2P_3$ in color 4. □

Claim 25. *There is no triangle of Type* 2 *in* G.

Proof. For a contradiction, suppose $c(v_2v_3) = 1$, $c(v_3v_4) = 2$ and $c(v_2v_4) = 3$. In order to avoid a rainbow S_3^+, $c(\{v_2, v_3, v_4\}, \{v_5, v_6, \ldots, v_{k+5}\}) \subseteq \{1,2,3\}$. Without loss of generality, we may assume that $c(v_1v_3) = 1$ and $c(v_5v_6)$=4. Suppose $3 \notin C(\{v_2, v_4\}, \{v_5, v_6\})$. For avoiding a monochromatic $2P_3$ and by Lemma

16(1), we have $c(\{v_2, v_4\}, \{v_5, v_6\}) = 2$. Then $2 \notin C(v_4, \{v_7, v_8, \ldots, v_{k+5}\})$. Moreover, if $c(v_4v_i) = 1$ for some $i \geq 7$, then $c(v_5v_i) \notin \{1, 2, \ldots, k\}$. Thus $1 \notin C(v_4, \{v_7, v_8, \ldots, v_{k+5}\})$. Hence, $c(v_4, \{v_7, v_8, \ldots, v_{k+5}\}) = 3$. Now $2 \notin C(v_5, \{v_7, v_8, \ldots, v_{k+5}\})$ (otherwise we have a monochromatic $2P_3$), $4 \notin C(v_5, \{v_7, v_8, \ldots, v_{k+5}\})$ (by Claim 24), and to avoid a rainbow S_3^+ we have $c(v_5, \{v_7, v_8, \ldots, v_{k+5}\}) = 3$. Now $\{v_2v_4v_7,\ v_8v_5v_9\}$ forms a monochromatic $2P_3$, a contradiction. Hence $3 \in C(\{v_2, v_4\}, \{v_5, v_6\})$.

We first suppose $c(v_4v_5) = 3$, and then $c(v_4v_6) = 3$, $c(v_3v_5) = 2$ or 3. If $c(v_3v_5) = 3$, then $c(v_3v_6) = 3$, and $c(v_1, \{v_4, v_5, v_6\}) = 1$ by Lemma 16(2). Now $c(\{v_2, v_3\}, \{v_7, v_8, \ldots, v_{k+5}\}) = 2$, which implies that $\{v_7v_2v_8, v_4v_3v_9\}$ forms a monochromatic $2P_3$. Thus $c(v_3v_5) = c(v_3v_6) = 2$. By Lemma 16(2), $c(v_1, \{v_4, v_5, v_6\}) = c_1$, where $c_1 \in \{1, 2\}$. If $c_1 = 1$, then $1 \notin C(v_2, \{v_7, v_8, \ldots, v_{k+5}\})$, and thus there exist two vertices $v_i, v_j \in \{v_7, v_8, \ldots, v_{k+5}\}$ such that $c(v_2v_i) = c(v_2v_j) = 2$ or 3. But in both cases we can find a monochromatic $2P_3$. Thus $c_1 = 2$. Then $c(v_1, \{v_7, v_8, \ldots, v_{k+5}\}) = 1$ for avoiding a monochromatic $2P_3$. In this case, $1, 2 \notin C(v_3, \{v_7, v_8, \ldots, v_{k+5}\})$, so $c(v_3, \{v_7, v_8, \ldots, v_{k+5}\}) = 3$. Then there is a monochromatic $2P_3$. Thus $c(v_4v_5) \neq 3$ and $c(v_4v_6) \neq 3$.

Moreover, we have $c(v_4v_5) \neq 1$, otherwise $c(v_4v_6) \notin \{1, 2, \ldots, k\}$. Thus $c(v_4v_5) = c(v_4v_6) = 2$. Next we suppose $c(v_2v_5) = 3$, and then $c(v_2v_6) = 3$. In addition, there exists at most one edge with color 2 in $E(v_1, \{v_7, v_8, \ldots, v_{k+5}\})$. Thus we may assume that $c(v_1, \{v_7, v_8\}) = 1$. Now $c(v_3v_5) = c(v_3v_6) = 3$. Then $1, 3 \notin C(\{v_2, v_3\}, v_9)$, i.e., $c(\{v_2, v_3\}, v_9) = 2$, which implies $\{v_5v_4v_6, v_2v_9v_3\}$ forms a monochromatic $2P_3$. □

By Claims 24 and 25, there is a triangle of Type 3 in G. Without loss of generality, let $c(v_1v_2) = 1$, $c(v_1v_3) = 2$, $c(v_2v_3) = 3$ and $c(v_4v_5) = 4$. Suppose $3 \notin C(\{v_2, v_3\}, \{v_4, v_5\})$, so we may assume that $c(\{v_2, v_3\}, \{v_4, v_5\}) = 1$ by Lemma 16(1). In order to avoid a monochromatic $2P_3$, we have $c(v_1, \{v_6, v_7, \ldots, v_{k+5}\}) = 2$, $1 \notin C(\{v_2, v_3, v_4, v_5\}, \{v_6, v_7, \ldots, v_{k+5}\})$, and there exists at most one edge with color 2 in $E(v_2, \{v_6, v_7, \ldots, v_{k+5}\})$. Thus we may further assume that $c(v_2, \{v_6, v_7, \ldots, v_{k+4}\}) = 3$. Then $c(\{v_4, v_5\}, v_6) = 3$ for avoiding a rainbow S_3^+. But then $\{v_3v_2v_7, v_4v_6v_5\}$ forms a monochromatic $2P_3$. Thus $3 \in C(\{v_2, v_3\}, \{v_4, v_5\})$, say $c(v_3v_4) = 3$, so $c(v_3v_5) = 3$ and by Lemma 16(2) we have $c(v_1, \{v_4, v_5\}) = 2$.

Next we consider $c(v_2v_4)$. In order to avoid a rainbow S_3^+, we have $c(v_2v_4) \in \{1, 2\}$. If $c(v_2v_4) = 2$, then $c(v_2v_5) = 2$. So $2 \notin C(\{v_1, v_2\}, \{v_6, v_7, \ldots, v_{k+5}\})$ and there exists at most one edge with color 3 in $E(v_2, \{v_6, v_7, \ldots, v_{k+5}\})$. Thus we have $c(v_1, \{v_6, v_7, \ldots, v_{k+5}\}) = 1$ and we may assume $c(v_2, \{v_6, v_7, \ldots, v_{k+4}\}) = 1$, resulting in a monochromatic $2P_3$. Therefore, $c(v_2v_4) = 1$ and by symmetry $c(v_2v_5) = 1$.

Now there exists at most one edge with color 1 in $E(v_1, \{v_6, v_7, \ldots, v_{k+5}\})$, so we may assume that $c(v_1, \{v_6, v_7, \ldots, v_{k+4}\}) = 2$. Moreover, there exists at most one edge with color 2 (respectively, color 3) in $E(v_2, \{v_6, v_7, \ldots, v_{k+5}\})$. Thus, there are at least $k-2$ (≥ 2) edges in $E(v_2, \{v_6, v_7, \ldots, v_{k+5}\})$ using color 1, so we may assume $c(v_2v_6) = 1$. We claim that $c(v_1v_{k+5}) = 2$, otherwise if $c(v_1v_{k+5}) = 1$, then $c(v_4v_{k+5}) = c(v_5v_{k+5}) = 2$, resulting in a monochromatic $2P_3$. Now we may assume that $c(v_2v_7) = 1$ without loss of generality. In addition, if $c(v_2v_i) = 2$ (or 3) for some $i \geq 8$, then $c(v_6v_i) = c(v_7v_i) = 1$, resulting in a monochromatic $2P_3$. Thus $c(v_2, \{v_8, v_9, \ldots, v_{k+5}\}) = 1$. To avoid a triangle of Type 2, we have $2 \notin C(v_3, \{v_6, v_7, \ldots, v_{k+5}\})$. If $c(v_3v_i) = 1$ for some $i \geq 6$, then $c(v_4v_i) = c(v_5v_i) = 3$. And then $c(v_3, \{v_6, v_7, \ldots, v_{k+5}\} \setminus \{v_i\}) = 1$, resulting in a monochromatic $2P_3$. Thus $c(v_3, \{v_6, v_7, \ldots, v_{k+5}\}) = 3$. Let $A_1 = \{v_2\}$, $A_2 = \{v_1\}$, $A_3 = \{v_3\}$ and $A_4 = \{v_4, v_5, \ldots, v_{k+5}\}$. Then we have $c(A_i, A_4) = i$ for $i \in \{1, 2, 3\}$.

Claim 26. $1, 2, 3 \notin C(A_4)$.

Proof. For a contradiction, suppose $c(uv) = 1$ for some $u, v \in A_4$. Note that for every $i \in \{1, 2, 3\}$, if there exists an edge e with color i in $G[A_4]$, then every edge adjacent to e in $G[A_4]$ cannot use color i for avoiding a monochromatic $2P_3$. Thus $2, 3 \notin C(\{u, v\}, A_4 \setminus \{u, v\})$ by Claim 25, i.e., $C(\{u, v\}, A_4 \setminus \{u, v\}) \subseteq \{4, 5, \ldots, k\}$. By Claims 24 and 25, we have $c(uv_i) = c(vv_i)$ for every $v_i \in A_4 \setminus \{u, v\}$. Thus each color appears on at most three edges in $E(u, A_4 \setminus \{u, v\})$ to avoid a monochromatic $2P_3$. Since $|A_4 \setminus \{u, v\}| = k$ and $|C(\{u, v\}, A_4 \setminus \{u, v\})| \leq k-3$, we have $k \geq 5$ and we may assume that $c(\{u, v\}, \{v_{i_1}, v_{i_2}\}) = c_1$ and $c(\{u, v\}, \{v_{i_3}, v_{i_4}\}) = c_2$, where $v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}$ are four distinct vertices in $A_4 \setminus \{u, v\}$ and $4 \leq c_1 < c_2 \leq k$. In order to avoid a rainbow S_3^+, we have $C(\{v_{i_1}, v_{i_2}\}, \{v_{i_3}, v_{i_4}\}) \subseteq \{c_1, c_2\}$. Without loss of generality, let $c(v_{i_1}v_{i_3}) = c_1$, then $c(v_{i_1}v_{i_4}) \neq c_1$, i.e., $c(v_{i_1}v_{i_4}) = c_2$. But then we can find a monochromatic $2P_3$ no matter $c(v_{i_2}v_{i_4}) = c_1$ or c_2. □

If $G[A_4]$ contains a rainbow triangle $\mathcal{C}$, then $C(\mathcal{C}) \subseteq \{4, \ldots, k\}$ by Claims 24 and 25, resulting in a rainbow S_3^+. Thus $G[A_4]$ is a rainbow triangle-free coloring of K_{k+2} with $k-3$ colors by Claim 26. By Theorem 10, there is a monochromatic $2P_3$, a contradiction. ■

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