EFFICIENT DOMINATION IN CAYLEY GRAPHS OF GENERALIZED DIHEDRAL GROUPS

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Abstract

An independent subset D of the vertex set V of the graph Γ is an *efficient dominating set* for Γ if each vertex $v \in V \setminus D$ has precisely one neighbour in D. In this article, we classify the connected cubic Cayley graphs on generalized dihedral groups which admit an efficient dominating set.

Keywords: efficient domination set, Cayley graph, generalized dihedral group.

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1. Introduction

In 1973 Biggs [3] initiated the study of perfect codes and in particular perfect 1-codes in graphs which are nowadays more commonly referred to as efficient dominating sets (see Section 2 for the definition). Even though he was mainly interested in the existence of perfect codes in distance-transitive graphs, numerous papers investigating the existence of efficient dominating sets in graphs in general have been published since, especially in the last two decades (see for instance [5, 16] and the references therein).

Since the general problem of determining whether a graph has an efficient dominating set is an NP-complete problem (see [2]), researchers have focused on various restricted classes of graphs. Among them vertex-transitive graphs have received a considerable amount of attention (see for instance [6, 8–10, 13, 15, 17–19, 21, 24, 26]). One might think that the assumption on vertex-transitivity could be strong enough to make the question of whether such graphs admit efficient dominating sets much easier. However, no general result in this direction has been obtained thus far which probably makes the problem, proposed in [17], of characterizing the vertex-transitive graphs admitting an efficient dominating set way too difficult to solve in general. In fact, even for the nicest possible situation in which the graph in question admits a regular action of a cyclic group (such graphs are called *circulants*) only a few partial results have been obtained thus far (see [6, 9, 10, 13, 18, 21]). To better understand the situation, it is reasonable to study particular classes of vertex-transitive graphs.

One possible direction to take is of course to restrict the valence of the graphs in question. This approach was taken in [17] where the authors focused on the cubic vertex-transitive graphs. Nevertheless, even with this strong restriction they were only able to characterize vertex-transitive graphs of order a power of 2 admitting an efficient dominating set. Another possibility is to consider only the Cayley graphs (see Section 2 for the definition) which are the most natural vertex-transitive graphs. But as mentioned, even for the Cayley graphs of cyclic groups, only a handful of partial results have been obtained. It thus makes sense to combine the two restrictions and focus on Cayley graphs of small valence. In [6] the cubic and quartic Cayley graphs of abelian groups admitting efficient dominating sets have been classified. One of the most natural next families of groups to consider are the generalized dihedral groups (see Section 2 for a definition). That this class of (cubic) Cayley graphs is indeed worth investigating is indicated by the number of such graphs among all (small) cubic vertex-transitive graphs. In the following table we list the number of all connected cubic vertex-transitive graphs of order n (#VT) for each $4 \leq n \leq 160$ divisible by 4 (so as to be potential candidates for graphs admitting an efficient dominating set) and state how many of them are Cayley graphs (#Cay) and how many of these are Cayley

graphs of a generalized dihedral group (#GD). These numbers were obtained by using the computer software Magma [4] and the census of all connected cubic vertex-transitive graphs, constructed in 2013 by Potočnik, Spiga and Verret [23].

n	4	8	12	16	20	24	28	32	36	40	44	48	52	56	60	64	68	72	76	80
#VT	1	2	4	4	7	11	6	10	12	12	7	32	10	16	38	26	12	37	11	33
#Cay	1	2	4	4	5	11	5	10	12	10	7	32	9	13	30	26	11	36	11	29
#GD	1	2	3	4	4	7	5	8	8	9	7	14	8	12	14	14	10	19	11	19

n	84	88	92	96	100	104	108	112	116	120	124	128	132	136	140	144	148	152	156	160
#VT	30	17	13	90	25	22	42	38	18	105	17	83	35	28	35	124	22	27	46	104
#Cay	27	16	13	90	23	20	41	35	17	100	17	82	35	26	34	123	21	26	45	94
#GD	18	16	13	27	17	19	22	24	16	33	17	26	26	23	26	39	20	26	30	37

There is another reason why the investigation of cubic Cayley graphs of generalized dihedral groups is of interest. As was proved in [1] it turns out that this class of graphs contains all so-called honeycomb toroidal graphs (see Section 2 for the definition) which turn out to be useful in the theory of interconnection networks (see [1, 7, 11, 12, 20, 22, 25] and the references therein).

The main result of this paper is a complete classification of connected cubic Cayley graphs of generalized dihedral groups admitting an efficient dominating set. It turns out that some of these graphs are also Cayley graphs of abelian groups, and so the results of [6] can be applied. The remaining cubic Cayley graphs of generalized dihedral groups are the above mentioned honeycomb toroidal graphs, which are considered in Section 5.

2. Preliminaries

Throughout the paper all graphs are assumed to be simple, finite and connected, unless otherwise specified.

Let $\phi\Gamma$ be a graph with vertex set V. For a vertex $v \in V$ we let $N(v)$ denote the *neighborhood* of v in $\phi\Gamma$ which consists of all the vertices at distance 1 from v. Similarly, $N[v] = N(v) \cup \{v\}$ denotes the *closed neighborhood* of v in $\phi\Gamma$. A subset $D \subset V$ is an *efficient dominating set* for $\phi\Gamma$ if the sets $N[v]$, where $v \in D$, partition the vertex-set V. In other words, D is an efficient dominating set for $\phi\Gamma$ if for each vertex v of $\phi\Gamma$ it holds that either $v \in D$ and v is not adjacent to any vertex of D, or v is not in D but is adjacent to precisely one vertex of D. Following [5] we abbreviate "efficient dominating set" to e.d.s. throughout the paper.

Observe that a necessary (but not sufficient) condition for a k-regular graph of order n to admit an e.d.s. is that $k + 1$ divides n. Therefore, a cubic graph can only admit an e.d.s. if its order is divisible by 4, which explains why we have restricted to orders divisible by 4 in the table from the previous section.

Let G be a group and $S \subset G$ an inverse closed subset not containing the identity of G. Then the *Cayley graph* $\mathrm{Cay}(G; S)$ of G with respect to S is the graph with vertex set G in which each $g \in G$ is adjacent to all vertices of the form gs, where $s \in S$. In this paper we will mainly be concerned with Cayley graphs of the class of groups we now define.

A group G is a *generalized dihedral group* if it has an index 2 abelian subgroup A and an involution $t \in G \setminus A$ such that $tat = a^{-1}$ holds for all $a \in A$. In other words, G is the semidirect product $A \rtimes \langle t \rangle$, where A is abelian and t is an involution inverting each $a \in A$ by conjugation. Observe that the only abelian generalized dihedral groups are the elementary abelian 2-groups.

We now introduce a particular family of cubic graphs that will play one of the central roles in this paper. The graphs in question are known in the literature under various different names such as *brick products*, *generalized honeycomb tori* and *honeycomb toroidal graphs* (see for instance [1, 7, 11, 12, 20, 22, 25]). In this paper we stick to the name honeycomb toroidal graphs.

Following [1] we first define the honeycomb toroidal graphs as follows. For each $m \geq 1$, $n \geq 2$ and $1 \leq s \leq 2n-1$, where $m+s$ is even, let $\mathrm{HTG}(m, 2n, s)$ be the graph of order $2mn$ with vertex set $V = \{(i, j) : 0 \leq i \leq m-1, 0 \leq j \leq 2n-1\}$ in which each vertex (i, j) is adjacent to $(i, j+1)$ and in addition each vertex (i, j), where $i \neq m-1$ and i and j are of different parity, is adjacent to $(i+1, j)$, while the vertices $(m-1, j)$ for j of different parity than $m-1$ are adjacent to $(0, j+s)$. In all of the above computations, operations on the first coordinate are carried out modulo m while on the second modulo $2n$.

For our purposes it will be more convenient to describe the adjacencies in the HTG graphs in a slightly different way. To distinguish between the two equivalent definitions we slightly change the notation of these graphs. For each $m \geq 1$, $n \geq 2$ and $1 \leq k \leq 2n-1$, where k is odd, let $\mathrm{Htg}(m, 2n, k)$ be the graph of order $2mn$ with vertex set $V = \{(i, j) : 0 \leq i \leq m-1, 0 \leq j \leq 2n-1\}$ and the following adjacencies:

$$(i, j) \sim (i, j+1); \ 0 \leq i \leq m-1, 0 \leq j \leq 2n-1$$

and

$$(i, j) \sim \begin{cases} (i+1, j+1); & 0 \leq i < m-1, 0 \leq j \leq 2n-1, j \text{ even}, \\ (0, j+k); & i = m-1, \ 0 \leq j \leq 2n-1, \ j \text{ even}. \end{cases}$$

Again, the computations on the first coordinate are performed modulo m and on the second modulo $2n$. It is straightforward to check that the map ψ from $\mathrm{HTG}(m, 2n, s)$ to $\mathrm{Htg}(m, 2n, s+1-m)$ mapping each vertex (i, j) to $(i, i+j+1)$ is an isomorphism of graphs, which shows that $\mathrm{HTG}(m, 2n, s)$ is isomorphic to $\mathrm{Htg}(m, 2n, s+1-m)$ (where $s+1-m$ is computed modulo $2n$). The two

different presentations of the graph HTG$(5,6,1) \cong$ Htg$(5,6,3)$ are given in Figure 1. Note that k being odd implies that Htg$(m,2n,k)$ is a cubic graph. For each $0 \leq i \leq m-1$ we let $L_i = \{(i,j) \colon 0 \leq j \leq 2n-1\} \subseteq V(\text{Htg}(m,2n,k))$ and call it the *i-th layer* of Htg$(m,2n,k)$. The following result, which is just a restatement of [1, Thorem 3.4] using the graphs Htg instead of HTG, will be important in Section 5.
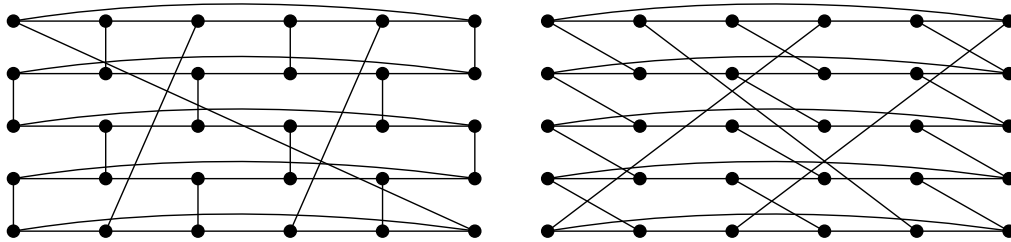

Figure 1. The graph HTG$(5,6,1) \cong$ Htg$(5,6,3)$ in its two presentations.

Proposition 1 [1]**.** *Let $m \geq 1$, $n \geq 2$ and $1 \leq k \leq 2n-1$, where k is odd, and let $G = \left\langle t,x,y \mid t^2, x^n, y^m = x^{m+(k-1)/2}, xy = yx, txt = x^{-1}, tyt = y^{-1} \right\rangle$ be the corresponding generalized dihedral group of order $2mn$. Then the honeycomb toroidal graph Htg$(m,2n,k)$ is isomorphic to the Cayley graph Cay$(G;S)$ of the generalized dihedral group G with respect to $S = \{t, tx, ty\}$.*

It is not difficult to verify that one of the possible isomorphisms φ from Htg$(m,2n,k)$ to Cay$(G;S)$ from the above proposition is the one where for each $0 \leq i \leq m-1$ and $0 \leq j \leq n-1$ we set

$$(1) \qquad \varphi((i,2j)) = tx^{i-j}y^{-i} \quad \text{and} \quad \varphi((i,2j+1)) = x^{j-i+1}y^{i}.$$

We end this section by recalling the notion of an *LCF notation* of a cubic graph possessing a *Hamilton cycle* (a cycle passing through all vertices of the graph) which is due to Lederberg, Coxeter and Frucht (see [14] for details). Let $\phi\Gamma$ be a cubic graph of order 2ℓ and suppose $(v_0, v_1, \ldots, v_{2\ell-1})$ is a Hamilton cycle of $\phi\Gamma$. Then each vertex v_i has exactly one additional neighbor $v_{f(i)}$ other than v_{i-1} and v_{i+1} in $\phi\Gamma$ (where the indices are computed modulo 2ℓ). The LCF notation of the graph $\phi\Gamma$ (with respect to this Hamilton cycle) is then the sequence $[d_0, d_1, \ldots, d_{2\ell-1}]$, where d_i is the unique integer from $-\ell < d_i \leq \ell$ such that $f(i) - i \equiv d_i \pmod{2\ell}$. In the case that the sequence is periodic, say for some divisor ℓ' of ℓ with $\ell = q\ell'$ we have $d_i = d_{i+\ell'}$ for all i, then this sequence is shortened to the exponential notation $[d_0, d_1, \ldots, d_{\ell'-1}]^q$. Since an LCF notation, say $[d_0, d_1, \ldots, d_{2\ell-1}]$, of a cubic graph possessing a Hamilon cycle uniquely determines the graph in question we denote the corresponding graph having vertex-set $\mathbb{Z}_{2\ell}$ by LCF$([d_0, d_1, \ldots, d_{2\ell-1}])$.

3.   CUBIC CAYLEY GRAPHS OF GENERALIZED DIHEDRAL GROUPS

Throughout this section let $G = A \rtimes \langle t \rangle$ be a generalized dihedral group where A is abelian and $t \in G \setminus A$ is an involution such that $tat = a^{-1}$ holds for all $a \in A$. Observe that each element of the coset $tA = At$ is an involution. We now investigate the possible connected cubic Cayley graphs of G. For the rest of this section we thus make the assumption that $\phi\Gamma = \mathrm{Cay}(G; S)$ for some inverse closed subset $S \subset G$ with $|S| = 3$ and $\langle S \rangle = G$. The trivial case where $A = \mathbb{Z}_2$ results in the complete graph K_4 which clearly admits an e.d.s. For the rest of this section we thus assume that $|A| > 2$. Since $\phi\Gamma$ is connected, S contains at least one element from tA, and so there are exactly three essentially different possibilities for S depending on the size of the intersection $|S \cap tA|$. We analyze each of them separately.

Case 1. $|S \cap tA| = 1$. In this case S consists of two elements from A and one from tA. Then either $A = \langle a \rangle$ is cyclic (in which case G is the dihedral group of order $2|A|$) and with no loss of generality $S = \{a, a^{-1}, t\}$ or $A \cong \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, b \rangle$ (in which case G is the elementary abelian 2-group of order 8) and with no loss of generality $S = \{a, b, t\}$. In both possibilities $\phi\Gamma$ is isomorphic to a prism, and so is a Cayley graph of an abelian group. By [6, Proposition 3.2], $\phi\Gamma$ admits an e.d.s. if and only if $|A|$ is divisible by 4.

Case 2. $|S \cap tA| = 2$. In this case $S \cap A$ consists of a single involution, say a, implying that A is of even order, say $2n$, and that, with no loss of generality, we can assume $S = \{a, t, tb\}$ for some $b \in A$. Of course, connectedness of $\phi\Gamma$ implies $A = \langle a, b \rangle$. Now, if $\langle b \rangle = A$ then $a = b^n$, and so $\phi\Gamma \cong \mathrm{Cay}(\mathbb{Z}_{4n}; \{\pm 1, 2n\})$ is a Möbius ladder. By [6, Proposition 3.2], $\phi\Gamma$ admits an e.d.s. if and only if n is odd, that is, if $|A| \equiv 2 \pmod 4$. If however $\langle b \rangle$ is an index 2 subgroup of A (so that $A \cong \mathbb{Z}_n \times \mathbb{Z}_2$) then $\phi\Gamma$ is again isomorphic to a prism, and so [6, Proposition 3.2] implies that $\phi\Gamma$ admits an e.d.s. if and only if n is even, that is if $|A|$ is divisible by 4.

Case 3. $S \subset tA$. Without loss of generality we can assume that $S = \{t, ta, tb\}$ for some $a, b \in A$. Observe that in this case $\phi\Gamma$ is a bipartite graph. As mentioned in Section 2, $\phi\Gamma$ can only admit an e.d.s. if its order is a multiple of 4. However, since each vertex from A dominates three vertices from the coset tA and only one from A while each vertex from tA dominates three vertices from the coset A and only one from tA, it is clear that for each e.d.s. D of $\phi\Gamma$ we must have $|D \cap A| = |D \cap tA|$. Since $|D| = |G|/4 = |A|/2$ it thus follows that $|D \cap A| = |A|/4$, implying that the order of A must be divisible by 4.

In the rest of the paper we thoroughly investigate the graphs arising from the above Case 3 with $|A|$ divisible by 4. Note that we have two essentially different possibilities depending on whether at least one of a, b and ba^{-1} generates A (in

which case G is a dihedral group) or not. We deal with each of the two possibilities in separate sections.

4. The Case $\langle a \rangle = A$

In this section we investigate the existence of an e.d.s. in a Cayley graph $\phi\Gamma = \mathrm{Cay}(G; \{t, ta, tb\})$, where $G = \langle t, a, b \rangle$ is a (generalized) dihedral group with $A = \langle a, b \rangle$ an abelian group of order divisible by 4 such that $a \neq b$, $tat = a^{-1}$, $tbt = b^{-1}$ and that one of a, b and ba^{-1} generates A. Observe that the latter assumption implies that G is in fact a dihedral group and that one of the subgraphs $\mathrm{Cay}(G; \{t, ta\}), \mathrm{Cay}(G; \{t, tb\})$ and $\mathrm{Cay}(G; \{ta, tb\})$ of $\phi\Gamma$ corresponds to a Hamilton cycle of $\phi\Gamma$. With no loss of generality we assume that $A = \langle a \rangle$, so that the edges corresponding to t and ta give rise to a Hamilton cycle of $\phi\Gamma$.

Denote the order of A (and thus of a) by n and let $1 < k \leq n - 1$ be such that $b = a^k$. Then $tb = ta^k = (tat)^{k-1}ta$. Thus, if $2k - 1 < n$ it is not difficult to see that $\phi\Gamma \cong \mathrm{LCF}([2k - 1, -(2k - 1)]^n)$. Similarly, if $n < 2k - 1 < 2n - 1$ then $\phi\Gamma \cong \mathrm{LCF}([2k' - 1, -(2k' - 1)]^n)$ where $k' = n - k + 1$ (an example with $n = 8$ and $k = 6$ is shown in Figure 2). It thus suffices to determine which of the graphs $\mathrm{LCF}([2k - 1, -(2k - 1)]^n)$, where n is divisible by 4 and $1 < 2k - 1 < n$, admit an e.d.s.
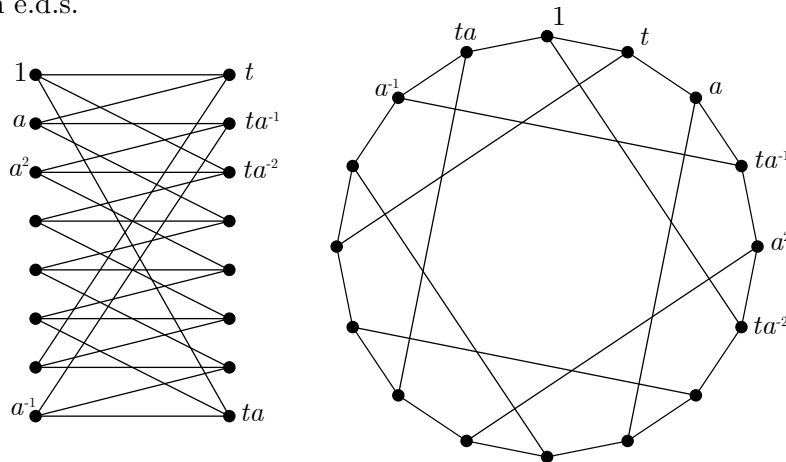

Figure 2. The graph $\mathrm{Cay}(\langle t, a \mid a^8, t^2, tat = a^{-1} \rangle; \{t, ta, ta^6\}) \cong \mathrm{LCF}([5, -5]^8)$.

Proposition 2. *Let n be any positive integer divisible by 4 and let k be an integer such that $1 < k \leq n/2$. Write $n = 2^r\ell$, where $r \geq 2$ and ℓ is an odd integer. Then the graph $\phi\Gamma = \mathrm{LCF}([2k - 1, -(2k - 1)]^n)$ does not admit an e.d.s. if and only if $2k - 1 \equiv \pm 1 \pmod{2^{r+1}}$.*

Proof. Observe first that the order of $\phi\Gamma$ is $2n = 2^{r+1}\ell$ and recall that we assume the vertex-set of $\phi\Gamma$ is $\mathbb{Z}_{2n}$. We first show that in the case when $2k - 1 \equiv \pm 1$

$\pmod{2^{r+1}}$ the graph $\phi\Gamma$ does not admit an e.d.s. We first deal with the case $2k - 1 \equiv 1 \pmod{2^{r+1}}$ and leave the analogous case of $2k - 1 \equiv -1 \pmod{2^{r+1}}$ to the reader.

By way of contradiction suppose that $2k - 1 \equiv 1 \pmod{2^{r+1}}$ and that D is an e.d.s. of $\phi\Gamma$. For each $0 \leq i < 2^{r+1}$ let $V_i = \left\{ j \in \mathbb{Z}_{2n} : j \equiv i \pmod{2^{r+1}} \right\}$ and set $D_i = D \cap V_i$ and $d_i = |D_i|$. To simplify the notation we write D_{-i} and d_{-i} instead of $D_{2^{r+1}-i}$ and $d_{2^{r+1}-i}$, respectively. Recall that each vertex j of $\phi\Gamma$ is adjacent to $j - 1, j + 1$ and to $j + 2k - 1$ in the case that j is even, and is adjacent to $j - 1, j + 1$ and to $j - 2k + 1$ in the case that j is odd. Since D is an e.d.s., the ℓ vertices of V_0 are thus dominated in such a way that d_0 of them are themselves in D_0, while $2d_1$ of them are dominated by vertices from D_1 and d_{-1} by vertices from D_{-1}. Similarly the ℓ vertices of V_1 are dominated in such a way that d_1 of them are themselves in D_1, while d_2 of them are dominated by vertices of D_2 and $2d_0$ by vertcies of D_0. We therefore find that

$$\ell = d_{-1} + d_0 + 2d_1 = 2d_0 + d_1 + d_2.$$

Continuing this way we get the following system of 2^{r+1} equations:

$$(2) \qquad \begin{aligned} \ell &= d_{-1} + d_0 + 2d_1 \\ \ell &= 2d_0 + d_1 + d_2 \\ \ell &= d_1 + d_2 + 2d_3 \\ \ell &= 2d_2 + d_3 + d_4 \\ &\vdots \\ \ell &= d_{-5} + d_{-4} + 2d_{-3} \\ \ell &= 2d_{-4} + d_{-3} + d_{-2} \\ \ell &= d_{-3} + d_{-2} + 2d_{-1} \\ \ell &= 2d_{-2} + d_{-1} + d_0. \end{aligned}$$

From $2d_0 + d_1 + d_2 = \ell = d_1 + d_2 + 2d_3$ we get $2d_0 = 2d_3$ and consequently $d_0 = d_3$. Combining every other pair of the above system of equations, we thus find that for each even i, $0 \leq i \leq 2^{r+1} - 4$, we can replace the equation $\ell = 2d_i + d_{i+1} + d_{i+2}$ by $0 = d_i - d_{i+3}$ in (2) to obtain the following system of 2^{r+1} equations:

$$(3) \qquad \begin{aligned} \ell &= d_{-1} + d_0 + 2d_1 \\ 0 &= d_0 - d_3 \\ \ell &= d_1 + d_2 + 2d_3 \\ 0 &= d_2 - d_5 \\ &\vdots \\ \ell &= d_{-5} + d_{-4} + 2d_{-3} \\ 0 &= d_{-4} - d_{-1} \\ \ell &= d_{-3} + d_{-2} + 2d_{-1} \\ \ell &= 2d_{-2} + d_{-1} + d_0. \end{aligned}$$

Let us denote the right-hand sides of these 2^{r+1} equations by $c_0, c_1, \ldots, c_{2^{r+1}-1}$ in this order. We now evaluate the expression (recall that $r \geq 2$, and so $2^{r+1} - 2 \equiv 2$ (mod 4))

(4)
$$c_0 - c_1 - 2c_2 + 2c_3 + 3c_4 - 3c_5 - 4c_6 + 4c_7$$
$$\pm \cdots + (2^r - 1)c_{2^{r+1}-4} - (2^r - 1)c_{2^{r+1}-3} - 2^r c_{2^{r+1}-2}.$$

On one hand, (3) implies that (4) evaluates to

$$\ell(1 - 2 + 3 - 4 + 5 \pm \cdots + 2^r - 1 - 2^r) = -\ell 2^{r-1}.$$

To evaluate (4) in a different way observe first that d_0 only appears in c_0 and c_1, and d_1 only in c_0 and c_2. Thus, d_0 and d_1 both have coefficient 0 in the evaluations of (4). Next, observe that for an even i with $2 \leq i \leq 2^{r+1} - 4$, the term d_i appears only in c_i and c_{i+1}, and so the coefficient of d_i in the evaluation of (4) is $\pm((i + 2)/2 - (i + 2)/2) = 0$. For odd i with $3 \leq i \leq 2^{r+1} - 3$, the term d_i appears in c_{i-2}, c_{i-1} and c_{i+1}, and so the coefficient of d_i in the evaluation of (4) is
$$\pm(-(i-1)/2 + 2(i+1)/2 - (i+3)/2) = 0.$$

The coefficient of d_{-2} in the evaluation of (4) is clearly -2^r while the coefficient of d_{-1} is $1 + 2^r - 1 - 2 \cdot 2^r = -2^r$. Therefore, (4) evaluates to $-2^r d_{-2} - 2^r d_{-1}$, and so
$$-\ell 2^{r-1} = -2^r(d_{-2} + d_{-1}),$$

contradicting the fact that ℓ is odd. This finally proves that in the case when $2k + 1 \equiv 1 \pmod{2^{r+1}}$ the graph $\phi\Gamma$ does not admit an e.d.s.

To complete the proof we now show that in all of the remaining cases we can indeed find an e.d.s. for the graph $\phi\Gamma$. Suppose then that $2k - 1 \not\equiv \pm 1 \pmod{2^{r+1}}$. Since $r \geq 2$ there thus exists the smallest integer r_0 with $2 \leq r_0 \leq r$ such that $2k - 1 \not\equiv \pm 1 \pmod{2^{r_0+1}}$. Note that the order of $\phi\Gamma$ is a multiple of 2^{r_0+1}. Now, if $r_0 = 2$, then $2k - 1 \equiv \pm 3 \pmod 8$, and so we can assume (otherwise take $-2k + 1$ instead of $2k - 1$) that $2k - 1 \equiv 3 \pmod 8$. In this case simply take $D = \{8i : 0 \leq i \leq 2^{r-2} - 1\} \cup \{5 + 8i : 0 \leq i \leq 2^{r-2} - 1\}$. It is easy to verify that in this case D is an e.d.s. for $\phi\Gamma$.

We are thus left with the possibility $r_0 > 2$. By assumption, $2k - 1 \equiv \pm 1 \pmod{2^{r_0}}$. With no loss of generality assume $2k - 1 \equiv -1 \pmod{2^{r_0}}$. Since $2k - 1 \not\equiv -1 \pmod{2^{r_0+1}}$, it thus follows that $2k - 1 \equiv 2^{r_0} - 1 \pmod{2^{r_0+1}}$. In this case let

$$D' = \left\{0, 4, 8, \ldots, 2^{r_0} - 4, 2^{r_0} + 1, 2^{r_0} + 5, 2^{r_0} + 9, \ldots, 2^{r_0+1} - 3\right\}$$

and then set
$$D = \left\{j + 2^{r_0+1}i : 0 \leq i < 2^{r-r_0}\ell, \ j \in D'\right\}.$$

We now indicate how one can verify that D is an e.d.s. for $\phi\Gamma$. To this end take any vertex of $\phi\Gamma$, say $v' = j' + 2^{r_0+1}i'$ where $0 \leq j' < 2^{r_0+1}$ and $0 \leq i' < 2^{r-r_0}\ell$. We deal with the case that $j' \equiv 0 \pmod 4$ and leave the remaining three cases to the reader. Observe first that if $j' < 2^{r_0}$ then $j' \in D'$, implying that $v' \in D$. On the other hand, if $2^{r_0} \leq j' < 2^{r_0+1}$ then $j' \notin D'$, and so $v' \notin D$, but in this case v' is dominated by $v' + 1 \in D$. We now only need to show that v' is not dominated by some other vertex from D. Of course, since v' is even, the only other vertex of $\phi\Gamma$ that could possibly dominate v' is its neighbor $v = v' + 2k - 1$. But as $2k - 1 \equiv 2^{r_0} - 1 \pmod{2^{r_0+1}}$ we see that $v \equiv 2^{r_0} + j' - 1 \pmod{2^{r_0+1}}$, and so $v \notin D$ as $2^{r_0} + j' - 1 \equiv 3 \pmod 4$. ∎

The above proposition provides the first part of the classification of cubic Cayley graphs of generalized dihedral groups admitting an e.d.s.

Corollary 3. *Let $n \geq 3$ be an integer, let $G = \langle t, a \mid t^2, a^n, tat = a^{-1} \rangle$ be the dihedral group of order $2n$ and let $b = a^k$ for some $1 < k \leq n - 1$. Write $n = 2^r\ell$, where ℓ is odd. Then the Cayley graph $\mathrm{Cay}(G; \{t, ta, tb\})$ admits an e.d.s. if and only if $r \geq 2$ and $2k - 1 \not\equiv \pm 1 \pmod{2^{r+1}}$.*

Proof. Denote $\mathrm{Cay}(G; \{t, ta, tb\})$ by $\phi\Gamma$. The necessity of $r \geq 2$ was established in Section 3. Now, if $2k - 1 < n$ then $\phi\Gamma \cong \mathrm{LCF}([2k-1, -(2k-1)]^n)$, while in the case that $n < 2k - 1$ we have that $\phi\Gamma \cong \mathrm{LCF}([2(n-k+1)-1, -(2(n-k+1)-1)]^n)$. However, as 2^{r+1} divides $2n$, the condition $2k - 1 \equiv \pm 1 \pmod{2^{r+1}}$ is equivalent to $2(n - k + 1) - 1 \equiv \pm 1 \pmod{2^{r+1}}$, and so Proposition 2 applies. ∎

To illustrate the above result we consider all cubic Cayley graphs of the dihedral group $G = \langle t, a \mid t^2, a^{12}, tat = a^{-1} \rangle$ of order 24 with connection set of the form $S = \{t, ta, ta^k\}$, where $2 \leq k \leq 6$ (note that by the above remarks we only need to consider k up to 6 since $S' = \{t, ta, ta^{n-k+1}\}$ and S gives rise to isomorphic graphs). It is not difficult to see (but one can also use Magma [4]) that the five different possibilities for $2 \leq k \leq 6$ give rise to pairwise nonisomorphic graphs. By Corollary 3 the Cayley graphs $\mathrm{Cay}(G; \{t, ta, ta^2\})$, $\mathrm{Cay}(G; \{t, ta, ta^3\})$ and $\mathrm{Cay}(G; \{t, ta, ta^6\})$ all admit an e.d.s., while none of the Cayley graphs $\mathrm{Cay}(G; \{t, ta, ta^4\})$ and $\mathrm{Cay}(G; \{t, ta, ta^5\})$ admits an e.d.s. We remark that the graph $\mathrm{Cay}(G; \{t, ta, ta^2\})$ is in fact isomorphic to the prism of order 24.

5. The Honeycomb Toroidal Graphs

In view of the results from the previous section it remains to investigate the existence of an e.d.s. in a Cayley graph $\phi\Gamma = \mathrm{Cay}(G; \{t, ta, tb\})$, where $G = \langle t, a, b \rangle$ is a generalized dihedral group with $A = \langle a, b \rangle$ an abelian group of order divisible by 4 such that $a \neq b$, $tat = a^{-1}$, $tbt = b^{-1}$ and none of a, b and ba^{-1}

generates A. By Proposition 1, the graph $\phi\Gamma$ is isomorphic to a honeycomb toroidal graph. In fact, if n is the order of a and m is the smallest nonnegative integer such that $b^m \in \langle a \rangle$, say $b^m = a^{m+k'}$ for an appropriate $0 \le k' \le n-1$, then $\phi\Gamma \cong \mathrm{Htg}(m, 2n, 2k'+1)$. Observe that, since a does not generate A, $m \ge 2$. We thus only need to classify the honeycomb toroidal graphs $\mathrm{Htg}(m, 2n, k)$ admitting an e.d.s., subject to the conditions that $m \ge 2$ and mn is divisible by 4.

We first state two results giving isomorphisms between the Honeycomb toroidal graphs that will be important in the remainder of this section. The straightforward verification that the map given in the statement of the first of these is indeed a graph isomorphism is left to the reader.

Lemma 4. *Let $m \ge 2$, $n \ge 2$ and $1 \le k < 2n$ be integers with k odd. Let k' be the unique integer such that $1 \le k' < 2n$ and $k' \equiv 2 - k - 2m \pmod{2n}$, and let $\phi\Gamma_1 = \mathrm{Htg}(m, 2n, k)$ and $\phi\Gamma_2 = \mathrm{Htg}(m, 2n, k')$. Then the mapping $\varphi \colon V(\phi\Gamma_1) \to V(\phi\Gamma_2)$, defined by the rule $\varphi((i,j)) = (i, 2i - j)$, $0 \le i < m$, $0 \le j < 2n$, where $2i - j$ is computed modulo $2n$, is an isomorphism of graphs.*

Lemma 5. *Let $m \ge 2$, $n \ge 2$ and $1 \le k < 2n$ be integers with k odd. Set $m' = \gcd(n, (k-1)/2)$ and $m'' = \gcd(n, m + (k-1)/2)$ and then let $n' = mn/m'$ and $n'' = mn/m''$. Then there exist odd integers k' and k'' with $1 \le k' < 2n'$ and $1 \le k'' < 2n''$ such that $\mathrm{Htg}(m, 2n, k) \cong \mathrm{Htg}(m', 2n', k') \cong \mathrm{Htg}(m'', 2n'', k'')$.*

Proof. By Proposition 1, the graph $\mathrm{Htg}(m, 2n, k)$ is isomorphic to the Cayley graph $\phi\Gamma = \mathrm{Cay}(G; \{t, ta, tb\})$ of the generalized dihedral group $G = \langle t, a, b \mid t^2, a^n, b^m = a^{m+(k-1)/2}, ab = ba, tat = a^{-1}, tbt = b^{-1} \rangle$. In fact, the isomorphism φ from (1) maps the layer L_0 of $\mathrm{Htg}(m, 2n, k)$ to the set of all vertices from $\langle t, a \rangle$ of $\phi\Gamma$, showing that the parameter n from $\mathrm{Htg}(m, 2n, k)$ does indeed correspond to the order of a. Exchanging the roles of a and b in the definition of G the same reasoning (again using φ from (1)) shows that $\phi\Gamma$ must also be isomorphic to a honeycomb toroidal graph $\mathrm{Htg}(\tilde{m}, 2\tilde{n}, \tilde{k})$ in which the layer $\tilde{L}_0$ corresponds to the set of vertices from $\langle t, b \rangle$ of $\phi\Gamma$. This implies that $\tilde{n}$ coincides with the order of b while (in order to get a graph of the same order) $\tilde{m}\tilde{n} = mn$ must hold. Since $b^m = a^{m+(k-1)/2}$ while $b^i \notin \langle a \rangle$ for $1 \le i < m$, it is thus clear that $\tilde{n} = mn/\gcd(n, m + (k-1)/2) = n''$ and consequently $\tilde{m} = m''$.

The second part of the proof is done analogously where this time we relabel the elements of G such that $t' = ta$ and then $a' = a^{-1}b$ and $b' = a^{-1}$ (so as to get $\{t', t'a', t'b'\} = \{t, ta, tb\}$). Again using the corresponding isomorphism φ from (1) we now see that $\phi\Gamma \cong \mathrm{Htg}(\tilde{m}, 2\tilde{n}, \tilde{k})$ for some odd integer $\tilde{k}$ with $1 \le \tilde{k} < 2\tilde{n}$, where $\tilde{n}$ is the order of $a' = a^{-1}b$. It is easy to see that this order is $mn/\gcd(n, (k-1)/2) = n'$, and so the result follows. ∎

We now investigate the structure of a possible e.d.s. of a honeycomb toroidal graph $\mathrm{Htg}(m, 2n, k)$, where $m \ge 2$. In some of the arguments we will be using

the following terminology. Let D be a dominating set for a graph $\phi\Gamma$ (that is, every vertex of $\phi\Gamma$ is either an element of D or is adjacent to at least one vertex from D). In the case that a vertex v of $\phi\Gamma$ is either an element of D but is also adjacent to at least one (other) vertex from D, or v is adjacent to at least two vertices from D, we say that v is *doubly dominated* (by D). Observe that if D is an e.d.s. of $\phi\Gamma$ then no vertex of $\phi\Gamma$ is doubly dominated by D.

Lemma 6. *Let $m \geq 2$, $n \geq 3$ and $1 \leq k < 2n$ be integers with k odd such that the graph $\phi\Gamma = \mathrm{Htg}(m, 2n, k)$ admits an e.d.s. D. Then the following hold.*

(i) *If $(i, j) \in D$ for some $0 \leq i < m$ and $0 \leq j < 2n$, then none of the vertices $(i, j + j')$ where $j' \in \{-2, -1, 1, 2\}$ is in D.*

(ii) *For each $0 \leq i < m$ and $0 \leq j < 2n$, there exists $0 \leq j' \leq 4$ such that $(i, j + j') \in D$.*

(iii) *If $(i, j) \in D$ for some $0 \leq i < m$ and $0 \leq j < 2n$, then for one of $j' \in \{3, 4, 5\}$ the vertex $(i, j + j')$ is in D.*

Proof. (i) Since for any such j' the vertices (i, j) and (i, j') are either adjacent or have a common neighbor some vertex of $\phi\Gamma$ would be doubly dominated if (i, j) and (i, j') were both in D.

(ii) We provide an argument for the case when $1 \leq i \leq m - 2$. The case when $i \in \{0, m - 1\}$ can be done analogously. Suppose to the contrary that none of the vertices $(i, j + j')$, $0 \leq j' \leq 4$ is in D. Since $\phi\Gamma$ is cubic $(i, j + 1)$ and $(i, j + 3)$ can only be dominated if either both $(i - 1, j)$ and $(i - 1, j + 2)$ (if j is even) or both $(i + 1, j + 2)$ and $(i + 1, j + 4)$ (if j is odd) are in D. But this contradicts (i).

(iii) This follows immediately from (i) and (ii). ∎

Corollary 7. *Let $m \geq 2$ and $k \in \{1, 3, 5\}$. Then the graph $\mathrm{Htg}(m, 6, k)$ does not admit an e.d.s.*

Proof. Let $\phi\Gamma = \mathrm{Htg}(m, 6, k)$. Suppose to the contrary that $D \subset V(\phi\Gamma)$ is an e.d.s. of $\phi\Gamma$. Since $\phi\Gamma$ is cubic, $|D| = |V|/4 = 3m/2$. In particular, m has to be even and there exists at least one layer L_i such that $|L_i \cap D| \leq 1$. However, this contradicts to the item (iii) of Lemma 6. ∎

Lemma 8. *Let $m \geq 2$, $n \geq 2$ and $1 \leq k < 2n$ be integers with k odd such that the graph $\mathrm{Htg}(m, 2n, k)$ admits an e.d.s. D. Then the following hold.*

(i) *If for some $0 \leq i < m$ and $0 \leq j < 2n$ with j even the vertices (i, j) and $(i, j + 3)$ are both in D, then $(i + 1, j - 1), (i + 1, j + 4) \in D$ if $i \neq m - 1$, and $(0, j + k - 2), (0, j + k + 3) \in D$ if $i = m - 1$.*

(ii) *If for some $0 \leq i < m$ and $0 \leq j < 2n$ with j odd the vertices (i, j) and $(i, j + 5)$ are both in D, then $(i + 1, j + 1), (i + 1, j + 4) \in D$ if $i \neq m - 1$, and $(0, j + k), (0, j + k + 3) \in D$ if $i = m - 1$.*

Proof. We provide an argument for the case when $i \neq m - 1$ and leave the analogous case of $i = m - 1$ to the reader. Suppose first that $(i, j), (i, j + 3) \in D$ for some even j (note that, by Lemma 6 and Corollary 7, this implies that $n \geq 4$). Consider the vertex $(i + 1, j + 3)$ and observe that it cannot be contained in D, since otherwise $(i, j + 2)$ would be doubly dominated. Therefore, $(i + 1, j + 3)$ must be dominated by one of its neighbours. But if $(i, j + 2)$ (respectively, $(i + 1, j + 2)$) is in D, then $(i, j + 3)$ (respectively, $(i + 1, j + 1)$) is doubly dominated, and so $(i + 1, j + 4) \in D$ (see Figure 3). Since $(i + 1, j + 1)$ is already dominated by (i, j), none of $(i + 1, j + 1)$ and $(i + 1, j)$ is in D, and so Lemma 6 implies that $(i + 1, j - 1) \in D$.
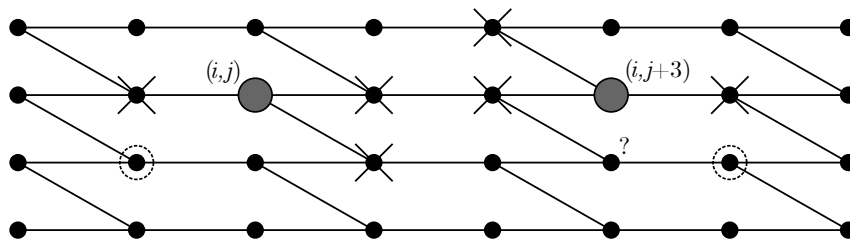
Figure 3. The situation from the proof of Lemma 8.

Suppose now that $(i, j), (i, j + 5) \in D$ for some odd j. Similarly as above observe first that in order to dominate the vertex $(i, j + 3)$ and avoid double domination we must have that $(i + 1, j + 4) \in D$. But then $(i + 1, j + 2)$ can only be dominated by $(i + 1, j + 1) \in D$ (otherwise we again get double domination). ∎

We now show that if m and n are both even, one can always find an e.d.s. of $\mathrm{Htg}(m, 2n, k)$.

Lemma 9. *Let $m \geq 2$, $n \geq 2$ and $1 \leq k < 2n$ be integers with k odd. If m and n are both even, then the graph $\mathrm{Htg}(m, 2n, k)$ admits an e.d.s.*

Proof. Observe that, since n is even, the number of vertices in each layer L_i of $\phi\Gamma = \mathrm{Htg}(m, 2n, k)$ is a multiple of 4. Since there is an even number of layers, it is easy to explicitly construct an e.d.s. For each even i, $0 \leq i < m$, set

$$D_i = \{(i, 4j) \colon 0 \leq j < n/2\},$$

and, for each odd i, set

$$D_i = \{(i, 4j + 3) \colon 0 \leq j < n/2\}.$$

It is straightforward to check that the union of all D_i, where $0 \leq i \leq m - 1$, is an e.d.s. for $\mathrm{Htg}(m, 2n, k)$. ∎

We now show that a partial converse of Lemma 9 holds. In particular, for an e.d.s. to exist n has to be even.

Lemma 10. *Let $m \geq 2$, $n \geq 2$ and $1 \leq k < 2n$ be integers with k odd. If n is odd, then the graph $\mathrm{Htg}(m, 2n, k)$ does not admit an e.d.s.*

Proof. Suppose to the contrary that the statement of the lemma does not hold and let $n \geq 3$ be the smallest odd integer such that for some $m \geq 2$ and $1 \leq k < 2n$ with k odd the graph $\phi\Gamma = \mathrm{Htg}(m, 2n, k)$ admits an e.d.s. D. By Corollary 7, it follows that $n \geq 5$. Now, let $H = \langle k - 1 \rangle$ denote the subgroup of $\mathbb{Z}_{2n}$ generated by $k - 1$. Throughout this proof we regard the elements of H as integers from $\{0, 1, \ldots, 2n - 1\}$ but make all computations with them modulo $2n$. This should cause no confusion.

Since $|D| = mn/2$ and n is odd, the number m of layers of $\phi\Gamma$ is even. Moreover, since $2n$ is not divisible by 4, Lemma 6 implies that there exists a vertex (i, j) of $\phi\Gamma$ such that $(i, j), (i, j + 3) \in D$. By Lemma 4, we can assume that j is even (if j is odd then the images $\varphi((i, j + 3)) = (i, 2i - j - 3)$ and $\varphi((i, j)) = (i, 2i - j)$ are of the desired form as then $2i - j - 3$ is even). Without loss of generality we can also assume that $i = 0$ and $j = 2$ (otherwise relabel the vertices of $\phi\Gamma$ accordingly). Lemma 8 then implies that $(1, 1), (1, 6) \in D$. Since the second coordinate in $(1, 1)$ is odd we can again use Lemma 8 to find that also $(2, 2), (2, 5) \in D$. We can now continue in this way to establish that for each even i, $0 \leq i < m$, we have $(i, 2), (i, 5) \in D$, while for each odd i, $0 \leq i < m$, we have $(i, 1), (i, 6) \in D$ (see Figure 4).
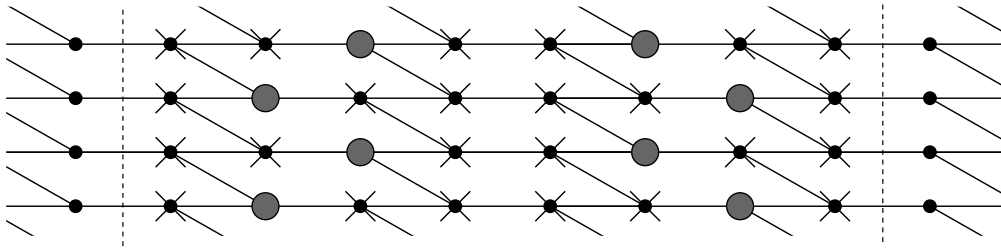

Figure 4. The situation from the proof of Lemma 10.

In particular, $(m-1, 1), (m-1, 6) \in D$ (recall that m is even). Again applying Lemma 8 we have that $(0, k + 1), (0, k + 4) \in D$. We can now repeat the whole argument with $(0, k + 1)$ and $(0, k + 4)$ as the starting pair of vertices. In this way we get that for each even i we have

$$(5) \qquad D_i' = \{(i, j) \colon j \in (2 + H) \cup (5 + H)\} \subseteq D,$$

while for each odd i we have

(6) $$D'_i = \{(i,j)\colon j \in (1+H) \cup (6+H)\} \subseteq D.$$

Observe that the vertices from $D' = \bigcup_{i=1}^{m-1} D'_i$ dominate precisely all of the vertices (i,j), where $0 \le i < m$ and $j \in j' + H$ for some $0 \le j' \le 7$. Denote this set of vertices by V'.

Let us now determine the possible values for $d = \gcd(k-1, 2n)$. Since n and k are both odd, it follows that $d \equiv 2 \pmod 4$. Moreover, $H = \langle d \rangle = \{dj\colon 0 \le j \le 2n/d\}$. If $d = 2$, then (5) implies that $(0,4) \in D$ which contradicts Lemma 6 as we already have $(0,2) \in D$. Similarly, if $d = 6$, then (6) implies that $(1,7) \in D$, which again contradicts Lemma 6 as we already have $(1,6) \in D$. Suppose now that $d = 10$ and consider the vertex $(0,8)$. By (6) we have that $(1,11), (m-1, 7-k) \in D$ (since $6 - (k-1) = 7 - k$). These two vertices dominate $(0,10)$ and $(0,7)$, respectively, and so it follows that none of $(0,7), (0,8), (0,9)$ is in D. The only way to dominate $(0,8)$ is thus to insist that $(1,9) \in D$. But then $(1,10)$ is doubly dominated, a contradiction. This shows that in fact $d \ge 14$ holds.

Let $\phi\Gamma'$ denote the subgraph of $\phi\Gamma$ induced on the set $V(\phi\Gamma)\setminus V'$ and note that $D'' = D \setminus D'$ is an e.d.s. of $\phi\Gamma'$. Moreover, none of the vertices of $\phi\Gamma'$ of valency 2 (that is, vertices of the form (i,j), where $0 \le i < m$ and $j \in (-1+H) \cup (8+H)$) is contained in D'', as otherwise either $(i, j+1)$ or $(i, j-1)$ would be doubly dominated. This shows that D'' is also an e.d.s. of the graph $\phi\Gamma''$, obtained from $\phi\Gamma'$ by adding all of the edges $(i,j)(i,j+9)$ where $0 \le i < m$ and $j \in -1+H$. It is not difficult to see that the graph $\phi\Gamma''$ is isomorphic to $\mathrm{Htg}(m, n'', k'')$, where $n'' = 2n - 8(2n/d)$, and k'' is some odd integer with $1 \le k'' < 2n''$ (in fact, one can verify that $k'' = (d-8)(k-1)/d + 1$). But since $2n \equiv 2 \pmod 4$ also $n'' \equiv 2 \pmod 4$, and so by minimality of n the graph $\phi\Gamma''$ does not admit an e.d.s., a contradiction. ∎

We are now ready to classify the honeycomb toroidal graphs admitting an e.d.s.

Theorem 11. *Let $m \ge 2$, $n \ge 2$ and $1 \le k < 2n$ be integers with k odd and let $\phi\Gamma = \mathrm{Htg}(m, 2n, k)$. Let $m' = \gcd(n, (k-1)/2)$ and $m'' = \gcd(n, m + (k-1)/2)$ and then write $n = 2^r \ell$, $m' = 2^{r'} \ell'$ and $m'' = 2^{r''} \ell''$, where ℓ, ℓ' and ℓ'' are all odd. Then the graph $\phi\Gamma$ admits an e.d.s. if and only if one of the following holds.*

(i) *m and n are both even,*

(ii) *m is odd and n is even and either*

  - *$1 \le r' < r$, or*
  - *$r' = 0$ and $r'' < r$.*

Proof. Recall that, by Lemma 5, $\phi\Gamma \cong \text{Htg}(m', 2n', k') \cong \text{Htg}(m'', 2n'', k'')$ for some odd $1 \leq k' < 2n'$ and $1 \leq k'' < 2n''$, where $n' = mn/m'$ and $n'' = mn/m''$. Now, if n is odd then Lemma 10 implies that $\phi\Gamma$ does not admit an e.d.s.

For the rest of the proof we thus assume that n is even. If m is also even, then Lemma 9 implies that $\phi\Gamma$ admits an e.d.s. It thus remains to investigate the situation when n is even (and consequently $r > 0$) and m is odd. We now distinguish three cases depending on the value of r' (note that $0 \leq r' \leq r$). If $1 \leq r' < r$, then m' and n' are both even, and so Lemma 9 implies that $\text{Htg}(m', 2n', k')$ (and thus also $\phi\Gamma$) admits an e.d.s. On the other hand, if $r' = r$, then n' is odd, and so Lemma 10 implies that $\text{Htg}(m', 2n', k')$ (and thus also $\phi\Gamma$) does not admit an e.d.s.

The last case to consider is thus when $r' = 0$. Since n is even, the fact that $r' = 0$ implies that $(k-1)/2$ must be odd. Since m is also odd, it follows that m'' is even, that is, $r'' > 0$. Depending on whether $1 \leq r'' < r$ or $r'' = r$, we can thus apply Lemma 9 or Lemma 10 to $\text{Htg}(m'', 2n'', k'')$. Since the above cases cover all the possibilities for m, n and k, this completes the proof. ∎

We remark that all of the possibilities from the proof of the above theorem are indeed possible. For instance, to see that the four essentially different possibilities in the case that n is even and m is odd can indeed occur, consider the four graphs $\text{Htg}(3, 48, k)$ where $k \in \{1, 3, 5, 11\}$. In all cases $r = 3$ but for $\text{Htg}(3, 48, 1)$ we get $r' = 3$ (and so the graph does not admit an e.d.s.), for $\text{Htg}(3, 48, 3)$ we get $r' = 0$ and $r'' = 2 < r$ (and so the graph admits an e.d.s.), for $\text{Htg}(3, 48, 5)$ we get $r' = 1$ (and so the graph admits an e.d.s.), while for $\text{Htg}(3, 48, 11)$ we get $r' = 0$ and $r'' = 3 = r$ (and so the graph does not admit an e.d.s.).

It is now also easy to give a classification of the corresponding cubic Cayley graphs of generalized dihedral groups that admit an e.d.s.

Corollary 12. *Let $G = \langle t, a, b \rangle$ be a generalized dihedral group of order $2mn$ for some integers $m, n \geq 2$, where $A = \langle a, b \rangle$ is an abelian group of order mn, a is of order n and $tat = a^{-1}$ and $tbt = b^{-1}$. Suppose that none of a, b and ba^{-1} generates the subgroup A. Then the Cayley graph $\text{Cay}(G; \{t, ta, tb\})$ admits an e.d.s. if and only if at least one of a, b and ba^{-1} generates a subgroup of A which is of even order and of even index.*

Proof. Let $\phi\Gamma = \text{Cay}(G; \{t, ta, tb\})$. Since $\langle a \rangle$ is a subgroup of index m in A, there exists a unique odd integer $1 \leq k \leq 2n - 1$ such that $b^m = a^{m+(k-1)/2}$. By Proposition 1, the graph $\phi\Gamma$ is isomorphic to the honeycomb toroidal graph $\text{Htg}(m, 2n, k)$. Recall that one of the corresponding isomorphisms is the isomorphism φ from (1). We can now apply Theorem 11. Clearly, n and m are the order and index of $\langle a \rangle$ in A, respectively. Moreover, using the defining relations for t, a and b in G or inspecting the nature of the action of φ it is easy to see that the order of ba^{-1} is $2mn/\gcd(2n, k-1)$, which is exactly the parameter n'

from Theorem 11 (and so m' is the index of $\langle ba^{-1} \rangle$ in A). Similarly, the order of b is $2mn / \gcd(2n, 2m + k - 1)$, which is exactly the parameter n'' from Theorem 11 (and so m'' is the index of $\langle b \rangle$ in A). Since the two items of (ii) in Theorem 11 correspond to the case when n' and m' are both even and to the case when n'' and m'' are both even (recall from the proof of Theorem 11 that we cannot have $r' = r'' = 0$), respectively, the result follows. ∎

With the above result in hand let us now examine all of the cubic Cayley graphs of generalized dihedral groups of order 24. One can check that, up to isomorphism, the only possibility to obtain a generalized dihedral group $\langle t, a, b \mid t^2, a^n, b^m = a^{m+(k-1)/2}, ab = ba, tat = a^{-1}, tbt = b^{-1} \rangle$ of order 24 with $m, n \geq 2$ such that none of a, b and ba^{-1} generates $\langle a, b \rangle$ is if $m = 2$, $n = 6$ and $k = 5$. The corresponding Cayley graph $\mathrm{Cay}(G; \{t, ta, tb\})$ admits an e.d.s. by Corollary 12 since $\langle a \rangle$ is an index 2 subgroup (and thus of even order) of $\langle a, b \rangle$. Together with the results at the end of the previous section this finally shows that out of the seven pairwise nonisomorphic connected cubic Cayley graphs of generalized dihedral groups of order 24 four of them admit an e.d.s. while three of them do not (the two from the previous section and the Möbius ladder of order 24).

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