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HAMILTONIAN EXTENDABLE GRAPHS

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Abstract

A graph is called *Hamiltonian extendable* if there exists a Hamiltonian path between any two nonadjacent vertices. In this paper, we give an explicit formula of the minimum number of edges for Hamiltonian extendable graphs and we also characterize the degree sequence for Hamiltonian extendable graphs with minimum number of edges. Besides, we completely characterize the pairs of forbidden subgraphs for 2-connected graphs to be Hamiltonian extendable.

Keywords: Hamiltonian extendable, forbidden subgraph.

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1. Introduction

All graphs considered here are finite, undirected and simple. For notation or terminology not defined, see [3]. We denote by $V(G)$, $E(G)$, $\Delta(G)$, $\delta(G)$, $\alpha(G)$, $\kappa(G)$, the vertex set, the edge set, the maximum degree, the minimum degree, independence number, vertex connectivity of a graph G, respectively. We denote by $N_G(v)$ ($N(v)$ for short) and $d_G(v)$ ($d(v)$ for short) the neighborhood and the degree of a vertex v in G, respectively. For $v \in V(G)$, set $N_G[v] = N_G(v) \cup \{v\}$, and for $S \subseteq V(G)$, let $N_G(S) = \left(\bigcup_{v\in S} N_G(v)\right)\setminus S$. Let $S \subseteq V(G)$ and $S' \subseteq E(G)$. The *induced subgraph* of G by S and S' is denoted by $G[S]$ and $G[S']$, respectively. We use $G - S$ and $G - S'$ to denote the subgraph $G[V(G) - S]$ and $G[E(G) - S']$, respectively. If the number of components of $G - S$ is greater than the number of components of G, then S is a *vertex cut* of G, and we call S an $|S|$-*vertex cut* of G. We call the vertex in S a *cut-vertex* of G if $|S| = 1$. If $E(G[S]) = \emptyset$, then

S is an *independent set* of G. A path with end-vertices u and v of G is denoted by $P_G(u, v)$.

A graph is *traceable* if it has a Hamiltonian path, i.e., a path containing all vertices of the graph. A graph is *Hamiltonian* if it has a Hamiltonian cycle, i.e., a cycle containing all vertices of the graph. A graph is *Hamiltonian connected* if any pair of vertices are connected by a Hamiltonian path. We denote by K_n and $K_{m,n}$ the *complete graph* with order n and the *complete bipartite graph* with partite sets of cardinalities m and n, respectively. A *clique* is a complete subgraph of a graph. We use P_n and C_n to denote the path and the cycle with order n, respectively.

Matching extension has been studied by many authors, see [10, 13]. In [6], the authors introduce a different kind of matching extendability. A graph G is *matching extendable* if for any two nonadjacent vertices $x, y \in V(G)$, there exists an (almost) perfect matching in $G - \{x, y\}$. Since a perfect matching is a 1-factor, it is natural to consider the similar problem of 2-factor. In [7] the authors study 2-factor extendable graphs. In general, one may study the similar problem for k-factors. We concentrate on the case of Hamiltonian cycle.

We say that two nonadjacent vertices $x, y \in V(G)$ can be *extended* to a Hamiltonian cycle (or k-factor) if $G + xy$ has a Hamiltonian cycle (or k-factor) containing xy. A graph is *Hamiltonian* (or *k-factor*) *extendable* if each pair of nonadjacent vertices can be extended to a Hamiltonian cycle (or k-factor). A Hamiltonian (or k-factor) extendable graph with minimum number of edges is a *minimum Hamiltonian* (or *k-factor*) *extendable graph*. The size of a minimum Hamiltonian (or k-factor) extendable graph of order n is denoted by $Exp_h(n)$ (or $Exp_k(n)$). In [6] and [7] the authors gave the explicit formulas for $Exp_1(n)$ and $Exp_2(n)$ for all n.

Theorem 1 (Costa, de Werra and Picouleau)**.**

(1) [6] *If $n \geq 8$ and n is even, then $Exp_1(n) = \frac{3n}{2} - 1$; if $n \geq 9$ and n is odd, then $Exp_1(n) = n - 1$.*

(2) [7] *If $n \geq 10$, then $Exp_2(n) = \lceil \frac{11n}{8} \rceil$.*

In [11] Moon has proved that each Hamiltonian connected graph with order at least 4 has at least $\frac{3n}{2}$ edges and the bound is sharp. We determine $Exp_h(n)$ for any integer $n \geq 3$. In fact, Costa, de Werra, Picouleau [8] and we obtained the following result (Theorem 2) independently.

Theorem 2. *It holds that*

(1) $Exp_h(3) = 2$, $Exp_h(4) = 4$, $Exp_h(5) = 6$,

(2) $Exp_h(n) = \lceil \frac{3n}{2} \rceil$, *for* $n \geq 6$.

We would like to characterize the Hamiltonian extendable graphs G of order n with $|E(G)| = Exp_h(n) = \lceil \frac{3n}{2} \rceil$, for $n \geq 6$. Although we could not completely characterize them, we may give some properties of those graphs. By $K_n x K_m$ we denote the graph G with $V(G) = V(K_n) \cup \{x\} \cup V(K_m)$, $E(G) = E(K_n) \cup E(K_m) \cup \{xu : u \in V(K_n) \cup V(K_m)\}$, while $V(K_n)$ and $V(K_m)$ are supposed to be disjoint (even if $n = m$) and $x \notin V(K_n) \cup V(K_m)$. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of G, $d(v_1) \le d(v_2) \le \cdots \le d(v_{n-1}) \le d(v_n)$. Then the sequence $(d(v_1), d(v_2), \ldots, d(v_n))$ is called a *degree sequence* of G. The following is our second main result.

Theorem 3. *Let G be a Hamiltonian extendable graph of order $n \ge 6$ and $|E(G)| = \lceil \frac{3n}{2} \rceil$. Then the following statements hold.*

(1) *If there is no vertex of degree 2, and if n is even, then G is 3-regular; if n is odd, then the degree sequence of G is $(3, 3, \ldots, 3, 4)$.*

(2) *If $n \ge 7$, then $\Delta(G) \le 4$; if $n = 6$, then either $G \cong K_3 w K_2$ or $\Delta(G) \le 4$.*

(3) *If there exists a vertex of degree 2 that has two neighbors of degree exactly 4 in G, then the degree sequence of G is $(2, 3, 3, \ldots, 3, 4, 4)$.*

(4) *If there exists a vertex of degree 2 that has one neighbor of degree 3 and the other neighbor of degree 4, then either $G \cong F_4$ or else $G \cong F_5$, where F_4, F_5 are depicted in Figure 1.*

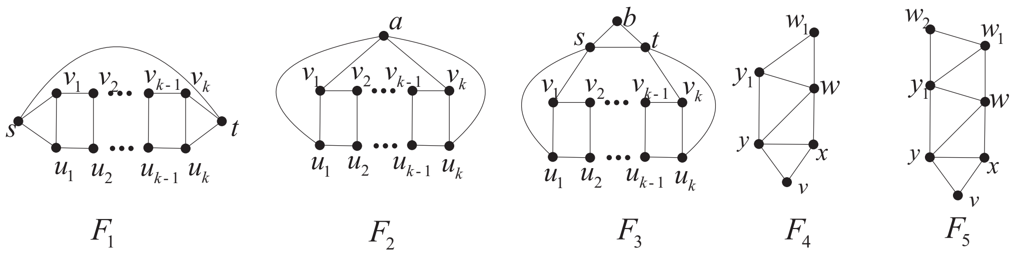

Figure 1. Five Hamiltonian extendable graphs.

Let $\mathcal{H}$ be a set of connected graphs. A graph G is said to be $\mathcal{H}$-free if G does not contain H as an induced subgraph for all H in $\mathcal{H}$. We call each graph H in $\mathcal{H}$ a forbidden subgraph. If $\mathcal{H} = \{H\}$, then G is H-free. If $|\mathcal{H}| = 2$, then we call $\mathcal{H}$ a forbidden pair. For two sets $\mathcal{H}_1$, $\mathcal{H}_2$, we write $\mathcal{H}_1 \preceq \mathcal{H}_2$ if for each graph H'' in $\mathcal{H}_2$, there exists a graph H' in $\mathcal{H}_1$ such that H' is an induced subgraph of H''. By the definition of the relation"$\preceq$", if $\mathcal{H}_1 \preceq \mathcal{H}_2$, then every $\mathcal{H}_1$-free graph is also $\mathcal{H}_2$-free.

Every Hamiltonian connected graph is Hamiltonian extendable. However, the converse is not true in general. In [2, 4, 9, 15], the authors have partially characterized pairs of forbidden subgraphs for a 3-connected graph to be Hamiltonian connected. Note that the 3-connected condition is necessary for Hamiltonian connected graphs while not necessary for Hamiltonian extendable graphs. In this paper, we completely characterize a set of forbidden subgraphs with $|\mathcal{H}| \leq 2$ to force a 2-connected graph to be Hamiltonian extendable.

By $N_{i,j,k}$ we denote the graph obtained by attaching three vertex disjoint paths of lengths i, j and k to a triangle. For $i, j > 0$, $N_{i,j,0}$ is denoted by $B_{i,j}$ and $N_{i,0,0}$ by Z_i. By the definition of Hamiltonian extendable graphs, the two vertices of any 2-vertex cut of a Hamiltonian extendable graph are adjacent.

Theorem 4. *Let A be a connected graph and G be a 2-connected A-free graph such that the two vertices of any 2-vertex cut of G are adjacent. Then G is Hamiltonian extendable if and only if A is an induced subgraph of P_3.*

Theorem 5. *Let R, S be two connected graphs other than an induced subgraph of P_3 and let G be a 2-connected $\{R, S\}$-free graph of order at least 7 such that the two vertices of any 2-vertex cut of G are adjacent. Then G is Hamiltonian extendable if and only if $\{R, S\} \preceq \{K_{1,3}, B_{1,1}\}$ or $\{R, S\} \preceq \{K_{1,3}, Z_2\}$.*

2. Properties of Hamiltonian Extendable Graphs — Proofs of Theorems 2 and 3

We shall give some basic properties of Hamiltonian extendable graphs. We define $V_2(G) = \{v \in V(G) : d_G(v) = 2\}$, $V_{\geq i}(G) = \{v \in V(G) : d_G(v) \geq i\}$ and $E(X, Y) = \{uv \in E(G) | u \in X, v \in Y\}$, for $X, Y \subseteq V(G)$ with $X \cap Y = \emptyset$.

Property 6. If G is a Hamiltonian extendable graph, then the following statements hold.

(1) G is connected.
(2) Every vertex of degree 2 in G lies in a triangle for $|V(G)| > 3$.
(3) G has a cut-vertex x if and only if $G \cong K_s x K_t$. Furthermore, G has no cut-edge unless $G \cong K_{n-2} x K_1$.
(4) Two vertices with degree 2 are adjacent if and only if $G \cong K_{n-3} w K_2$.

Proof. The proof is left to the reader. ■

Lemma 7. *Let G be a Hamiltonian extendable graph of order $n \geq 6$ and $\delta(G) = 2$.*

(1) *If there exist two vertices in $V_2(G)$ such that they have at least one common neighbor, then either $|E(G)| > \lceil \frac{3n}{2} \rceil$ or $G \cong K_3 w K_2$ and $|E(G)| = \lceil \frac{3n}{2} \rceil$.*
(2) *If any two vertices in $V_2(G)$ have no common neighbor in G, then each vertex in $V_2(G)$ has at least one neighbor of degree at least 4 in G.*

Proof. (1) Assume that there exist two vertices $u, v \in V_2(G)$ such that $N(u) \cap N(v) \neq \emptyset$. If $uv \in E(G)$, by Property 6(4), $G \cong K_{n-3}wK_2$, where w is the common neighbor of u, v. Thus, if $n \geq 7$, then $|E(G)| > \lceil \frac{3n}{2} \rceil$; if $n = 6$, then $|E(G)| = \lceil \frac{3n}{2} \rceil$ and $G \cong K_3wK_2$. Now we suppose that any two vertices of degree 2 in G are nonadjacent.

First we suppose that $|N(u) \cap N(v)| = 2$. By Property 6(2), the two vertices in $N(u) \cap N(v)$ are adjacent. Then $G - \{u, v\}$ is a clique, otherwise, the nonadjacent vertices in $G - \{u, v\}$ cannot be extended, contradicting that G is Hamiltonian extendable. Then $|E(G)| = C(n-2, 2) + 4$. Since $n \geq 6$, $|E(G)| > \lceil \frac{3n}{2} \rceil$.

Next we suppose that $|N(u) \cap N(v)| = 1$. Let $N(u) \cap N(v) = \{w\}$ and $x \in N(u) \setminus \{w\}$, $y \in N(v) \setminus \{w\}$. By Property 6(2), $\{wx, wy\} \subset E(G)$. Then $yx \in E(G)$. Otherwise, y and x should be the two end-vertices of a Hamiltonian path of G by definition, and hence xu, uw, wv, vy, yx are in a Hamiltonian cycle of $G + yx$, which is impossible. We claim that $d(w) = n - 1$. Otherwise, we suppose that there exists a vertex $z \in V(G) \setminus \{u, v, x, y\}$ such that $wz \notin E(G)$. Since G is Hamiltonian extendable, w and z should be the two end-vertices of a Hamiltonian path of G and hence wu, wv and wz are in a Hamiltonian cycle of $G + wz$, a contradiction. Then $E(\{x, y\}, V(G) \setminus \{x, y, u, v, w\}) \neq \emptyset$, otherwise, w is a cut-vertex of G, contradicting Property 6(3).

We show that $|V_2(G)| \leq 3$. Suppose, for the contrary, that $|V_2(G)| \geq 4$. Let $\{u, v, a, b\} \subseteq V_2(G)$. Since $d(w) = n-1$, $\{aw, bw\} \subseteq E(G)$. Suppose that $V(G) = \{u, v, x, y, w, a, b\}$. Recall that $E(\{x, y\}, V(G) \setminus \{x, y, u, v, w\}) \neq \emptyset$. Without loss of generality, we suppose that $xa \in E(G)$. Since $d(b) = 2$, either $bx \in E(G)$ or $by \in E(G)$. However, if $bx \in E(G)$, then the nonadjacent vertices a, v cannot be extended; if $by \in E(G)$, then the nonadjacent vertices a, u cannot be extended, contradicting that G is Hamiltonian extendable. Thus $V(G) \setminus \{u, v, x, y, w, a, b\} \neq \emptyset$. Let $s \in V(G) \setminus \{u, v, x, y, w, a, b\}$. Since $d(v) = 2$ and $vs \notin E(G)$, v and s should be the two end-vertices of a Hamiltonian path of G and hence wu, wa and wb are in a Hamiltonian cycle of $G + vs$, which is impossible.

If $|V_2(G)| = 2$, then $2|E(G)| = \sum_{r \in V(G)} d(r) \geq 2 * 2 + (n-1) + 3 + 4 + 3(n-5)$. Then $|E(G)| > \lceil \frac{3n}{2} \rceil$, since $n \geq 6$. If $|V_2(G)| = 3$, let $V_2(G) = \{u, v, a\}$. Then $G[V(G) \setminus \{u, v, w, a\}]$ is a clique, otherwise, the nonadjacent vertices in $G[V(G) \setminus \{u, v, w, a\}]$ cannot be extendable (otherwise, wu, wv, wa should be in a Hamiltonian cycle containing the nonadjacent vertices, which is impossible), a contradiction. Since $d(w) = n - 1$, $G - \{u, v, a\}$ is a clique and $|E(G)| \geq C(n-3, 2) + 2 * 3$. If $n = 6$ and $|E(G)| = 9$, then by symmetry, $ay \in E(G)$. Then u and y cannot be extendable (otherwise, yu, ya, yv should be in a Hamiltonian cycle containing the nonadjacent vertices, which is impossible), a contradiction. If $n \geq 7$, then $|E(G)| > \lceil \frac{3n}{2} \rceil$.

(2) Assume to the contrary that there exists a vertex $u_0 \in V_2(G)$ such that u_0 has two neighbors x, y with $d(x) \leq 3$ and $d(y) \leq 3$. By Property 6(2), $xy \in E(G)$.

Since any two vertices in $V_2(G)$ have no common neighbor in G, $d(x) = d(y) = 3$. Take $x_1 \in N(x) \setminus \{u_0, y\}$ and $y_1 \in N(y) \setminus \{u_0, x\}$. Then $x_1 \neq y_1$, otherwise, $x_1 = y_1$ is a cut-vertex of G, contradicting Property 6(3). Then x_1 and y cannot be extended (otherwise, yu_0, u_0x, xx_1, x_1y should be in a Hamiltonian cycle of $G + x_1y$, which is impossible), a contradiction. This proves Lemma 7. ∎

***Proof of Theorem* 2.** (1) Trivially P_3 is the minimum Hamiltonian extendable graph with order 3. Then Z_1 is the minimum Hamiltonian extendable graph with order 4. Otherwise, if G is a minimum Hamiltonian extendable graph of order 4 and $|E(G)| < 4$, by Property 6(1), G is a tree. Then G is either P_4 or $K_{1,3}$. However, none of them is Hamiltonian extendable. Note that K_2wK_2 is the unique graph of a minimum Hamiltonian extendable graph with order 5. Otherwise, if G is a minimum Hamiltonian extendable graph of order 5 and $|E(G)| \leq 5$, then by Property 6(3), $\delta(G) \geq 2$. Then G is C_5, which is not Hamiltonian extendable.

(2) Let G be a Hamiltonian extendable graph with $n \geq 6$. Firstly, we prove $|E(G)| \geq \lceil \frac{3n}{2} \rceil$. By Property 6(1), G is connected. If $\delta(G) = 1$, then by Property 6(3), $G \cong K_{n-2}xK_1$ and $|E(G)| > \lceil \frac{3n}{2} \rceil$, since $n \geq 6$. If $\delta(G) \geq 3$, then $|E(G)| \geq \lceil \frac{3n}{2} \rceil$. Hence we may suppose that $\delta(G) = 2$.

If there exist two vertices in $V_2(G)$ satisfying that they have at least one common neighbor in G, then by Lemma 7(1), $|E(G)| \geq \lceil \frac{3n}{2} \rceil$. Now we suppose that any two vertices in $V_2(G)$ have no common neighbor in G. Then by Lemma 7(2), each vertex in $V_2(G)$ has at least one neighbor of degree at least 4 in G.

For each vertex u in $V_2(G)$, choose exactly one vertex $v_4(u)$ of degree at least 4 in $N(u)$ and set $V_4^2(G) = \{v_4(u) \in N(u) : d(v_4(u)) \geq 4,\ u \in V_2(G)\}$. Then $V(G) = V_2(G) \cup V_4^2(G) \cup (V_{\geq 3}(G) \setminus V_4^2(G))$ and $|V_2(G)| \leq |V_4^2(G)|$. Therefore, $2|E(G)| = \sum_{r \in V(G)} d(r) \geq 2|V_2(G)| + 4|V_4^2(G)| + 3(n - |V_2(G)| - |V_4^2(G)|) = 3n$. This proves $|E(G)| \geq \lceil \frac{3n}{2} \rceil$.

To conclude the proof of Theorem 2 it remains to construct a Hamiltonian extendable graph of order n, with $|E(G)| = \lceil \frac{3n}{2} \rceil$, for arbitrary $n \geq 6$. Start with two paths $P_1 = v_1v_2 \cdots v_k$ and $P_2 = u_1u_2 \cdots u_k$. Then add the edges u_iv_i for each $i \in \{1, 2, \ldots, k\}$. If $n = 2k + 1$, then add a vertex a such that $\{av_1, av_k, au_1, au_k\} \subset E(G)$. If $n = 2k + 2$, then add two vertices s, t such that $\{sv_1, su_1, tv_k, tu_k, st\} \subset E(G)$. These two graphs are denoted by F_1, F_2 in Figure 1. In [11], Moon has proved that these two graphs are Hamiltonian connected, hence Hamiltonian extendable. This proves Theorem 2. ∎

***Proof of Theorem* 3.** Let G be a Hamiltonian extendable graph with order $n \geq 6$ and $|E(G)| = \lceil \frac{3n}{2} \rceil$. If $\delta(G) = 1$, then by Property 6(3), $|E(G)| > \lceil \frac{3n}{2} \rceil$, a contradiction. Thus, $\delta(G) \geq 2$.

(1) Since $|E(G)| = \lceil \frac{3n}{2} \rceil$ and $V_2(G) = \emptyset$, we have $\delta(G) \geq 3$ and therefore G is 3-regular if n is even, or its degree sequence is equal to $(3, 3, \ldots, 3, 4)$ if n is odd.

(2) By (1), it clearly holds for $\delta(G) \geq 3$. Hence it suffices to consider the case that $\delta(G) = 2$. If there exists a pair of vertices in $V_2(G)$ such that they have at least one common neighbor, then by Lemma 7(1), $G \cong K_3wK_2$. In the following, we suppose that each pair of vertices in $V_2(G)$ have no common neighbor. By Lemma 7(2), $V(G) = V_2(G) \cup V_4^2(G) \cup (V_{\geq 3}(G) \setminus V_4^2(G))$, where $V_4^2(G)$ has defined in the proof of Theorem 2.

By Lemma 7(1), if $n = 6$ and $\Delta(G) \geq 5$, then $G \cong K_3wK_2$. Now we show that if $n \geq 7$, then $\Delta(G) \leq 4$. For a contradiction, suppose that there exists a vertex $s \in V(G)$ such that $d(s) \geq 5$. Then there exists a vertex in $N(s)$ of degree 2, otherwise, by Lemma 7(2) and $\delta(G) = 2$, $|V_2(G)| \leq |V_4^2(G)|$, and hence $2|E(G)| = \sum_{v \in V(G)} d(v) \geq 5 + 3 * 5 + 2|V_2(G)| + 4|V_4^2(G)| + 3(n - 6 - |V_2(G)| - |V_4^2(G)|) \geq 3n + 2$, a contradiction. Then $2|E(G)| = \sum_{v \in V(G)} d(v) \geq 5 + 2 + 3 * 4 + 2(|V_2(G)| - 1) + 3(n - 6 - (|V_2(G)| - 1)) = 3n + 2 - |V_2(G)|$. Since $|E(G)| = \lceil \frac{3n}{2} \rceil$, we have

$$\sum_{v \in V(G)} d(v) = \begin{cases} 3n + 1 & \text{if } n \text{ is odd,} \\ n & \text{if } n \text{ is even.} \end{cases}$$

Therefore, if n is even, then $|V_2(G)| = 2$; if n is odd, then $|V_2(G)| = 1$. However, if n is even and $|V_2(G)| = 2$, then by Lemma 7(2), $\sum_{v \in V(G)} d(v) \geq 5 + 2 + 3 * 4 + 2 + 4 + 3(n - 6 - 2) = 3n + 1$, a contradiction. This implies that n is odd and $|V_2(G)| = 1$. Since $\sum_{v \in V(G)} d(v) = 3n + 1$, the degree sequence of G is $(2, 3, 3, \ldots, 3, 5)$.

In the following, we show that G with degree sequence $(2, 3, 3, \ldots, 3, 5)$ is not Hamiltonian extendable. Let $N(s) = \{v_1, v_2, v_3, v_4, v_5\}$ and $d(v_1) = 2$, then the remaining vertices in G have degree exactly 3. By Property 6(2), without loss of generality we may suppose that $v_1v_2 \in E(G)$. Then v_2 is adjacent to one of $\{v_3, v_4, v_5\}$, otherwise, there exists a vertex $v_0 \in N(v_2) \setminus N[s]$, then v_0s cannot be extended (otherwise, sv_1, v_1v_2, v_2v_0, v_0s should be in a Hamiltonian cycle of $G + v_0s$, which is impossible), contradicting that G is Hamiltonian extendable. Without loss of generality, let $v_2v_3 \in E(G)$.

Now we prove that $v_3v_4 \notin E(G)$. By contradiction, suppose that $v_3v_4 \in E(G)$. Then $v_4v_5 \notin E(G)$, otherwise, v_5 is a cut-vertex of G, contradicting Property 6(3). Therefore, there exists a vertex $v_4' \in N(v_4) \setminus N[s]$. Then $v_4's$ cannot be extended (otherwise, sv_1, v_1v_2, v_2v_3, v_3v_4, v_4v_4', $v_4's$ should be in a Hamiltonian cycle of $G + v_4's$, which is impossible), contradicting that G is Hamiltonian extendable. This implies that $v_3v_4 \notin E(G)$. By symmetry, $v_3v_5 \notin E(G)$.

Therefore, there is a vertex $t \in N(v_3) \setminus N[s]$. Then ts cannot be extended (otherwise, sv_1, v_1v_2, v_2v_3, v_3t and ts should be in a Hamiltonian cycle of $G + ts$, which is impossible), a contradiction. This implies that G with degree sequence

$(2, 3, 3, \ldots, 3, 5)$ is not Hamiltonian extendable, contradicting the assumption that G is Hamiltonian extendable. This proves (2).

(3) By (2), $V(G) = V_2(G) \cup V_4^2(G) \cup (V_{\geq 3}(G) \setminus V_4^2(G))$. Since each pair of vertices in $V_2(G)$ have no common neighbor, $2|E(G)| = \sum_{v \in V(G)} d(v) \geq 2 + 2 * 4 + 2(|V_2(G)| - 1) + 3(n - 3 - (|V_2(G)| - 1)) = 3n + 2 - |V_2(G)|$. Therefore, if n is even, then $|V_2(G)| = 2$; if n is odd, then $|V_2(G)| = 1$. However, if n is even and $|V_2(G)| = 2$, then by Lemma 7(2), $2|E(G)| = \sum_{v \in V(G)} d(v) \geq 2 + 2 * 4 + 2 + 4 + 3(n - 5) = 3n + 1$, a contradiction. Then n is odd and $|V_2(G)| = 1$. Therefore, the degree sequence of G is $(2, 3, 3, \ldots, 3, 4, 4)$. This proves (3).

(4) Let $v \in V_2(G)$ and $N_G(v) = \{x, y\}$ with $d_G(x) = 3$ and $d_G(y) = 4$. By Property 6(2), $xy \in E(G)$. Let $w \in N(x) \setminus \{v, y\}$. Then $wy \in E(G)$, otherwise, w and y should be the two end-vertices of a Hamiltonian path of G and hence wy, yv, vx and xw should be in a Hamiltonian cycle of $G + wy$, which is impossible. Let $y_1 \in N(y) \setminus \{v, x, w\}$. Then $y_1 w \in E(G)$. Otherwise, y_1 and w should be the two end-vertices of a Hamiltonian path of G and hence $wx, xv, vy, yy_1, y_1 w$ should be in a Hamiltonian cycle of $G + y_1 w$, which is impossible. Then $d(w) \geq 3$. Since any pair of vertices in $V_2(G)$ have no common neighbor and $y \in N(y_1) \cap N(v)$, we have $d(y_1) \geq 3$. By (2), $d(w) \leq 4$ and $d(y_1) \leq 4$. Therefore, $d(w), d(y_1) \in \{3, 4\}$. Then $d(w) = 4$, otherwise, y_1 is a cut-vertex, contradicting Property 6(3). Let $w_1 \in N(w) \setminus \{x, y, y_1\}$. Then $y_1 w_1 \in E(G)$. Otherwise, w_1 and y_1 should be the two end-vertices of a Hamiltonian path of G and hence $w_1 w, wx, xv, vy, yy_1, y_1 w_1$ should be in a Hamiltonian cycle of $G + y_1 w_1$, which is impossible.

If $d(y_1) = 3$, then $V(G) = \{x, y, v, w, w_1, y_1\}$, otherwise, w_1 is a cut-vertex of G, contradicting Property 6(3). Thus, $G \cong F_4$, where F_4 is depicted in Figure 1. Now we suppose that $d(y_1) = 4$. Then $d(w_1) \geq 3$, otherwise, y_1 is a cut-vertex of G, contradicting Property 6(3). By (2), $d(w_1) = 3$ or 4. If $d(w_1) = 4$, then y_1 is not adjacent to a vertex of degree 2, otherwise, the vertex of degree 2 should be adjacent to w_1 and w_1 is a cut-vertex of G, contradicting Property 6(3). Then by Lemma 7(2), $2|E(G)| = \sum_{v \in V(G)} d(v) \geq 2 + 3 + 4 + 4 + 4 + 2(|V_2(G)| - 1) + 4(|V_2(G)| - 1) + 3(n - 5 - 2(|V_2(G)| - 1)) = 3n + 2$, a contradiction. This implies that $d(w_1) = 3$. Let $w_2 \in N(w_1) \setminus \{w, y_1\}$. Then $w_2 y_1 \in E(G)$. Otherwise, w_2 and y_1 should be the two end-vertices of a Hamiltonian path of G and then $y_1 y, yv, vx, xw, ww_1, w_1 w_2, w_2 y_1$ should be in a Hamiltonian cycle of $G + w_2 y_1$, which is impossible. Then $d(w_2) = 2$, otherwise, w_2 is a cut-vertex, contradicting Property 6(3). This implies that $G \cong F_5$, where F_5 is depicted in Figure 1. This proves (4). ∎

Remark 8. The degree sequences in Theorem 3(1) and (3) are best possible in a sense. We may construct Hamiltonian extendable graphs having the degree sequences in Theorem 3(1) and (3). F_1 and F_2 depicted in Figure 1 are Hamiltonian extendable graphs with degree sequence $(3, 3, \ldots, 3, 3)$ and $(3, 3, \ldots, 3, 4)$, respectively. Now we construct a graph F_3 from F_2 in Figure 1 by replacing the vertex a of F_2 with a triangle $bstb$ such that $\{sv_1, su_1, tv_k, tu_k\} \subset E(F_3)$. Then F_3 is a Hamiltonian extendable graph with degree sequence $(2, 3, 3, \ldots, 3, 4, 4)$.

3. Forbidden Subgraphs for Hamiltonian Extendable Graphs — Proofs of Theorems 4 and 5

We will characterize pairs of forbidden subgraphs for Hamiltonian extendable graphs in this section. By Property 6(3), a characterization has been given for Hamiltonian extendable graphs with a cut-vertex. Then we consider 2-connected Hamiltonian extendable graphs. We shall use some results in [5], [15], [4].

Theorem 9 (Chvátal and Erdős [5])**.** *Let G be a connected graph with $\alpha(G) \leq \kappa(G) + 1$ (or $\alpha(G) \leq \kappa(G)$, $\alpha(G) \leq \kappa(G) - 1$, respectively). Then G is traceable (or Hamiltonian, Hamiltonian connected, respectively).*

A graph G is obtained from H by *duplication* if we can obtain G by expanding some of the vertices of H to a clique, here expanding a vertex v to a clique C is the operation consisting of replacing v with C and adding additional edges between $u \in V(G) \setminus \{v\}$ and C if $uv \in E(G)$.

Theorem 10 (Shepherd [15])**.** *A connected graph G is $\{K_{1,3}, B_{1,1}\}$-free if and only if either $\alpha(G) = 2$ or G is obtained from a path or cycle by duplication.*

Theorem 11 (Broersma, Faudree, Huck, Trommel and Veldman [4])**.** *Let G be a 3-connected $\{K_{1,3}, Z_3\}$-free graph. Then G is Hamiltonian connected.*

Theorem 12. *Let G be a 2-connected graph with $\alpha(G) = 2$ such that the two vertices of any 2-vertex cut of G are adjacent. Then G is Hamiltonian extendable.*

Proof. For any two nonadjacent vertices $u, v \in V(G)$, it suffices to find a spanning (u, v)-path of G. Since the two vertices of any 2-vertex cut of G are adjacent, $G - \{u, v\}$ is connected and $\alpha(G - \{u, v\}) \leq 2$. By Theorem 9, $G - \{u, v\}$ has a Hamiltonian path $P_{G-\{u,v\}}(x, y)$, for some $x, y \in V(G) \setminus \{u, v\}$. Since $uv \notin E(G)$ and $\alpha(G) = 2$, $V(G) \setminus \{u, v\} = N(u) \cup N(v) = (N(u) \cap N(v)) \cup (N(u) \setminus N(v)) \cup (N(v) \setminus N(u))$.

Suppose first that x, y lie in three different sets $N(u) \cap N(v), N(u) \setminus N(v)$, $N(v) \setminus N(u)$, respectively, or x, y are both in $N(u) \cap N(v)$. Then by adding u and v to $P_{G-\{u,v\}}(x, y)$, we obtain a Hamiltonian path of G with end-vertices u, v. Now we assume that x and y are all in either $N(u) \setminus N(v)$ or $N(v) \setminus N(u)$. By symmetry, we may suppose that $\{x, y\} \subseteq N(u) \setminus N(v)$. Then $G[N(u) \setminus N(v)]$ is a clique, otherwise, any two nonadjacent vertices in $N(u) \setminus N(v)$ plus v is an independent set with cardinality 3, a contradiction. Then $P_{G-\{u,v\}}(x, y) \cup \{xy\}$ is a Hamiltonian cycle of $G - \{u, v\}$. Therefore, $P_{G-\{u,v\}}(x, y) \cup \{xy\}$ must have an edge whose end-vertices lie in $N(u)$ and $N(v)$, respectively. This produces a spanning (u, v)-path of G and Theorem 12 is proved. ■

Theorem 13. *Let G be a 2-connected and $\{K_{1,3}, B_{1,1}\}$-free graph such that the two vertices of any 2-vertex cut of G are adjacent. Then G is Hamiltonian extendable.*

Proof. By Theorems 10, 12 and by the assumption of this theorem, it suffices to consider that G is obtained from a path $P = v_1v_2 \cdots v_k$ by duplication. Since G is 2-connected, each vertex $v_i \in V(P) \setminus \{v_1, v_k\}$ is duplicated by a clique G_i with order at least 2. By the definition of duplication, for each $i \in \{1, 2, \ldots, k-1\}$, $G[V(G_i) \cup V(G_{i+1})]$ is a clique. Then G is Hamiltonian extendable. ■

Let F_6 be the unique connected graph with degree sequence $(2, 2, 2, 4, 4, 4)$, i.e., F_6 is obtained from a triangle $xyzx$ by subdividing each edge of the triangle with vertices $\{a, b, c\}$ and adding three new edges $\{ab, ac, bc\}$. Note that F_6 is a 2-connected and $\{K_{1,3}, Z_2\}$-free but it is not Hamiltonian extendable graph. Therefore, in the following we shall exclude this graph when we consider 2-connected and $\{K_{1,3}, Z_2\}$-free Hamiltonian extendable graphs. The length of a shortest path between u and v of G is called the *distance* between u and v and denoted by $d_G(u, v)$. The *diameter* of a graph G is the greatest distance between two vertices of G and denoted by $\mathrm{diam}(G)$.

Theorem 14. *Let $G \not\cong F_6$ be a 2-connected and $\{K_{1,3}, Z_2\}$-free graph satisfying that the two vertices of any 2-vertex cut of G are adjacent. Then G is Hamiltonian extendable.*

Proof. Let $G \not\cong F_6$ be a 2-connected and $\{K_{1,3}, Z_2\}$-free graph such that the two vertices of any 2-vertex cut of G are adjacent. Then $\mathrm{diam}(G) \leq 3$. Otherwise, we assume that $\mathrm{diam}(G) \geq 4$. Choose a shortest path $P = x_0x_1x_2 \cdots x_t$ ($t \geq 4$) such that $d_G(x_0, x_t) = \mathrm{diam}(G)$. First we show that x_0x_1 lies in a triangle. Otherwise, $d(x_0) \geq 3$ (because the two vertices of any 2-vertex cut of G are adjacent). Let $\{x_1, x', x''\} \subseteq N(x_0)$. Since G is $K_{1,3}$-free, $x'x'' \in E(G)$. Since $G[\{x_0, x', x'', x_1, x_2\}] \not\cong Z_2$, $E(G) \cap \{x_2x', x_2x''\} \neq \emptyset$. Without loss of generality, we suppose that $x'x_2 \in E(G)$. Since $d_G(x_0, x_t) = \mathrm{diam}(G)$, $\{x'x_3, x_1x_3\} \cap E(G) = \emptyset$. Then $G[\{x_2, x', x_1, x_3\}] \cong K_{1,3}$, a contradiction. This implies that x_0x_1 is in a triangle vx_1x_0v (say). Since $G[\{v, x_0, x_1, x_2, x_3\}] \not\cong Z_2$ and $d_G(x_0, x_t) = \mathrm{diam}(G)$, $vx_2 \in E(G)$. Then $G[\{v, x_1, x_2, x_3, x_4\}] \cong Z_2$, a contradiction.

By Theorem 11, if $\kappa(G) \geq 3$, then G is Hamiltonian connected and hence G is Hamiltonian extendable. It suffices to consider that $\kappa(G) = 2$ and G has a minimum 2-vertex cut, say, $\{u, v\}$. Since the two vertices of any 2-vertex cut of G are adjacent, $uv \in E(G)$. Since G is $K_{1,3}$-free, $G - \{u, v\}$ has exactly two components G_1, G_2 (say). Let $(N(u) \setminus N(v)) \cap V(G_i) = A_i$, $(N(v) \setminus N(u)) \cap$

$V(G_i) = B_i$, $N(u) \cap N(v) \cap V(G_i) = C_i$ and $V(G_i) \setminus (A_i \cup B_i \cup C_i) = D_i$, where $i \in \{1, 2\}$. Since G is 2-connected and $\{u, v\}$ is a minimum 2-vertex cut of G, $N(w) \cap V(G_i) \neq \emptyset$, for any $w \in \{u, v\}$, $i \in \{1, 2\}$.

Claim 15. *The following statements hold.*

(1) $G[A_i \cup C_i]$ *and* $G[B_i \cup C_i]$ *are cliques, for* $i \in \{1, 2\}$.

(2) *At least one of the sets* A_1, A_2 *is empty and at least one of the sets* B_1, B_2 *is empty.*

(3) *At least one of the sets* A_i, B_j *is empty, for* $\{i, j\} = \{1, 2\}$.

(4) *If* $C_i = \emptyset$, *then* $|A_i| \geq 2$ *and* $|B_i| \geq 2$, *for* $i \in \{1, 2\}$.

(5) *Each* $b \in B_i$ *is adjacent to at least* $|A_i| - 1$ *vertices in* A_i *and each* $a \in A_i$ *is adjacent to at least* $|B_i| - 1$ *vertices in* B_i, *for* $i \in \{1, 2\}$.

(6) *If* $A_i \neq \emptyset$ *and* $B_i \neq \emptyset$, *then* $|A_i| \leq 2$, $|B_i| \leq 2$, *for* $i \in \{1, 2\}$.

(7) *If* $C_i \neq \emptyset$ *and* $A_i \neq \emptyset$, *then* $|A_i| = 1$; *if* $C_i \neq \emptyset$ *and* $B_i \neq \emptyset$, *then* $|B_i| = 1$, *for* $i \in \{1, 2\}$.

Proof. (1) They follow from the assumption that G is $K_{1,3}$-free. Assume that there exist two nonadjacent vertices $\{x, y\} \subseteq A_1 \cup C_1$ (by symmetry), then $G[\{u, x, y, z\}] \cong K_{1,3}$, where $z \in N(u) \cap V(G_2)$, a contradiction.

(2) First we prove that at least one of the sets A_1, A_2 is empty. Otherwise, let $a_1 \in A_1$ and $a_2 \in A_2$, then $G[\{u, a_1, a_2, v\}] \cong K_{1,3}$, a contradiction. Similarly, at least one of the sets B_1, B_2 is empty.

(3) First we show that at least one of the sets A_1, B_2 is empty. Otherwise, let $a_1 \in A_1$ and $b_2 \in B_2$. By (2), $A_2 = \emptyset$ and $B_1 = \emptyset$. Since $\{u, v\}$ is a minimum vertex cut, $N(v) \cap V(G_1) \neq \emptyset$ and $N(u) \cap V(G_2) \neq \emptyset$. Then $C_1 \neq \emptyset$ and $C_2 \neq \emptyset$. Then $G[\{a_1, c_1, u, c_2, b_2\}] \cong Z_2$, where $c_i \in C_i$, $i \in \{1, 2\}$, a contradiction. Similarly, if at least one of the sets A_2, B_1 is empty.

(4) Analogously we show that if $C_1 = \emptyset$, then $|A_1| \geq 2$. Since $C_1 = \emptyset$, $A_1 \neq \emptyset$ and $B_1 \neq \emptyset$. By (2), $A_2 = \emptyset$ and $B_2 = \emptyset$. Then $C_2 \neq \emptyset$ and let $c_2 \in C_2$. Furthermore, $D_1 = \emptyset$, otherwise, let $d_1 \in D_1$, then $G[\{c_2, u, v, w, d_1\}] \cong Z_2$, where $w \in N(d_1) \cap (A_1 \cup B_1)$, a contradiction. If $|A_1| = 1$, say $A_1 = \{a_1\}$, then $|B_1| \geq 2$, otherwise, since G is 2-connected, $d_G(a_1) = 2$, contradicting Property 6(2). Then by Property 6(2), there is $\{b_1, b_1'\} \subseteq B_1$ such that $\{a_1 b_1, a_1 b_1'\} \subseteq E(G)$. Then $G[\{b_1, b_1', a_1, u, c_2\}] \cong Z_2$, a contradiction.

(5) Analogously we prove that each $b \in B_1$ is adjacent to at least $|A_1| - 1$ vertices in A_1. Otherwise, assume that there are two vertices $a_1, a_1' \in A_1$ and a vertex $b_1' \in B_1$ such that $\{b_1' a_1, b_1' a_1'\} \cap E(G) = \emptyset$. Then $G[\{a_1, a_1', u, v, b_1'\}] \cong Z_2$, a contradiction.

(6) Analogously we prove that $|A_1| \leq 2$. Otherwise, by (5), for each $b_1 \in B_1$, there exist at least two vertices $a_1, a_1' \in A_1$ such that $\{b_1 a_1, b_1 a_1'\} \subseteq E(G)$. Then $G[\{b_1, a_1, a_1', v, w\}] \cong Z_2$, where $w \in N(v) \cap V(G_2)$, a contradiction.

(7) Analogously we show that if $C_1 \neq \emptyset$ and $A_1 \neq \emptyset$, then $|A_1| = 1$. Otherwise, let $a_1, a_1' \in A_1$ and $c_1 \in C_1$. Then $G[\{c_1, a_1, a_1', v, w\}] \cong Z_2$, where $w \in N(v) \cap V(G_2)$, a contradiction. This completes the proof of Claim 15. $\square$

If $\text{diam}(G) = 1$, then G is a complete graph and hence Hamiltonian extendable. Then we assume that $2 \leq \text{diam}(G) \leq 3$. Since $\text{diam}(G) \leq 3$, at least one of the sets D_1, D_2 is empty.

Now we distinguish the following two cases.

Case 1. $D_1 = \emptyset$ and $D_2 = \emptyset$. Note that the case when $A_1 = \emptyset$, $B_1 \neq \emptyset$ and the case when $A_1 \neq \emptyset$, $B_1 = \emptyset$ are symmetric; by Claim 15(2), the case when $A_1 = \emptyset$, $B_1 = \emptyset$ and the case when $A_1 \neq \emptyset$, $B_1 \neq \emptyset$ are symmetric.

Therefore, up to symmetry, first we suppose that $A_1 = \emptyset$ and $B_1 \neq \emptyset$. Then $C_1 \neq \emptyset$ because G is 2-connected. By Claim 15(2) and (3), $B_2 = \emptyset$, $A_2 = \emptyset$. Since $\{u, v\}$ is a minimum vertex cut of G, we have $C_2 \neq \emptyset$. Note that $G[B_1 \cup C_1]$ and $G[C_2]$ are cliques by Claim 15(1). We can check that G is Hamiltonian extendable.

Now we suppose that $A_1 = \emptyset$ and $B_1 = \emptyset$. Then $C_1 \neq \emptyset$. If $A_2 = \emptyset$ and $B_2 = \emptyset$, then $C_2 \neq \emptyset$. Note that $G[C_1]$ and $G[C_2]$ are cliques. Then G is Hamiltonian extendable. Then we suppose that at least one of the sets A_2, B_2 is nonempty. By symmetry, we suppose that $B_2 \neq \emptyset$. Then $|C_1| = 1$, otherwise, we suppose that $\{c_1, c_1'\} \subseteq C_1$, then by Claim 15(1), $G[\{u, c_1, c_1', w, b_2\}] \cong Z_2$, where $w \in N(u) \cap V(G_2)$, $b_2 \in B_2$, a contradiction. If $A_2 = \emptyset$, then $C_2 \neq \emptyset$ because G is 2-connected. Note that $G[B_2 \cup C_2]$ is a clique by Claim 15(1). We can check that G is Hamiltonian extendable. Then we suppose that $A_2 \neq \emptyset$. We shall consider two cases whether C_2 is empty or not. If $C_2 = \emptyset$, by Claim 15(4) and (6), $|A_2| = 2$ and $|B_2| = 2$. Then G is a graph of order exactly 7, and by Claim 15(1) and (5), it is Hamiltonian extendable. Then we suppose that $C_2 \neq \emptyset$. By Claim 15(7), $|A_2| = 1$ and $|B_2| = 1$. Let $A_2 = \{a_2\}$ and $B_2 = \{b_2\}$. If $|C_2| = 1$, say, $C_2 = \{c_2\}$, by Claim 15(1), $\{a_2c_2, b_2c_2\} \subseteq E(G)$. Then $a_2b_2 \in E(G)$, otherwise, $G \cong F_6$, a contradiction. Then G is a Hamiltonian extendable graph of 6 vertices. Now we suppose that $|C_2| \geq 2$. Note that $G[V(G_2)] \cong K_{|V(G_2)|}$ or $K_{|V(G_2)|} - \{a_2b_2\}$. Then G is Hamiltonian extendable.

Case 2. $D_1 = \emptyset$ and $D_2 \neq \emptyset$ or $D_1 \neq \emptyset$ and $D_2 = \emptyset$. By symmetry, it suffices to consider the case that $D_1 = \emptyset$ and $D_2 \neq \emptyset$. Since $\text{diam}(G) \leq 3$, $D_2 = (N(A_2 \cup B_2 \cup C_2) \cap V(G_2)) \setminus (A_2 \cup B_2 \cup C_2)$.

Claim 16. *The following statements hold.*

(1) $E(D_2, A_2) = \emptyset$, $E(D_2, B_2) = \emptyset$. *Furthermore,* $E(D_2, C_2) \neq \emptyset$.

(2) $A_1 = \emptyset$, $B_1 = \emptyset$. *Furthermore,* $|C_1| = 1$.

(3) $A_2 = \emptyset$, $B_2 = \emptyset$.

(4) *There exists no pair of vertices in D_2 that have common neighbor in C_2. Furthermore, D_2 is an independent set.*

Proof. (1) By symmetry, we show that $E(D_2, A_2) = \emptyset$. Otherwise, assume that there are two vertices $a_2 \in A_2$ and $d \in D_2$ such that $a_2d \in E(G)$. If $C_1 = \emptyset$, by Claim 15(4), $|A_1| \geq 2$. Let $\{a_1, a_1'\} \subseteq A_1$. Then $G[\{a_1, a_1', u, a_2, d\}] \cong Z_2$, a contradiction. If $C_1 \neq \emptyset$, say, $c_1 \in C_1$, then $G[\{c_1, u, v, a_2, d\}] \cong Z_2$, a contradiction. Furthermore, since G_2 is connected and $D_2 \neq \emptyset$, $E(D_2, C_2) \neq \emptyset$.

(2) By symmetry, we first show that $A_1 = \emptyset$. Assume to the contrary that $A_1 \neq \emptyset$. If $C_1 = \emptyset$, by Claim 15(4), $|A_1| \geq 2$. Let $\{a_1, a_1'\} \subseteq A_1$, then $G[\{a_1, a_1', u, c_2, d\}] \cong Z_2$, where $c_2 \in C_2$ and $d \in D_2$, a contradiction. If $C_1 \neq \emptyset$, say, $c_1 \in C_1$, then $G[\{a_1, c_1, u, c_2, d\}] \cong Z_2$, a contradiction. Furthermore, let $\{c_1, c_1'\} \subseteq C_1$. Since $c_1c_1' \in E(G)$ by Claim 15(1), $G[\{c_1, c_1', u, c_2, d\}] \cong Z_2$, a contradiction.

(3) By symmetry, we show that $A_2 = \emptyset$. Otherwise, we suppose that $a_2 \in A_2$. Note that there exists $c_2 \in C_2$ by Claim 16(1). Since $a_2c_2 \in E(G)$ by Claim 15(1) and $a_2d \notin E(G)$ by Claim 16(1), $G[\{c_2, a_2, v, d\}] \cong K_{1,3}$, where $d \in D_2$, a contradiction.

(4) Assume to the contrary that there are $\{d_1, d_2\} \subseteq D_2$ and $c_2 \in C_2$ such that $\{d_1c_2, d_2c_2\} \subseteq E(G)$. Since $G[\{d_1, d_2, c_2, u\}] \ncong K_{1,3}$, $d_1d_2 \in E(G)$. Then $G[\{d_1, d_2, c_2, u, w\}] \cong Z_2$, where $w \in N(u) \cap V(G_1)$, a contradiction. If $d_1d_2 \in E(G)$ and $d_1c_2 \in E(G)$, then $d_2c_2 \notin E(G)$. Then $G[\{u, v, c_2, d_1, d_2\}] \cong Z_2$, a contradiction. Then D_2 is an independent set. This proves Claim 16. □

Since G is 2-connected, by Claim 16(4), for each $d \in D_2$, $|N(d) \cap C_2| \geq 2$. Since G is $K_{1,3}$-free and D_2 is an independent set by Claim 16(4), for any pair of vertices $d_1, d_2 \in D_2$, $N(d_1) \cap N(d_2) = \emptyset$. Since $G[C_2]$ is a clique by Claim 15(1), we can check that G is Hamiltonian extendable. This proves Theorem 14. ■

In order to show the necessity of Theorems 4, 5, we give some graphs which are not Hamiltonian extendable (see Figure 2) as follows.

• G_0 is the graph obtained from $K_{3,3}$ by replacing one of the vertices with a clique K_{n-5} such that each vertex in $V(K_{n-5})$ is adjacent to each vertex in $\{u, v, w\}$.

• $G_1 \cong K_{m,m}$, $m \geq 4$.

• G_2 is the graph obtained from K_{n-3} and an independent set $\{u, v, w\}$. Take two vertices $\{x, y\}$ in K_{n-3} such that each vertex in $\{u, v, w\}$ is adjacent to each vertex in $\{x, y\}$.

• G_3 is the graph obtained from K_{n-3}. Take two vertices $\{x, y\}$ in K_{n-3} and add three additional vertices u, v, w and the edges uv, ux, vx, vw, vy and wy.

• G_4 is the graph with the vertex set $\{u_i, v_i : 0 \leq i \leq 6k+3\}$ and the edge set $\{u_iu_{i+1}, v_iv_{i+2}, u_iv_i : 0 \leq i \leq 6k+3\}$, indices are taken modulo $6k+4$, $k \geq 1$.

Note that in [1, 14], the authors show that there is no Hamiltonian path from u_0 to u_2 in G_4.

• G_5 is the graph obtained from a cycle $v_1v_2v_3\cdots v_{3k}v_1$ (k is even) by adding vertex set $\{w_i : 0 \le i \le k-1\}$ and adding edge set $\{v_{3l+1}v_{3l+3} : 0 \le l \le k-1\} \cup \left\{v_{3t+2}v_{3t+2+\frac{3k}{2}} : 0 \le t \le \frac{k-2}{2}\right\} \cup \{w_iv_{3i+3}, w_iv_{3i+4} : 0 \le i \le k-1\}$. Here indices are taken modulo $3k$. Note that there is no Hamiltonian path between w_0 and w_1 in G_5.

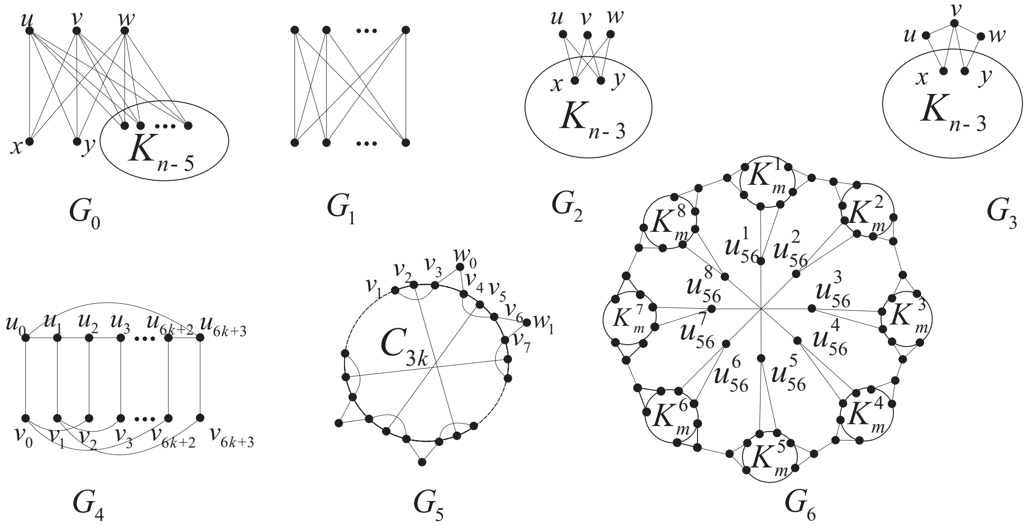

Figure 2. Seven graphs which are not Hamiltonian extendable graphs.

• G_6 is the graph obtained from eight disjoint cliques $\{K_m^1, K_m^2, \ldots, K_m^8\}$ with $m \ge 6$. Let $\{v_{i1}, v_{i2}, \ldots, v_{i6}\} \subseteq V(K_m^i)$, $1 \le i \le 8$. For each i, add three additional vertices $\{u_{12}^i, u_{34}^i, u_{56}^i\}$ such that $\{u_{12}^iv_{i1}, u_{12}^iv_{i2}, u_{34}^iv_{i3}, u_{34}^iv_{i4}, u_{56}^iv_{i5}, u_{56}^iv_{i6}\} \subseteq E(G_6)$ and $\{u_{12}^iu_{34}^{i+1}\} \subseteq E(G_6)$, where indices are taken modulo 8, and add edges $u_{56}^iu_{56}^{i+4}$, for each $i \in \{1,2,3,4\}$. Note that there is no Hamiltonian path between u_{56}^1 and u_{56}^3 in G_6.

***Proof of Theorem* 4.** Let G be a connected P_3-free graph. Then G is a complete graph and hence Hamiltonian extendable. Conversely, all graphs in Figure 2 are 2-connected and are not Hamiltonian extendable graphs satisfying that the two vertices of any 2-vertex cut of G are adjacent. Then A is an induced subgraph of them. Since G_1, G_4 have no common induced cycle, A is a tree. Since G_3 is $K_{1,3}$-free, A is a path. Since the maximal induced path of G_1 is P_3, A is an induced subgraph of P_3. ■

***Proof of Theorem* 5.** By Theorems 13 and 14, the sufficiency clearly holds. It remains to show the necessity. All graphs in Figure 2 are 2-connected and are not

Hamiltonian extendable graphs satisfying that the two vertices of any 2-vertex cut of G are adjacent. Then each graph contains at least one of $\{R, S\}$ as an induced subgraph. Without loss of generality, we suppose that G_1 contains R as an induced subgraph. Then R is $K_{1,t}$ $(t \geq 3)$ or contains C_4.

First we suppose that R contains C_4 or $K_{1,t}$ $(t \geq 5)$. Since G_2, G_3, G_4, G_5, G_6 are $\{C_4, K_{1,t}\}$-free $(t \geq 5)$, they contain S as an induced subgraph. Since G_4, G_5 have no common induced cycle, S is a tree. Since G_5 is $K_{1,3}$-free, S is a path. Since the maximal induced path of G_2 is P_3, S is an induced subgraph of P_3, a contradiction.

Next we suppose that R is $K_{1,4}$. Since G_0, G_3, G_4, G_5, G_6 are $K_{1,4}$-free, they must contain S as an induced subgraph. Note that G_4, G_5 have no common induced cycle. Then S is a tree. Since G_3 is $K_{1,3}$-free, S is a path. Since the maximal induced path of G_0 is P_3, S is an induced subgraph of P_3, a contradiction.

Finally we suppose that R is $K_{1,3}$. Since G_3, G_5, G_6 are $K_{1,3}$-free, they must contain S as an induced subgraph. Then S is a path or contains a cycle. Note that the maximal induced path of G_3 is P_4. If S is a path, then S is an induced subgraph of P_4. Then $\{R, S\} \preceq \{K_{1,3}, P_4\}$. Now we suppose that S contains a cycle. Since G_5 is K_4-free, S contains no K_4. Note that the maximal common induced cycle of G_3, G_5, G_6 is K_3. Furthermore, the maximal common induced subgraph of G_3, G_5, G_6 contain exactly one K_3. Since the maximal induced subgraphs containing one K_3 of G_3, G_5, G_6 are Z_2 and $B_{1,1}$, S is an induced subgraph of Z_2 or $B_{1,1}$. Therefore, $\{R, S\} \preceq \{K_{1,3}, B_{1,1}\}$ or $\{R, S\} \preceq \{K_{1,3}, Z_2\}$. Note that $\{K_{1,3}, P_4\} \preceq \{K_{1,3}, B_{1,1}\}$. This proves the necessity. ■

4. Concluding Remark

We conclude this paper with the following remarks.

• In Theorem 5, we assume that graphs have the order at least 7. In fact, from the proof of Theorem 5, we know that even if we consider graphs with sufficiently large order, Theorem 5 also holds because the order of all graphs in the proof of Theorem 5 may be infinite.

• Since a Hamiltonian extendable graph does not need to be Hamiltonian, we hope that some sufficient conditions for a Hamiltonian extendable graph will be improved. Look at the graph G'_0 obtained from a non-complete graph G' of order $\frac{n}{2}$ and additional $\frac{n}{2}$ vertices which are joining with all vertices in G'. Obviously, G'_0 is not Hamiltonian extendable graph with $\delta(G'_0) = \frac{n}{2}$. This shows the condition $\delta(G') \geq \frac{n}{2}$, which guarantees that a graph of order $n \geq 3$ is Hamiltonian, may not guarantee a graph to be Hamiltonian extendable.

• Now we compare the Hamiltonian connected graphs and Hamiltonian extendable graphs. In [12], Ore gave some sufficient conditions for graphs of order n to be Hamiltonian connected.

Theorem 17 (Ore, [12]). *If G satisfies one of the following statements,*
(1) $\delta(G) \geq \frac{n+1}{2}$,
(2) *any two nonadjacent vertices x, y satisfy that $d(x) + d(y) \geq n + 1$,*
(3) $|E(G)| \geq C(n-1, 2) + 3$,
(4) $\delta(G) \geq 3$, *$n \neq 6$ and* $|E(G)| = C(n-1, 2) + 2$,
then G is Hamiltonian connected.

Therefore, if G satisfies one of these conditions, then it is Hamiltonian extendable. Taking the above examples G_0' into consideration, the conditions (1), (2) are best possible for Hamiltonian extendability. In fact, combining the conditions (3), (4), every graph with $|E(G)| \geq C(n-1, 2) + 2$ is Hamiltonian extendable: it suffices to consider the case when $|E(G)| = C(n-1, 2) + 2$ and either $\delta(G) = 2$ or $n = 6$, $\delta(G) \geq 3$. In the first case, if G is the graph obtained from K_{n-1} by adding a vertex which is adjacent to exactly two vertices in $V(K_{n-1})$, then G is Hamiltonian extendable; in the second case, since $\delta(G) \geq 3$ and $n = 6$, G has no cut-vertex. Furthermore, since $\delta(G) \geq 3$ and $|E(G)| = 12$, G has no 2-vertex cut. This implies that G is 3-connected. Let $\{u, v, w\}$ be a minimum vertex cut of G and $V(G) \setminus \{u, v, w\} = \{x, y, z\}$. If $G - \{u, v, w\}$ has three components, then G is the graph obtained from $K_{3,3}$ by adding edges uv, vw, uw (because $|E(G)| = 12$). Hence G is Hamiltonian extendable. Now we suppose that $G - \{u, v, w\}$ has two components and let $yz \in E(G)$. Then either $G[\{u, v, w\}] \cong P_3$ and $|E(\{u, v, w\}, \{x, y, z\})| = 9$ or $G[\{u, v, w\}] \cong K_3$ and $|E(\{u, v, w\}, \{x, y, z\})| = 8$. Then we can check that G is Hamiltonian extendable. Note that the graph obtained from $K_{n-1} - \{xy\}$, where $xy \in E(K_{n-1})$, by adding an additional vertex v such that $\{vx, vy\} \subset E(G)$ is not Hamiltonian extendable. This implies that the condition $|E(G)| \geq C(n-1, 2) + 2$ for Hamiltonian extendable is also best possible.

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