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ORIENTED CHROMATIC NUMBER OF CARTESIAN PRODUCTS $P_m \Box P_n$ AND $C_m \Box P_n$

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Abstract

We consider oriented chromatic number of Cartesian products of two paths $P_m \Box P_n$ and of Cartesian products of paths and cycles, $C_m \Box P_n$. We say that the oriented graph \vec{G} is colored by an oriented graph \vec{H} if there is a homomorphism from \vec{G} to \vec{H} . In this paper we show that there exists an oriented tournament \vec{H}_{10} with ten vertices which colors every orientation of $P_8 \Box P_n$ and every orientation of $C_m \Box P_n$, for m = 3, 4, 5, 6, 7 and $n \ge 1$. We also show that there exists an oriented graph \vec{T}_{16} with sixteen vertices which colors every orientation of $C_m \Box P_n$.

Keywords: graphs, oriented coloring, oriented chromatic number. 2010 Mathematics Subject Classification: 05C15.

1. INTRODUCTION

An oriented graph is a digraph \vec{G} obtained from an undirected graph G by assigning to each edge one of two possible directions. We say that \vec{G} is an orientation of G and G is the underlying graph of \vec{G} . A tournament \vec{T} is an orientation of a complete graph. If there is a homomorphism $\phi: V(\vec{G}) \to V(\vec{T})$, then we say that \vec{G} is colored by \vec{T} or that \vec{T} colors \vec{G} . We also say that \vec{T} is a coloring graph (tournament). The oriented chromatic number of the oriented graph \vec{G} , denoted by $\vec{\chi}(\vec{G})$, is the smallest integer k such that \vec{G} is colored by a tournament with k colors (vertices). The oriented chromatic number $\vec{\chi}(G)$ of an undirected graph G is the maximal chromatic number over all possible orientations of G. The oriented chromatic number of a family of graphs is the maximal oriented chromatic number over all possible graphs of the family. The *upper oriented chromatic number* $\vec{\chi}^+(G)$ of an undirected graph G is the minimum order of an oriented graph \vec{H} such that every orientation \vec{G} of G admits a homomorphism to \vec{H} .

It is easy to see that for every undirected graph G, $\chi(G) \leq \chi(G) \leq \chi^+(G)$, see [19]. The *Cartesian product* $G \Box H$ of two undirected graphs G and H is the graph with the vertex set $V(G) \times V(H)$, where two vertices are adjacent if and only if they are equal in one coordinate and adjacent in the other. We use P_k to denote the path on k vertices. Sopena [19] considered upper oriented chromatic number of strong, Cartesian and direct products of graphs.

Theorem 1 [19]. If G and H are two undirected graphs, then $\vec{\chi}^+(G\Box H) \leq \vec{\chi}^+(G) \cdot \vec{\chi}^+(H) \cdot \min\{\chi(G), \chi(H)\}.$

Oriented coloring has been studied in recent years [1, 2, 6, 8-10, 12, 14, 16-20, 22], see [15] for a survey of the main results. Several authors established or bounded chromatic numbers for some families of graphs, such as oriented planar graphs [12, 14], outerplanar graphs [12, 17, 18], graphs with bounded degree three [10, 17, 20], k-trees [17], Halin graphs [5, 9], graphs with given excess [8] or grids [3, 4, 6, 13, 22].

In this paper we focus on the oriented chromatic number of Cartesian products of two paths, called 2-dimensional grids $G_{m,n} = P_m \Box P_n$, and Cartesian products of cycles and paths, called stacked prism graphs $Y_{m,n} = C_m \Box P_n$.

Theorem 2 [16,21]. Let G be an undirected graph. Then:

(a) If G is a forest with at least three vertices, then $\vec{\chi}^+(G) = 3$.

- (b) $\overrightarrow{\chi}^+(C_5) = 5$. Moreover, every orientation of C_5 can be colored by \overrightarrow{H}_2 (see Figure 1(b)).
- (c) For each $k \leq 3$, $k \neq 5$, we have $\overrightarrow{\chi}^+(C_k) = 4$. Moreover, every orientation of a cycle C_k with $k \leq 3$ and $k \neq 5$ can be colored by \overrightarrow{H}_1 (see Figure 1(a)).

Theorems 1 and 2 imply that $\overrightarrow{\chi}^+(P_m\Box P_n) \leq 3\cdot 3\cdot 2 = 18$. Furthermore, we know that

- $\overrightarrow{\chi}(P_m \Box P_n) \leq 11$, for every $m, n \geq 2$ [6],
- there exists an orientation of $P_4 \Box P_5$ which requires 7 colors for oriented coloring [6],
- there exists an orientation of $P_7 \Box P_{212}$ which requires 8 colors for oriented coloring [3],
- $\overrightarrow{\chi}(P_2 \Box P_2) = 4$, $\overrightarrow{\chi}(P_2 \Box P_3) = 5$ and $\overrightarrow{\chi}(P_2 \Box P_n) = 6$, for $n \ge 6$ [6],
- $\overrightarrow{\chi}(P_3 \Box P_n) = 6$, for every $3 \le n \le 6$, and $\overrightarrow{\chi}(P_3 \Box P_n) = 7$, for every $n \ge 7$ [6,22],

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Figure 1. Coloring graphs \overrightarrow{H}_1 and \overrightarrow{H}_2 .

- $\overrightarrow{\chi}(P_4 \Box P_4) = 6$ and $\overrightarrow{\chi}(P_4 \Box P_n) = 7$, for every $n \ge 5$ [6,22],
- $\overrightarrow{\chi}(P_5 \Box P_n) \leq 9$, for every $n \geq 5$ [4].

Since $\overrightarrow{\chi}^+(C_5) = 5$ and $\overrightarrow{\chi}^+(C_k) \leq 4$, for $k \neq 5$, by Theorem 1, we have

- $\overrightarrow{\chi}^+(C_5 \Box P_n) \le 2 \cdot 3 \cdot 5 = 30$, for $n \ge 3$,
- $\overrightarrow{\chi}^+(C_m \Box P_n) \le 2 \cdot 3 \cdot 4 = 24$, for $m \ne 5, n \ge 3$.

In this paper we show that there exists an oriented tournament \overline{H}_{10} , see Figure 2, which colors every orientation of every grid $P_8 \Box P_n$ and every orientation of $C_m \Box P_n$, with m = 3, 4, 5, 6, 7 and $n \ge 1$. We also show that there exists an oriented graph \overrightarrow{T}_{16} which colors every orientation of $C_m \Box P_n$, for $m \ge 8$ and $n \ge 1$. These imply that

- $\overrightarrow{\chi}(P_8 \Box P_n) \leq \overrightarrow{\chi}^+(P_8 \Box P_n) \leq 10$, for every n,
- $\overrightarrow{\chi}(C_m \Box P_n) \leq \overrightarrow{\chi}^+(C_m \Box P_n) \leq 10$, for m = 3, 4, 5, 6, 7 and $n \geq 1$,
- $\overrightarrow{\chi}(C_m \Box P_n) \leq \overrightarrow{\chi}^+(C_m \Box P_n) \leq 16$, for $m \geq 8$ and $n \geq 1$.

2. Coloring Graphs

2.1. Paley tournament

Let p be a prime number such that $p \equiv 3 \mod 4$, and let $\mathbb{Z}_p = \{0, \ldots, p-1\}$ be the ring of integers modulo p. We denote by $QR_p = \{r : r \neq 0, r = s^2, \text{ for some} s \in \mathbb{Z}_p\}$ — the set of *nonzero quadratic residues of* \mathbb{Z}_p . All arithmetic operation in this section are made in the ring \mathbb{Z}_p .

Definition 3. The directed graph \overrightarrow{T}_p with the set of vertices $V(\overrightarrow{T}_p) = \mathbb{Z}_p$ and the set of arcs $A(\overrightarrow{T}_p) = \{(x, y) : x, y \in V(\overrightarrow{T}_p) \text{ and } y - x \in QR_p\}$ is called the *Paley tournament* of order p. Observe that \overrightarrow{T}_p is a tournament.

Lemma 4. If $a \in QR_p$ and $b \in \mathbb{Z}_p$, then the mapping $f : \overrightarrow{T}_p \to \overrightarrow{T}_p$ defined by $f(x) = a \cdot x + b$ is an automorphism.

Lemma 5 [7]. The Paley tournament \overrightarrow{T}_p is arc-transitive; i.e., for any two pairs of arcs (u, v), $(x, y) \in A(\overrightarrow{T}_p)$, there exists an automorphism h such that h(u) = x and h(v) = y.

Lemma 6. The Paley tournament \overrightarrow{T}_p is self-converse; i.e., \overrightarrow{T}_p and its converse \overrightarrow{T}_p^R are isomorphic.

Proof. Consider the function $f: \overrightarrow{T}_p^R \to \overrightarrow{T}_p$ defined by f(x) = -x. Then $(x, y) \in A(\overrightarrow{T}_p^R)$ if and only if $(-x, -y) \in A(\overrightarrow{T}_p)$.

2.2. Coloring graph \overrightarrow{H}_{10}

Consider the coloring graph \overrightarrow{H}_{10} obtained from the Paley tournament \overrightarrow{T}_{11} by removing the vertex 0, i.e., $V(\overrightarrow{H}_{10}) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $(u, v) \in A(\overrightarrow{H}_{10})$ if $(v-u) \in \{1, 3, 4, 5, 9\}$, see Figure 2.



Figure 2. Coloring graph \vec{H}_{10} .

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Lemma 7. (a) For every $a \in \{1, 3, 4, 5, 9\}$, the function $h_a(x) = ax \pmod{11}$ is an automorphism of \overrightarrow{H}_{10} .

- (b) For every $x \in \{1, 3, 4, 5, 9\}$ there is an automorphism h_a such that $h_a(x) = 1$.
- (c) For every $x \in \{2, 6, 7, 8, 10\}$ there is an automorphism h_a such that $h_a(x) = 10$.

Lemma 8. Let \overrightarrow{G} be an orientation of a grid and let v be one of its vertex. Then the following two statements are equivalent.

- (a) There exists an oriented coloring (homomorphism) $c: \overrightarrow{G} \to \overrightarrow{H}_{10}$.
- (b) There exists an oriented coloring (homomorphism) $c': \overrightarrow{G} \to \overrightarrow{H}_{10}$ such that $c'(v) \in \{1, 10\}.$

2.3. Tromp graph

Definition 9. Let \vec{G} be an oriented graph. We build the Tromp graph $\vec{Tr}(\vec{G})$ in the following way.

- Let \vec{G}' be an isomorphic copy of \vec{G} ,
- ∞, ∞' be two additional vertices.
- Let $t: V(\overrightarrow{G}) \cup \{\infty\} \to V(\overrightarrow{G}') \cup \{\infty'\}$ be an isomorphism with $t(\infty) = \infty'$. For every $u \in V(\overrightarrow{G}) \cup \{\infty\}$ by u' we denote t(u) and for every $u \in V(\overrightarrow{G}') \cup \{\infty'\}$ by u' we denote $t^{-1}(u)$. The pair (u, u') is called a pair of *twin vertices*.
- The set of vertices $V(\overrightarrow{Tr}(\overrightarrow{G})) = V(\overrightarrow{G}) \cup V(\overrightarrow{G}') \cup \{\infty, \infty'\}.$
- The set of arcs is defined by

$$\begin{split} &\forall_{u\in V(\overrightarrow{G})}(u,\infty),(\infty,u'),(u',\infty'),(\infty',u)\in A(\overrightarrow{Tr}(\overrightarrow{G})), \\ &\forall_{u,v\in V(\overrightarrow{G}),\;(u,v)\in A(\overrightarrow{G})}(u,v),(u',v'),(v,u'),(v',u)\in A(\overrightarrow{Tr}(\overrightarrow{G})). \end{split}$$

Let $\overrightarrow{T}_{16} = \overrightarrow{Tr}(\overrightarrow{T}_7)$ be the Tromp graph on sixteen vertices obtained from the Paley tournament \overrightarrow{T}_7 , see Figure 3.

Suppose that *i* and *j* are integers such that $i \ge 1$ and $j \ge 1$. Consider the star $K_{1,i}$ with the set of vertices $V(K_{1,i}) = \{x, v_1, v_2, \ldots, v_i\}$ and edges of the form $\{x, v_k\}$ for $1 \le k \le i$; and a Tromp graph $\overrightarrow{Tr}(\overrightarrow{G})$. Let \overrightarrow{K} be an orientation of the star $K_{1,i}$ and $c : \overrightarrow{K} \to \overrightarrow{Tr}(\overrightarrow{G})$ be a homomorphism. We say that the sequence of colors $(c(v_1), c(v_2), \ldots, c(v_i))$ chosen for leaves of the star *is compatible* with orientation \overrightarrow{K} if for every pair of vertices v_k, v_l it holds:

- $c(v_k) \neq c(v_l)$ if (v_k, x) and $(x, v_l) \in \vec{K}$ or if (v_l, x) and $(x, v_k) \in \vec{K}$, and
- $c(v_k) \neq c(v_l)'$ if (v_k, x) and $(v_l, x) \in \overrightarrow{K}$ or if (x, v_l) and $(x, v_k) \in \overrightarrow{K}$.



Figure 3. Coloring graph $\overrightarrow{T}_{16} = \overrightarrow{Tr}(\overrightarrow{T}_7)$.

Definition 10. We say that the Tromp graph \overrightarrow{T} has the property $P_c(i, j)$ if $|V(\overrightarrow{T})| \geq i$ and for every orientation \overrightarrow{K} of the star $K_{1,i}$ and every sequence of colors $(c(v_1), c(v_2), \ldots, c(v_k))$ chosen for leaves compatible with \overrightarrow{K} we can choose j different ways to color x, the central vertex of the star.

Lemma 11 [11]. The Tromp graph \overrightarrow{T}_{16} has the properties $P_c(1,7)$, $P_c(2,3)$ and $P_c(3,1)$.

3. GRIDS $G_{8,n} = P_8 \Box P_n$

Definition 12. The comb R_8 is an undirected graph with the set of vertices $V(R_8) = \{(1,1),\ldots,(8,1),(1,2),\ldots,(8,2)\}$ and edges of the form $\{(i,1),(i,2)\}$ for $1 \le i \le 8$, or $\{(i,2),(i+1,2)\}$ for $1 \le i < 8$; see Figure 4. The vertices $(1,1),\ldots,(8,1)$ form the first column of the comb R_8 , while $(1,2),\ldots,(8,2)$ form the second column.

Definition 13. A set $S \subseteq (V(\overrightarrow{H}_{10}))^8$ is closed under extension if



Figure 4. Comb R_8 .

- (a) for every orientation \overrightarrow{P} of the path $P_8 = (v_1, \ldots, v_8)$, there exists a coloring $c : \overrightarrow{P} \to \overrightarrow{H}_{10}$ such that $(c(v_1), \ldots, c(v_8)) \in S$,
- (b) for every orientation \overrightarrow{R} of the comb R_8 and for every sequence $(c_1, \ldots, c_8) \in S$, there exists a coloring $c : \overrightarrow{R} \to \overrightarrow{H}_{10}$ and an automorphism h_a of \overrightarrow{H}_{10} such that
 - (1) $(c(1,1),\ldots,c(8,1)) = (c_1,\ldots,c_8)$, and
 - (2) $h_a(c(1,2),\ldots,c(8,2)) \in S.$

Lemma 14. There exists a set $S \subseteq (V(\overrightarrow{H}_{10}))^8$ which is closed under extension.

Proof. In order to proof the lemma we use a computer. We have designed an algorithm that finds a proper set S. Let

 $S_{\max}(P_8) = \{(c_1, \dots, c_8) : c_1 \in \{1, 10\}, \text{ and } \forall_{2 \le i \le 8} c_i \in V(\overrightarrow{H}_{10}), \text{ and } c_{i-1} \neq c_i\}.$

Note, that for every sequence $t = (t_1, \ldots, t_8) \in S_{\max}(P_8)$, there exists an orientation \overrightarrow{P} of the path $P_8 = (v_1, \ldots, v_8)$ and a coloring $c : \overrightarrow{P} \to \overrightarrow{H}_{10}$ such that $(c(v_1), \ldots, c(v_8)) = t$. For a set T, a sequence $t = (t_1, \ldots, t_8) \in T$, and an orientation \overrightarrow{R} of the comb R_8 , we say that t can be extended in T on \overrightarrow{R} if there exists a coloring $c : \overrightarrow{R} \to \overrightarrow{H}_{10}$ and a homomorphism h_a such that

- $(c(1,1),\ldots,c(8,1)) = t$, and
- $h_a(c(1,2),\ldots,c(8,2)) \in T.$

The algorithm starts with $T = S_{\max}(P_8)$. In the while loop, for each sequence $t \in T$ and for each orientation \vec{R} of the comb R_8 , the algorithm checks if t can be extended in T on \vec{R} . If the sequence t can not be extended, then t is removed from T. After the while loop, the set T satisfies the condition (b) of Definition 13. It is easy to see that if T is not empty, then it also satisfies the condition (a). In this case S = T is returned. If T is empty, then the algorithm returns NO.

Algorithm Compute Set S

OUTPUT: a set $S \subset (V(\overrightarrow{H}_{10}))^8$ closed under extension or NO if such a set does not exist.

```
1.
    compute the set S_{\max}(P_8)
    T := S_{\max}(P_8)
2.
    SetIsReady := false
3.
4.
    while not SetIsReady
5.
       SetIsReady := true
       for every sequence t = (t_1, \ldots, t_8) \in T
6.
7.
         color the first column of the comb {\cal R}_8
                   by setting c(i,1)=t_i, for 1\leq i\leq 8
8.
9.
         SeqCanBeExtended := true
         for every orientation \overrightarrow{R} of the comb R_8
10.
            if t cannot be extended on \overrightarrow{R}
11.
                SeqCanBeExtended := false
12.
13.
         if not SeqCanBeExtended
14.
            T := T - t
15.
            SetIsReady := false
16.
      if T = \emptyset
17.
       return NO
18.
      else
       S := T
19.
20.
       return the set S
```

Using Algorithm ComputeSetS we have found a nonempty set S closed under extension. The set S is posted on the website https://inf.ug.edu.pl/grids/.

Theorem 15. Every orientation of every grid with eight rows can be colored by the coloring graph \vec{H}_{10} .

Proof. For a given orientation \overrightarrow{G} of G(8, n) and $i \leq n$, by $\overrightarrow{G}(i)$ we denote the induced subgraph of \overrightarrow{G} formed by the first *i* columns of \overrightarrow{G} . It is easy to show by induction that, for every *i*, there is a coloring $c : \overrightarrow{G}(i) \to \overrightarrow{H}_{10}$ such that $c(i\text{th column}) \in S$.

4. Stacked Prism Graphs $Y_{m,n} = C_m \Box P_n$

Theorem 16. Every orientation of $C_m \Box P_n$ with $m \ge 3$ and $n \ge 1$ can be colored by the Tromp graph \overrightarrow{T}_{16} .

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Figure 5. Stacked prism graph $Y_{m,n}$.

Proof. Let \overrightarrow{Y} be any orientation of stacked prism graph $Y_{m,n} = C_m \Box P_n$. We identify each vertex $u \in \overrightarrow{Y}$ with the pair of its coordinates $(i, j), 1 \leq i \leq m, 1 \leq j \leq n$. We shall show that \overrightarrow{Y} can be colored by \overrightarrow{T}_{16} . We color the vertices of \overrightarrow{Y} row by row. For the first row, clearly, it is always possible to color any oriented cycle by homomorphism to \overrightarrow{T}_{16} , because \overrightarrow{T}_{16} has the properties $P_c(2,3)$ and $P_c(1,7)$. Now, suppose that i > 1 and the rows from 1 to i - 1 are already colored. To color the vertex (1, i) we choose a color which is compatible

- with the color of vertex (2, i 1) in the star $\{(2, i), (1, i), (2, i 1)\},\$
- with the color of vertex (m, i-1) in the star $\{(m, i), (1, i), (m, i-1)\},\$

which is always possible using the property $P_c(1,7)$. Using the property $P_c(2,3)$ it is always possible to color vertex (2,i) by the color compatible with color of the vertex (3, i - 1) in the star $\{(3, i), (2, i), (3, i - 1)\}$. Then we continue this method to color vertices $(3, i), \ldots, (m - 2, i)$. To color the vertex (m - 1, i) we choose a color which is compatible with the colors of vertices (m, i - 1) and (1, i) in the star $\{(m, i), (1, i), (m, i - 1), (m - 1, i)\}$. This is possible, because the colors of vertices (1, i) and (m, i - 1) are compatible in the star $\{(m, i), (1, i), (m, i - 1)\}$. Finally we color the vertex (m, i) using the property $P_c(3, 1)$. Similarly we can color the following rows.

Theorem 17. Every orientation of stacked prism graph $Y_{m,n} = C_m \Box P_n$ with $3 \le m \le 7$ can be colored by the coloring graph \overrightarrow{H}_{10} .

Proof. The proof of the theorem is similar to the proof of Theorem 15 and follows from Lemma 20.

Definition 18. For $m \ge 3$, the *m*-sunlet graph Sun_m is an undirected graph with the set of vertices $V(Sun_m) = \{(1, 1), \ldots, (m, 1), (1, 2), \ldots, (m, 2)\}$ and edges of the form $\{(i, 1), (i, 2)\}$ for $1 \le i \le m$, or $\{(i, 2), (i + 1, 2)\}$ for $1 \le i < m$, or $\{(m, 2), (1, 2)\}$; see Figure 6.



Figure 6. *m*-sunlet graph.

Definition 19. A set $S \subseteq (V(\overrightarrow{H}_{10}))^m$ is cycle-closed under extension if

- (a) for every orientation \overrightarrow{C} of the cycle $C_m = (v_1, \ldots, v_m)$, there exists a coloring $c: \overrightarrow{C} \to \overrightarrow{H}_{10}$ such that $(c(v_1), \ldots, c(v_m)) \in S$,
- (b) for every orientation \overrightarrow{Sun} of the *m*-sunlet graph Sun_m and for every sequence $(c_1, \ldots, c_m) \in S$, there exists a coloring $c : \overrightarrow{Sun} \to \overrightarrow{H}_{10}$ and an automorphism h_a of \overrightarrow{H}_{10} such that
 - (1) $(c(1,1),\ldots,c(m,1)) = (c_1,\ldots,c_m)$, and
 - (2) $h_a(c(1,2),\ldots,c(m,2)) \in S.$

Lemma 20. For each m = 3, 4, 5, 6, 7, there exists a nonempty set $S_m \subseteq (V(\overrightarrow{H}_{10}))^m$, which is cycle-closed under extension.

Proof. In order to proof the lemma we use a computer. We have designed an algorithm, similar to the Algorithm ComputeSetS, that finds a set cycle-closed under extension. The algorithm, for a given m, uses the m-sunlet Sun_m instead of a comb R_8 . Using the algorithm we have found that for each $m = 3, \ldots, 7$, there exists a nonempty set cycle-closed under extension.

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