Discussiones Mathematicae
Graph Theory 42 (2022) 799–810
https://doi.org/10.7151/dmgt.2307

ORIENTED CHROMATIC NUMBER OF CARTESIAN PRODUCTS $P_m \Box P_n$ AND $C_m \Box P_n$

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Abstract

We consider oriented chromatic number of Cartesian products of two paths $P_m \Box P_n$ and of Cartesian products of paths and cycles, $C_m \Box P_n$. We say that the oriented graph $\overrightarrow{G}$ is colored by an oriented graph $\overrightarrow{H}$ if there is a homomorphism from $\overrightarrow{G}$ to $\overrightarrow{H}$. In this paper we show that there exists an oriented tournament $\overrightarrow{H}_{10}$ with ten vertices which colors every orientation of $P_8 \Box P_n$ and every orientation of $C_m \Box P_n$, for $m = 3, 4, 5, 6, 7$ and $n \geq 1$. We also show that there exists an oriented graph $\overrightarrow{T}_{16}$ with sixteen vertices which colors every orientation of $C_m \Box P_n$.

Keywords: graphs, oriented coloring, oriented chromatic number.

2010 Mathematics Subject Classification: 05C15.

1. Introduction

An *oriented graph* is a digraph $\overrightarrow{G}$ obtained from an undirected graph G by assigning to each edge one of two possible directions. We say that $\overrightarrow{G}$ is an *orientation* of G and G is the *underlying graph* of $\overrightarrow{G}$. A *tournament* $\overrightarrow{T}$ is an orientation of a complete graph. If there is a homomorphism $\phi : V(\overrightarrow{G}) \to V(\overrightarrow{T})$, then we say that $\overrightarrow{G}$ is *colored by* $\overrightarrow{T}$ or that $\overrightarrow{T}$ *colors* $\overrightarrow{G}$. We also say that $\overrightarrow{T}$ is a *coloring graph* (tournament). The *oriented chromatic number* of the oriented graph $\overrightarrow{G}$, denoted by $\overrightarrow{\chi}(\overrightarrow{G})$, is the smallest integer k such that $\overrightarrow{G}$ is colored by a tournament with k colors (vertices). The *oriented chromatic number* $\overrightarrow{\chi}(G)$ of an undirected graph G is the maximal chromatic number over all possible orientations of G. The oriented chromatic number of a family of graphs is the maximal oriented chromatic number over all possible graphs of the family. The *upper oriented chromatic number* $\overrightarrow{\chi}^+(G)$ of an undirected graph G is the minimum order of an oriented graph $\overrightarrow{H}$ such that every orientation $\overrightarrow{G}$ of G admits a homomorphism to $\overrightarrow{H}$.

It is easy to see that for every undirected graph G, $\chi(G) \leq \overrightarrow{\chi}(G) \leq \overrightarrow{\chi}^+(G)$, see [19]. The *Cartesian product* $G\Box H$ of two undirected graphs G and H is the graph with the vertex set $V(G) \times V(H)$, where two vertices are adjacent if and only if they are equal in one coordinate and adjacent in the other. We use P_k to denote the path on k vertices. Sopena [19] considered upper oriented chromatic number of strong, Cartesian and direct products of graphs.

Theorem 1 [19]**.** *If G and H are two undirected graphs, then* $\overrightarrow{\chi}^+(G\Box H) \leq \overrightarrow{\chi}^+(G) \cdot \overrightarrow{\chi}^+(H) \cdot \min\{\chi(G), \chi(H)\}$.

Oriented coloring has been studied in recent years [1, 2, 6, 8–10, 12, 14, 16–20, 22], see [15] for a survey of the main results. Several authors established or bounded chromatic numbers for some families of graphs, such as oriented planar graphs [12,14], outerplanar graphs [12,17,18], graphs with bounded degree three [10, 17, 20], k-trees [17], Halin graphs [5, 9], graphs with given excess [8] or grids [3, 4, 6, 13, 22].

In this paper we focus on the oriented chromatic number of Cartesian products of two paths, called 2-*dimensional grids* $G_{m,n} = P_m\Box P_n$, and Cartesian products of cycles and paths, called *stacked prism graphs* $Y_{m,n} = C_m\Box P_n$.

Theorem 2 [16, 21]**.** *Let G be an undirected graph. Then:*

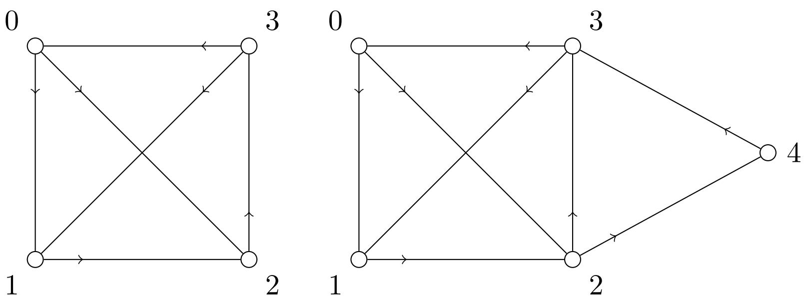
(a) *If G is a forest with at least three vertices, then* $\overrightarrow{\chi}^+(G) = 3$.

(b) $\overrightarrow{\chi}^+(C_5) = 5$. *Moreover, every orientation of C_5 can be colored by* $\overrightarrow{H}_2$ (*see Figure* 1(b)).

(c) *For each $k \leq 3$, $k \neq 5$, we have* $\overrightarrow{\chi}^+(C_k) = 4$. *Moreover, every orientation of a cycle C_k with $k \leq 3$ and $k \neq 5$ can be colored by* $\overrightarrow{H}_1$ (*see Figure* 1(a)).

Theorems 1 and 2 imply that $\overrightarrow{\chi}^+(P_m\Box P_n) \leq 3 \cdot 3 \cdot 2 = 18$. Furthermore, we know that

- $\overrightarrow{\chi}(P_m\Box P_n) \leq 11$, for every $m, n \geq 2$ [6],
- there exists an orientation of $P_4\Box P_5$ which requires 7 colors for oriented coloring [6],
- there exists an orientation of $P_7\Box P_{212}$ which requires 8 colors for oriented coloring [3],
- $\overrightarrow{\chi}(P_2\Box P_2) = 4$, $\overrightarrow{\chi}(P_2\Box P_3) = 5$ and $\overrightarrow{\chi}(P_2\Box P_n) = 6$, for $n \geq 6$ [6],
- $\overrightarrow{\chi}(P_3\Box P_n) = 6$, for every $3 \leq n \leq 6$, and $\overrightarrow{\chi}(P_3\Box P_n) = 7$, for every $n \geq 7$ [6, 22],



(a) $\overrightarrow{H}_1$ (b) $\overrightarrow{H}_2$

Figure 1. Coloring graphs $\overrightarrow{H}_1$ and $\overrightarrow{H}_2$.

- $\overrightarrow{\chi}(P_4 \square P_4) = 6$ and $\overrightarrow{\chi}(P_4 \square P_n) = 7$, for every $n \geq 5$ [6, 22],
- $\overrightarrow{\chi}(P_5 \square P_n) \leq 9$, for every $n \geq 5$ [4].

Since $\overrightarrow{\chi}^+(C_5) = 5$ and $\overrightarrow{\chi}^+(C_k) \leq 4$, for $k \neq 5$, by Theorem 1, we have

- $\overrightarrow{\chi}^+(C_5 \square P_n) \leq 2 \cdot 3 \cdot 5 = 30$, for $n \geq 3$,
- $\overrightarrow{\chi}^+(C_m \square P_n) \leq 2 \cdot 3 \cdot 4 = 24$, for $m \neq 5$, $n \geq 3$.

In this paper we show that there exists an oriented tournament $\overrightarrow{H}_{10}$, see Figure 2, which colors every orientation of every grid $P_8 \square P_n$ and every orientation of $C_m \square P_n$, with $m = 3, 4, 5, 6, 7$ and $n \geq 1$. We also show that there exists an oriented graph $\overrightarrow{T}_{16}$ which colors every orientation of $C_m \square P_n$, for $m \geq 8$ and $n \geq 1$. These imply that

- $\overrightarrow{\chi}(P_8 \square P_n) \leq \overrightarrow{\chi}^+(P_8 \square P_n) \leq 10$, for every n,
- $\overrightarrow{\chi}(C_m \square P_n) \leq \overrightarrow{\chi}^+(C_m \square P_n) \leq 10$, for $m = 3, 4, 5, 6, 7$ and $n \geq 1$,
- $\overrightarrow{\chi}(C_m \square P_n) \leq \overrightarrow{\chi}^+(C_m \square P_n) \leq 16$, for $m \geq 8$ and $n \geq 1$.

2. COLORING GRAPHS

2.1. Paley tournament

Let p be a prime number such that $p \equiv 3 \bmod 4$, and let $\mathbb{Z}_p = \{0, \ldots, p-1\}$ be the ring of integers modulo p. We denote by $QR_p = \{r : r \neq 0,\ r = s^2$, for some $s \in \mathbb{Z}_p\}$ — the set of *nonzero quadratic residues of* $\mathbb{Z}_p$. All arithmetic operation in this section are made in the ring $\mathbb{Z}_p$.

Definition 3. The directed graph $\overrightarrow{T}_p$ with the set of vertices $V(\overrightarrow{T}_p) = \mathbb{Z}_p$ and the set of arcs $A(\overrightarrow{T}_p) = \{(x, y) : x, y \in V(\overrightarrow{T}_p)$ and $y - x \in QR_p\}$ is called the *Paley tournament* of order p. Observe that $\overrightarrow{T}_p$ is a tournament.

Lemma 4. *If $a \in QR_p$ and $b \in \mathbb{Z}_p$, then the mapping $f : \overrightarrow{T}_p \to \overrightarrow{T}_p$ defined by $f(x) = a \cdot x + b$ is an automorphism.*

Lemma 5 [7]**.** *The Paley tournament $\overrightarrow{T}_p$ is arc-transitive; i.e., for any two pairs of arcs (u, v), $(x, y) \in A(\overrightarrow{T}_p)$, there exists an automorphism h such that $h(u) = x$ and $h(v) = y$.*

Lemma 6. *The Paley tournament $\overrightarrow{T}_p$ is self-converse; i.e., $\overrightarrow{T}_p$ and its converse $\overrightarrow{T}_p^R$ are isomorphic.*

Proof. Consider the function $f : \overrightarrow{T}_p^R \to \overrightarrow{T}_p$ defined by $f(x) = -x$. Then $(x, y) \in A(\overrightarrow{T}_p^R)$ if and only if $(-x, -y) \in A(\overrightarrow{T}_p)$. ∎

2.2. Coloring graph $\overrightarrow{H}_{10}$

Consider the coloring graph $\overrightarrow{H}_{10}$ obtained from the Paley tournament $\overrightarrow{T}_{11}$ by removing the vertex 0, i.e., $V(\overrightarrow{H}_{10}) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $(u, v) \in A(\overrightarrow{H}_{10})$ if $(v - u) \in \{1, 3, 4, 5, 9\}$, see Figure 2.

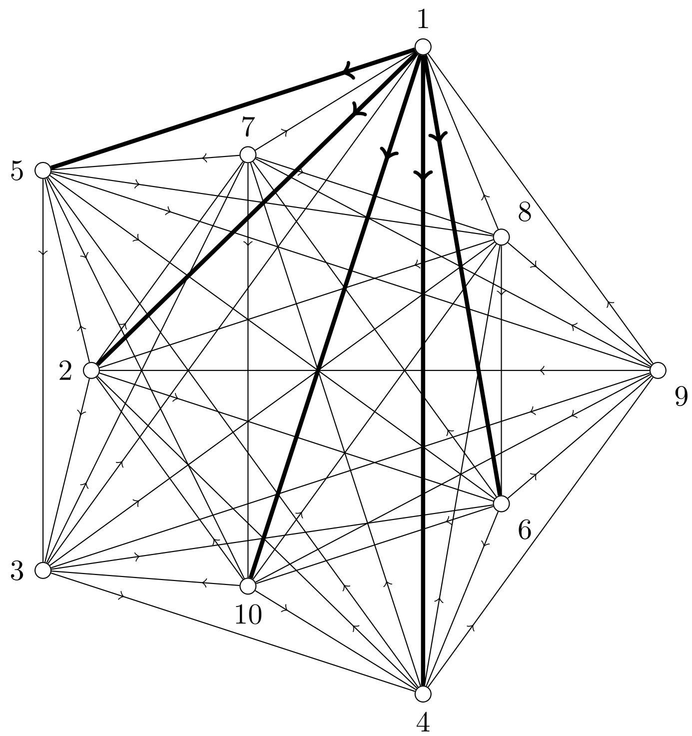


Figure 2. Coloring graph $\overrightarrow{H}_{10}$.

Lemma 7. (a) *For every* $a \in \{1,3,4,5,9\}$, *the function* $h_a(x) = ax \pmod{11}$ *is an automorphism of* $\overrightarrow{H}_{10}$.

(b) *For every* $x \in \{1,3,4,5,9\}$ *there is an automorphism* h_a *such that* $h_a(x) = 1$.

(c) *For every* $x \in \{2,6,7,8,10\}$ *there is an automorphism* h_a *such that* $h_a(x) = 10$.

Lemma 8. *Let* $\overrightarrow{G}$ *be an orientation of a grid and let* v *be one of its vertex. Then the following two statements are equivalent.*

(a) *There exists an oriented coloring (homomorphism)* $c : \overrightarrow{G} \to \overrightarrow{H}_{10}$.

(b) *There exists an oriented coloring (homomorphism)* $c' : \overrightarrow{G} \to \overrightarrow{H}_{10}$ *such that* $c'(v) \in \{1, 10\}$.

2.3. Tromp graph

Definition 9. Let $\overrightarrow{G}$ be an oriented graph. We build the Tromp graph $\overrightarrow{Tr}(\overrightarrow{G})$ in the following way.

- Let $\overrightarrow{G}'$ be an isomorphic copy of $\overrightarrow{G}$,
- ∞, ∞' be two additional vertices.
- Let $t : V(\overrightarrow{G}) \cup \{\infty\} \to V(\overrightarrow{G}') \cup \{\infty'\}$ be an isomorphism with $t(\infty) = \infty'$. For every $u \in V(\overrightarrow{G}) \cup \{\infty\}$ by u' we denote $t(u)$ and for every $u \in V(\overrightarrow{G}') \cup \{\infty'\}$ by u' we denote $t^{-1}(u)$. The pair (u, u') is called a pair of *twin vertices*.
- The set of vertices $V(\overrightarrow{Tr}(\overrightarrow{G})) = V(\overrightarrow{G}) \cup V(\overrightarrow{G}') \cup \{\infty, \infty'\}$.
- The set of arcs is defined by

$$\forall_{u \in V(\overrightarrow{G})} (u, \infty), (\infty, u'), (u', \infty'), (\infty', u) \in A(\overrightarrow{Tr}(\overrightarrow{G})),$$

$$\forall_{u,v \in V(\overrightarrow{G}),\, (u,v) \in A(\overrightarrow{G})} (u, v), (u', v'), (v, u'), (v', u) \in A(\overrightarrow{Tr}(\overrightarrow{G})).$$

Let $\overrightarrow{T}_{16} = \overrightarrow{Tr}(\overrightarrow{T}_7)$ be the Tromp graph on sixteen vertices obtained from the Paley tournament $\overrightarrow{T}_7$, see Figure 3.

Suppose that i and j are integers such that $i \geq 1$ and $j \geq 1$. Consider the star $K_{1,i}$ with the set of vertices $V(K_{1,i}) = \{x, v_1, v_2, \ldots, v_i\}$ and edges of the form $\{x, v_k\}$ for $1 \leq k \leq i$; and a Tromp graph $\overrightarrow{Tr}(\overrightarrow{G})$. Let $\overrightarrow{K}$ be an orientation of the star $K_{1,i}$ and $c : \overrightarrow{K} \to \overrightarrow{Tr}(\overrightarrow{G})$ be a homomorphism. We say that the sequence of colors $(c(v_1), c(v_2), \ldots, c(v_i))$ chosen for leaves of the star *is compatible* with orientation $\overrightarrow{K}$ if for every pair of vertices v_k, v_l it holds:

- $c(v_k) \neq c(v_l)$ if (v_k, x) and $(x, v_l) \in \overrightarrow{K}$ or if (v_l, x) and $(x, v_k) \in \overrightarrow{K}$, and
- $c(v_k) \neq c(v_l)'$ if (v_k, x) and $(v_l, x) \in \overrightarrow{K}$ or if (x, v_l) and $(x, v_k) \in \overrightarrow{K}$.

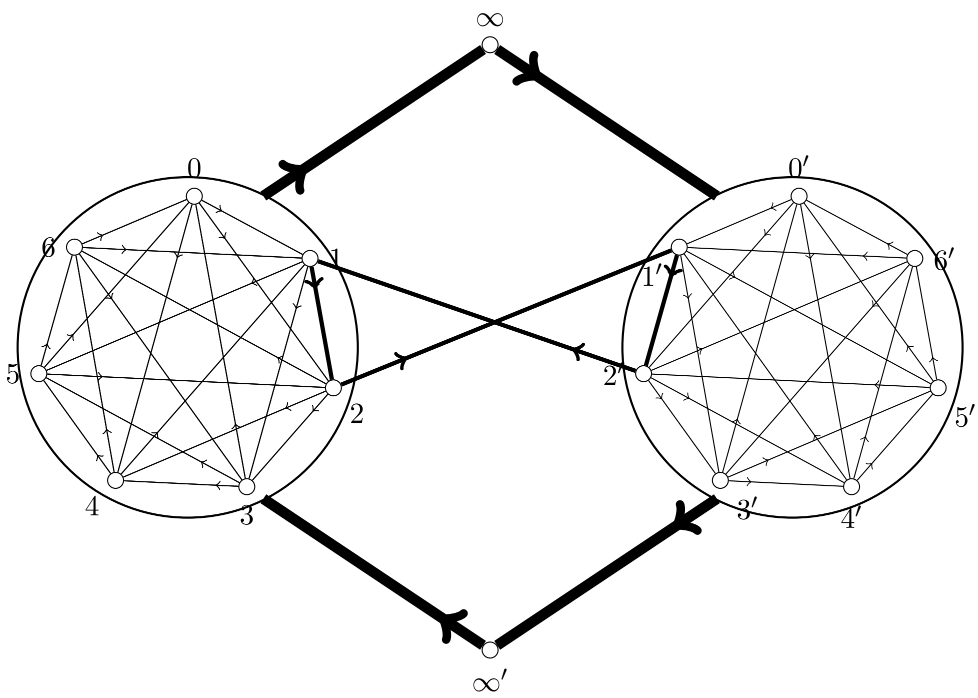


Figure 3. Coloring graph $\overrightarrow{T}_{16} = \overrightarrow{Tr}(\overrightarrow{T}_7)$.

Definition 10. We say that the Tromp graph $\overrightarrow{T}$ has the property $P_c(i, j)$ if $|V(\overrightarrow{T})| \geq i$ and for every orientation $\overrightarrow{K}$ of the star $K_{1,i}$ and every sequence of colors $(c(v_1), c(v_2), \ldots, c(v_k))$ chosen for leaves compatible with $\overrightarrow{K}$ we can choose j different ways to color x, the central vertex of the star.

Lemma 11 [11]**.** *The Tromp graph* $\overrightarrow{T}_{16}$ *has the properties* $P_c(1, 7)$*,* $P_c(2, 3)$ *and* $P_c(3, 1)$*.*

3. Grids $G_{8,n} = P_8 \square P_n$

Definition 12. The *comb* R_8 is an undirected graph with the set of vertices $V(R_8) = \{(1,1), \ldots, (8,1), (1,2), \ldots, (8,2)\}$ and edges of the form $\{(i,1),(i,2)\}$ for $1 \leq i \leq 8$, or $\{(i,2),(i+1,2)\}$ for $1 \leq i < 8$; see Figure 4. The vertices $(1,1), \ldots, (8,1)$ form the first column of the comb R_8, while $(1,2), \ldots, (8,2)$ form the second column.

Definition 13. A set $S \subseteq (V(\overrightarrow{H}_{10}))^8$ is *closed under extension* if

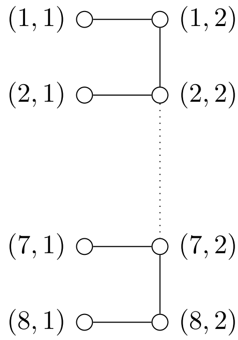


Figure 4. Comb R_8.

(a) for every orientation $\overrightarrow{P}$ of the path $P_8 = (v_1, \ldots, v_8)$, there exists a coloring $c : \overrightarrow{P} \to \overrightarrow{H}_{10}$ such that $(c(v_1), \ldots, c(v_8)) \in S$,

(b) for every orientation $\overrightarrow{R}$ of the comb R_8 and for every sequence $(c_1, \ldots, c_8) \in S$, there exists a coloring $c : \overrightarrow{R} \to \overrightarrow{H}_{10}$ and an automorphism h_a of $\overrightarrow{H}_{10}$ such that

(1) $(c(1,1), \ldots, c(8,1)) = (c_1, \ldots, c_8)$, and

(2) $h_a(c(1,2), \ldots, c(8,2)) \in S$.

Lemma 14. *There exists a set $S \subseteq (V(\overrightarrow{H}_{10}))^8$ which is closed under extension.*

Proof. In order to proof the lemma we use a computer. We have designed an algorithm that finds a proper set S. Let

$$S_{\max}(P_8) = \big\{(c_1, \ldots, c_8) : c_1 \in \{1, 10\}, \text{ and } \forall_{2 \le i \le 8}\ c_i \in V(\overrightarrow{H}_{10}), \text{ and } c_{i-1} \neq c_i\big\}.$$

Note, that for every sequence $t = (t_1, \ldots, t_8) \in S_{\max}(P_8)$, there exists an orientation $\overrightarrow{P}$ of the path $P_8 = (v_1, \ldots, v_8)$ and a coloring $c : \overrightarrow{P} \to \overrightarrow{H}_{10}$ such that $(c(v_1), \ldots, c(v_8)) = t$. For a set T, a sequence $t = (t_1, \ldots, t_8) \in T$, and an orientation $\overrightarrow{R}$ of the comb R_8, we say that t can be extended in T on $\overrightarrow{R}$ if there exists a coloring $c : \overrightarrow{R} \to \overrightarrow{H}_{10}$ and a homomorphism h_a such that

- $(c(1,1), \ldots, c(8,1)) = t$, and
- $h_a(c(1,2), \ldots, c(8,2)) \in T$.

The algorithm starts with $T = S_{\max}(P_8)$. In the while loop, for each sequence $t \in T$ and for each orientation $\overrightarrow{R}$ of the comb R_8, the algorithm checks if t can be extended in T on $\overrightarrow{R}$. If the sequence t can not be extended, then t is removed from T. After the while loop, the set T satisfies the condition (b) of Definition 13. It is easy to see that if T is not empty, then it also satisfies the condition (a). In this case $S = T$ is returned. If T is empty, then the algorithm returns NO.

Algorithm ComputeSet$S$

OUTPUT: a set $S \subset (V(\overrightarrow{H}_{10}))^8$ closed under extension or NO if such a set does not exist.

```
compute the set S_max(P_8)
T := S_max(P_8)
SetIsReady := false
while not SetIsReady
  SetIsReady := true
  for every sequence t = (t_1, ..., t_8) ∈ T
    color the first column of the comb R_8
            by setting c(i,1) = t_i, for 1 ≤ i ≤ 8
    SeqCanBeExtended := true
    for every orientation R⃗ of the comb R_8
      if t cannot be extended on R⃗
         SeqCanBeExtended := false
    if not SeqCanBeExtended
      T := T − t
      SetIsReady := false
 if T = ∅
  return NO
 else
  S := T
  return the set S
```

Using Algorithm ComputeSet$S$ we have found a nonempty set S closed under extension. The set S is posted on the website `https://inf.ug.edu.pl/grids/`. ■

Theorem 15. *Every orientation of every grid with eight rows can be colored by the coloring graph* $\overrightarrow{H}_{10}$.

Proof. For a given orientation $\overrightarrow{G}$ of $G(8,n)$ and $i \leq n$, by $\overrightarrow{G}(i)$ we denote the induced subgraph of $\overrightarrow{G}$ formed by the first i columns of $\overrightarrow{G}$. It is easy to show by induction that, for every i, there is a coloring $c : \overrightarrow{G}(i) \to \overrightarrow{H}_{10}$ such that $c(i\text{th column}) \in S$. ■

4. Stacked Prism Graphs $Y_{m,n} = C_m \Box P_n$

Theorem 16. *Every orientation of* $C_m \Box P_n$ *with* $m \geq 3$ *and* $n \geq 1$ *can be colored by the Tromp graph* $\overrightarrow{T}_{16}$.

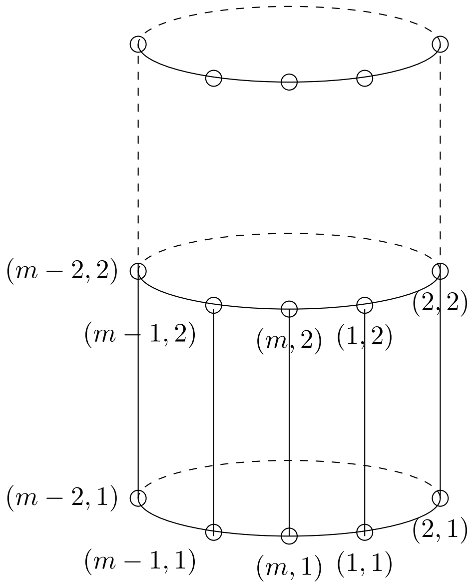


Figure 5. Stacked prism graph $Y_{m,n}$.

Proof. Let $\overrightarrow{Y}$ be any orientation of stacked prism graph $Y_{m,n} = C_m \Box P_n$. We identify each vertex $u \in \overrightarrow{Y}$ with the pair of its coordinates (i, j), $1 \le i \le m$, $1 \le j \le n$. We shall show that $\overrightarrow{Y}$ can be colored by $\overrightarrow{T}_{16}$. We color the vertices of $\overrightarrow{Y}$ row by row. For the first row, clearly, it is always possible to color any oriented cycle by homomorphism to $\overrightarrow{T}_{16}$, because $\overrightarrow{T}_{16}$ has the properties $P_c(2, 3)$ and $P_c(1, 7)$. Now, suppose that $i > 1$ and the rows from 1 to $i-1$ are already colored. To color the vertex $(1, i)$ we choose a color which is compatible

- with the color of vertex $(2, i-1)$ in the star $\{(2, i), (1, i), (2, i-1)\}$,
- with the color of vertex $(m, i-1)$ in the star $\{(m, i), (1, i), (m, i-1)\}$,

which is always possible using the property $P_c(1, 7)$. Using the property $P_c(2, 3)$ it is always possible to color vertex $(2, i)$ by the color compatible with color of the vertex $(3, i-1)$ in the star $\{(3, i), (2, i), (3, i-1)\}$. Then we continue this method to color vertices $(3, i), \ldots, (m-2, i)$. To color the vertex $(m-1, i)$ we choose a color which is compatible with the colors of vertices $(m, i-1)$ and $(1, i)$ in the star $\{(m, i), (1, i), (m, i-1), (m-1, i)\}$. This is possible, because the colors of vertices $(1, i)$ and $(m, i-1)$ are compatible in the star $\{(m, i), (1, i), (m, i-1)\}$ Finally we color the vertex (m, i) using the property $P_c(3, 1)$. Similarly we can color the following rows. ■

Theorem 17. *Every orientation of stacked prism graph* $Y_{m,n} = C_m \Box P_n$ *with* $3 \le m \le 7$ *can be colored by the coloring graph* $\overrightarrow{H}_{10}$.

Proof. The proof of the theorem is similar to the proof of Theorem 15 and follows from Lemma 20. ■

Definition 18. For $m \geq 3$, the *m-sunlet graph Sun_m* is an undirected graph with the set of vertices $V(Sun_m) = \{(1,1), \ldots, (m,1), (1,2), \ldots, (m,2)\}$ and edges of the form $\{(i,1),(i,2)\}$ for $1 \leq i \leq m$, or $\{(i,2),(i+1,2)\}$ for $1 \leq i < m$, or $\{(m,2),(1,2)\}$; see Figure 6.

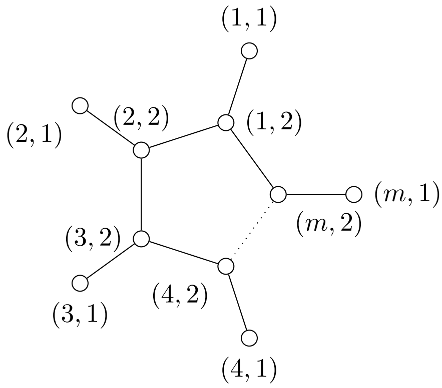


Figure 6. m-sunlet graph.

Definition 19. A set $S \subseteq (V(\overrightarrow{H}_{10}))^m$ is *cycle-closed under extension* if

(a) for every orientation $\overrightarrow{C}$ of the cycle $C_m = (v_1, \ldots, v_m)$, there exists a coloring $c : \overrightarrow{C} \to \overrightarrow{H}_{10}$ such that $(c(v_1), \ldots, c(v_m)) \in S$,

(b) for every orientation $\overrightarrow{Sun}$ of the m-sunlet graph Sun_m and for every sequence $(c_1, \ldots, c_m) \in S$, there exists a coloring $c : \overrightarrow{Sun} \to \overrightarrow{H}_{10}$ and an automorphism h_a of $\overrightarrow{H}_{10}$ such that

  (1) $(c(1,1), \ldots, c(m,1)) = (c_1, \ldots, c_m)$, and

  (2) $h_a(c(1,2), \ldots, c(m,2)) \in S$.

Lemma 20. *For each $m = 3, 4, 5, 6, 7$, there exists a nonempty set $S_m \subseteq (V(\overrightarrow{H}_{10}))^m$, which is cycle-closed under extension.*

Proof. In order to proof the lemma we use a computer. We have designed an algorithm, similar to the Algorithm ComputeSet$S$, that finds a set cycle-closed under extension. The algorithm, for a given m, uses the m-sunlet Sun_m instead of a comb R_8. Using the algorithm we have found that for each $m = 3, \ldots, 7$, there exists a nonempty set cycle-closed under extension. ■

Acknowledgements

The author would like to thank the anonymous reviewer for pointing out that the Tromp graph $\overrightarrow{T}_{16}$ can be used to color every orientation of every stacked prism graphs.

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Received 5 June 2019
Revised 31 January 2020
Accepted 3 February 2020