

## ASCENDING SUBGRAPH DECOMPOSITIONS OF ORIENTED GRAPHS THAT FACTOR INTO TRIANGLES

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### Abstract

In 1987, Alavi, Boals, Chartrand, Erdős, and Oellermann conjectured that all graphs have an ascending subgraph decomposition (ASD). In a previous paper, Wagner showed that all oriented complete balanced tripartite graphs have an ASD. In this paper, we will show that all orientations of an oriented graph that can be factored into triangles with a large portion of the triangles being transitive have an ASD. We will also use the result to obtain an ASD for any orientation of complete multipartite graphs with  $3n$  partite classes each containing 2 vertices (a  $K(2 : 3n)$ ) or 4 vertices (a  $K(4 : 3n)$ ).

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### 1. INTRODUCTION

In [1], Alavi, Boals, Chartrand, Erdős, and Oellermann defined a type of graph decomposition called an ascending subgraph decomposition (ASD).

**Definition.** A graph  $G$  with  $\binom{n+1}{2} + r$  edges ( $0 \leq r \leq n$ ) has an ascending subgraph decomposition if there exists a partition of the edge set of  $G$  such that the graphs  $G_1, G_2, \dots, G_n$  induced by the sets of edges in the partition satisfy the properties that  $G_i$  is isomorphic to a subgraph of  $G_{i+1}$  for all  $1 \leq i \leq n-1$  and  $|E(G_i)| = i$  for all  $i = 1, 2, \dots, n-1$  and  $|E(G_n)| = n + r$ .

For digraphs, we can define an ASD of a digraph if we can similarly partition the directed edges (or arcs).

We will also need the following definition of a 2-factorization of a graph.

**Definition.** A graph  $G$  on  $N$  vertices has a 2-factorization if the edge set of  $G$  can be partitioned into subsets of  $N$  edges where all  $N$  vertices in the subgraph induced by each set of edges in the partition have degree 2.

For oriented graphs, we use a similar definition as above, but we refer to arcs instead of edges and the sum of the indegree and outdegree of each vertex in the induced subgraphs is 2.

We let  $K(m : N)$  denote a complete multipartite graph with  $N$  partite classes containing  $m$  vertices each. This paper will only consider the case when  $m = 2$  or 4 and  $N = 3n$ .

See [2] for all terms and notation not specifically defined in this paper.

## 2. STRATEGY

In [7], we obtained a 2-factorization of a tournament on  $6n + 3$  vertices into only triangles to construct an ASD. The first terms in the ASD were matchings of increasing size. The next terms included directed paths of length 2. The last terms of the ASD included triangles. In satisfying the isomorphic subgraph requirement of the ASD, there was no problem with the arc direction until triangles are included as there are 2 types of oriented triangles—transitive and cyclic. We used the following definition of an ascending sequence of specific height and cap to help build the ASD.

**Definition.** Let  $S$  be a finite multiset  $\{(x_i, y_i)\}_{i=1}^m$  where  $x_i$  and  $y_i$  are non-negative integers for all  $i = 1, 2, \dots, m$ . We say that  $S$  has an ascending sequence of height  $h$  and cap  $c$  if there exists a sequence  $S' = \{(a_j, b_j)\}_{j=1}^{h+c-1}$  where  $h + c - 1 \leq m$  satisfying the following.

1.  $a_j + b_j = j$  for all  $j = 1, 2, \dots, h$ ,
2.  $a_j + b_j = h$  for all  $j = h, h + 1, \dots, h + c - 1$ ,
3.  $a_j \leq a_{j+1}$  and  $b_j \leq b_{j+1}$  for all  $j = 1, 2, \dots, h + c - 2$ ,
4.  $a_j \leq x_j$  and  $b_j \leq y_j$  for some ordering of  $S$ .

Each ordered pair  $(x_i, y_i)$  corresponds to the 2-factor  $F_i$  that contained  $x_i$  transitive triangles and  $y_i$  cyclic triangles. We used the ascending sequence in our construction of the ASD to ensure that the isomorphic subgraph condition is satisfied. A similar approach was used in [8] and [9].

In this paper, the main result generalizes the ASD construction given the ascending sequence of sufficient height corresponding to the 2-factorization.

This result will allow us to avoid a separate specialized construction of the ASD for similar cases. In this paper, we apply the main result to oriented equipartite graphs  $K(2 : 3n)$  for  $n \geq 3$  and  $K(4 : 3n)$  for  $n \geq 2$ . We will use the following

Oberwolfach result by Liu in [5] to obtain the specific 2-factorization into triangles (so  $t = 3$ ). We will apply the main result to that 2-factorization.

**Theorem 2.1** (Liu). *The complete multipartite graph  $K(m : n)$  can be partitioned into 2-factors where each 2-factor consists of cycles of length  $t \geq 3$  if and only if  $t|mn$ ,  $m(n-1)$  is even,  $t$  is even when  $n = 2$ , and  $(m, n, t)$  is none of  $(2, 3, 3)$ ,  $(6, 3, 3)$ ,  $(2, 6, 3)$ ,  $(6, 2, 6)$ .*

The future goal is to apply the result to the more general case of an oriented  $K(m : 3n)$ .

### 3. MAIN RESULT

The following theorem by Fu and Hu in [3] will also be useful for proving our main result.

**Theorem 3.1** (Fu and Hu). *Let  $G$  be a graph with  $\binom{n+1}{2}$  edges and  $E(G)$  be the disjoint union of matchings  $M_1, M_2, \dots, M_k$  such that all but at most one have at least  $n$  edges. Then  $G$  has an ASD with each subgraph a matching.*

Note that digraphs of even order  $3N$  that have a 2-factorization into triangles but that have smaller size, can be shown to have an ASD. If the last term of the ASD would be  $D_N$  or a lower subscript, we can apply Theorem 3.1 since matchings of size at most  $N$  would suffice. If the last term of the ASD would be  $D_{N+i}$  where  $1 \leq i \leq N$ , then matchings would not suffice and directed paths of length 2 (which exist as a subgraph of either orientation of a triangle) along with matchings would be used to complete the ASD. However, the focus of this paper is more dense digraphs.

In the following theorem, we will assume that a digraph of order  $3N$  ( $N$  even) and size at most  $\binom{3N}{2}$  has a 2-factorization into only triangles. We will prove that if the multiset of ordered pairs associated with the triangles in 2-factorization leads to an ascending sequence of sufficient height and cap 1, we may find an ASD for the digraph. By sufficient height, we mean that the height is the smallest positive integer  $h$  such that the size of the digraph is no more than  $\binom{h+2N+2}{2}$ . To ensure that  $h \geq 1$ , we require a size at least  $\binom{2N+2}{2}$ . As such, in the statement of the following theorem, we require  $m \geq \frac{2N^2+1}{3N} + 1 = \frac{2N}{3} + \frac{1}{3N} + 1$  for a digraph of size  $3Nm$ . The theorem is written for a more general case so it may be applied to other cases beyond orientations of  $K(2 : 3n)$  and  $K(4 : 3n)$ .

**Theorem 3.2.** *Let  $N$  be an even positive integer and  $m$  be an integer where  $\frac{2N}{3} + \frac{1}{3N} + 1 \leq m \leq \frac{3N}{2} - 1$ . Let  $D$  be a digraph of order  $3N$  and size  $3Nm$  having a 2-factorization where the  $i^{\text{th}}$  2-factor  $F_i$  consists of  $x_i$  transitive triangles and*

$y_i$  cyclic triangles and  $x_i + y_i = N$  for  $i = 1, 2, \dots, m$ . Let the positive integer  $h$  be such that  $3Nm = \binom{h+2N+1}{2} + r$  where  $0 \leq r \leq h + 2N$ . If the sequence  $S = \{(x_i, y_i)\}_{i=1}^m$  has an ascending sequence of height  $h$  and cap 1, then the digraph  $D$  has an ascending subgraph decomposition.

**Proof.** Consider such a 2-factorization with factors  $F_1, F_2, \dots, F_m$ . Without loss of generality, suppose that the ascending sequence of height  $h$  and cap 1 is

$$(a_{m-h+1}, b_{m-h+1}), (a_{m-h+2}, b_{m-h+2}), \dots, (a_{m-1}, b_{m-1}), (a_m, b_m).$$

So,  $a_i + b_i = i - (m - h)$  for  $i = m - h + 1, m - h + 2, \dots, m$ .

First, we will construct the terms  $D_N, D_{N+1}, \dots, D_{2N-1}, D_{2N}$  using the factors  $F_1, F_2, \dots, F_{\frac{N}{2}}, F_{\frac{N}{2}+1}$ . Notice that each triangle can be decomposed into a directed path of length 2 and a single arc. For each factor  $F_i$  where  $1 \leq i \leq \frac{N}{2}$ , we form  $D_{2N-i+1}$  and  $D_{N+i-1}$ . The term  $D_{2N-i+1}$  consists of  $N - i + 1$  disjoint directed paths of length 2 and a matching of size  $i - 1$ . The term  $D_{N+i-1}$  consists of  $i - 1$  disjoint directed paths of length 2 and a matching of size  $N - i + 1$ . For  $D_{\frac{3N}{2}}$ , take from  $F_{\frac{N}{2}+1}$ ,  $\frac{N}{2}$  directed paths of length 2 and a matching of size  $\frac{N}{2}$ . The remaining arcs in factor  $F_{\frac{N}{2}+1}$  can be decomposed into two matchings  $M_0$  of size  $N$  and  $M'$  of size  $\frac{N}{2}$ .

We shall now construct the terms  $D_{2N+1}, D_{2N+2}, \dots, D_{2N+h}$  and the terms  $D_{N-h}, D_{N-h+1}, \dots, D_{N-1}$  from the factors  $F_{m-h+1}, F_{m-h+2}, \dots, F_m$ . We will use the ascending sequence to construct these digraphs. For  $1 \leq j \leq h$ , we will construct the terms  $D_{2N+j}$  and  $D_{N-j}$  using the factor  $F_{m-h+j}$ . From factor  $F_{m-h+j}$  take  $a_{m-h+j}$  transitive triangles and  $b_{m-h+j}$  cyclic triangles (for a total of  $j$  triangles), which are guaranteed by the ascending sequence, along with  $N - j$  disjoint directed paths of length 2. These arcs form the term  $D_{2N+j}$ . The remaining unused arcs of factor  $F_{m-h+j}$  is a matching of size  $N - j$  which is the term  $D_{N-j}$ . Thus, we have used the arcs in factors  $F_{m-h+1}, F_{m-h+2}, \dots, F_m$  to construct the terms  $D_{2N+1}, D_{2N+2}, \dots, D_{2N+h}$  and the terms  $D_{N-h}, D_{N-h+1}, \dots, D_{N-1}$ .

As the remaining terms  $D_1, D_2, \dots, D_{N-h-1}$  will all be matchings, we will use Theorem 3.1. Note a total of  $\binom{N-h}{2} + r$  arcs remain in factors  $F_{\frac{N}{2}+2}, \dots, F_{m-h}$  and matchings  $M_0$  and  $M'$ . Since the factors can be broken into 3 matchings of size  $N$  each, we have a total of  $3(m - h - \frac{N}{2} - 1) + 2$  matchings where all except one (matching  $M'$ ) have size  $N$ . We may remove  $r$  arcs (and place those extra arcs in the last term of the ASD  $D_{2N+h}$ ) in such a way so all but at most one of the matchings have  $N \geq N - h$  arcs allowing us to apply Theorem 3.1 to obtain the terms  $D_1, D_2, \dots, D_{N-h-1}$ .

That completes our ASD and the proof. ■

The next two sections are applications of the main result.

4. APPLICATION OF THE MAIN RESULT TO ORIENTED  $K(2 : 3n)$ 

We now apply the main result to show that any oriented  $K(2 : 3n)$  for  $n \geq 1$  has an ASD. Note that  $K(2 : 3)$  and  $K(2 : 6)$  do not have 2-factorizations into triangles by Theorem 2.1. Showing that any orientation of the tripartite graph  $K(2 : 3)$  has an ASD is an easy exercise (and was proven in [9]). For the case of oriented  $K(2 : 6)$ , we will use the following result in [4] by Huang, Kotzig, and Rosa.

**Theorem 4.1** (Huang, Kotzig and Rosa). *The complete multipartite graph  $K(2 : 6)$  has a 2-factorization where each 2-factor consists of two triangles and one  $C_6$ .*

We now will prove that every orientation of  $K(2 : 6)$  has an ASD.

**Proposition 4.2.** *Every orientation of the complete multipartite graph  $K(2 : 6)$  has an ascending subgraph decomposition.*

**Proof.** Consider any orientation of  $K(2 : 6)$ . Use Theorem 4.1 to obtain a 2-factorization where each 2-factor consists of two triangles and one  $C_6$ . There are 5 such factors; label them  $F_1, \dots, F_5$ . For each  $i$ , assign to the factor  $F_i$  the ordered pairs  $(x_i, y_i)$  where  $x_i$  and  $y_i$  are the number of transitive and cyclic triangles in factor  $F_i$ , respectively.

By the pigeonhole principle, at least 3 of the ordered pairs are either from the set  $\{(2, 0), (1, 1)\}$  or the set  $\{(0, 2), (1, 1)\}$ . The argument proceed similarly for either case, so without loss of generality, suppose there are three ordered pairs from the set  $\{(2, 0), (1, 1)\}$ . Of those three, at least two are the same ordered pair. Relabel the factors so that factors  $F_1$  and  $F_2$  have the same ordered pair (i.e., we have  $x_1 = x_2$  and  $y_1 = y_2$ ) and factor  $F_3$  has the (possibly) different ordered pair from the set  $\{(2, 0), (1, 1)\}$ .

We will now construct the terms of the ASD,  $D_1, \dots, D_{10}$ , as follows.

1. Terms  $D_6$  and  $D_1$  are constructed from factor  $F_5$  where  $D_6$  consists of a matching of size 4 and a directed path of length 2, and  $D_1$  is a single arc. Note that the remaining unused arcs will be placed in the last term  $D_{10}$ .
2. Terms  $D_7$  and  $D_5$  are constructed from factor  $F_4$  where  $D_7$  consists of a matching of size 3 and two directed paths of length 2, and  $D_5$  is a matching of size 5.
3. Terms  $D_8$  and  $D_4$  are constructed from factor  $F_3$  where  $D_8$  consists of a matching of size 3, a directed path of length 2, and a transitive triangle; and  $D_4$  is a matching of size 4.
4. Terms  $D_9$  and  $D_3$  are constructed from factor  $F_2$  where  $D_9$  consists of a matching of size 3 and two triangles, of which  $x_2$  are transitive and  $y_2$  are cyclic, and  $D_3$  is a matching of size 3.

5. Terms  $D_{10}$  and  $D_2$  are constructed from factor  $F_1$  where  $D_{10}$  consists of a matching of size 3 and two triangles, of which  $x_1$  are transitive and  $y_1$  are cyclic and one more arc from factor  $F_1$  (and the 5 unused arcs from factor  $F_5$ ), and  $D_2$  is a matching of size 2. Note that since  $x_1 = x_2$  and  $y_1 = y_2$ ,  $D_{10}$  contains an isomorphic copy of  $D_9$ . ■

Before proving the case for  $n \geq 3$ , we need the following lemmas. The first of which was proven in [7].

**Lemma 4.3.** *Let  $n$  and  $N$  be positive integers with  $n \leq N$ . If the multiset  $S = \{(x_1, y_1), (x_2, y_2), \dots, (x_{2n}, y_{2n})\}$  consisted of ordered pairs of nonnegative integers with  $x_i + y_i = N$  for all  $i = 1, 2, \dots, 2n$ , then  $S$  has an ascending sequence of height  $n$  and cap 1.*

We need to know the minimum number of transitive triangles in our decomposition so that from the resulting multiset of ordered pairs, we can find an ascending sequence of the desired height and cap. We will use the following lemma from [8], that was proven using a result from [6].

**Lemma 4.4.** *Let  $T$  be a tournament of order  $n \geq 2$  with  $V(T) = [n]$ . Then at least  $\frac{3(n-3)}{4(n-2)}$  portion of the triangles in  $T$  are transitive if  $n$  is odd and has at least  $\frac{3(n-2)}{4(n-1)}$  portion of the triangles in  $T$  are transitive if  $n$  is even.*

For our result, we will only use the lesser lower bound of  $\frac{3(n-3)}{4(n-2)}$  which applies for all tournaments of order  $n$ .

In the general case, the terms in our ASD contain mostly transitive triangles. The following lemma gives a lower bound on the portion of transitive triangles in  $K(m : 3n)$ .

**Lemma 4.5.** *Any orientation of  $K(m : 3n)$  has at least  $\frac{3(3n-3)}{4(3n-2)}$  portion of the triangles being transitive.*

**Proof.** Consider any orientation of  $K(m : 3n)$ . If a single vertex is chosen from each of the  $3n$  partite classes, a tournament on  $3n$  vertices results. By Lemma 4.4, the resulting tournament has at least  $\frac{3(3n-3)}{4(3n-2)}$  portion of the triangles being transitive. Note that since each partite class contains the same number of the vertices, each different triangle in  $K(m : 3n)$  is contained in  $m^{3n-3}$  such tournaments. Therefore, the portion of triangles being transitive in  $K(m : 3n)$  must also be at least  $\frac{3(3n-3)}{4(3n-2)}$ . ■

The following technical lemma will be used to obtain required ascending sequence of the ordered pairs representing the number each type of triangle in each 2-factor.

**Lemma 4.6.** *Let  $n \geq 2$  be an integer. Let  $S$  be a multiset of ordered pairs  $\{(x_i, y_i)\}_{i=1}^{3n-1}$  where  $x_i$  and  $y_i$  are nonnegative integers with the following properties.*

1.  $x_i + y_i = 2n$  for all  $i = 1, 2, \dots, 3n - 1$ .
- 2.

$$\sum_{i=1}^{3n-1} x_i \geq \frac{9n(n-1)(3n-1)}{6n-4} =: f(n).$$

*Then  $S$  contains an ascending sequence of height  $2n - 1$  and cap 1.*

**Proof.** First, order the multiset so that  $x_i \geq x_{i+1}$  for all  $i$ .

We will consider two cases.

*Case I.* Suppose  $x_{3n-4} \geq n$ . Then,  $x_1 \geq x_2 \geq \dots \geq x_{2n-2} \geq n$ . Let  $S'$  be the ordered multiset  $\{(x_i - n, y_i)\}_{i=1}^{2n-2}$ . By Lemma 4.3,  $S'$  has an ascending sequence of height  $n - 1$  and cap 1. Without loss of generality, suppose the sequence is  $(a_1, b_1), \dots, (a_{n-1}, b_{n-1})$ . But then,

$$(1, 0), (2, 0), \dots, (n, 0), (a_1 + n, b_1), \dots, (a_{n-1} + n, b_{n-1})$$

is our ascending sequence of height  $2n - 1$  and cap 1 in  $S$ .

*Case II.* Suppose  $x_{3n-4} \leq n - 1$ . For this second case, we will prove, by induction on  $n$ , that we can construct the ascending sequence of height  $2n - 1$  and cap 1 using the first  $2n - 1$  elements in the ordered multiset  $S$ .

If  $n = 1$ , our ascending sequence of height  $2n - 1$  and cap 1 is  $(1, 0)$  if  $x_1 \neq 0$  or  $(0, 1)$  if  $x_1 = 0$ .

Now let  $n \geq 2$  and suppose that we can find an ascending sequence of height  $2(n - 1) - 1$  and cap 1 using the first  $2(n - 1) - 1$  elements of any ordered multiset  $S$  of  $3(n - 1) - 1$  elements with  $x_i \geq x_{i+1}$  for all values of  $i$  that satisfies the conditions of the lemma. Now consider an ordered multiset  $S$  with  $3n - 1$  elements that satisfies the conditions of the lemma. We will now prove the following claim.

**Claim.** *The value of  $x_{2n-1} \geq 2$ .*

**Proof.** Suppose instead that  $x_{2n-1} \leq 1$ . Since

$$x_i \leq \begin{cases} 2n & \text{if } i \leq 2n - 2, \\ 1 & \text{if } i \geq 2n - 1, \end{cases}$$

we have that

$$\sum_{i=1}^{3n-1} x_i \leq 2n(2n - 2) + (n + 1) = 4n^2 - 3n + 1.$$

So, by Property 2 of the lemma,

$$4n^2 - 3n + 1 \geq \sum_{i=1}^{3n-1} x_i \geq \frac{9n(n-1)(3n-1)}{6n-4}.$$

But for  $n \geq 2$ ,

$$\frac{9n(n-1)(3n-1)}{6n-4} > 4n^2 - 3n + 1,$$

which is a contradiction. Therefore,  $x_{2n-1} \geq 2$ . □

We will now consider a new ordered multiset  $S'' = \{(x'_i, y'_i)\}_{i=1}^{3n-4}$  formed from  $S$  as follows. If  $x_i \geq 2$ , then  $x'_i = x_i - 2$  and  $y'_i = y_i$ . If  $x_i = 1$ , then  $x'_i = 0$  and  $y'_i = y_i - 1$ . If  $x_i = 0$ , then  $x'_i = 0$  and  $y'_i = y_i - 2$ . Clearly,  $S''$  satisfies Property 1 of the lemma. Property 2 is proven in the following claim.

**Claim.** *The ordered multiset  $S'$  satisfies*

$$\sum_{i=1}^{3n-4} x'_i \geq \frac{9(n-1)(n-2)(3n-4)}{6n-10} = f(n-1).$$

**Proof.** By the definition of  $x'_i$  and the fact that  $x_{3n-1} \leq x_{3n-2} \leq x_{3n-3} \leq x_{3n-4} \leq n-1$ , we have

$$\sum_{i=1}^{3n-4} x'_i \geq \sum_{i=1}^{3n-4} (x_i - 2) \geq f(n) - 2(3n-4) - 3(n-1).$$

Let  $g(n) = f(n) - 2(3n-4) - 3(n-1) - f(n-1) = \frac{63n^2 - 147n + 76}{2(3n-5)(3n-2)}$ . Since  $g(n) > 0$  for all integers  $n \geq 2$ , we have

$$\sum_{i=1}^{3n-4} x'_i \geq \frac{9(n-1)(n-2)(3n-4)}{6n-10}. \quad \square$$

Thus,  $S''$  satisfies Property 2 of the lemma. By induction, we can find an ascending sequence of height  $2(n-1) - 1$  and cap 1 using the first  $2(n-1) - 1$  elements of  $S''$ . Suppose this sequence is  $(a_{k_1}, b_{k_1}), (a_{k_2}, b_{k_2}), \dots, (a_{k_{2n-3}}, b_{k_{2n-3}})$  where  $k_i \in [2n-3]$ ,  $a_{k_i} \leq x'_{k_i} = x_{k_i} - 2$ , and  $b_{k_i} \leq y'_{k_i} = y_{k_i}$  for all  $i$ . Since  $x_{2n-2} \geq x_{2n-1} \geq 2$ , we have

$$(1, 0), (2, 0), (a_{k_1} + 2, b_{k_1}), (a_{k_2} + 2, b_{k_2}), \dots, (a_{k_{2n-3}} + 2, b_{k_{2n-3}})$$

as our ascending sequence of height  $2n-1$  and cap 1 in  $S$ .

This completes the proof. ■



**Theorem 4.7.** *Any orientation of  $K(2 : 3n)$  for  $n \geq 3$  has an ascending subgraph decomposition.*

**Proof.** Consider any orientation of  $K(2 : 3n)$  where  $n \geq 3$ . Obtain a 2-factorization into triangles by applying Theorem 2.1. Since at least  $\frac{3(3n-3)}{4(3n-2)}$  portion of the triangles in the oriented  $K(2 : 3n)$  are transitive by Lemma 4.5, we know that at least one of the 2-factorizations has at least  $\frac{3(3n-3)}{4(3n-2)}$  portion of the triangles being transitive. Thus, we choose such a factorization that has at least that portion of transitive triangles. Apply Lemma 4.6 to obtain an ascending sequence of height  $2n - 1$  and cap 1. Apply Theorem 3.2 to obtain the ascending subgraph decomposition. ■

## 5. APPLICATION OF THE MAIN RESULT TO ORIENTED $K(4 : 3n)$

We now apply the main result to show that any oriented  $K(4 : 3n)$  for  $n \geq 1$  has an ASD. Again, the case when  $n = 1$  is an easy exercise. For the case when  $n \geq 2$ , we use a lemma similar to Lemma 4.6 to obtain an ascending sequence of height  $4n - 3$  and cap 1.

**Lemma 5.1.** *Let  $n \geq 2$  be an integer. Let  $S$  be a multiset of ordered pairs  $\{(x_i, y_i)\}_{i=1}^{6n-2}$  where  $x_i$  and  $y_i$  are nonnegative integers with the following properties.*

1.  $x_i + y_i = 4n$  for all  $i = 1, 2, \dots, 6n - 2$ .
- 2.

$$\sum_{i=1}^{6n-2} x_i \geq \frac{18n(n-1)(3n-1)}{3n-2} =: f(n).$$

*Then  $S$  contains an ascending sequence of height  $4n - 3$  and cap 1.*

**Proof.** First order the multiset so that that  $x_i \geq x_{i+1}$  for all  $i$ .

We will consider two cases.

*Case I.* Suppose  $x_{6n-8} \geq 2n$ . Then,  $x_1 \geq x_2 \geq \dots \geq x_{4n-6} \geq 2n$ . Let  $S'$  be the ordered multiset  $\{(x_i - 2n, y_i)\}_{i=1}^{4n-6}$ . By Lemma 4.3,  $S'$  has an ascending sequence of height  $2n - 3$  and cap 1. Without loss of generality, suppose the sequence is  $(a_1, b_1), \dots, (a_n, b_n)$ . But then,

$$(1, 0), (2, 0), \dots, (2n, 0), (a_1 + 2n, b_1), \dots, (a_{2n-3} + 2n, b_{2n-3})$$

is our ascending sequence of height  $4n - 3$  and cap 1 in  $S$ .

*Case II.* Suppose  $x_{6n-8} \leq 2n - 1$ . For this second case, we will prove, by induction on  $n$ , that we can construct the ascending sequence of height  $4n - 3$  and cap 1 using the first  $4n - 3$  elements in the ordered multiset  $S$ .

If  $n = 2$ , Property 2 of the lemma gives that  $\sum_{i=1}^{10} x_i \geq 45$ . From the assumption of Case II,  $x_4 \leq 3$  and likewise  $x_i \leq 3$  for  $i \geq 5$ . These together imply that  $x_1 = x_2 = x_3 = 8$  and also  $x_4 = x_5 = x_6 = x_7 = x_8 = x_9 = x_{10} = 3$  our ascending sequence of height 5 and cap 1 can be  $(1, 0), (2, 0), (3, 0), (4, 0), (5, 0)$ .

Now let  $n \geq 3$  and suppose that we can find an ascending sequence of height  $4(n-1) - 3$  and cap 1 using the first  $4(n-1) - 3$  elements of any ordered multiset  $S$  of  $6(n-1) - 2$  elements with  $x_i \geq x_{i+1}$  for all values of  $i$  that satisfies the conditions of the lemma. Now consider an ordered multiset  $S$  with  $6n-2$  elements that satisfies the conditions of the lemma. We will now prove the following claim.

**Claim.** *The value of  $x_{4n-3} \geq 4$ .*

**Proof.** Suppose instead that  $x_{4n-3} \leq 3$ . Since

$$x_i \leq \begin{cases} 4n & \text{if } i \leq 4n-4, \\ 3 & \text{if } i \geq 4n-3, \end{cases}$$

we have that

$$\sum_{i=1}^{6n-2} x_i \leq 4n(4n-4) + 3(2n+2) = 16n^2 - 10n + 6.$$

So, by Property 2 of the lemma,

$$16n^2 - 10n + 6 \geq \sum_{i=1}^{6n-2} x_i \geq \frac{18n(n-1)(3n-1)}{3n-2}.$$

But for  $n \geq 3$ ,

$$\frac{18n(n-1)(3n-1)}{3n-2} > 16n^2 - 10n + 6,$$

which is a contradiction. Therefore,  $x_{4n-3} \geq 4$ . □

We will now consider a new ordered multiset  $S'' = \{(x'_i, y'_i)\}_{i=1}^{6n-8}$  formed from  $S$  as follows.

1. If  $x_i \geq 4$ , then  $x'_i = x_i - 4$  and  $y'_i = y_i$ .
2. If  $x_i = 3$ , then  $x'_i = 0$  and  $y'_i = y_i - 1$ .
3. If  $x_i = 2$ , then  $x'_i = 0$  and  $y'_i = y_i - 2$ .
4. If  $x_i = 1$ , then  $x'_i = 0$  and  $y'_i = y_i - 3$ .
5. If  $x_i = 0$ , then  $x'_i = 0$  and  $y'_i = y_i - 4$ .

Clearly,  $S''$  satisfies Property 1 of the lemma. Property 2 is proven in the following claim.

**Claim.** *The ordered multiset  $S'$  satisfies.*

$$\sum_{i=1}^{6n-8} x'_i \geq \frac{12(n-1)(n-2)(3n-4)}{3n-5} = f(n-1).$$

**Proof.** By definition of  $x'_i$  and since  $x_{6n-2} \leq x_{6n-3} \leq \dots \leq x_{6n-7} \leq 2n-1$ , we have

$$\sum_{i=1}^{6n-8} x'_i \geq \sum_{i=1}^{6n-8} (x_i - 4) \geq f(n) - 4(6n-8) - 6(2n-1).$$

Let  $g(n) = f(n) - 4(6n-8) - 6(2n-1) - f(n-1)$  or

$$g(n) = \frac{18n(n-1)(3n-1)}{3n-2} - \frac{18(n-1)(n-2)(3n-4)}{3n-5} - 36n + 38.$$

Simplifying, for all integers  $n \geq 3$ , we have

$$g(n) = 4 \left( \frac{1}{3n-5} - \frac{1}{3n-2} + 2 \right) > 0.$$

Thus, we have

$$\sum_{i=1}^{6n-8} x'_i \geq \frac{18(n-1)(n-2)(3n-4)}{3n-5} = f(n-1),$$

which completes the proof of the claim.  $\square$

Thus,  $S''$  satisfies Property 2 of the lemma. By induction, we can find an ascending sequence of height  $4(n-1)-3$  and cap 1 using the first  $4(n-1)-3$  elements of  $S''$ . Suppose this sequence is  $(a_{k_1}, b_{k_1}), (a_{k_2}, b_{k_2}), \dots, (a_{k_{4n-7}}, b_{k_{4n-7}})$  where  $k_i \in [4n-7]$ ,  $a_{k_i} \leq x'_{k_i} = x_{k_i} - 4$ , and  $b_{k_i} \leq y'_{k_i} = y_{k_i}$  for all  $i$ . Since  $x_{4n-6} \geq x_{4n-5} \geq x_{4n-4} \geq x_{4n-3} \geq 4$ , we have

$$(1, 0), (2, 0), (3, 0), (4, 0), (4 + a_{k_1}, b_{k_1}), (4 + a_{k_2}, b_{k_2}), \dots, (4 + a_{k_{4n-7}}, b_{k_{4n-7}})$$

as our ascending sequence of height  $4n-3$  and cap 1 in  $S$ .

This completes the proof.  $\blacksquare$

**Theorem 5.2.** *Any orientation of  $K(4 : 3n)$  for  $n \geq 2$  has an ascending subgraph decomposition.*

**Proof.** Consider any orientation of  $K(4 : 3n)$  where  $n \geq 2$ . Obtain a 2-factorization into triangles by applying Theorem 2.1. Since at least  $\frac{3(3n-3)}{4(3n-2)}$  portion of the triangles in the oriented  $K(4 : 3n)$  are transitive by Lemma 4.5, we know that at least one of the 2-factorizations has at least  $\frac{3(3n-3)}{4(3n-2)}$  portion of the

triangles being transitive. Thus, we choose such a factorization that has at least that portion of transitive triangles. Apply Lemma 5.1 to obtain an ascending sequence of height  $4n - 3$  and cap 1. Apply Theorem 3.2 to obtain the ascending subgraph decomposition. ■

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### REFERENCES

- [1] Y. Alavi, A.J. Boals, G. Chartrand, P. Erdős and O. Oellermann, *The ascending subgraph decomposition problem*, Congr. Numer. **58** (1987) 7–14.
- [2] G. Chartrand, L. Lesniak and P. Zhang, *Graphs & Digraphs*, 4th Ed. (Chapman & Hall/CRC, Boca Raton, 2005).
- [3] H.-L. Fu and W.-H. Hu, *Ascending subgraph decompositions of regular graphs*, Discrete Math. **253** (2002) 11–18.  
[https://doi.org/10.1016/S0012-365X\(01\)00445-9](https://doi.org/10.1016/S0012-365X(01)00445-9)
- [4] C. Huang, A. Kotzig and A. Rosa, *On a variation of the Oberwolfach problem*, Discrete Math. **27** (1979) 261–277.  
[https://doi.org/10.1016/0012-365X\(79\)90162-6](https://doi.org/10.1016/0012-365X(79)90162-6)
- [5] J. Liu, *The equipartite Oberwolfach problem with uniform tables*, J. Combin. Theory Ser. A **101** (2003) 20–34.  
[https://doi.org/10.1016/S0097-3165\(02\)00011-0](https://doi.org/10.1016/S0097-3165(02)00011-0)
- [6] J. Moon, *Topics on Tournaments* (Holt, Rinehart and Winston, New York, 1968).
- [7] B.C. Wagner, *Ascending subgraph decompositions of tournaments of order  $6n + 3$* , Graphs Combin. **29** (2013) 1951–1959.  
<https://doi.org/10.1007/s00373-012-1236-1>
- [8] B.C. Wagner, *Ascending subgraph decompositions of tournaments of orders  $6n + 2$  and  $6n + 1$* , Graphs Combin. **32** (2016) 813–822.  
<https://doi.org/10.1007/s00373-015-1591-9>
- [9] B.C. Wagner, *Ascending subgraph decompositions in oriented complete balanced tripartite graphs*, Graphs Combin. **29** (2013) 1549–1555.  
<https://doi.org/10.1007/s00373-012-1208-5>

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