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ON ANTIPODAL AND DIAMETRICAL PARTIAL CUBES

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Abstract

We prove that any diametrical partial cube of diameter at most 6 is antipodal. Because any antipodal graph is harmonic, this gives a partial answer to a question of Fukuda and Handa [*Antipodal graphs and oriented matroids*, Discrete Math. 111 (1993) 245–256] whether any diametrical partial cube is harmonic, and improves a previous result of Klavžar and Kovše [*On even and harmonic-even partial cubes*, Ars Combin. 93 (2009) 77–86].

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1. Introduction

If x, y are two vertices of a connected graph G, then y is said to be a *relative antipode* of x if $d_G(x,y) \geq d_G(x,z)$ for every neighbor z of x, where d_G denotes the usual distance in G; and it is said to be an *absolute antipode* of x if $d_G(x,y) = \mathrm{diam}(G)$ (the *diameter* of G). The graph G is said to be *antipodal* if every vertex x of G has exactly one relative antipode $\overline{x}$; it is *diametrical* if every vertex x of G has exactly one absolute antipode $\overline{x}$; and it is *harmonic* (or *automorphically diametrical* [19]) if it is diametrical and the *antipodal map* $x \mapsto \overline{x}$, $x \in V(G)$, is an automorphism of G, i.e., $\overline{x}\overline{y} \in E(G)$ whenever $xy \in E(G)$.

Bipartite antipodal graphs were introduced by Kotzig [13] under the name of S-graphs. Later Glivjak, Kotzig and Plesnik [7] proved in particular that *a graph G is antipodal if and only if for any $x \in V(G)$ there is an $\overline{x} \in V(G)$ such that*

$$d_G(x,y) + d_G(y,\overline{x}) = d_G(x,\overline{x}) \quad \text{for all} \quad y \in V(G), \tag{1}$$

where d_G denotes the usual distance in G. The definition was extended to the non-bipartite case by Kotzig and Laufer [14]. Several papers followed.

On the other hand diametrical graphs were introduced by Mulder [16] in the case of median graphs. They were later studied by Parthasarathy and Nandakumar [17] under the name of self-centered unique eccentric point graphs, then by Göbel and Veldman [8] under the name of even graphs, by Fukuda and Handa [6] who proved that the tope graphs of oriented matroids are harmonic partial cubes (i.e., isometric subgraphs of hypercubes), and finally by Klavžar and Kovše [11] who gave a partial solution to a problem set in [6]. Partial cubes, which were introduced by Firsov [5] and characterized by Djoković [4] and Winkler [20], have been extensively studied, see [11, 12, 15, 18] for recent papers. In [11, 12, 18], antipodal, diametrical and harmonic partial cubes play a very important role.

Our paper deals with the problem of determining whether any diametrical partial cube is antipodal. We prove (Theorem 3.2) that this property is true for all partial cubes of diameter at most 6. It follows, by the fact that the diameter of a partial cube is always at most equal to its isometric dimension with the equality if and only if it is antipodal, that if a diametrical partial cube was not antipodal, then its diameter would be at least 7 and its isometric dimension at least 8.

At the end of their paper [6], Fukuda and Handa asked whether any diametrical partial cube is harmonic. Because any antipodal graph is harmonic (the converse is also true for partial cubes), Theorem 3.2 is a partial answer to this question, which improves a result [11, Theorem 4.1] of Klavžar and Kovše. Note that a bipartite graph of diameter 4 that is not a partial cube may be diametrical but not harmonic, as is shown by the graph in Figure 1.

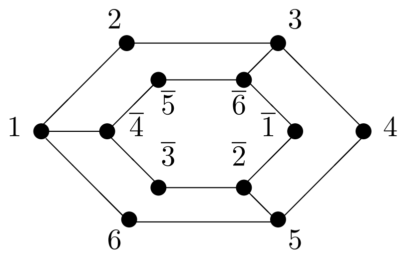

Figure 1. A diametrical bipartite graph that is not harmonic.

2. Preliminaries

The graphs we consider are undirected, without loops or multiple edges, and are finite and connected. For a set S of vertices of a graph G we denote by $G[S]$ the subgraph of G induced by S, and $G - S = G[V(G) - S]$. A *path* P with $V(P) = \{x_0, \ldots, x_n\}$, $x_i \neq x_j$ if $i \neq j$, and $E(P) = \{x_ix_{i+1} : 0 \leq i < n\}$ is denoted by $\langle x_0, \ldots, x_n\rangle$. If x and y are two vertices of a path P, then we denote by $P[x, y]$ the subpath of P whose endvertices are x and y. A *cycle* C with $V(C) = \{x_1, \ldots, x_n\}$, $x_i \neq x_j$ if $i \neq j$, and $E(C) = \{x_ix_{i+1} : 1 \leq i < n\} \cup \{x_nx_1\}$, is denoted by $\langle x_1, \ldots, x_n, x_1\rangle$.

The usual *distance* between two vertices x and y of a graph G, that is, the length of any (x, y)-*geodesic* (= shortest (x, y)-path) in G, is denoted by $d_G(x, y)$. A connected subgraph H of G is *isometric* in G if $d_H(x, y) = d_G(x, y)$ for all vertices x and y of H. The (*geodesic*) *interval* $I_G(x, y)$ between two vertices x and y of G consists of the vertices of all (x, y)-geodesics in G.

In the *geodesic convexity*, that is, the convexity on the vertex set of a graph G which is induced by the *geodesic interval operator* I_G, a subset C of $V(G)$ is convex provided that it contains the geodesic interval $I_G(x, y)$ for all $x, y \in C$. A subset H of $V(G)$ is a *half-space* if H and $V(G) \setminus H$ are convex.

For an edge ab of a graph G, let

$$W_{ab} = \{x \in V(G) : d_G(a, x) < d_G(b, x)\},$$

$$U_{ab} = \{x \in W_{ab} : x \text{ has a neighbor in } W_{ba}\}.$$

Note that the sets W_{ab} and W_{ba} are disjoint and that $V(G) = W_{ab} \cup W_{ab}$ if G is bipartite.

Two edges xy and uv are in the Djoković-Winkler relation Θ if

$$d_G(x, u) + d_G(y, v) \neq d_G(x, v) + d_G(y, u).$$

The relation Θ is clearly reflexive and symmetric.

Remark 2.1. If G is bipartite, then, by [9, Lemma 11.2], the notation can be chosen so that the edges xy and uv are in relation Θ if and only if

$$d_G(x, u) = d_G(y, v) = d_G(x, v) - 1 = d_G(y, u) - 1,$$

or equivalently if and only if

$$y \in I_G(x, v) \text{ and } x \in I_G(y, u).$$

From now on, we will always use this way of defining the relation Θ. Note that, in this way, the edges xy and yx are not in relation Θ because $y \notin I_G(x, x)$ and $x \notin I_G(y, y)$. In other word, each time the relation Θ is used, the notation of an edge induces an orientation of this edge.

We recall that, by Djoković [4, Theorem 1] and Winkler [20], *a connected bipartite graph G is a partial cube, that is, an isometric subgraph of some hypercube, if it has the following equivalent properties.*

(Conv.) *For every edge ab of G, the sets W_{ab} and W_{ba} are convex.*

(Trans.) *The relation Θ is transitive, and thus is an equivalence relation.*

It follows in particular that *the non-trivial* (i.e., distinct from $\emptyset$ and $V(G)$) *half-spaces of a partial cube G are the sets W_{ab}, $ab \in E(G)$.* In the following lemma we recall two well-known properties of partial cubes that we will need later.

Lemma 2.2. *Let G be a partial cube. We have the following properties.*

(i) *Let x, y be two vertices of G, P an (x, y)-geodesic and W an (x, y)-path of G. Then each edge of P is Θ-equivalent to some edge of W.*

(ii) *A path P in G is a geodesic if and only if any two distinct edges of P are not Θ-equivalent.*

3. Diametrical Versus Antipodal Partial Cubes

If A is a set of vertices of an antipodal graph G, we write

$$\overline{A} = \{\bar{x} : x \in A\}.$$

Note that, by (1), a graph G is antipodal if and only if

$$I_G(x, \overline{x}) = V(G) \quad \text{for all} \quad x \in V(G). \tag{2}$$

Clearly any antipodal graph is harmonic. The following result, which is an implicit consequence of two results [6, Proposition 4.1 and Theorem 4.2] of Fukuda and Handa (see also [18, Theorem 4.1] for a direct proof), shows that the converse is also true for partial cubes.

Proposition 3.1. *Any harmonic partial cube is antipodal.*

In this section we consider the question of Fukuda and Handa [6]. We recall that they asked whether any diametrical partial cube is harmonic, and thus antipodal by the proposition above. Klavžar and Kovše gave a first answer [11, Theorem 4.1] to this question by proving that *all diametrical partial cubes of isometric dimension at most* 6 *are harmonic.* Our main result is the following theorem.

Theorem 3.2. *Any diametrical partial cube of diameter at most* 6 *is antipodal.*

We need several lemmas to prove this theorem. Recall that the *isometric dimension* of a finite partial cube G, i.e.,the least non-negative integer n such that G is an isometric subgraph of an n-cube, coincides with the number of Θ-classes of $E(G)$. We denote it by $\mathrm{idim}(G)$. By Lemma 2.2 we clearly have

$$\mathrm{diam}(G) \leq \mathrm{idim}(G).$$

Lemma 3.3 (Polat [18, Lemma 3.2])**.** *Let G be a diametrical partial cube. Then G is antipodal if and only if* $\mathrm{diam}(G) = \mathrm{idim}(G)$.

Corollary 3.4. *Let d be a non-negative integer. If any diametrical partial cube whose diameter is at most d is antipodal, then any diametrical partial cube whose isometric dimension is at most $d+1$ is antipodal.*

Proof. Let G be a diametrical partial cube with $\mathrm{idim}(G) \leq d+1$. Then $\mathrm{diam}(G) \leq d+1$ because $\mathrm{diam}(G) \leq \mathrm{idim}(G)$. We are done if $\mathrm{diam}(G) \leq d$ by assumption. If $\mathrm{diam}(G) = d+1$, then we have $d+1 = \mathrm{diam}(G) \leq \mathrm{idim}(G) \leq d+1$, and thus $\mathrm{diam}(G) = \mathrm{idim}(G)$. Therefore G is antipodal by Lemma 3.3. ■

As an immediate consequence of Theorem 3.2 and Corollary 3.4, we have the following result.

Corollary 3.5. *Any diametrical partial cube whose isometric dimension is at most* 7 *is antipodal.*

This corollary is then an improvement of the above result of Klavžar and Kovše.

Lemma 3.6. *A diametrical partial cube G is antipodal if and only if, for any edge ab of G, the antipode of each vertex in W_{ab} (respectively, in W_{ba}) belongs to W_{ba} (respectively, W_{ab}), i.e., $\overline{W_{ab}} = W_{ba}$.*

Proof. G is antipodal if and only if $\mathrm{idim}(G) = \mathrm{diam}(G)$ by Lemma 3.3. Hence if and only if each edge of G is Θ-equivalent to exactly one edge of each $(x, \overline{x})$-geodesic for any $x \in V(G)$, and thus if and only if $\overline{W_{ab}} = W_{ba}$ for any edge ab of G. ■

We first give a consequence of this lemma. Recall that a graph G is said to be *self-centered* if any of its vertices is central, i.e., if $\mathrm{rad}(G) = \mathrm{diam}(G)$, or in other words, if each vertex of G has at least one absolute antipode. Diametrical graphs are special self-centered graphs.

Corollary 3.7. *A self-centered partial cube G is antipodal if and only if, for each edge ab of G, all absolute antipodes of any vertex in W_{ab} belong to W_{ba}.*

Proof. The necessity is a consequence of [18, Lemma 4.4] stating that *if G is an antipodal partial cube, then $\overline{W_{ab}} = W_{ba}$ for every edge ab of G.* Conversely, assume that, for each edge ab of G, all absolute antipodes of any vertex in W_{ab} belong to W_{ba}. By the fact that any partial cube has the Separation Property S_2, i.e., any two vertices can be separated by a half-space, it follows that, if some vertex x of G has two absolute antipodes y and z, then $y \in W_{ab}$ and $z \in W_{ba}$ for some $ab \in E(G)$, contrary to the assumption. Therefore G is diametrical, and thus antipodal by Lemma 3.6. ■

We sum up some of the results above as follows.

Theorem 3.8. *For a diametrical partial cube G, the following assertions are equivalent.*

(i) *G is antipodal.*

(ii) *G is harmonic.*

(iii) $\mathrm{diam}(G) = \mathrm{idim}(G)$.

(iv) $\overline{W_{ab}} = W_{ba}$ *for any edge ab of G.*

(v) *The antipodal map of G is an isomorphism of $G[W_{ab}]$ onto $G[W_{ba}]$ for any edge ab of G.*

Proof. It remains to prove the equivalence of (v) with the other assertions. We recall that the antipodal map of a diametrical graph is the function which maps every vertex of this graph to its antipode. If G satisfies (v), then the antipodal map α of G is obviously an automorphism of G, and thus G is harmonic. Conversely, assume that G is harmonic, and thus satisfies (iv). Then, on the one hand, α is an automorphism of G, and on the other hand, if ab is any edge of G, then $\alpha(x) \in W_{ba}$ for all $x \in W_{ab}$. Therefore α is an isomorphism of $G[W_{ab}]$ onto $G[W_{ba}]$. ■

From now on we use the following notation. If G is a partial cube, ab an edge of G, and n a positive integer, then

$$U^n_{ab} = \big\{x \in W_{ab} : d_G(x, W_{ba}) = n\big\}.$$

In particular $U^1_{ab} = U_{ab}$. The set U^n_{ba} is defined analogously.

We now state a lemma which is essential to prove the main result of this paper. Its proof, which is rather long, is the subject of next section.

Lemma 3.9. *Let G be a diametrical partial cube of diameter $d \geq 1$, and ab an edge of G. Then we have the following properties.*

(i) $\overline{u} \in W_{ba}$ *if* $u \in U^1_{ab} \cup U^2_{ab}$.

(ii) *If $d \geq 4$, then $U^{d-1}_{ab} = \emptyset$ (and a fortiori $U^d_{ab} = \emptyset$).*

(iii) *If $d \leq 6$, then $\overline{u} \in W_{ba}$ for every $u \in W_{ab}$.*

***Proof of Theorem* 3.2.** Let ab be an edge of some diametrical partial cube G of diameter $d \leq 6$. Then $\overline{W_{ab}} = W_{ba}$ by Lemma 3.9(iii). Therefore G is antipodal by Lemma 3.6. ■

4. Proof of Lemma 3.9

Recall that, in a diametrical partial cube G, the absolute antipode of any vertex of G is unique, and the degree of any vertex of G is at least 2 if G is distinct from K_1 and K_2 (see [8]). We will use these properties in several parts of this proof.

4.1. Proof of (i)

Let $u \in W_{ab}$. We denote by $\langle u_0, \ldots, u_n \rangle$ a geodesic with $u_0 \in U^1_{ba}$, $u_n = u \in U^n_{ab}$ and $u_i \in U^i_{ab}$ for $1 \leq i \leq n$.

(i.1) Suppose that $u \in U^1_{ab}$, i.e., $u = u_1$, and that $\overline{u_1} \in W_{ab}$. Because W_{ab} is a half-space, it follows that $u_1 \in I_G(u_0, \overline{u_1})$, and thus $d_G(u_0, \overline{u_1}) = d + 1$, contrary to the fact that $d = \mathrm{diam}(G)$. Hence $\overline{u_1} \in W_{ba}$.

(i.2) Suppose that $d \geq 2$ and $u \in U^2_{ab}$, i.e., $u = u_2$, and that $\overline{u_2} \in W_{ab}$. Because $d_G(u_1, \overline{u_2})$ cannot be equal to $d + 1$, it follows that $d_G(u_1, \overline{u_2}) = d - 1$. Hence, as above, $d_G(u_0, \overline{u_2}) = d_G(u_1, \overline{u_2}) + 1 = d$, contrary to the fact that $\overline{u_2}$ has exactly one absolute antipode, namely u_2. Therefore $\overline{u_2} \in W_{ba}$.

4.2. Proof of (ii)

Suppose that $U^{d-1}_{ab} \neq \emptyset$, and let $u \in U^{d-1}_{ab}$. Denote by $\langle u_0, \ldots, u_{d-1} \rangle$ a geodesic such that $u_0 \in U^1_{ba}$, $u_{d-1} = u$ and $u_i \in U^i_{ab}$ for $1 \leq i \leq d - 1$. By what we proved above $\overline{u_1} \in W_{ba}$. Then, because W_{ab} is a half-space, it follows that $u_0 \in I_G(u_1, \overline{u_1})$, and that $I_G(u_0, \overline{u_1}) \subseteq W_{ba}$. Let $\langle x_1, \ldots, x_d \rangle$ be a $(u_0, \overline{u_1})$-geodesic with $x_1 = u_0$ and $x_d = \overline{u_1}$. Then $d_G(u_{d-1}, x_2) = d$, i.e., $x_2 = \overline{u_{d-1}}$, since otherwise $d_G(u_{d-1}, x_2)$ would be equal to $d - 2$, contrary to the assumption that $d_G(u_{d-1}, W_{ba}) = d - 1$. It follows that $d_G(u_{d-1}, x_3) = d - 1$, and thus that $d_G(u_{d-1}, x_4) = d - 2$ by the uniqueness of the absolute antipode of each vertex in a diametrical graph, once again contrary to the assumption that $d_G(u_{d-1}, W_{ba}) = d - 1$. Consequently $U^{d-1}_{ab} = \emptyset$.

4.3. Proof of (iii) for $d \leq 5$

The result is clear by (i) and (ii) if $d \leq 4$. Assume that $d = 5$. Suppose that $\overline{u} \in W_{ab}$ for some $u \in W_{ab}$.

(4.3.1) $u, \overline{u} \in U^3_{ab}$ by (i) and (ii). Hence the distances from u and $\overline{u}$ to any element of U^1_{ab} are at least 2.

(4.3.2) $U^2_{ba} = \emptyset$. Suppose that $W_{ba} \neq U_{ba}$. Then there exists a vertex $x \in U^2_{ba}$. Clearly $d_G(u, x) < 5$ since $x \neq \overline{u}$ because $\overline{u} \in W_{ab}$ by hypothesis. Let P be a (u, x)-geodesic, and x_u the vertex of U_{ab} that lies on P. Because $d_G(x_u, x) \geq 2$ and $u \in U^3_{ab}$, it follows that $d_G(u, x_u) = 2$ and $d_G(x_u, x) = 2$, and thus $d_G(u, x) = 4$. Put $P = \langle x_0, x_1, x_2, x_3, x_4 \rangle$ with $x_0 = u$, $x_2 = x_u$, $x_3 \in U_{ba}$ and $x_4 = x$.

Likely we have $d_G(\overline{u}, x) = 4$. Because $d_G(\overline{u}, x_3)$ cannot be equal to 5, it follows that $d_G(\overline{u}, x_3) = 3$, and thus $d_G(\overline{u}, x_2) = 2$ since $x_2 \in I_G(v, x_3)$ for every $v \in W_{ab}$. Hence $d_G(u, \overline{u}) \leq d_G(u, x_2) + d_G(\overline{u}, x_2) = 4$, contrary to $d_G(u, \overline{u}) = 5$.

We infer that $W_{ba} = U_{ba}$.

(4.3.3) By (4.3.2), the map $v \to \overline{v}$, $v \in U_{ab}$, is a bijection from U_{ab} to U_{ba}, and thus $\overline{U_{ab}} = W_{ba}$. On the other hand, u has a neighbor $w \in U^2_{ab}$. It follows, by the uniqueness of the antipode, that $\overline{w}$ cannot belong to W_{ba}, contrary to (i).

Consequently $\overline{u} \in W_{ba}$.

4.4. Proof of (iii) for $d = 6$

Suppose that $\overline{u} \in W_{ab}$ for some $u \in W_{ab}$.

(4.4.1) $u, \overline{u} \in U^3_{ab} \cup U^4_{ab}$ by (i) and (ii). Hence the distances from u and $\overline{u}$ to any element of U^1_{ab} are at least 2.

(4.4.2) Suppose that $U^4_{ba} \neq \emptyset$, and let $x \in U^4_{ba}$ and $y \in U^3_{ab}$. Then $d_G(x, y) = 6$. Hence $U^4_{ba} = \{x\}$ and $U^3_{ab} = \{y\}$, and moreover $U^4_{ab} = \emptyset$ since otherwise $d_G(x, U^4_{ab})$ would be equal to 7 contrary to $\operatorname{diam}(G) = 6$. Therefore $\overline{W_{ab}} \subseteq W_{ba}$ by (i), and we are done.

Consequently, from now on we assume that $U^4_{ba} = \emptyset$.

(4.4.3) $u, \overline{u} \in U^3_{ab}$. Assume that $u \in U^4_{ab}$. Suppose that $U^2_{ba} \neq \emptyset$. Then $d_G(u, U^2_{ba}) = 5$ and $d_G(u, U^3_{ba}) = 6$. Hence $U^3_{ba} = \emptyset$ since $\overline{u} \in W_{ab}$ by hypothesis. Moreover two vertices $x, y \in U^2_{ba}$ cannot be adjacent, since otherwise we would have $d_G(u, x) = d_G(u, y) = 5$, contrary to the fact that G is a bipartite graph. It follows, because any vertex in a diametrical graph distinct from K_1 and K_2 has degree at least 2, that any vertex in U^2_{ba} must be adjacent to two vertices in U^1_{ba}.

Hence there exists a path $\langle a, b, c, b', a' \rangle$ such that $a, a' \in U^1_{ab}$, $b, b' \in U^1_{ba}$ and $c \in U^2_{ba}$, and such that $d_G(u, a) = d_G(u, a') = 3$, $d_G(u, b) = d_G(u, b') = 4$ and $d_G(u, c) = 5$. Because $d_G(b, b') = 2$, it follows that a and a' have a common neighbor d. Note that d and c are not adjacent since $c \in U^2_{ba}$, and thus the 6-cycle $\langle a, b, c, b', a', d, a \rangle$ is isometric in G. The distance between u and d is either 2 or 4.

Suppose that $d_G(u, d) = 2$. Because $d_G(u, \overline{u}) = 6$, it follows that $d_G(\overline{u}, d) = 4$ or 5.

Case 1. $d_G(\overline{u}, d) = 5$. Then $d_G(\overline{u}, a) = d_G(\overline{u}, a') = 4$, $d_G(\overline{u}, b) = d_G(\overline{u}, b') = 5$, and thus $d_G(\overline{u}, c) = 4$ because c is not the antipode of $\overline{u}$. Hence there exists a geodesic $\langle e, f, c \rangle$ such that $e \in U^1_{ab}$, $f \in U^1_{ba}$, $d_G(\overline{u}, e) = 2$ and thus $d_G(\overline{u}, f) = 3$. It follows that $d_G(u, f) = 4$ since $d_G(u, c) = 5$ and f is not the antipode of u, and thus $d_G(u, e) = 3$. Hence $d_G(u, \overline{u}) \leq d_G(u, e) + d_G(\overline{u}, e) = 5$, contrary to $d_G(u, \overline{u}) = 6$.

Case 2. $d_G(\overline{u}, d) = 4$. Then, clearly, $d_G(\overline{u}, x) = d_G(u, x)$ for all $x \in \{a, b, c, b', a'\}$.

Hence, without loss of generality, we can suppose that $d_G(u, d) = 4$, since otherwise we would use $\overline{u}$ instead of u in the following.

Let P be a (u, a)-geodesic and P' a (u, a')-geodesic. Then $Q = P \cup \langle a, d \rangle$ and $Q' = P' \cup \langle a', d \rangle$ are two (u, d)-geodesics. Hence, by Lemma 2.2(i), every edge of

Q' is Θ-equivalent to an edge of Q. In particular the edge da', which cannot be Θ-equivalent to the edge ad, is Θ-equivalent to some edge xy of P. It follows that bc, which Θ-equivalent to da' since the 6-cycle $\langle a, b, c, b', a', d, a\rangle$ is isometric in G, is also Θ-equivalent to the edge xy, but this is impossible, by Lemma 2.2(ii), because $P \cup \langle a, b, c\rangle$ is a geodesic.

We infer that $U^2_{ba} = \emptyset$, and thus that $W_{ba} = U^1_{ba}$. Consequently, the map $v \to \overline{v}$, $v \in U^1_{ab}$, is a bijection of U^1_{ab} onto U^1_{ba}, and thus $\overline{U^1_{ab}} = W_{ba}$ by (i). On the other hand, because $u \in U^4_{ab}$ by assumption, there exists a vertex $w \in U^2_{ab}$. It follows, by the uniqueness of the antipode, that $\overline{w}$ cannot belong to W_{ba}, contrary to (i).

Therefore the assumption is false, and thus $u \notin U^4_{ab}$, and similarly $\overline{u} \notin U^4_{ab}$.

(4.4.4) $d_G(u, U^3_{ba}) = d_G(\overline{u}, U^3_{ba}) = 5$. This is clear because $u, \overline{u} \in U^3_{ab}$ by (4.4.3).

(4.4.5) *Two vertices in U^3_{ba} cannot be adjacent.* Suppose that two vertices $x, y \in U^3_{ba}$ are adjacent. It follows, by (4.4.4), that $d_G(u, x) = d_G(u, y) = 5$, contrary to the fact that x and y are two adjacent vertices of the bipartite graph G.

(4.4.6) *Any element of U^3_{ba} is adjacent to at least two elements of U^2_{ba}.* This is a consequence of (4.4.2), (4.4.5) and of the fact that the degree of any vertex of a diametrical graph distinct from K_1 and K_2 is at least 2.

(4.4.7) $d_G(u, U^2_{ba}) = d_G(\overline{u}, U^2_{ba}) = 4$. $d_G(u, U^2_{ba}) \geq 4$ since $u \in U^3_{ab}$. Suppose that there exists a vertex $v \in U^2_{ba}$ such that $d_G(u, v) = 5$. We have two cases depending on whether v has one or more than one neighbor in U^1_{ba}. Note that v has no neighbor $w \in U^3_{ab}$, since otherwise $d_G(u, w)$ would be equal to 4, contrary to (4.4.4).

Case 1. v has at least two neighbors in U^1_{ba}. Because $d_G(u, U_{ab} \geq 2$ and $d_G(u, v) = 5$, there exists a path $\langle a, b, v, b', a'\rangle$ such that $a, a' \in U^1_{ab}$, $b, b' \in U^1_{ba}$, $d_G(u, a) = d_G(u, a') = 3$ and $d_G(u, b) = d_G(u, b') = 4$. Then b and b' cannot be adjacent since G is bipartite, and thus the path $\langle b, v, b'\rangle$ is a geodesic. Hence $d_G(a, a') = d_G(b, b') = 2$. Let c be a common neighbor of a and a'.This vertex c belongs to W_{ab} since this set is convex. Moreover $c \neq u$ because $u \in U^3_{ab}$. Then $d_G(u, c) = 2$ or 4. If $d_G(u, c) = 2$, then $d_G(\overline{u}, c) = 4$ or 5 because $d_G(u, \overline{u}) = 6$. We then have three subcases.

Subcase 1.1. $d_G(u, c) = 2$ and $d_G(\overline{u}, c) = 5$. Then $d_G(\overline{u}, a) = d_G(\overline{u}, a') = 4$, $d_G(\overline{u}, b) = d_G(\overline{u}, b') = 5$, and thus $d_G(\overline{u}, v) = 4$ because $\overline{u}$ has a unique absolute antipode u. It follows that there is a path $\langle v, d, e\rangle$ such that $d \in U^1_{ba}$ and $e \in U^1_{ab}$, and with $d_G(\overline{u}, d) = 3$ and $d_G(\overline{u}, e) = 2$. Because $d_G(u, v) = 5$ by hypothesis, this implies that $d_G(u, d) = 4$ and $d_G(u, e) = 3$, which is impossible since $d_G(\overline{u}, e) = 2$ and $d_G(u, \overline{u}) = 6$.

Subcase 1.2. $d_G(u, c) = 2$ and $d_G(\overline{u}, c) = 4$. It follows that the distances from a, b, v, b', a' to $\overline{u}$ are the same that the distances from these vertices to u. Denote by P and P' a $(\overline{u}, a)$-geodesic and a $(\overline{u}, a')$-geodesic, respectively. Then $P \cup \langle a, c\rangle$ and $P' \cup \langle a', c\rangle$ are two $(\overline{u}, c)$-geodesics. By Lemma 2.2(i) and since $\langle a, c, a'\rangle$ is a geodesic, it follows that the edge ca' is Θ-equivalent to some edge xy of P. On the other hand ca' is also Θ-equivalent to bv. Hence bv is Θ-equivalent to xy, contrary, by Lemma 2.2(ii), to the fact that $P \cup \langle a, b, v\rangle$ is a geodesic.

Subcase 1.3. $d_G(u, c) = 4$. The proof is the same as in Subcase 1.2 by replacing $\overline{u}$ by u.

We infer that v cannot have two neighbors in U^1_{ba}.

Case 2. v has exactly one neighbor in U^1_{ba}. Because v cannot have a neighbor in U^3_{ba} as we saw above, it must have a neighbor w in U^2_{ba}. Then there is a path $\langle a, b, v, w, b', a'\rangle$ such that $a, a' \in U^1_{ab}$, $b, b' \in U^1_{ba}$, $d_G(u, a) = 3$, $d_G(u, b) = 4$, $d_G(u, v) = 5$, $d_G(u, w) = 4$, $d_G(u, b') = 3$ and $d_G(u, a') = 2$. Obviously $d_G(a, a') = d_G(b, b') = 1$ or 3. Moreover, because $d_G(u, \overline{u}) = 6$, we must have $d_G(\overline{u}, a') = 4$, $d_G(\overline{u}, b') = 5$, $d_G(\overline{u}, w) = 4$, $d_G(\overline{u}, v) = 5$, $d_G(\overline{u}, b) = 4$ and $d_G(\overline{u}, a) = 3$. We distinguish two subcases.

Subcase 2.1. $d_G(a, a') = d_G(b, b') = 3$. Let $\langle a, c, d, a'\rangle$ be an (a, a')-geodesic. We have two cases.

Subsubcase 2.1.1. u and d are not adjacent. Then $d_G(u, d) = 3$, whereas $d_G(u, c)$ may be 2 or 4. Denote by P and P' a (u, a)-geodesic and a (u, a')-geodesic, respectively. The path $P' \cup \langle a', d\rangle$ is a geodesic. According to whether $d_G(u, c)$ is 2 or 4, i.e., according to whether c lies or does not lie on P, $P[u, c] \cup \langle c, d\rangle$ is a geodesic, or $P \cup \langle a, c\rangle$ and $P' \cup \langle a', d, c\rangle$ are geodesics. In both cases, by Lemma 2.2(i) and since $\langle a, c, d, a'\rangle$ is a geodesic, it follows that the edge da' is Θ-equivalent to some edge xy of P distinct from ac. On the other hand da' is also Θ-equivalent to bv, because the cycle $\langle a, b, v, w, b', a', d, c, a\rangle$ is clearly isometric in G. Hence bv is Θ-equivalent to xy, contrary, by Lemma 2.2(ii), to the fact that $P \cup \langle a, b, v\rangle$ is a geodesic. Consequently u and d are adjacent.

Subsubcase 2.1.2. u and d are adjacent. Because $d_G(u, d) = 1$, and since $d_G(u, \overline{u}) = 6$, it follows that $d_G(\overline{u}, d) = 5$, and thus $d_G(\overline{u}, c) = 4$ since $d_G(\overline{u}, a) = 3$. Denote by P and P' a $(\overline{u}, a)$-geodesic and a $(\overline{u}, a')$-geodesic, respectively. Then $P \cup \langle a, c, d\rangle$ and $P' \cup \langle a', d\rangle$ are geodesics. As above, the edge da', which Θ-equivalent to bv, is Θ-equivalent to some edge xy of P. Hence bv is Θ-equivalent to xy, contrary to the fact that $P \cup \langle a, b, v\rangle$ is a geodesic. Consequently, u and d cannot be adjacent.

Subcase 2.2. $d_G(a, a') = d_G(b, b') = 1$. Because $d_G(\overline{u}, b') = d_G(\overline{u}, v) = 5$ and $d_G(\overline{u}, w) = 4$ as we saw above, it follows that there exists a geodesic $\langle a'', b'', w\rangle$ with $a'' \in U^1_{ab}$, $b'' \in U^1_{ba}$, $d_G(\overline{u}, b'') = 3$ and $d_G(\overline{u}, a'') = 2$. Moreover, a' and a'' have a common neighbor e since $d_G(a', a'') = d_G(b', b'') = 2$. Then $d_G(\overline{u}, e) = 3$ since $d_G(\overline{u}, a') = 4$ and $d_G(\overline{u}, a'') = 2$.

Suppose that a and a'' are adjacent, and then that so are b and b''. Then the edge $a''e$ is Θ-equivalent to both the edges aa' and wb'. Because aa' is Θ-equivalent to bb', it follows that wb' and bb' are also Θ-equivalent, contrary to the fact that these edges are adjacent. Therefore a and a'' are not adjacent.

Let P and P' be a $(\overline{u}, a)$-geodesic and a $(\overline{u}, a'')$-geodesic, respectively. Then $P \cup \langle a, a' \rangle$ and $P' \cup \langle a'', e, a' \rangle$ are $(\overline{u}, a')$-geodesics. Because $\langle a, a', e, a'' \rangle$ is a geodesic since a and a'' are not adjacent, it follows, by Lemma 2.2(i), that $a''e$ is Θ-equivalent to some edge xy of P. On the other hand, $a''e$ is Θ-equivalent to wb', and thus to vb. Hence vb is also Θ-equivalent to xy, contrary, by Lemma 2.2(ii), to the fact that $P \cup \langle a, b, v \rangle$ is a geodesic.

We infer that there exists no vertex $v \in U^2_{ba}$ such that $d_G(u, v) = 5$, and analogously such that $d_G(\overline{u}, v) = 5$.

(4.4.8) *Two vertices in U^2_{ba} cannot be adjacent.* By (4.4.7), two adjacent vertices of U^2_{ba} should be at distance 4 from u. This is impossible because G is bipartite.

(4.4.9) *Two elements of U^1_{ba} can have at most one common neighbor in U^2_{ba}.* Suppose that two vertices $y, y' \in U^1_{ba}$ have two common neighbors $z, z' \in U^2_{ba}$. Denote by x and x' the neighbors in U^1_{ab} of y and y', respectively. Then $d_G(x, x') = d_G(y, y') = 2$. Denote by x'' a common neighbors of x and x'. Then x'' is not adjacent to z and z' because $z, z' \in U^2_{ba}$. Hence the edge $x''x'$ is Θ-equivalent to both yz and yz'. It follows that yz and yz' are also Θ-equivalent, which is impossible since they are adjacent.

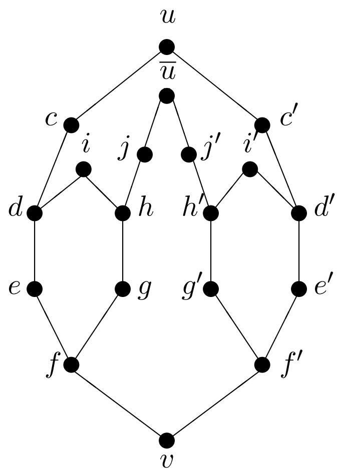

Figure 2. The graph F.

(4.4.10) $U^3_{ba} = \emptyset$. Suppose that there exists a vertex $v \in U^3_{ba}$. By (4.4.4), $d_G(u, v) = d_G(\overline{u}, v) = 5$, and by (4.4.6), v has at least two neighbors $f, f' \in U^2_{ba}$. Then the distances of f and f' to u and $\overline{u}$ are equal to 4, and thus f and f' are linked to u and $\overline{u}$ by geodesics of length 4. Hence there exists a subgraph F of G (see Figure 2) where the following properties are satisfied.

- $u, \overline{u} \in U^3_{ab}$, $c, c', j, j' \in U^2_{ab}$, $i, i' \in U^1_{ab} \cup U^2_{ab}$, $d, d', h, h' \in U^1_{ab}$, $e, e', g, g' \in U^1_{ba}$, $f, f' \in U^2_{ba}$, $v \in U^3_{ba}$;
- all paths below are geodesics:

$$\begin{aligned}
&P = \langle u, c, d, i, h, j, \overline{u}\rangle \text{ and } P' = \langle u, c', d', i', h', j', \overline{u}\rangle,\\
&Q = \langle u, c, d, e, f, v\rangle \text{ and } Q' = \langle u', c', d', e', f', v\rangle,\\
&R = \langle \overline{u}, j, h, g, f, v\rangle \text{ and } R' = \langle \overline{u}, j', h', g', f', v\rangle,\\
&L = \langle u, c, d, e, f, g\rangle \text{ and } L' = \langle u, c', d', e', f', g'\rangle,\\
&M = \langle \overline{u}, j, h, g, f, e\rangle \text{ and } M' = \langle \overline{u}, j', h', g', f', e'\rangle;
\end{aligned}$$

- several of the vertices of F may coincide according to the possible cases that are listed below:
  (0) no vertices coincide and the cycles $C_u = \langle u, c, d, e, f, v, f', e', d', c', u\rangle$ and $C_{\overline{u}} = \langle \overline{u}, j, h, g, f, v, f', g', h', j', \overline{u}\rangle$ are isometric in G;
  (1) $c = c'$;
  (2) $j = j'$;
  (3) $c = c'$, $j = j'$;
  (4) $j = j'$, $h = h'$, $g = g'$ and the cycle C_u is isometric in G;
  (5) $c = c'$, $d = d'$, $e = e'$ and the cycle $C_{\overline{u}}$ is isometric in G;
  (6) $c = c'$, $j = j'$, $h = h'$ and $g = g'$;
  (7) $j = j'$, $c = c'$, $d = d'$ and $e = e'$;
  (8) $c = c'$, $j = j'$, $d = d'$, $e = e'$, $h = h'$ and $g = g'$.

The distances between u and $\overline{u}$ and the other vertices of this graph are then perfectly defined by the geodesics above. Also note that F is not isometric in G, for example the distances between c and j', c' and j, d and h' and d' and h cannot be equal to 6, because the antipode of each of these vertices belongs to W_{ba} by (i).

We easily see that if the cycle C_u is not isometric in G, then we have a subgraph of F that is isomorphic to the graph F that fulfills the conditions of case (1), (5), (6), (7) or (8); and likely if the cycles $C_{\overline{u}}$ is not isometric in G.

We now show that G cannot contain such a subgraph F. By the remark above, it suffices to consider each case.

(0) Because P and P' are $(\overline{u}, u)$-geodesics, it follows, by Lemma 2.2(i), that the edge uc' is Θ-equivalent to some edge of P, and more precisely because C_u is an isometric cycle of G, to some edge xy of $P[d, \overline{u}]$. On the other hand, uc' is Θ-equivalent to fv, since C_u is isometric in G.

If xy is di or ih, then, by the transitivity of the relation Θ, and because the 6-cycle $\langle d, e, f, g, h, i\rangle$ is clearly isometric in G, uc' is Θ-equivalent to the edges ef or fg, but none of these edges is Θ-equivalent to fv since fv is adjacent to both of them, contrary to the transitivity of Θ.

If xy is hj or $j\overline{u}$, then, by Lemma 2.2(ii), fv cannot be Θ-equivalent to xy since these edges are edges of the geodesic R, contrary to the transitivity of Θ.

We infer that case (0) is impossible.

(1) $c = c'$. Then $d_G(e, e') = d_G(d, d') = 2$. Let w be a common neighbor of e and e'. We will show below that w is adjacent to v, but it cannot be adjacent to c because $c \in U^2_{ab}$ and $w \in U^1_{ba} \cup U^2_{ba}$. It follows that the cycle $\langle c, d, e, w, e', d', c\rangle$ is isometric in G. Hence cd' is Θ-equivalent to ew. Moreover cd' is also Θ-equivalent to fv if $d_G(d', f) = 4$, or to ef if $d_G(d', f) = 2$. Because ew is not Θ-equivalent to ef since these edges are adjacent, it follows that cd' is Θ-equivalent to fv, which entails that w and v are adjacent.

On the other hand, because $P[c, \overline{u}]$ and $P'[c, \overline{u}]$ are $(c, \overline{u})$-geodesics, it follows, by Lemma 2.2(i), that cd' is Θ-equivalent to some edge xy of $P[d, \overline{u}]$. Hence, if $xy = di$ or ih, then xy is Θ-equivalent to fg or ef, respectively, and thus cannot be Θ-equivalent to fv since these edges are adjacent. If $xy = hj$ or $j\overline{u}$, then, by Lemma 2.2(ii), xy cannot be Θ-equivalent to fv because these edges are edges of the geodesic R.

Therefore case (1) is also impossible.

(2) $j = j'$. This case is analogous to case (1), and thus is impossible.

(3) $c = c'$ and $j = j'$. This case is a combination of cases (1) and (2), and thus we can easily show that it is impossible.

(4) $j = j'$, $h = h'$, $g = g'$ and C_u is isometric in G. Following what we did in case (0), we can show that uc' is Θ-equivalent to di or ih. On the other hand, uc' is Θ-equivalent to fv since C_u is isometric in G. Hence, if $xy = di$ or ih, then xy is Θ-equivalent to fg or ef, respectively, and thus cannot be Θ-equivalent to fv since these edges are adjacent. Therefore case (4) is also impossible.

(5) $c = c'$, $d = d'$, $e = e'$ and $C_{\overline{u}}$ is isometric in G. This case is analogous to case (4), and thus is impossible.

(6) $c = c'$, $j = j'$, $h = h'$ and $g = g'$. This case is a combination of cases (1) and (4), and thus we can easily show that it is impossible.

(7) $j = j'$, $c = c'$, $d = d'$ and $e = e'$. This case is a combination of cases (2) and (5), and it is analogous to case (6), and thus is impossible.

(8) $c = c'$, $j = j'$, $d = d'$, $e = e'$, $h = h'$ and $g = g'$. Then di is Θ-equivalent to both fg and $f'g$, but these two edges cannot be Θ-equivalent since they are adjacent. Hence case (6) is impossible.

Therefore we can infer that $U^3_{ba} = \emptyset$.

(4.4.11) *Any element of U^2_{ba} is adjacent to at least two elements of U^1_{ba}.* This is a consequence of (4.4.8), (4.4.10) and of the fact that the degree of any vertex of a diametrical graph of diameter greater than 1 is at least 2.

(4.4.12) *If a vertex $x \in U^2_{ba}$ has exactly two neighbors $y, z \in U^1_{ba}$, then x is the only common neighbor of y and z.* Suppose that y and z have another common neighbor v. Then $d_G(\overline{v}, y) = d_G(\overline{v}, z) = 5$, and thus $d_G(\overline{v}, x) = 6$ since, by (4.4.10), y and z are the only neighbors of x. This is impossible because $x \neq v$ by hypothesis.

Note that, by this property, the neighbors in U^1_{ab} of y and z have a common neighbor in U^2_{ab}.

(4.4.13) *If two vertices $x, y \in U^1_{ba}$ have a common neighbor $v \in U^2_{ba}$, then the neighbors of x and y in U^1_{ab} have exactly one common neighbor.*

Let x' and y' be the neighbors in U^1_{ab} of x and y, respectively. It follows that $d_G(x', y') = 2$, and thus x' and y' have at least a common neighbor. Suppose that x' and y' have two common neighbors w and w'. Then vy is Θ-equivalent to both $x'w$ and $x'w'$, but $x'w$ and $x'w'$ are not Θ-equivalent since they are adjacent, contrary to the transitivity of the relation Θ.

(4.4.14) *Forbidden isometric subgraphs of partial cubes.* We present three bipartite graphs that are not partial cubes, and thus that cannot be isometric subgraphs of a partial cube. It is well-known that $K_{2,3}$ is not a partial cube. More generally, if $K_{2,3} = \langle a, c, b\rangle \cup \langle a, d, b\rangle \cup \langle a, e, b\rangle$, then the graph $L_1 = \langle a, c_a, c, c_b, b\rangle \cup \langle a, d, b\rangle \cup \langle a, e, b\rangle$ obtained by subdividing the edges ca and cb of $K_{2,3}$, is also an expansion of this graph (see [2] or [18]), and thus is not a partial cube. Likely the graph $L_2 = \langle a, c_a, c, c_b, b\rangle \cup \langle a, d_a, d, d_b, b\rangle \cup \langle a, e, b\rangle$ obtained by subdividing the edges da and db of L_1, is also an expansion of this graph, and thus is not a partial cube.

It follows that $K_{2,3}, L_1$ and L_2 cannot be isometric subgraphs of a partial cube. We use these properties later.

(4.4.15) *For any $v \in U^2_{ba}$, there exists some $w \in U^2_{ab}$ that is adjacent to two vertices in U^1_{ab} whose neighbors in U^1_{ba} are adjacent to v.*

Suppose that there is a vertex $v \in U^2_{ba}$ that does not satisfy the statement above. Let $b, d \in U^1_{ba}$ be two of its neighbors. Then the neighbors a and c in U^1_{ab} of b and d, respectively, have a common neighbor e. By the above hypothesis, $e \notin U^2_{ab}$, and thus $e \in U^1_{ab}$, and its neighbor f in U^1_{ba} is adjacent to both b and d. Because $d_G(u, v) = 4$ by (4.4.7), we can assume without loss of generality that $d_G(u, a) = 2$. Then $d_G(u, c) \neq 2$, since otherwise we would have some subgraph isomorphic to the graph L_2, which is impossible by (4.4.14).

Clearly $d_G(\overline{f}, b) = d_G(\overline{f}, d) = 5$ since $d_G(\overline{f}, f) = 6$. Because $d_G(\overline{f}, v) \neq 6$ by the uniqueness of the antipode, it follows that v must be adjacent to another vertex $h \in U^1_{ba}$ that is not adjacent to f. Let g be the neighbor of h in U^1_{ab}.

Then $d_G(g,a) = d_G(g,c) = 2$, since h is not adjacent to b and to d by (4.4.8). Let i and k be the common neighbors of g and a, and of g and c, respectively. Note that i and k may coincide, but must be distinct from e since otherwise h would be adjacent to f, contrary to the condition above. By the hypothesis above, i and k cannot belong to U^2_{ab}, and thus $i, k \in U^1_{ab}$. Denote by j and l the neighbors in U^1_{ba} of i and k, respectively. We distinguish two cases.

Case 1. $i \neq k$. As $d_G(u,c)$, $d_G(u,g)$ is not equal to 2, and thus $d_G(u,c) = d_G(u,g) = 4$. Then, because $d_G(u,l)$ cannot be equal to 6, it follows that we must have $d_G(u,l) = 4$, and thus $d_G(u,k) = 3$. Let $P_a = \langle a,w,u\rangle$ and P_k be some (u,a)-geodesic and (u,k)-geodesic, respectively. By Lemma 2.2(i) and because $\langle a,e,c,k\rangle$ is a geodesic since $i \neq k$ by hypothesis, the edge ck is Θ-equivalent to some edge xy of P_a. On the other hand ck is also Θ-equivalent to dl, and thus to vh. Hence vh and xy are Θ-equivalent. Therefore $d_G(h,y) = d_G(v,x)$, and thus $d_G(g,y) = d_G(v,x) - 1$. If $xy = aw$, then $d_G(g,w) = 1$, and thus $d_G(g,u) = 2$, contrary to $d_G(g,u) = 4$ as we saw above. If $xy = wu$, then once again $d_G(g,u) = 2$ and not 4.

Case 2. $i = k$, and thus $j = l$. Then the graph $\langle v,b,j\rangle \cup \langle v,h,j\rangle \cup \langle v,d,j\rangle$ is an induced subgraph of G that is isomorphic to $K_{2,3}$, which is impossible by (4.4.14).

Consequently the vertex v must satisfy (4.4.15).

From (4.4.9), (4.4.13) and (4.4.15) we infer the following result.

(4.4.16) *Any $v \in U^2_{ba}$ is the unique common neighbors of at least two of its neighbors $x, y \in U^1_{ba}$. Moreover the neighbors of x and y in U^1_{ab} have exactly one common neighbor in U^2_{ab}.*

(4.4.17) Let Φ be a map of $W_{ba} = U^1_{ba} \cup U^2_{ba}$ to W_{ab} defined as follows.

1. If $x \in U^1_{ba}$, then $\Phi(x)$ is the unique neighbor of x in U^1_{ab}.
2. If $x \in U^2_{ba}$, then, by (4.4.15), there exists at least a $y \in U^2_{ab}$ that is adjacent to two vertices in U^1_{ab} whose neighbors in U^1_{ba} are adjacent to x. Put $\Phi(x) = y$.

Such a map is not unique, because if a vertex $x \in U^2_{ba}$ has exactly two neighbors in U^1_{ba}, then the choice of $\Phi(x)$ is unique, but this is generally not the case if x has more than two neighbors in U^1_{ba}.

(4.4.18) *The map Φ is injective.* Suppose that two vertices $v_1, v_2 \in U^2_{ba}$ have the same image w by Φ. Then there exist four geodesics $\langle w,a,b,v_1\rangle$, $\langle w,a',b',v_1\rangle$, $\langle w,c,d,v_2\rangle$, $\langle w,c',d',v_2\rangle$, with $a,a',c,c' \in U^1_{ab}$ and $b,b',d,d' \in U^1_{ba}$. Note that one of the neighbors b,b' of v_1 may coincide with one of the neighbors d,d' of v_2, for example we may have $b' = d'$, and thus $a' = c'$. By (4.4.7), $d_G(u,v_1) = d_G(u,v_2) = 4$. Hence the distances from u to b,b',d and d' may be 3 or 5.

Suppose that $d_G(u,b) = d_G(u,b') = 5$, and thus that $d_G(u,a) = d_G(u,a') = 4$. Then $d_G(u,w) = 3$ or 5. Because $d_G(u,v_1) = 4$, it follows that there exists a geodesic $\langle g, h, v_1\rangle$ such that $g \in U^1_{ab}$, $h \in U^1_{ba}$ and $d_G(u, g) = 2$. Then $g \neq a$ and $h \neq b$. It follows that $d_G(g, a) = d_G(g, a') = 2$. Let i be a common neighbor of g and a, and i' a common neighbor of g and a'. We can assume that, if g and w are adjacent, then $i = i' = w$. We distinguish two cases.

Case 1. $i = i'$. Then both the edges v_1b and v_1b' are Θ-equivalent to the edge gi, but themselves cannot be Θ-equivalent since they are adjacent, contrary to the transitivity of the relation Θ.

Case 2. $i \neq i'$. By our assumption, g and w are not adjacent. Then the graph $\langle a, w, a'\rangle \cup \langle a, i, g, i', a'\rangle \cup \langle a, b, v_1, b', a'\rangle$ is an isometric subgraph of G that is isomorphic to the graph L_1, which is impossible by (4.4.14).

Therefore $d_G(u, a) = 2$. Likely, at least $d_G(u, c)$ or $d_G(u, c')$ is equal to 2, say $d_G(u, c) = 2$. Moreover $d_G(u, w)$ is then equal to 1 or 3.

Suppose that $d_G(u, w) = 3$. Because $d_G(b, d) = d_G(a, c) = 2$, b and d have a common neighbor j. By (4.4.7), $d_G(u, j) = 4$ if $j \in U^2_{ba}$. If $j \in U^1_{ba}$, and if k is the neighbor of j in U^1_{ab}, then k is a common neighbor of a and c, and thus $d_G(u, k) = 3$ because $d_G(u, a) = d_G(u, c) = 2$, and since it cannot be 1, because $u \in U^3_{ab}$. Hence $d_G(u, j) = 4$, as in the case above.

Let P_a and P_c be some (u, a)-geodesic and (u, c)-geodesic, respectively. $P_a \cup \langle a, w\rangle$ and $P_c \cup \langle c, w\rangle$ are geodesics. Hence, by Lemma 2.2(i) and because the edges wc and aw are not Θ-equivalent, the edge wc is Θ-equivalent to some edge xy of P_a. On the other hand, the 6-cycle $\langle a, b, j, d, c, w, a\rangle$ is isometric in G, because w and j are not adjacent since $w \in U^2_{ab}$. It follows that wc is Θ-equivalent to bj, but, by Lemma 2.2(ii), bj and xy cannot be Θ-equivalent since they are edges of the geodesic $P_a \cup \langle a, b, j\rangle$, contrary to the transitivity of the relation Θ.

Therefore, $d_G(u, w) \neq 3$, and thus $d_G(u, w) = 1$, and likely $d_G(\overline{u}, w) = 1$. Hence $d_G(u, \overline{u}) = 2$, contrary to $d_G(u, \overline{u}) = 6$.

Consequently $\Phi(v_1) \neq \Phi(v_2)$, and thus the map Φ is injective.

(4.4.19) *The map Φ is not surjective.* This is clearly the case, by the definition of Φ, if some vertex in U^2_{ab} has only one neighbor in U^1_{ab}. Suppose that all vertices in U^2_{ab} has at least two neighbors in U^1_{ab}, and suppose that Φ is surjective.

u has a neighbor w in U^2_{ab} since $u \in U^3_{ab}$. Because Φ is surjective by hypothesis, w has two neighbors $a, c \in U^1_{ab}$ such that, if b and d are the neighbors in U^1_{ba} of a and c, respectively, then b and d must be adjacent to some $v \in U^2_{ba}$ so that $\Phi(v) = w$.

On the other hand, because $d_G(u, \overline{u}) = 6$ and $d_G(u, w) = 1$, we must have $d_G(\overline{u}, w) = 5$, $d_G(\overline{u}, a) = d_G(\overline{u}, c) = 4$, and thus $d_G(\overline{u}, b) = d_G(\overline{u}, d) = 5$. Moreover $d_G(\overline{u}, v) = 4$ by (4.4.7). It follows that there exist a geodesic $\langle e, f, v\rangle$ such that $e \in U^1_{ab}$, $f \in U^1_{ba}$ and $d_G(\overline{u}, e) = 2$. Then $e \neq c$ and $f \neq d$. It follows that $d_G(e, a) = d_G(e, c) = 2$. Let i be a common neighbor of e and a, and j a common neighbor of e and c. We distinguish two cases.

Case 1. $i = j$. Then both the edges aw and ai are Θ-equivalent to the edge vd, but they cannot be Θ-equivalent since they are adjacent.

Case 2. $i \neq j$. Then the subgraph $\langle c, j, e\rangle \cup \langle c, w, a, i, e\rangle \cup \langle c, d, v, f, e\rangle$ is an isometric subgraph of G which is isomorphic to the graph L_2, contrary to (4.4.14).

As we obtain a contradiction in both cases, we deduce that the map Φ is not surjective.

(4.4.20) Because the map $\Phi : W_{ba} \to U^1_{ab} \cup U^2_{ab}$ is injective but not surjective, it follows that the antipode of some element of $U^1_{ab} \cup U^2_{ab}$ does not belong to W_{ba}, contrary to condition (i) of Lemma 3.9.

Therefore the hypothesis we made at the beginning of the proof, i.e., that $\overline{u}$ does not belong to W_{ba}, is false, and thus $\overline{u} \in W_{ba}$. This completes the proof of the lemma.

5. Concluding Remarks

The process used in the proof of Lemma 3.9 seems difficult to extend to a diametrical partial cube of diameter greater than 6. However such an extension may not be impossible, at least for partial cubes of very small diameter. Note that the length of the proofs between the cases $d < 5$, $d = 5$ and $d = 6$ increases considerably, and this also seems true between $d = 6$ and $d = 7$, provided that any diametrical partial cube of diameter 7 is antipodal, which we still do not know. By analogy with other results with similar increase of difficulty, this may suggest the existence of a counterexample, that is, of a non-antipodal diametrical partial cube of diameter $d \geq 7$.

On the other hand, note that the self-centered (actually diametrical) partial cubes that can be obtained from K_1 by a sequence of the so-called diametrical expansions (see [1, Section 3]) are the antipodal partial cubes by [18, Theorem 4.9], since the diametrical expansions of [1] correspond to the antipodality-respectful expansions of [18]. Hence we do not obtain a non-antipodal diametrical partial cube by this process.

These two remarks bring us to ask the following question.

Question 5.1. *Does there exists a non-antipodal diametrical* (*or even a non-antipodal self-centered*) *partial cube*?

We state a metaresult concerning the possible existence of non-antipodal diametrical partial cubes. First recall that Göbel and Veldman [8, Proposition 19] (see also Kotzig and Laufer [14, Theorem 2] for antipodal graphs only) proved that *the Cartesian product $G \Box H$ of two graphs G and H is diametrical* (*respectively, antipodal*) *if and only if both G and H are diametrical* (*respectively, antipodal*).

Because K_2 is antipodal, we infer that *the prism* $G \Box K_2$ *over a diametrical graph* G *is a diametrical graph that is antipodal if and only if so is* G. We then have the following consequence.

Proposition 5.2. *If there exists a non-antipodal diametrical partial cube of diameter* d *for some* $d \geq 7$*, then there exists a non-antipodal diametrical partial cube of diameter* d' *for every* $d' \geq d$.

We complete this section with a side result. Recall that a graph G is said to be *distance-balanced* if $|W_{ab}| = |W_{ba}|$ for every edge ab of G. Handa [10], who introduced this concept, observed that any harmonic graph, and thus any antipodal graph by Proposition 3.1, is distance-balanced, but that there exist distance-balanced partial cubes that are not diametrical. We have the following result.

Proposition 5.3. *Let* G *be diametrical partial cube that is distance-balanced. Then* G *is antipodal if and only if* $\overline{W_{ab}} \subseteq W_{ba}$ *or* $\overline{W_{ba}} \subseteq W_{ab}$ *for any edge* ab *of* G.

Proof. The necessity is clear by Lemma 3.6. Conversely, let G be a diametrical partial cube that is distance-balanced, and that satisfies the property of the statement, and let $ab \in E(G)$. Suppose, without loss of generality, that $\overline{W_{ab}} \subseteq W_{ba}$. Then $\overline{W_{ab}} = W_{ba}$ by the uniqueness of the absolute antipode in a diametrical graph and the fact that G is distance-balanced. Therefore G is antipodal by Lemma 3.6. ■

By analogy to Question 5.1 we have the following question.

Question 5.4. *Does there exists a distance-balanced diametrical partial cube that is not antipodal?*

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