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MINIMALLY STRONG SUBGRAPH (k, ℓ)-ARC-CONNECTED DIGRAPHS

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Abstract

Let $D = (V, A)$ be a digraph of order n, S a subset of V of size k and $2 \leq k \leq n$. A subdigraph H of D is called an S-strong subgraph if H is strong and $S \subseteq V(H)$. Two S-strong subgraphs D_1 and D_2 are said to be arc-disjoint if $A(D_1) \cap A(D_2) = \emptyset$. Let $\lambda_S(D)$ be the maximum number of arc-disjoint S-strong digraphs in D. The strong subgraph k-arc-connectivity is defined as $\lambda_k(D) = \min\{\lambda_S(D) \mid S \subseteq V, |S| = k\}$. A digraph $D = (V, A)$ is called minimally strong subgraph (k, ℓ)-arc-connected if $\lambda_k(D) \geq \ell$ but for any arc $e \in A$, $\lambda_k(D - e) \leq \ell - 1$. Let $\mathfrak{G}(n, k, \ell)$ be the set of all minimally strong subgraph (k, ℓ)-arc-connected digraphs with order n. We define $G(n, k, \ell) = \max\{|A(D)| \mid D \in \mathfrak{G}(n, k, \ell)\}$ and $g(n, k, \ell) = \min\{|A(D)| \mid D \in \mathfrak{G}(n, k, \ell)\}$.

In this paper, we study the minimally strong subgraph (k, ℓ)-arc-connected digraphs. We give a characterization of the minimally strong subgraph $(3, n-2)$-arc-connected digraphs, and then give exact values and bounds for the functions $g(n, k, \ell)$ and $G(n, k, \ell)$.

Keywords: strong subgraph k-connectivity, strong subgraph k-arc-connectivity, subdigraph packing.

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1. Introduction

1.1. Motivation and concepts

The generalized k-connectivity $\kappa_k(G)$ of a graph $G = (V, E)$ was introduced by Hager [8] in 1985 ($2 \le k \le |V|$). For a graph $G = (V, E)$ and a set $S \subseteq V$ of at least two vertices, an *S-Steiner tree* or, simply, an *S-tree* is a subgraph T of G which is a tree with $S \subseteq V(T)$. Two S-trees T_1 and T_2 are said to be *edge-disjoint* if $E(T_1) \cap E(T_2) = \emptyset$. Two edge-disjoint S-trees T_1 and T_2 are said to be *internally disjoint* if $V(T_1) \cap V(T_2) = S$. The *generalized local connectivity* $\kappa_S(G)$ is the maximum number of internally disjoint S-trees in G. For an integer k with $2 \le k \le n$, the *generalized k-connectivity* is defined as

$$\kappa_k(G) = \min\{\kappa_S(G) \mid S \subseteq V(G), |S| = k\}.$$

Observe that $\kappa_2(G) = \kappa(G)$. Li, Mao and Sun [10] introduced the following concept of generalized k-edge-connectivity. The *generalized local edge-connectivity* $\lambda_S(G)$ is the maximum number of edge-disjoint S-trees in G. For an integer k with $2 \le k \le n$, the *generalized k-edge-connectivity* is defined as

$$\lambda_k(G) = \min\{\lambda_S(G) \mid S \subseteq V(G), |S| = k\}.$$

Observe that $\lambda_2(G) = \lambda(G)$. Generalized connectivity of graphs has become a well-established area in graph theory, see a recent monograph [9] by Li and Mao on generalized connectivity of undirected graphs.

To extend generalized k-connectivity to directed graphs, Sun, Gutin, Yeo and Zhang [14] observed that in the definition of $\kappa_S(G)$, one can replace "an S-tree" by "a connected subgraph of G containing S." Therefore, Sun *et al.* [14] defined *strong subgraph k-connectivity* by replacing "connected" with "strongly connected" (or, simply, "strong") as follows. Let $D = (V, A)$ be a digraph of order n, S a subset of V of size k and $2 \le k \le n$. A subdigraph H of D is called an *S-strong subgraph* if H is strong and $S \subseteq V(H)$. Two S-strong subgraphs D_1 and D_2 are said to be *arc-disjoint* if $A(D_1) \cap A(D_2) = \emptyset$. Two arc-disjoint S-strong subgraphs D_1 and D_2 are said to be *internally disjoint* if $V(D_1) \cap V(D_2) = S$. Let $\kappa_S(D)$ be the maximum number of internally disjoint S-strong subgraphs in D. The *strong subgraph k-connectivity* is defined as

$$\kappa_k(D) = \min\{\kappa_S(D) \mid S \subseteq V, |S| = k\}.$$

As a natural counterpart of the strong subgraph k-connectivity, Sun and Gutin [11] introduced the concept of strong subgraph k-arc-connectivity. Let $D = (V(D), A(D))$ be a digraph of order n, $S \subseteq V$ a k-subset of $V(D)$ and $2 \le k \le n$. Let $\lambda_S(D)$ be the maximum number of arc-disjoint S-strong subgraphs in D. The *strong subgraph k-arc-connectivity* is defined as

$$\lambda_k(D) = \min\{\lambda_S(D) \mid S \subseteq V(D), |S| = k\}.$$

The strong subgraph k-(arc-)connectivity is not only a natural extension of the concept of generalized k-(edge-)connectivity, but also relates to important problems in graph theory. For $k = 2$, $\kappa_2(\overleftrightarrow{G}) = \kappa(G)$ [14] and $\lambda_2(\overleftrightarrow{G}) = \lambda(G)$ [11]. Hence, $\kappa_k(D)$ and $\lambda_k(D)$ could be seen as generalizations of connectivity and edge-connectivity of undirected graphs, respectively. For $k = n$, $\kappa_n(D) = \lambda_n(D)$ is the maximum number of arc-disjoint spanning strong subgraphs of D. Moreover, we know that $\kappa_S(D)$ and $\lambda_S(D)$ denote the number of internally-disjoint and arc-disjoint strong subgraphs containing a given set S, respectively. Hence, these parameters are relevant to the subdigraph packing problem, see [2–5, 7, 13]. For a recent survey on the topic of strong subgraph connectivity, the readers can see [12].

A digraph $D = (V(D), A(D))$ is called *minimally strong subgraph* (k, ℓ)*-arc-connected* if $\lambda_k(D) \geq \ell$ but for any arc $e \in A(D)$, $\lambda_k(D - e) \leq \ell - 1$. Note that $2 \leq k \leq n, 1 \leq \ell \leq n - 1$ by the definition of $\lambda_k(D)$ and Theorem 3. Let $\mathfrak{G}(n, k, \ell)$ be the set of all minimally strong subgraph (k, ℓ)-arc-connected digraphs with order n. We define

$$G(n, k, \ell) = \max\{|A(D)| \mid D \in \mathfrak{G}(n, k, \ell)\}$$

and

$$g(n, k, \ell) = \min\{|A(D)| \mid D \in \mathfrak{G}(n, k, \ell)\}.$$

We further define

$$Ex(n, k, \ell) = \{D \mid D \in \mathfrak{G}(n, k, \ell), |A(D)| = G(n, k, \ell)\}$$

and

$$ex(n, k, \ell) = \{D \mid D \in \mathfrak{G}(n, k, \ell), |A(D)| = g(n, k, \ell)\}.$$

In [11], Sun and Gutin first studied the minimally strong subgraph (k, ℓ)-arc-connected digraphs and gave some characterizations for some special cases (Proposition 7 and Theorem 8). In this paper, we continue to study the minimally strong subgraph (k, ℓ)-arc-connected digraphs. We first give a characterization of the minimally strong subgraph $(3, n - 2)$-arc-connected digraphs (Theorem 4), then give exact values and bounds for the functions $g(n, k, \ell)$ and $G(n, k, \ell)$ (Theorem 6 and Proposition 10).

1.2. Prelimilaries

We will use the following Tillson's decomposition theorem.

Theorem 1 [15]**.** *The arcs of* $\overleftrightarrow{K}_n$ *can be decomposed into Hamiltonian cycles if and only if* $n \neq 4, 6$.

The following proposition will also be used in our argument.

Proposition 2 [11]**.** *Let D be a digraph of order n, and let $k \geq 2$ be an integer. Then*

$$\lambda_{k+1}(D) \leq \lambda_k(D) \text{ for every } k \leq n-1, \tag{1}$$

$$\lambda_k(D') \leq \lambda_k(D) \text{ where } D' \text{ is a spanning subgraph of } D, \tag{2}$$

$$\kappa_k(D) \leq \lambda_k(D) \leq \min\{\delta^+(D), \delta^-(D)\}, \tag{3}$$

Sun and Gutin [11] obtained a sharp lower bound and a sharp upper bound of $\lambda_k(D)$ for $2 \leq k \leq n$.

Theorem 3. *Let $2 \leq k \leq n$. For a strong digraph D of order n, we have*

$$1 \leq \lambda_k(D) \leq n-1.$$

Moreover, both bounds are sharp, and the upper bound holds if and only if $D \cong \overleftrightarrow{K}_n$, where $k \notin \{4,6\}$, or, $k \in \{4,6\}$ and $k < n$.

2. Characterization of the Minimally Strong Subgraph $(3, n-2)$-Arc-Connected Digraphs

For a digraph D, its *reverse* D^{rev} is a digraph with same vertex set and such that $xy \in A(D^{\text{rev}})$ if and only if $yx \in A(D)$.

Theorem 4. *A digraph D is minimally strong subgraph $(3, n-2)$-arc-connected if and only if D is a digraph obtained from the complete digraph $\overleftrightarrow{K}_n$ by deleting an arc set M such that $\overleftrightarrow{K}_n[M]$ is a union of vertex-disjoint cycles which cover all but at most one vertex of $\overleftrightarrow{K}_n$.*

Proof. Let D be a digraph obtained from the complete digraph $\overleftrightarrow{K}_n$ by deleting an arc set M such that $\overleftrightarrow{K}_n[M]$ is a union of vertex-disjoint cycles which cover all but at most one vertex of $\overleftrightarrow{K}_n$. To prove the theorem it suffices to show that (a) D is minimally strong subgraph $(3, n-2)$-arc-connected, that is, $\lambda_3(D) \geq n-2$ but for any arc $e \in A(D)$, $\lambda_3(D-e) \leq n-3$, and (b) if a digraph H is minimally strong subgraph $(3, n-2)$-arc-connected then it must be constructed from $\overleftrightarrow{K}_n$ as the digraph D above. Thus, the remainder of the proof has two parts.

Part (a). We just consider the case that $\overleftrightarrow{K}_n[M]$ is a union of vertex-disjoint cycles which cover all vertices of $\overleftrightarrow{K}_n$, since the argument for the other case is similar. For any $e \in A(\overleftrightarrow{K}_n) \setminus M$, observe that e must be adjacent to at least one element of M, so $\lambda_3(D-e) \leq \min\{\delta^+(D-e), \delta^-(D-e)\} = n-3$ by (3). Hence, it suffices to show that $\lambda_3(D) = n-2$ in the following. So we will show that for $S = \{x, y, z\} \subseteq V(D)$, there are at least $n-2$ arc-disjoint S-strong subgraphs in D.

Case 1. x, y, z belong to the same cycle, say $\mathcal{C} = u_1u_2 \cdots u_tu_1$, of $\overleftrightarrow{K}_n[M]$.

Subcase 1.1. S induces a path of length two in $\mathcal{C}$. Without loss of generality, assume that $x = u_1, y = u_2, z = u_3$.

For the case that $t = 3$, we construct the following $n-2$ arc-disjoint S-strong subgraphs: let D_1 be the cycle $zyxz$; for any $u \in V(D) \setminus S$, let D_u be the subdigraph of D with vertex set $S \cup \{u\}$ and arc set $\{xu, ux, yu, uy, zu, uz\}$.

For the case that $t = 4$, we construct the following $n-2$ arc-disjoint S-strong subgraphs: let D_1 be the cycle $zyxz$; let D_2 be the subdigraph of D with vertex set $V(\mathcal{C})$ and arc set $\{xu_t, zx, yu_t, u_ty, u_tz\}$; for any $u \in V(D) \setminus V(\mathcal{C})$, let D_u be the subdigraph of D with vertex set $S \cup \{u\}$ and arc set $\{xu, ux, yu, uy, zu, uz\}$.

For the case that $t \geq 5$, we construct the following $n-2$ arc-disjoint S-strong subgraphs: let D_1 be the cycle $zyxz$; let D_2 be the cycle zxu_tyu_4z; let D_3 be the subdigraph of D with vertex set $S \cup \{u_4, u_t\}$ and arc set $\{xu_4, u_4x, zu_t, u_tz, u_tu_4, u_4y, yu_t\}$; for any $u \in V(D) \setminus (\{u_4, u_t\} \cup S)$, let D_u be the subdigraph of D with vertex set $S \cup \{u\}$ and arc set $\{xu, ux, yu, uy, zu, uz\}$.

Subcase 1.2. Exactly two elements of S are adjacent. Without loss of generality, assume that $x = u_1, y = u_2$. We know $t \geq 5$ in this case.

If $t = 5$, then $z = u_4$. We construct the following $n-2$ arc-disjoint S-strong subgraphs: let D_1 be the cycle $zyxz$; let D_2 be the cycle zxu_tyz; let D_3 be the subdigraph of D with vertex set $V(\mathcal{C})$ and arc set $\{xu_3, u_3x, u_3y, yu_t, u_tz, zu_3\}$; for any $u \in V(D) \setminus V(\mathcal{C})$, let D_u be the subdigraph of D with vertex set $S \cup \{u\}$ and arc set $\{xu, ux, yu, uy, zu, uz\}$.

We now consider the case that $t \geq 6$. If $z = u_4$, then we construct the following $n-2$ arc-disjoint S-strong subgraphs: let D_1 be the cycle $zyxz$; let D_2 be the cycle zxu_tyz; let D_3 be the subdigraph of D with vertex set $\{x, y, u_3, z, u_t\}$ and arc set $\{xu_3, u_3x, u_3y, yu_t, u_tz, zu_3\}$; let D_4 be the subdigraph of D with vertex set $\{x, y, z, u_5, u_t\}$ and arc set $\{xu_5, u_5x, u_5y, yu_5, u_5z, u_tu_5, zu_t\}$; for any $u \in V(D) \setminus \{x, y, u_3, z, u_5, u_t\}$, let D_u be the subdigraph of D with vertex set $S \cup \{u\}$ and arc set $\{xu, ux, yu, uy, zu, uz\}$.

If $z = u_{t-1}$, then we construct the following $n-2$ arc-disjoint S-strong subgraphs: let D_1 be the cycle $zyxz$; let D_2 be the cycle zxu_tyz; let D_3 be the subdigraph of D with vertex set $\{x, y, u_3, u_{t-2}, z, u_t\}$ and arc set $\{xu_3, u_3x, u_3y, yu_t, u_tz, zu_{t-2}, u_{t-2}u_3\}$; let D_4 be the subdigraph of D with vertex set $\{x, y, u_3, u_{t-2}, z, u_t\}$ and arc set $\{xu_{t-2}, u_{t-2}x, u_{t-2}y, yu_{t-2}, u_{t-2}u_t, u_tu_{t-2}, u_tu_3, u_3u_t, u_3z, zu_3\}$; for any $u \in V(D) \setminus \{x, y, u_3, u_{t-2}, z, u_t\}$, let D_u be the subdigraph of D with vertex set $S \cup \{u\}$ and arc set $\{xu, ux, yu, uy, zu, uz\}$.

If $z \notin \{u_4, u_{t-1}\}$, say $z = u_5$, then we construct the following $n-2$ arc-disjoint S-strong subgraphs: let D_1 be the cycle $zyxz$; let D_2 be the cycle zxu_tyz; let D_3 be the subdigraph of D with vertex set $\{x, y, u_3, u_4, z, u_{t-1}\}$ and arc set $\{xu_3, u_3x, u_3y, yu_{t-1}, u_{t-1}z, zu_4, u_4u_3\}$; let D_4 be the subdigraph of D with vertex set $\{x, y, z, u_{t-1}, u_t\}$ and arc set $\{xu_{t-1}, u_{t-1}x, u_tz, zu_t, yu_t, u_tu_{t-1}, u_{t-1}y\}$; let D_5 be the subdigraph of D with vertex set $\{x, y, u_3, u_4, z, u_{t-1}\}$ and arc set $\{xu_4, u_4x, u_4y, yu_4, u_4u_{t-1}, u_{t-1}u_4, u_{t-1}u_3, u_3u_{t-1}, u_3z, zu_3\}$; for any $u \in V(D) \setminus \{x, y, u_3, u_4, z, u_{t-1}, u_t\}$, let D_u be the subdigraph of D with vertex set $S \cup \{u\}$ and arc set $\{xu, ux, yu, uy, zu, uz\}$.

Subcase 1.3. Any two elements of S are nonadjacent. Without loss of generality, assume that $x = u_1$. We know $t \geq 6$ in this case.

If $t = 6$, then we can assume that $y = u_3, z = u_5$. We construct the following $n-2$ arc-disjoint S-strong subgraphs: let D_1 be the cycle $zyxz$; let $D_2 = D_1^{\text{rev}}$; let D_3 be the subdigraph of D with vertex set $S \cup \{u_2, u_t\}$ and arc set $\{xu_6, u_6y, yu_2, u_2x, u_2z, zu_2\}$; let D_4 be the subdigraph of D with vertex set $S \cup \{u_4, u_t\}$ and arc set $\{zu_4, u_4y, yu_6, u_6z, u_4x, xu_4\}$; for any $u \in V(D) \setminus S$, let D_u be the subdigraph of D with vertex set $S \cup \{u\}$ and arc set $\{xu, ux, yu, uy, zu, uz\}$.

In the following we assume that $t \geq 7$. We consider the case that exactly one pair of elements, say x and z, of S does not have a common neighbor in the cycle $\mathcal{C}$. Without loss of generality, assume that $y = u_3, z = u_5$ (observe that x and y have a common neighbor u_2, y and z have a common neighbor u_4, but z and x do not have a common neighbor in the cycle $\mathcal{C}$). We construct the following $n-2$ arc-disjoint S-strong subgraphs: let D_1 be the cycle $zyxz$; let $D_2 = D_1^{\text{rev}}$; let D_3 be the subdigraph of D with vertex set $S \cup \{u_2, u_t\}$ and arc set $\{xu_t, u_ty, yu_2, u_2x, u_2z, zu_2\}$; let D_4 be the subdigraph of D with vertex set $S \cup \{u_4, u_t\}$ and arc set $\{zu_4, u_4y, yu_t, u_tz, u_4x, xu_4\}$; let D_5 be the subdigraph of D with vertex set $S \cup \{u_6, u_t\}$ and arc set $\{xu_6, u_6x, u_6y, yu_6, zu_t, u_tu_6, u_6z\}$; for any $u \in V(D) \setminus (S \cup \{u_2, u_4, u_6, u_t\})$, let D_u be the subdigraph of D with vertex set $S \cup \{u\}$ and arc set $\{xu, ux, yu, uy, zu, uz\}$.

We now consider the case that exactly one pair of elements, say x and y, of S has a common neighbor in the cycle $\mathcal{C}$. Without loss of generality, assume that $y = u_3, z = u_6$ (we know x and y have a common neighbor u_2, y and z do not have a common neighbor, z and x do not have a common neighbor in the cycle $\mathcal{C}$). We construct the following $n-2$ arc-disjoint S-strong subgraphs: let D_1 be the cycle $zyxz$; let $D_2 = D_1^{\text{rev}}$; let D_3 be a subdigraph of D with vertex set $S \cup \{u_2, u_t\}$ and arc set $\{xu_t, u_ty, yu_2, u_2x, u_2z, zu_2\}$; let D_4 be a subdigraph of D with vertex set $S \cup \{u_4, u_7\}$ and arc set $\{u_4y, yu_7, u_7u_4, u_4x, xu_4, u_4z, zu_4\}$; let D_5 be a subdigraph of D with vertex set $S \cup \{u_5, u_7\}$ and arc set $\{u_5u_7, u_7z, zu_5, u_5x, xu_5, u_5y, yu_5\}$; let D_6 be a subdigraph of D with vertex set $S \cup \{u_7, u_t\}$ and arc set $\{u_7y, yu_t, u_tu_7, u_7x, xu_7, u_tz, zu_t\}$; for any $u \in V(D) \setminus (S \cup \{u_2, u_4, u_5, u_7, u_t\})$, let D_u be the subdigraph of D with vertex set $S \cup \{u\}$ and arc set $\{xu, ux, yu, uy, zu, uz\}$.

We consider the remaining case that any pair of elements of S does not have a common neighbor in the cycle $\mathcal{C}$. Without loss of generality, assume that $y = u_4, z = u_7$ (we know x and y do not a common neighbor u_2, y and z do not have a common neighbor, z and x do not have a common neighbor in the cycle $\mathcal{C}$). We construct the following $n-2$ arc-disjoint S-strong subgraphs: let D_1 be the cycle $zyxz$; let $D_2 = D_1^{\rm rev}$; let D_3 be a subdigraph of D with vertex set $S\cup\{u_2, u_t\}$ and arc set $\{xu_t, u_tu_2, u_2x, u_2y, yu_2, u_2z, zu_2\}$; let D_4 be a subdigraph of D with vertex set $S\cup\{u_3, u_t\}$ and arc set $\{u_3u_t, u_ty, yu_3, u_3x, xu_3, u_3z, zu_3\}$; let D_5 be a subdigraph of D with vertex set $S\cup\{u_5, u_t\}$ and arc set $\{u_5y, yu_t, u_tu_5, u_5x, xu_5, u_5z, zu_5\}$; let D_6 be a subdigraph of D with vertex set $S\cup\{u_6, u_t\}$ and arc set $\{u_6u_t, u_tz, zu_6, xu_6, u_6x, yu_6, u_6y\}$; let D_7 be a subdigraph of D with vertex set $S\cup\{u_8, u_t\}$ and arc set $\{u_8z, zu_t, u_tu_8, xu_8, u_8x, yu_8, u_8y\}$; for any $u \in V(D)\setminus (S\cup\{u_2, u_3, u_5, u_6, u_8, u_t\})$, let D_u be the subdigraph of D with vertex set $S\cup\{u\}$ and arc set $\{xu, ux, yu, uy, zu, uz\}$.

Case 2. Exactly two elements of S belong to the same cycle, say $\mathcal{C}_1 = u_1u_2\cdots u_tu_1$, of $\overleftrightarrow{K}_n[M]$, and the remaining element belongs to the other cycle $\mathcal{C}_2 = v_1v_2\cdots v_hv_1$. Without loss of generality, assume that $x, y \in V(\mathcal{C}_1), z = v_1$.

Subcase 2.1. x and y are adjacent. Without loss of generality, assume that $x = u_1, y = u_2$. We just consider the case that $t \geq 4$ and $h \geq 3$, since the arguments for the other cases are similar and simpler. We construct the following $n-2$ arc-disjoint S-strong subgraphs: let D_1 be the cycle zyv_hxz; let $D_2 = D_1^{\rm rev}$; let D_3 be a subdigraph of D with vertex set $S\cup\{u_t\}$ and arc set $\{xu_t, u_ty, yx, u_tz, zu_t\}$; let D_4 be a subdigraph of D with vertex set $S\cup\{u_3, u_t\}$ and arc set $\{yu_t, u_tu_3, u_3y, u_3x, xu_3, u_3z, zu_3\}$; let D_5 be a subdigraph of D with vertex set $S\cup\{v_2, v_h\}$ and arc set $\{zv_h, v_hv_2, v_2z, v_2x, xv_2, yv_2, v_2y\}$; for any $u \in V(D)\setminus (S\cup\{u_3, u_t, v_2, v_h\})$, let D_u be the subdigraph of D with vertex set $S\cup\{u\}$ and arc set $\{xu, ux, yu, uy, zu, uz\}$.

Subcase 2.2. x and y are nonadjacent. Without loss of generality, assume that $x = u_1$.

We first consider the case that $t = 4$, and observe that $y = u_3$ now. Furthermore, assume that $h \geq 3$ since the argument for the remaining case is similar and simpler. We construct the following $n-2$ arc-disjoint S-strong subgraphs: let D_1 be the cycle $xyzx$; let $D_2 = D_1^{\rm rev}$; let D_3 be a subdigraph of D with vertex set $S\cup\{u_2, u_t\}$ and arc set $\{xu_t, u_ty, yu_2, u_2x, u_2z, zu_2\}$; let D_4 be a subdigraph of D with vertex set $S\cup\{v_2, u_t\}$ and arc set $\{xv_2, v_2x, yv_2, v_2y, u_tz, zu_t, u_tv_2, v_2u_t\}$; let D_5 be a subdigraph of D with vertex set $S\cup\{v_2, v_h\}$ and arc set $\{xv_h, v_hx, yv_h, v_hy, v_hv_2, v_2z, zv_h\}$; for any $u \in V(D)\setminus (S\cup\{u_2, u_t, v_2, v_h\})$, let D_u be the subdigraph of D with vertex set $S\cup\{u\}$ and arc set $\{xu, ux, yu, uy, zu, uz\}$.

Now we assume that $t \geq 5$. We first consider the case that x and y have exactly one common neighbor in the cycle $\mathcal{C}_1$. With a similar argument to that of the case that $t \geq 7$ and exactly one pair of elements, say x and y, of S has a common neighbor in the cycle $\mathcal{C}$ in Subcase 1.3, we can construct $n-2$ arc-disjoint S-strong subgraphs.

We next consider the case that x and y do not have a common neighbor in the cycle $\mathcal{C}_1$. If $h \geq 3$, then with a similar argument to that of the case that $t \geq 7$ and any pair of elements of S doesnot have a common neighbor in the cycle $\mathcal{C}$ in Subcase 1.3, we can construct $n-2$ arc-disjoint strong subgraphs containing S. Otherwise, we have $h = 2$. Without loss of generality, assume that $y = u_4$. We construct the following $n-2$ arc-disjoint S-strong subgraphs: let D_1 be the cycle $xyzx$; let $D_2 = D_1^{\rm rev}$; let D_3 be a subdigraph of D with vertex set $S \cup \{u_2, u_t\}$ and arc set $\{xu_t, u_tu_2, u_2x, u_2y, yu_2, u_2z, zu_2\}$; let D_4 be a subdigraph of D with vertex set $S \cup \{u_3, u_t\}$ and arc set $\{u_3u_t, u_ty, yu_3, u_3x, xu_3, u_3z, zu_3\}$; let D_5 be a subdigraph of D with vertex set $S \cup \{u_5, u_t\}$ and arc set $\{u_5y, yu_t, u_tu_5, u_5x, xu_5, u_5z, zu_5\}$; let D_6 be a subdigraph of D with vertex set $S \cup \{u_t, v_h\}$ and arc set $\{xv_h, v_hx, v_hu_t, u_tv_h, v_hy, yv_h, u_tv_1, v_1u_t\}$; for any $u \in V(D) \setminus (S \cup \{u_2, u_3, u_t, u_5, v_h\})$, let D_u be the subdigraph of D with vertex set $S \cup \{u\}$ and arc set $\{xu, ux, yu, uy, zu, uz\}$.

Case 3. The elements of S belong to distinct cycles, say $x \in V(\mathcal{C}_1)$, $y \in V(\mathcal{C}_2)$, $z \in V(\mathcal{C}_3)$, of $\overleftrightarrow{K}_n[M]$.

Subcase 3.1. $|V(\mathcal{C}_i)| \geq 3$ for all $1 \leq i \leq 3$. With a similar argument to the case that $t \geq 7$ and exactly one pair of elements, say x and y, of S has a common neighbor in the cycle $\mathcal{C}$ in Subcase 1.3, we can construct $n-2$ arc-disjoint S-strong subgraphs.

Subcase 3.2. $|V(\mathcal{C}_{i_0})| = 2$ for some $1 \leq i_0 \leq 3$. With a similar argument to the case that x, y do not have a common neighbor in the cycle $\mathcal{C}_1$ and $h = 2$ in last paragraph of Subcase 2.2, we can construct $n-2$ arc-disjoint S-strong subgraphs.

Subcase 3.3. $|V(\mathcal{C}_{i_0})| = |V(\mathcal{C}_{j_0})| = 2$ for some $1 \leq i_0,\ j_0 \leq 3$. Without loss of generality, we assume that $i_0 = 2, j_0 = 3$ and furthermore, $u_1x, xu_2 \in E(\mathcal{C}_1)$, $u_3y, yu_3 \in E(\mathcal{C}_2), u_4z, zu_4 \in E(\mathcal{C}_3)$. We construct the following $n-2$ arc-disjoint S-strong subgraphs: let D_1 be the cycle $xyzx$; let $D_2 = D_1^{\rm rev}$; let D_3 be a subdigraph of D with vertex set $S \cup \{u_1, u_2\}$ and arc set $\{u_1u_2, u_2x, xu_1, u_2y, yu_2, u_2z, zu_2\}$; let D_4 be the cycle $xu_4yu_1zu_3x$; let $D_5 = D_4^{\rm rev}$; for any $u \in V(D) \setminus (S \cup \{u_1, u_2, u_3, u_4\})$, let D_u be a subdigraph of D with vertex set $S \cup \{u\}$ and arc set $\{xu, ux, yu, uy, zu, uz\}$.

Subcase 3.4. $|V(\mathcal{C}_i)| = 2$ for all $1 \leq i \leq 3$. This case is easy and we omit the details.

Part (b). Let H be minimally strong subgraph $(3, n-2)$-arc-connected. By Theorem 3, we have that $H \ncong \overleftrightarrow{K}_n$, that is, H can be obtained from a complete digraph $\overleftrightarrow{K}_n$ by deleting a nonempty arc set M. To end our argument, we need the following claim. Let us start from a simple yet useful observation, which follows from (3).

Proposition 5. *No pair of arcs in M has a common head or tail.*

Thus, $\overleftrightarrow{K}_n[M]$ must be a union of vertex-disjoint cycles or paths, otherwise, there are two arcs of M such that they have a common head or tail, a contradiction with Proposition 5.

Claim 1. *$\overleftrightarrow{K}_n[M]$ does not contain a path of order at least two.*

Proof. Suppose that $\overleftrightarrow{K}_n[M]$ contains a path of order at least two. Let $M' \supseteq M$ be a set of arcs obtained from M by adding some arcs from $\overleftrightarrow{K}_n - M$ such that the digraph $\overleftrightarrow{K}_n[M']$ contains no path of order at least two. For example, if $\overleftrightarrow{K}_n[M]$ contains a path $u_1, \ldots, u_\ell$ with $\ell \geq 2$, then add the arc $u_\ell u_1$ to M'. Note that $\overleftrightarrow{K}_n[M']$ is a supergraph of $\overleftrightarrow{K}_n[M]$ and is a union of vertex-disjoint cycles which cover all but at most one vertex of $\overleftrightarrow{K}_n$. By Part (a), we have that $\lambda_3(\overleftrightarrow{K}_n - M') = n-2$, so H is not minimally strong subgraph $(3, n-2)$-arc-connected, a contradiction. □

It follows from Claim 1 and its proof that $\overleftrightarrow{K}_n[M]$ must be a union of vertex-disjoint cycles which cover all but at most one vertex of $\overleftrightarrow{K}_n$, which completes the proof of Part (b). ■

3. RESULTS FOR $g(n,k,\ell)$, $G(n,k,\ell)$, $ex(n,k,\ell)$ AND $Ex(n,k,\ell)$

The following result concerns the precise value for $g(n,k,\ell)$.

Theorem 6. *For any triple (n,k,ℓ) with $2 \leq k \leq n, 1 \leq \ell \leq n-1$ such that $(n,k,\ell) \notin \{(4,4,3),(6,6,5)\}$, we have*

$$g(n,k,\ell) = n\ell.$$

Proof. By Theorem 3 and the definition of $g(n,k,\ell)$, we have $(n,k,\ell) \notin \{(4,4,3), (6,6,5)\}$.

For all digraphs D and $k \geq 2$, we have $\lambda_k(D) \leq \delta^+(D)$ and $\lambda_k(D) \leq \delta^-(D)$ by (3). Hence for each D with $\lambda_k(D) = \ell$, we have that $\delta^+(D), \delta^-(D) \geq \ell$, so $|A(D)| \geq n\ell$ and then $g(n,k,\ell) \geq n\ell$.

We first consider the case that $n \notin \{4,6\}$. Let $D \cong \overleftrightarrow{K}_n$. By Theorem 1, D can be decomposed into $n-1$ Hamiltonian cycles H_i $(1 \leq i \leq n-1)$. Let D_ℓ be the spanning subdigraph of D with arc sets $A(D_\ell) = \bigcup_{1\leq i\leq \ell} A(H_i)$. Clearly, we have $\lambda_k(D_\ell) \geq \ell$ for $2 \leq k \leq n, 1 \leq \ell \leq n-1$. Furthermore, by (3), we have $\lambda_k(D_\ell) \leq \ell$ since the in-degree and out-degree of each vertex in D_ℓ are both ℓ. Hence, $\lambda_k(D_\ell) = \ell$ for $2 \leq k \leq n, 1 \leq \ell \leq n-1$. For any $e \in A(D_\ell)$, we have $\delta^+(D_\ell - e) = \delta^-(D_\ell - e) = \ell - 1$, so $\lambda_k(D_\ell - e) \leq \ell - 1$ by (3). Thus,

D_ℓ is minimally strong subgraph (k,ℓ)-arc-connected. As $|A(D_\ell)| = n\ell$, we have $g(n,k,\ell) \leq n\ell$. From the lower bound that $g(n,k,\ell) \geq n\ell$, we have $g(n,k,\ell) = n\ell$ for the case that $n \notin \{4,6\}$.

Now we assume that $n \in \{4,6\}$. We just consider the case that $n = 6$, since the remaining case is similar and simpler. Let D be a digraph with vertex set $V(D) = \{u_i \mid 1 \leq i \leq 6\}$ such that D is a union of four arc-disjoint cycles C_i, where $C_1 : u_1u_2u_3u_4u_5u_6u_1$, $C_2 = {C_1}^{\rm rev}$, $C_3 : u_1u_3u_5u_2u_4u_6u_1$ and $C_4 = {C_3}^{\rm rev}$.

Let D_ℓ $(1 \leq \ell \leq 4)$ be the spanning subdigraph of D with arc sets $A(D_\ell) = \bigcup_{1\leq i\leq \ell} A(C_i)$. Let $D_5 = \overleftrightarrow{K}_6$. Clearly, we have $\lambda_k(D_\ell) \geq \ell$ for $2 \leq k \leq 5, 1 \leq \ell \leq 5$. Furthermore, by (3), we have $\lambda_k(D_\ell) \leq \ell$ since the in-degree and out-degree of each vertex in D_ℓ are both ℓ. Hence, $\lambda_k(D_\ell) = \ell$ for $2 \leq k \leq 5, 1 \leq \ell \leq 5$. For any $e \in A(D_\ell)$, we have $\delta^+(D_\ell - e) = \delta^-(D_\ell - e) = \ell - 1$, so $\lambda_k(D_\ell - e) \leq \ell - 1$ by (3). Thus, D_ℓ is minimally strong subgraph (k,ℓ)-arc-connected. As $|A(D_\ell)| = n\ell$, we have $g(n,k,\ell) \leq n\ell$. Hence, $g(n,k,\ell) = n\ell$ holds for this case by the lower bound that $g(n,k,\ell) \geq n\ell$. For the case that $k = n = 6$, we have $1 \leq \ell \leq 4$, with a similar argument, we can also deduce that $g(n,k,\ell) = n\ell$. ■

A digraph D is *minimally strong* if D is strong but $D - e$ is not for every arc e of D. Sun and Gutin [11] gave the following characterizations.

Proposition 7 [11]**.** *The following assertions hold.*

(i) *A digraph D is minimally strong subgraph $(k,1)$-arc-connected if and only if D is minimally strong digraph.*

(ii) *Let $2 \leq k \leq n$. If $k \notin \{4,6\}$, or, $k \in \{4,6\}$ and $k < n$, then a digraph D is minimally strong subgraph $(k,n-1)$-arc-connected if and only if $D \cong \overleftrightarrow{K}_n$.*

Theorem 8 [11]**.** *A digraph D is minimally strong subgraph $(2,n-2)$-arc-connected if and only if D is a digraph obtained from the complete digraph $\overleftrightarrow{K}_n$ by deleting an arc set M such that $\overleftrightarrow{K}_n[M]$ is a union of vertex-disjoint cycles which cover all but at most one vertex of $\overleftrightarrow{K}_n$.*

To prove upper bounds on the number of arcs in a minimally strong subgraph (k,ℓ)-arc-connected digraph, we will use the following result, see e.g. Corollary 5.3.6 of [1].

Theorem 9. *Every strong digraph D on n vertices has a strong spanning subgraph H with at most $2n-2$ arcs and equality holds only if H is a symmetric digraph whose underlying undirected graph is a tree.*

Proposition 10. *We have* (i) $G(n,n,\ell) \leq 2\ell(n-1)$*;* (ii) *For every k $(2 \leq k \leq n)$, $G(n,k,1) = 2(n-1)$ and $Ex(n,k,1)$ consists of symmetric digraphs whose underlying undirected graphs are trees;* (iii) $G(n,k,n-2) = (n-1)^2$ *for $k \in \{2,3\}$.*

Proof. (i) Let $D = (V,A)$ be a minimally strong subgraph (n,ℓ)-arc-connected digraph, and let $D_1, \ldots, D_\ell$ be arc-disjoint strong spanning subgraphs of D. Since D is minimally strong subgraph (n, ℓ)-arc-connected and $D_1, \ldots, D_\ell$ are pairwise arc-disjoint, $|A| = \sum_{i=1}^{\ell} |A(D_i)|$. Thus, by Theorem 9, $|A| \leq 2\ell(n-1)$.

(ii) In the proof of Proposition 7, Sun and Gutin [11] showed that a digraph D is strong if and only if $\lambda_k(D) \geq 1$. Now let $\lambda_k(D) \geq 1$ and a digraph D has a minimal number of arcs. By Theorem 9, we have that $|A(D)| \leq 2(n-1)$, and if $D \in Ex(n, k, 1)$ then $|A(D)| = 2(n-1)$ and D is a symmetric digraph whose underlying undirected graph is a tree.

Part (iii) follows directly from Theorems 4 and 8. ■

By Theorems 4 and 8, we can get the following result on $ex(n, k, \ell)$ and $Ex(n, k, \ell)$.

Proposition 11. *The following assertions hold.*

(i) *For $k \in \{2, 3\}$, $Ex(n, k, n-2) = \{\overleftrightarrow{K_n} - M\}$ where M is an arc set such that $\overleftrightarrow{K}_n[M]$ is a union of vertex-disjoint cycles which cover all but exactly one vertex of $\overleftrightarrow{K}_n$.*

(ii) *For $k \in \{2, 3\}$, $ex(n, k, n-2) = \{\overleftrightarrow{K_n} - M\}$ where M is an arc set such that $\overleftrightarrow{K}_n[M]$ is a union of vertex-disjoint cycles which cover all vertices of $\overleftrightarrow{K}_n$.*

4. Discussion

In this paper, we give the characterization of minimally strong subgraph $(3, n-2)$-arc-connected digraphs. We determine the precise values for $g(n, k, \ell)$ completely and the precise values for $G(n, k, n-2)$ for $k \in \{2, 3\}$. So it would be interesting to determine $G(n, k, n-2)$ for every value of $k \geq 2$, as obtaining characterizations of all $(k, n-2)$-arc-connected digraphs for $2 \leq k \leq n$ seems a very difficult problem. It would also be interesting to find a sharp upper bound for $G(n, k, \ell)$ for all $k \geq 2$ and $\ell \geq 2$.

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References

[1] J. Bang-Jensen and G. Gutin, Digraphs: Theory, Algorithms and Applications, 2nd Edition (Springer, London, 2009).
https://doi.org/10.1007/978-1-84800-998-1

[2] J. Bang-Jensen and J. Huang, *Decomposing locally semicomplete digraphs into strong spanning subdigraphs*, J. Combin. Theory Ser. B **102** (2012) 701–714.
https://doi.org/10.1016/j.jctb.2011.09.001

[3] J. Bang-Jensen and M. Kriesell, *Disjoint sub(di)graphs in digraphs*, Electron. Notes Discrete Math. **34** (2009) 179–183.
https://doi.org/10.1016/j.endm.2009.07.030

[4] J. Bang-Jensen and A. Yeo, *Decomposing k-arc-strong tournaments into strong spanning subdigraphs*, Combinatorica **24** (2004) 331–349.
https://doi.org/10.1007/s00493-004-0021-z

[5] J. Bang-Jensen and A. Yeo, *Arc-disjoint spanning sub(di)graphs in digraphs*, Theoret. Comput. Sci. **438** (2012) 48–54.
https://doi.org/10.1016/j.tcs.2012.03.003

[6] J.A. Bondy and U.S.R. Murty, Graph Theory, GTM 244 (Springer, Berlin, 2008).

[7] J. Cheriyan and M.R. Salavatipour, *Hardness and approximation results for packing Steiner trees*, Algorithmica **45** (2006) 21–43.
https://doi.org/10.1007/s00453-005-1188-4

[8] M. Hager, *Pendant tree-connectivity*, J. Combin. Theory Ser. B **38** (1985) 179–189.
https://doi.org/10.1016/0095-8956(85)90083-8

[9] X. Li and Y. Mao, Generalized Connectivity of Graphs (Springer, Switzerland, 2016).
https://doi.org/10.1007/978-3-319-33828-6

[10] X. Li, Y. Mao and Y. Sun, *On the generalized (edge-)connectivity of graphs*, Australas. J. Combin. **58** (2014) 304–319.

[11] Y. Sun and G. Gutin, *Strong subgraph k-arc-connectivity*.
arXiv:1805.01687v1

[12] Y. Sun and G. Gutin, *Strong subgraph connectivity of digraphs: A survey*.
arXiv:1808.02740v1

[13] Y. Sun, G. Gutin and J. Ai, *Arc-disjoint strong spanning subdigraphs in compositions and products of digraphs*, Discrete Math. **342** (2019) 2297–2305.
https://doi.org/10.1016/j.disc.2019.05.003

[14] Y. Sun, G. Gutin, A. Yeo and X. Zhang, *Strong subgraph k-connectivity*, J. Graph Theory **92** (2019) 5–18.
https://doi.org/10.1002/jgt.22437

[15] T.W. Tillson, *A Hamiltonian decomposition of K^*_{2m}, $2m \geq 8$*, J. Combin. Theory Ser. B **29** (1980) 68–74.
https://doi.org/10.1016/0095-8956(80)90044-1

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