# TOTAL COLORING OF CLAW-FREE PLANAR GRAPHS 

Zuosong Liang<br>School of Management<br>Qufu Normal University<br>Rizhao 276826, China<br>e-mail: liangzuosong@126.com


#### Abstract

A total coloring of a graph is an assignment of colors to both its vertices and edges so that adjacent or incident elements acquire distinct colors. Let $\Delta(G)$ be the maximum degree of $G$. Vizing conjectured that every graph has a total $(\Delta+2)$-coloring. This Total Coloring Conjecture remains open even for planar graphs, for which the only open case is $\Delta=6$. Claw-free planar graphs have $\Delta \leq 6$. In this paper, we prove that the Total Coloring Conjecture holds for claw-free planar graphs.


Keywords: total coloring, total coloring conjecture, planar graph, claw.
2010 Mathematics Subject Classification: 05C15.

## 1. InTRODUCTION

All graphs considered here are finite, simple and nonempty. Let $G=(V, E)$ be a graph with vertex set $V$ and edge set $E$. The number of vertices of $G$ is called the order of $G$. For a vertex $v \in V$, the open neighborhood $N(v)$ of $v$ is defined as the set of vertices adjacent to $v$, i.e., $N(v)=\{u \mid u v \in E\}$. The closed neighborhood of $v$ is $N[v]=N(v) \cup\{v\}$. Every vertex in $N(v)$ is also called a neighbor of $v$. The degree of $v$ is equal to $|N(v)|$, denoted by $d_{G}(v)$ or simply $d(v)$. By $\delta(G)$ and $\Delta(G)$, we denote the minimum degree and the maximum degree of the graph $G$, respectively. For a subset $S \subseteq V$, the closed neighborhood of $S$ is $N[S]=\bigcup_{v \in S} N[v]$ and the closed 2-neighborhood of $S$ is $N_{2}[S]=N[N[S]]$. For a subset $X \subseteq V$, the subgraph induced by $X$ is denoted by $G[X]$. The set of edges between $X$ and $Y$ in $E$ is denoted by $E(X, Y)$ for $X, Y \subseteq V$. As usual, $K_{m, n}$ denotes a complete bipartite graph with classes of cardinality $m$ and $n$, and $K_{n}$ and $C_{n}$ denote the complete graph and cycle of order $n$. The graph $K_{1,3}$ is
also called a claw, and $K_{3}$ a triangle. Given a graph $F$, a graph $G$ is $F$-free if it does not contain $F$ as an induced subgraph. In particular, a $K_{1,3}$-free graph is claw-free. For a family $\left\{F_{1}, \ldots, F_{k}\right\}$ of graphs, we say that $G$ is $\left(F_{1}, \ldots, F_{k}\right)$-free if it is $F_{i}$-free for all $i$. By starting with a disjoint union of two graphs $G$ and $H$ and adding edges joining every vertex of $G$ to every vertex of $H$, one obtains the join of $G$ and $H$, denoted by $G \vee H$. The $n$-wheel is the graph $C_{n} \vee K_{1}$ and the double $n$-wheel is the graph $C_{n} \vee \bar{K}_{2}(n \geq 4)$.

Given a graph $G$, an element of $G$ is a member of $V(G) \cup E(G)$. Let two elements of a graph $G$ be adjacent if they are either adjacent or incident in the traditional sense. Given a graph $G$, a total $k$-coloring of $G$ is a function that takes each element to $\{1,2, \ldots, k\}$ such that adjacent distinct elements receive distinct colors. In 1968, Vizing [17] (see also [2]) made the following conjecture, known as the Total Coloring Conjecture.

Conjecture. Every graph has a total $(\Delta+2)$-coloring.
This conjecture is trivial for $\Delta \leq 2$. Rosenfeld [13] and Vijayaditya [16] proved it for $\Delta=3$. Kostochka proved the $\Delta=4[9]$ and $\Delta=5$ [11] cases. The conjecture remains open even for planar graphs, but more is known. Borodin [4] proved it for planar graphs with $\Delta \geq 9$. The $\Delta=8$ case was proved for planar graphs by Yap [21] and Andersen [1]. The $\Delta=7$ case was proved for planar graphs by Sanders and Zhao [14]. Thus the only open case for planar graphs is the $\Delta=6$ case. An extensive study on total coloring is done in $[3,5-8,10,18-20]$ and elsewhere. Claw-free planar graphs have maximum degree at most 6 [12]. In this paper, we prove that the Total Coloring Conjecture holds for claw-free planar graphs.

## 2. Total Coloring on the Claw-Free Planar Graphs

First we introduce some notation and lemmas which are useful for the total coloring of claw-free planar graphs. Let a $k$-vertex be a vertex of degree $k$. Let an at most $k$-vertex, or simply a $k^{-}$-vertex, be a vertex of degree at most $k$. Given a graph and integers $j_{1}, j_{2}, \ldots, j_{i}$, let a $\left(j_{1}^{-}, j_{2}^{-}, \ldots, j_{i}^{-}\right)$-vertex be an $i$-vertex $v$ of $G$ such that, for each $1 \leq m \leq i$, there is a $j_{m}^{-}$-vertex $y_{m}$ of $G$ and the vertices $y_{1}, y_{2}, \ldots, y_{i}$ are distinct neighbors of $v$.

Lemma 1 [15]. If $G$ is a (claw, $K_{4}$ )-free planar graph, then $\Delta(G) \leq 5$ and for every vertex $v$ of degree 5 in $G, G[N[v]]$ is a 5 -wheel.

Lemma 2. Let $v$ be a $\left(6^{-}, 6^{-}, 5^{-}\right)$-vertex, or a $(6,6,6)$-vertex such that $G[N[v]]$ is not a claw, in a graph $G$. If $G-v$ has a total 8 -coloring, then $G$ is total 8 -colorable.

Proof. Let $v$ be a $\left(6^{-}, 6^{-}, 6^{-}\right)$-vertex of a graph G. Given a total 8 -coloring $\phi$ of $G-v$, we will attempt to extend $\phi$ to a total 8 -coloring of $G$. Let $N(v)=$ $\left\{y_{1}, y_{2}, y_{3}\right\}$, and for $i \in\{1,2,3\}$ let $L_{v y_{i}}$ be the set of colors that are not used on $y_{i}$ or its incident edges and so are available for use on the edge $v y_{i}$. Then $\left|L_{v y_{i}}\right| \geq 2$ for each $i$, since $d_{G-v}\left(y_{i}\right) \leq 5$. Clearly, these edges can be properly colored unless $L_{v y_{i}}$ is the same set of two colors, say $L_{v y_{i}}=\{1,2\}$, for each $i$. This is impossible if $v$ is a $\left(6^{-}, 6^{-}, 5^{-}\right)$-vertex, so assume that $v$ is a $(6,6,6)$-vertex such that $G[N[v]]$ is not a claw. Without loss of generality, $y_{2} y_{3}$ is an edge of $G$ and $\phi\left(y_{2} y_{3}\right)=3$. Recolor $y_{2} y_{3}$ with color 1 , and then color $v y_{1}, v y_{2}$ and $v y_{3}$ with colors 1,2 and 3 , respectively. Once the edges incident with $v$ are colored, there are at most six colors that cannot be used on $v$, and so $v$ can be colored. This gives a total 8-coloring of $G$, as required.

Lemma 3. Let $v$ be a $\left(7^{-}, 6^{-}, 5^{-}, 4^{-}\right)$-vertex in a graph $G$ such that $G[N[v]]$ contains neither a claw nor a $K_{4}$. If $G-v$ has a total 8-coloring, then $G$ is total 8-colorable.

Proof. Let $N(v)=\left\{y_{1}, y_{2}, y_{3}, y_{4}\right\}$ where $d\left(y_{i}\right) \leq 8-i$ for each $i$. Given a total coloring $\phi$ of $G-v$ using a set $C$ of eight colors, let $F$ be the set of colors that are used on the vertices in $N(v)$, and for $i \in\{1,2,3,4\}$ let $L_{v y_{i}}$ be the set of colors that are available for use on the edge $v y_{i}$, as in the previous proof. Then $\left|L_{v y_{i}}\right| \geq i$ for each $i$, and so the edges $v y_{1}, v y_{2}, v y_{3}, v y_{4}$ can be properly colored in this order. If possible, do this so that at least one of these edges has a color in $F$. Call the new coloring $\phi^{\prime}$.

It is now possible to color $v$ unless every color is used on an element adjacent to $v$. For this to happen, it must be that $|F|=4$ and all the lists $L_{v y_{i}}$ are subsets of $C \backslash F$. In particular, $L_{v y_{4}}=C \backslash F$. Since $G[N[v]]$ contains neither a claw nor a $K_{4}, G$ must contain an edge $y_{r} y_{4}$ for some $r \in\{1,2,3\}$. Clearly $\phi\left(y_{r} y_{4}\right) \notin L_{v y_{4}}$, and so $\phi\left(y_{r} y_{4}\right) \in F$, while $\phi^{\prime}\left(v y_{r}\right) \in L_{v y_{r}} \subseteq L_{v y_{4}}$. Interchange the colors of $v y_{r}$ and $y_{r} y_{4}$. Since $v y_{r}$ now has a color in $F$, there are at most seven different colors that are unavailable for $v$, and so $v$ can now be colored. This gives a total 8 -coloring of $G$, as required.

Theorem 4. Every claw-free planar graph is total 8-colorable.
Proof. Let $G$ be a claw-free planar graph. We prove the theorem by induction on the size $m=|E(G)|$. Suppose that the theorem holds when $G$ has fewer than $m$ edges. In the following, we will prove the theorem when $G$ has $m$ edges.

If $G$ has a cut vertex $v$, we can easily see that the theorem holds. So we may assume that $G$ is 2 -connected. In addition, if $G$ has no $K_{4}$, then $\Delta(G) \leq 5$ by Lemma 1, and so the theorem holds since the Total Coloring Conjecture holds when $\Delta(G) \leq 5[11]$. So let $K=\left[x_{1} x_{2} x_{3} x_{4}\right]$ be a $K_{4}$ of $G$, where $x_{1}$ is inside the cycle $C=\left[x_{2} x_{3} x_{4}\right]$ in the embedding of $G$ in the plane. Let $G^{\prime}$ be the plane
graph induced by the vertices inside and on $C$, and choose $K$ so that $G^{\prime}$ has as fewer vertices as possible. Then $K$ is the only $K_{4}$ in $G^{\prime}$.

Since every claw-free planar graph has maximum degree at most 6 , the result follows from Lemma 2 if $G$ contains a $3^{-}$-vertex. Thus we may assume that $\delta(G) \geq 4$. Without loss of generality, we may assume that $x_{1}$ has a neighbor $u$ inside the triangle $T=\left[x_{1} x_{2} x_{3}\right]$ in the embedding of $G$ in the plane. Let $V_{\mathrm{in}}(T)$ denote the set of vertices inside the triangle $T$, and let $G_{T}=G\left[V_{\mathrm{in}}(T) \cup V(T)\right]$, the plane graph induced by the vertices inside and on $T$. We will make frequent use of the following facts.
(F1) $G_{T}$ is $K_{4}$-free. This is because $K$ is the only $K_{4}$ in $G^{\prime}$.
(F2) Every vertex of $V_{\mathrm{in}}(T)$ is adjacent to at most two vertices of $T$, by (F1).
(F3) Every vertex of $V_{\mathrm{in}}(T)$ has degree 4 or 5 , since $\delta(G) \geq 4$ and $\Delta\left(G_{T}\right) \leq 5$ by Lemma 1.
(F4) We may assume that no two 4-vertices in $V_{\text {in }}(T)$ are adjacent. Indeed, if they are, then each of them is a $\left(6^{-}, 6^{-}, 5^{-}, 4^{-}\right)$-vertex by (F2) and (F3), and so the result follows by Lemma 3.
(F5) For $i=1,2,3$, the neighbors of $x_{i}$ in $V_{\text {in }}(T)$ induce a complete graph. Otherwise, there would be a claw centered on $x_{i}$ (including the edge $x_{i} x_{4}$ ).
(F6) Every vertex of $T$ is adjacent to 0,1 or 2 adjacent vertices in $V_{\text {in }}(T)$. This follows from (F1) and (F5).
(F7) Every 5 -vertex in $V_{\text {in }}(T)$ is adjacent to 0 or 2 vertices of $T$. Indeed, let $v$ be a 5 -vertex in $V_{\text {in }}(T)$ that is adjacent to $x_{1}$ (say). By Lemma $1, G[N[v]]$ is a 5 -wheel. Thus $G[N[v]]$ contains two vertices $v_{1}, v_{2}$ that are adjacent to both $x_{1}$ and $v$ but not to each other. At least one of $v_{1}, v_{2}$ must be in $T$, by ( F 5 ). But the neighbors $x_{1}, v_{1}, v_{2}$ of $v$ cannot all be in $T$, by (F2). Thus $v$ is adjacent to exactly two vertices of $T$.

Recall that $x_{1}$ has a neighbor $u \in V_{\text {in }}(T)$. We consider two cases.
Case 1. $d(u)=5$. By (F7), $u$ is adjacent to exactly one of $x_{2}$ and $x_{3}$, say $x_{2}$. By Lemma 2, $G[N[u]]$ is a 5 -wheel. Let $x_{1} x_{2} u_{1} u_{2} u_{3} x_{1}$ be the 5 -cycle of $G[N[u]]$ (see Figure 1). Then $u$ and $u_{3}$ are the two vertices of $V_{\text {in }}(T)$ that are adjacent to $x_{1}$, and $u$ and $u_{1}$ are the two vertices of $V_{\text {in }}(T)$ that are adjacent to $x_{2}$.

Case 1.1. $d\left(u_{1}\right)=5$. Note that $u_{1}$ is adjacent to $x_{2}$ but cannot be adjacent to $x_{1}$, and so $u_{1}$ is adjacent to $x_{3}$ by (F7). So let $x_{2} x_{3} w u_{2} u x_{2}$ be the 5 -cycle of $G\left[N\left[u_{1}\right]\right]$ (see Figure 1, left). Each vertex of $T$ is now adjacent to two vertices of $V_{\text {in }}(T)$, and so cannot be adjacent to any further vertex of $V_{\text {in }}(T)$, by (F6). Thus any further neighbor of $u_{3}$ cannot be adjacent to $x_{1}$ and so must be adjacent to $u_{2}$, to avoid a claw centered on $u_{3}$. If $d\left(u_{2}\right)=4$, then the only possible further neighbor of $u_{3}$ is $w$, and so $u_{2}, u_{3}$ are adjacent 4 -vertices, contrary to (F4). If however $d\left(u_{2}\right)=5$, then the additional neighbor $q$ of $u_{2}$ can be adjacent only to $u_{2}, u_{3}, w$ of the vertices so far named (since it cannot be adjacent to $x_{1}$ or $x_{3}$ ), and so $q$ is a 3 -vertex by the claw-freeness, which is not permissible, by (F3).


Figure 1. Cases 1.1 and 1.2 of Theorem 5.
Case 1.2. $d\left(u_{1}\right)=4$. Then $d\left(u_{2}\right)=5$, by (F4). Let $u u_{1} w_{1} w_{2} u_{3} u$ be the 5 -cycle of $G\left[N\left[u_{2}\right]\right]$ (see Figure 1, right). Note that $x_{1}$ and $x_{2}$ each already have two neighbors in $V_{\mathrm{in}}(T)$, and so $u_{2}$ has no neighbors in $T$, by (F6) and (F7). Thus $w_{1} \in V_{\text {in }}(T)$ and so $d\left(w_{1}\right)=5$, by (F4), since $d\left(u_{1}\right)=4$. But it is impossible for $G\left[N\left[w_{1}\right]\right]$ to be a 5 -wheel since $d\left(u_{1}\right)=4$ and there is no edge $x_{2} w_{1}$.

Case 2. $d(u)=4$. Let the neighbors of $u$ be $x_{1}, u_{1}, u_{2}$ and $u_{3}$. We may assume that $u_{2}, u_{3} \in V_{\text {in }}(T)$ and possibly $u_{1}=x_{2}$ or $u_{1}=x_{3}$, by (F2). Then $d\left(u_{2}\right)=5$ and $d\left(u_{3}\right)=5$, by (F4). If $x_{1}$ is adjacent to the 5 -vertex $u_{2}$ or $u_{3}$, by choosing $u_{2}$ or $u_{3}$ instead of $u$, we are back in Case 1 . In addition, if $x_{2}$ or $x_{3}$ is adjacent to the 5 -vertex $u_{2}$ or $u_{3}$, by choosing $x_{2}$ or $x_{3}$ instead of $x_{1}$, we are also back in Case 1. Thus we assume that $N\left[\left\{u_{2}, u_{3}\right\}\right] \subseteq V_{\text {in }}(T)$. But since $G\left[N\left[u_{2}\right]\right]$ and $G\left[N\left[u_{3}\right]\right]$ are 5 -wheels, $u$ must be adjacent to at least four vertices of $N\left[\left\{u_{2}, u_{3}\right\}\right]$ as well as to $x_{1}$, and this is impossible since $d(u)=4$.

This completes the proof of Theorem 4.
Note that the only open case for the Total Coloring Conjecture in planar graphs is $\Delta=6$. By Theorem 4, immediately, we have the following theorem.

Theorem 5. The Total Coloring Conjecture holds in claw-free planar graphs.

## Acknowledgements

The author would like to thank the referees for valuable comments and suggestions. Especially, the author has followed closely the version suggested by one referee. This research was supported by the National Nature Science Foundation of China (No. 11601262).

## References

[1] L. Andersen, Total Colouring of Simple Graphs, Master's Thesis (University of Aalborg, 1993).
[2] M. Behzad, Graphs and Their Chromatic Numbers, PhD Thesis (Michigan State University, 1965).
[3] V.A. Bojarshinov, Edge and total coloring of interval graphs, Discrete Appl. Math. 114 (2001) 23-28. https://doi.org/10.1016/S0166-218X(00)00358-9
[4] O.V. Borodin, On the total coloring of planar graphs, J. Reine Angew. Math. 394 (1989) 180-185.
https://doi.org/10.1515/crll.1989.394.180
[5] O.V. Borodin, A.V. Kostochka and D.R. Woodall, List edge and list total colourings of multigraphs, J. Combin. Theory Ser. B 71 (1997) 184-204. https://doi.org/10.1006/jctb.1997.1780
[6] B.-L. Chen, H.-L. Fu and M.T. Ko, Total chromatic number and chromatic index of split graphs, J. Combin. Math. Combin. Comput. 17 (1995) 137-146.
[7] C.M.H. de Figueiredo, J. Meidanis and C.P. de Mello, Total-chromatic number and chromatic index of dually chordal graphs, Inform. Process. Lett. 70 (1999) 147-152. https://doi.org/10.1016/S0020-0190(99)00050-2
[8] Ł. Kowalik, J.S. Sereni and R. Škrekovski, Total-coloring of plane graphs with maximum degree nine, SIAM J. Discrete Math. 22 (2008) 1462-1479.
https://doi.org/10.1137/070688389
[9] A.V. Kostochka, The total coloring of a multigraph with maximal degree 4, Discrete Math. 17 (1977) 161-163. https://doi.org/10.1016/0012-365X(77)90146-7
[10] A.V. Kostochka, Exact upper bound for the total chromatic number of a graph, Proc. 24th Int. Wiss. Koll. Tech. Hochsch. Ilmenau (1979) 33-36, in Russian.
[11] A.V. Kostochka, The total chromatic number of any multigraph with maximum degree five is at most seven, Discrete Math. 162 (1996) 199-214. https://doi.org/10.1016/0012-365X(95)00286-6
[12] M.D. Plummer, Extending matchings in claw-free graphs, Discrete Math. 125 (1994) 301-307.
https://doi.org/10.1016/0012-365X(94)90171-6
[13] M. Rosenfeld, On the total coloring of certain graphs, Israel J. Math. 9 (1971) 396-402.
https://doi.org/10.1007/BF02771690
[14] D.P. Sanders and Y. Zhao, On total 9-coloring planar graphs of maximum degree seven, J. Graph Theory 31 (1999) 67-73.
https://doi.org/10.1002/(SICI)1097-0118(199905)31:1;67::AID-JGT6;3.0.CO;2-C
[15] E.F. Shan, Z.S. Liang and L.Y. Kang, Clique-transversal sets and clique-coloring in planar graphs, European J. Combin. 36 (2014) 367-376. https://doi.org/10.1016/j.ejc.2013.08.003
[16] N. Vijayaditya, On total chromatic number of a graph, J. London Math. Soc. (2) 3 (1971) 405-408.
https://doi.org/10.1112/jlms/s2-3.3.405
[17] V.G. Vizing, Some unsolved problems in graph theory, Uspekhi Mat. Nauk 23 (1968) 117-134, English translation in Russian Math. Surveys 23 (1968) 125-141, in Russian.
https://doi.org/10.1070/RM1968v023n06ABEH001252
[18] W.F. Wang, Total chromatic number of planar graphs with maximum degree ten, J. Graph Theory 54 (2007) 91-102. https://doi.org/10.1002/jgt. 20195
[19] B. Wang and J.-L. Wu, Total colorings of planar graphs with maximum degree seven and without intersecting 3-cycles, Discrete Math. 311 (2011) 2025-2030. https://doi.org/10.1016/j.disc.2011.05.038
[20] P. Wang and J.-L. Wu, A note on total colorings of planar graphs without 4-cycles, Discuss. Math. Graph Theory 24 (2004) 125-135.
https://doi.org/10.7151/dmgt. 1219
[21] H.-P. Yap, Total Colourings of Graphs, in: Lecture Notes in Math. 1623 (Springer, Berlin, Heidelberg, 1996).
https://doi.org/10.1007/BFb0092895
Received 14 May 2019
Revised 19 January 2020
Accepted 19 January 2020

