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# TOTAL COLORING OF CLAW-FREE PLANAR GRAPHS

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**Abstract**

A total coloring of a graph is an assignment of colors to both its vertices and edges so that adjacent or incident elements acquire distinct colors. Let $\Delta(G)$ be the maximum degree of $G$. Vizing conjectured that every graph has a total $(\Delta + 2)$-coloring. This Total Coloring Conjecture remains open even for planar graphs, for which the only open case is $\Delta = 6$. Claw-free planar graphs have $\Delta \leq 6$. In this paper, we prove that the Total Coloring Conjecture holds for claw-free planar graphs.

**Keywords:** total coloring, total coloring conjecture, planar graph, claw.

**2010 Mathematics Subject Classification:** 05C15.

## 1. Introduction

All graphs considered here are finite, simple and nonempty. Let $G = (V, E)$ be a graph with *vertex set* $V$ and *edge set* $E$. The number of vertices of $G$ is called the *order* of $G$. For a vertex $v \in V$, the *open neighborhood* $N(v)$ of $v$ is defined as the set of vertices adjacent to $v$, i.e., $N(v) = \{u \mid uv \in E\}$. The *closed neighborhood* of $v$ is $N[v] = N(v) \cup \{v\}$. Every vertex in $N(v)$ is also called a *neighbor* of $v$. The *degree* of $v$ is equal to $|N(v)|$, denoted by $d_G(v)$ or simply $d(v)$. By $\delta(G)$ and $\Delta(G)$, we denote the *minimum degree* and the *maximum degree* of the graph $G$, respectively. For a subset $S \subseteq V$, the *closed neighborhood* of $S$ is $N[S] = \bigcup_{v \in S} N[v]$ and the *closed* 2-*neighborhood* of $S$ is $N_2[S] = N[N[S]]$. For a subset $X \subseteq V$, the subgraph induced by $X$ is denoted by $G[X]$. The set of edges between $X$ and $Y$ in $E$ is denoted by $E(X, Y)$ for $X, Y \subseteq V$. As usual, $K_{m,n}$ denotes a complete bipartite graph with classes of cardinality $m$ and $n$, and $K_n$ and $C_n$ denote the complete graph and cycle of order $n$. The graph $K_{1,3}$ is also called a *claw*, and $K_3$ a *triangle*. Given a graph $F$, a graph $G$ is $F$-*free* if it does not contain $F$ as an induced subgraph. In particular, a $K_{1,3}$-free graph is *claw-free*. For a family $\{F_1, \dots, F_k\}$ of graphs, we say that $G$ is $(F_1, \dots, F_k)$-free if it is $F_i$-free for all $i$. By starting with a disjoint union of two graphs $G$ and $H$ and adding edges joining every vertex of $G$ to every vertex of $H$, one obtains the *join* of $G$ and $H$, denoted by $G \vee H$. The $n$-*wheel* is the graph $C_n \vee K_1$ and the *double* $n$-*wheel* is the graph $C_n \vee \overline{K}_2$ $(n \geq 4)$.

Given a graph $G$, an *element* of $G$ is a member of $V(G) \cup E(G)$. Let two elements of a graph $G$ be *adjacent* if they are either adjacent or incident in the traditional sense. Given a graph $G$, a *total* $k$-*coloring* of $G$ is a function that takes each element to $\{1, 2, \dots, k\}$ such that adjacent distinct elements receive distinct colors. In 1968, Vizing [17] (see also [2]) made the following conjecture, known as the Total Coloring Conjecture.

**Conjecture.** *Every graph has a total* $(\Delta + 2)$*-coloring.*

This conjecture is trivial for $\triangle \leq 2$. Rosenfeld [13] and Vijayaditya [16] proved it for $\Delta = 3$. Kostochka proved the $\Delta = 4$ [9] and $\Delta = 5$ [11] cases. The conjecture remains open even for planar graphs, but more is known. Borodin [4] proved it for planar graphs with $\Delta \geq 9$. The $\Delta = 8$ case was proved for planar graphs by Yap [21] and Andersen [1]. The $\Delta = 7$ case was proved for planar graphs by Sanders and Zhao [14]. Thus the only open case for planar graphs is the $\Delta = 6$ case. An extensive study on total coloring is done in [3, 5–8, 10, 18–20] and elsewhere. Claw-free planar graphs have maximum degree at most 6 [12]. In this paper, we prove that the Total Coloring Conjecture holds for claw-free planar graphs.

## 2. Total Coloring on the Claw-Free Planar Graphs

First we introduce some notation and lemmas which are useful for the total coloring of claw-free planar graphs. Let a $k$-*vertex* be a vertex of degree $k$. Let an *at most* $k$-*vertex*, or simply a $k^-$-vertex, be a vertex of degree at most $k$. Given a graph and integers $j_1, j_2, \dots, j_i$, let a $(j_1^-, j_2^-, \dots, j_i^-)$-vertex be an $i$-vertex $v$ of $G$ such that, for each $1 \leq m \leq i$, there is a $j_m^-$-vertex $y_m$ of $G$ and the vertices $y_1, y_2, \dots, y_i$ are distinct neighbors of $v$.

**Lemma 1** [15]**.** *If $G$ is a* (*claw*, $K_4$)*-free planar graph, then* $\Delta(G) \leq 5$ *and for every vertex $v$ of degree* 5 *in $G$,* $G[N[v]]$ *is a* 5*-wheel.*

**Lemma 2.** *Let $v$ be a* $(6^-, 6^-, 5^-)$*-vertex, or a* $(6, 6, 6)$*-vertex such that* $G[N[v]]$ *is not a claw, in a graph $G$. If* $G - v$ *has a total* 8*-coloring, then $G$ is total* 8*-colorable.*

***Proof.*** Let $v$ be a $(6^-, 6^-, 6^-)$-vertex of a graph G. Given a total 8-coloring $\phi$ of $G - v$, we will attempt to extend $\phi$ to a total 8-coloring of $G$. Let $N(v) = \{y_1, y_2, y_3\}$, and for $i \in \{1, 2, 3\}$ let $L_{vy_i}$ be the set of colors that are not used on $y_i$ or its incident edges and so are available for use on the edge $vy_i$. Then $|L_{vy_i}| \geq 2$ for each $i$, since $d_{G-v}(y_i) \leq 5$. Clearly, these edges can be properly colored unless $L_{vy_i}$ is the same set of two colors, say $L_{vy_i} = \{1, 2\}$, for each $i$. This is impossible if $v$ is a $(6^-, 6^-, 5^-)$-vertex, so assume that $v$ is a $(6, 6, 6)$-vertex such that $G[N[v]]$ is not a claw. Without loss of generality, $y_2y_3$ is an edge of $G$ and $\phi(y_2y_3) = 3$. Recolor $y_2y_3$ with color 1, and then color $vy_1$, $vy_2$ and $vy_3$ with colors 1, 2 and 3, respectively. Once the edges incident with $v$ are colored, there are at most six colors that cannot be used on $v$, and so $v$ can be colored. This gives a total 8-coloring of $G$, as required. ■

**Lemma 3.** *Let $v$ be a $(7^-, 6^-, 5^-, 4^-)$-vertex in a graph $G$ such that $G[N[v]]$ contains neither a claw nor a $K_4$. If $G - v$ has a total 8-coloring, then $G$ is total 8-colorable.*

***Proof.*** Let $N(v) = \{y_1, y_2, y_3, y_4\}$ where $d(y_i) \leq 8 - i$ for each $i$. Given a total coloring $\phi$ of $G - v$ using a set $C$ of eight colors, let $F$ be the set of colors that are used on the vertices in $N(v)$, and for $i \in \{1, 2, 3, 4\}$ let $L_{vy_i}$ be the set of colors that are available for use on the edge $vy_i$, as in the previous proof. Then $|L_{vy_i}| \geq i$ for each $i$, and so the edges $vy_1$, $vy_2$, $vy_3$, $vy_4$ can be properly colored in this order. If possible, do this so that at least one of these edges has a color in $F$. Call the new coloring $\phi'$.

It is now possible to color $v$ unless every color is used on an element adjacent to $v$. For this to happen, it must be that $|F| = 4$ and all the lists $L_{vy_i}$ are subsets of $C \setminus F$. In particular, $L_{vy_4} = C \setminus F$. Since $G[N[v]]$ contains neither a claw nor a $K_4$, $G$ must contain an edge $y_ry_4$ for some $r \in \{1, 2, 3\}$. Clearly $\phi(y_ry_4) \notin L_{vy_4}$, and so $\phi(y_ry_4) \in F$, while $\phi'(vy_r) \in L_{vy_r} \subseteq L_{vy_4}$. Interchange the colors of $vy_r$ and $y_ry_4$. Since $vy_r$ now has a color in $F$, there are at most seven different colors that are unavailable for $v$, and so $v$ can now be colored. This gives a total 8-coloring of $G$, as required. ■

**Theorem 4.** *Every claw-free planar graph is total 8-colorable.*

***Proof.*** Let $G$ be a claw-free planar graph. We prove the theorem by induction on the size $m = |E(G)|$. Suppose that the theorem holds when $G$ has fewer than $m$ edges. In the following, we will prove the theorem when $G$ has $m$ edges.

If $G$ has a cut vertex $v$, we can easily see that the theorem holds. So we may assume that $G$ is 2-connected. In addition, if $G$ has no $K_4$, then $\Delta(G) \leq 5$ by Lemma 1, and so the theorem holds since the Total Coloring Conjecture holds when $\Delta(G) \leq 5$ [11]. So let $K = [x_1x_2x_3x_4]$ be a $K_4$ of $G$, where $x_1$ is inside the cycle $C = [x_2x_3x_4]$ in the embedding of $G$ in the plane. Let $G'$ be the plane graph induced by the vertices inside and on $C$, and choose $K$ so that $G'$ has as fewer vertices as possible. Then $K$ is the only $K_4$ in $G'$.

Since every claw-free planar graph has maximum degree at most 6, the result follows from Lemma 2 if $G$ contains a $3^-$-vertex. Thus we may assume that $\delta(G) \geq 4$. Without loss of generality, we may assume that $x_1$ has a neighbor $u$ inside the triangle $T = [x_1x_2x_3]$ in the embedding of $G$ in the plane. Let $V_{\text{in}}(T)$ denote the set of vertices inside the triangle $T$, and let $G_T = G[V_{\text{in}}(T) \cup V(T)]$, the plane graph induced by the vertices inside and on $T$. We will make frequent use of the following facts.

(F1) $G_T$ is $K_4$-free. This is because $K$ is the only $K_4$ in $G'$.

(F2) Every vertex of $V_{\text{in}}(T)$ is adjacent to at most two vertices of $T$, by (F1).

(F3) Every vertex of $V_{\text{in}}(T)$ has degree 4 or 5, since $\delta(G) \geq 4$ and $\Delta(G_T) \leq 5$ by Lemma 1.

(F4) We may assume that no two 4-vertices in $V_{\text{in}}(T)$ are adjacent. Indeed, if they are, then each of them is a $(6^-, 6^-, 5^-, 4^-)$-vertex by (F2) and (F3), and so the result follows by Lemma 3.

(F5) For $i = 1, 2, 3$, the neighbors of $x_i$ in $V_{\text{in}}(T)$ induce a complete graph. Otherwise, there would be a claw centered on $x_i$ (including the edge $x_ix_4$).

(F6) Every vertex of $T$ is adjacent to 0, 1 or 2 adjacent vertices in $V_{\text{in}}(T)$. This follows from (F1) and (F5).

(F7) Every 5-vertex in $V_{\text{in}}(T)$ is adjacent to 0 or 2 vertices of $T$. Indeed, let $v$ be a 5-vertex in $V_{\text{in}}(T)$ that is adjacent to $x_1$ (say). By Lemma 1, $G[N[v]]$ is a 5-wheel. Thus $G[N[v]]$ contains two vertices $v_1$, $v_2$ that are adjacent to both $x_1$ and $v$ but not to each other. At least one of $v_1$, $v_2$ must be in $T$, by (F5). But the neighbors $x_1$, $v_1$, $v_2$ of $v$ cannot all be in $T$, by (F2). Thus $v$ is adjacent to exactly two vertices of $T$.

Recall that $x_1$ has a neighbor $u \in V_{\text{in}}(T)$. We consider two cases.

*Case* 1. $d(u) = 5$. By (F7), $u$ is adjacent to exactly one of $x_2$ and $x_3$, say $x_2$. By Lemma 2, $G[N[u]]$ is a 5-wheel. Let $x_1x_2u_1u_2u_3x_1$ be the 5-cycle of $G[N[u]]$ (see Figure 1). Then $u$ and $u_3$ are the two vertices of $V_{\text{in}}(T)$ that are adjacent to $x_1$, and $u$ and $u_1$ are the two vertices of $V_{\text{in}}(T)$ that are adjacent to $x_2$.

*Case* 1.1. $d(u_1) = 5$. Note that $u_1$ is adjacent to $x_2$ but cannot be adjacent to $x_1$, and so $u_1$ is adjacent to $x_3$ by (F7). So let $x_2x_3wu_2ux_2$ be the 5-cycle of $G[N[u_1]]$ (see Figure 1, left). Each vertex of $T$ is now adjacent to two vertices of $V_{\text{in}}(T)$, and so cannot be adjacent to any further vertex of $V_{\text{in}}(T)$, by (F6). Thus any further neighbor of $u_3$ cannot be adjacent to $x_1$ and so must be adjacent to $u_2$, to avoid a claw centered on $u_3$. If $d(u_2) = 4$, then the only possible further neighbor of $u_3$ is $w$, and so $u_2$, $u_3$ are adjacent 4-vertices, contrary to (F4). If however $d(u_2) = 5$, then the additional neighbor $q$ of $u_2$ can be adjacent only to $u_2$, $u_3$, $w$ of the vertices so far named (since it cannot be adjacent to $x_1$ or $x_3$), and so $q$ is a 3-vertex by the claw-freeness, which is not permissible, by (F3).

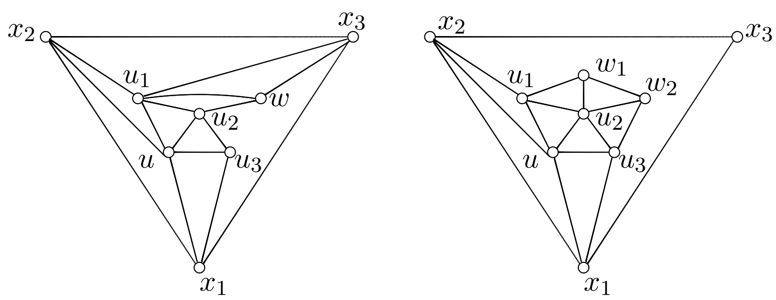


Figure 1. Cases 1.1 and 1.2 of Theorem 5.

*Case* 1.2. $d(u_1) = 4$. Then $d(u_2) = 5$, by (F4). Let $uu_1w_1w_2u_3u$ be the 5-cycle of $G[N[u_2]]$ (see Figure 1, right). Note that $x_1$ and $x_2$ each already have two neighbors in $V_{\text{in}}(T)$, and so $u_2$ has no neighbors in $T$, by (F6) and (F7). Thus $w_1 \in V_{\text{in}}(T)$ and so $d(w_1) = 5$, by (F4), since $d(u_1) = 4$. But it is impossible for $G[N[w_1]]$ to be a 5-wheel since $d(u_1) = 4$ and there is no edge $x_2w_1$.

*Case* 2. $d(u) = 4$. Let the neighbors of $u$ be $x_1$, $u_1$, $u_2$ and $u_3$. We may assume that $u_2, u_3 \in V_{\text{in}}(T)$ and possibly $u_1 = x_2$ or $u_1 = x_3$, by (F2). Then $d(u_2) = 5$ and $d(u_3) = 5$, by (F4). If $x_1$ is adjacent to the 5-vertex $u_2$ or $u_3$, by choosing $u_2$ or $u_3$ instead of $u$, we are back in Case 1. In addition, if $x_2$ or $x_3$ is adjacent to the 5-vertex $u_2$ or $u_3$, by choosing $x_2$ or $x_3$ instead of $x_1$, we are also back in Case 1. Thus we assume that $N[\{u_2, u_3\}] \subseteq V_{\text{in}}(T)$. But since $G[N[u_2]]$ and $G[N[u_3]]$ are 5-wheels, $u$ must be adjacent to at least four vertices of $N[\{u_2, u_3\}]$ as well as to $x_1$, and this is impossible since $d(u) = 4$.

This completes the proof of Theorem 4. ■

Note that the only open case for the Total Coloring Conjecture in planar graphs is $\Delta = 6$. By Theorem 4, immediately, we have the following theorem.

**Theorem 5.** *The Total Coloring Conjecture holds in claw-free planar graphs.*

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