

ALGORITHMIC ASPECTS OF THE INDEPENDENT 2-RAINBOW DOMINATION NUMBER AND INDEPENDENT ROMAN $\{2\}$ -DOMINATION NUMBER

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Abstract

A 2-rainbow dominating function (2RDF) of a graph G is a function g from the vertex set $V(G)$ to the family of all subsets of $\{1, 2\}$ such that for each vertex v with $g(v) = \emptyset$ we have $\bigcup_{u \in N(v)} g(u) = \{1, 2\}$. The minimum of $g(V(G)) = \sum_{v \in V(G)} |g(v)|$ over all such functions is called the 2-rainbow domination number. A 2RDF g of a graph G is independent if no two vertices assigned non empty sets are adjacent. The independent 2-rainbow domination number is the minimum weight of an independent 2RDF of G . A Roman $\{2\}$ -dominating function (R2DF) $f : V \rightarrow \{0, 1, 2\}$ of a graph $G = (V, E)$ has the property that for every vertex $v \in V$ with $f(v) = 0$ either there is $u \in N(v)$ with $f(u) = 2$ or there are $x, y \in N(v)$ with $f(x) = f(y) = 1$. The weight of f is the sum $f(V) = \sum_{v \in V} f(v)$. An R2DF f is called independent if no two vertices assigned non-zero values are adjacent. The independent Roman $\{2\}$ -domination number is the minimum weight of an independent R2DF on G .

We first show that the decision problem for computing the independent 2-rainbow (respectively, independent Roman $\{2\}$ -domination) number is NP-complete even when restricted to planar graphs. Then, we give a linear algorithm that computes the independent 2-rainbow domination number as well as the independent Roman $\{2\}$ -domination number of a given tree, answering problems posed in [M. Chellali and N. Jafari Rad, *Independent 2-rainbow domination in graphs*, *J. Combin. Math. Combin. Comput.* 94

(2015) 133–148] and [A. Rahmouni and M. Chellali, *Independent Roman $\{2\}$ -domination in graphs*, Discrete Appl. Math. 236 (2018) 408–414]. Then, we give a linear algorithm that computes the independent 2-rainbow domination number of a given unicyclic graph.

Keywords: independent 2-rainbow dominating function, independent Roman $\{2\}$ -dominating function, algorithm, 3-SAT.

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1. INTRODUCTION

For notation and terminology not given here we refer to [10]. Let $G = (V, E)$ be a graph with vertex set V of order n and edge set E . The *open neighborhood* of a vertex $v \in V$ is $N(v) = \{u \in V : uv \in E\}$ and the *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. The *degree* of v is $\deg(v) = |N(v)|$. A vertex of degree one is referred as a *leaf* and its unique neighbor is called a *support vertex*. A tree T of order $n \geq 2$ is called a *star* if $n = 2$ or $n \geq 3$ and T contains exactly one vertex that is not a leaf. A *double star* is a tree with precisely two vertices (as central vertices) that are not leaves. A path of order n is denoted by P_n . A *unicyclic* graph is a graph obtained from a tree T of order at least three by joining precisely two non-adjacent vertices of T . A *planar* graph is a graph that can be drawn on the plane in such a way that its edges intersect only at their endpoints.

A function $f : V \rightarrow \{0, 1, 2\}$ is a *Roman dominating function* (RDF) of a graph $G = (V, E)$ if every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The *weight* of an RDF f , denoted by $w(f)$, is the sum $f(V) = \sum_{v \in V} f(v)$. The minimum weight of an RDF on G is called the *Roman domination number* of G and is denoted by $\gamma_R(G)$. The mathematical concept of Roman domination, was defined and discussed by Stewart [16], and ReVelle and Rosing [13], and subsequently developed by Cockayne *et al.* [8]. For an RDF f on G , we denote by V_i (or V_i^f to refer to f) the set of all the vertices of G with label i under f . Thus an RDF f can be represented by a triple (V_0, V_1, V_2) , and we can use the notation $f = (V_0, V_1, V_2)$.

In a recent paper, Chellali *et al.* [6] introduced a new variant of Roman dominating functions. A *Roman $\{2\}$ -dominating function* (R2DF) $f : V \rightarrow \{0, 1, 2\}$ of G has the property that for every vertex $v \in V$ with $f(v) = 0$ either there is $u \in N(v)$ with $f(u) = 2$ or there are $x, y \in N(v)$ with $f(x) = f(y) = 1$. The weight of a Roman $\{2\}$ -dominating function f on G is the sum $f(V) = \sum_{v \in V} f(v)$ and the minimum weight of a Roman $\{2\}$ -dominating function f is the *Roman $\{2\}$ -domination number* of G , denoted by $\gamma_{R2}(G)$. Rahmouni and Chellali [12] introduced *independent Roman $\{2\}$ -dominating function* (IR2DF) as a Roman $\{2\}$ -dominating function $f = (V_0, V_1, V_2)$ for which $V_1 \cup V_2$ is an

independent set. The *independent Roman $\{2\}$ -domination number* of G , denoted by $i_{\{R2\}}(G)$, is the minimum weight of an IR2DF on G . They showed that the decision problem associated with $i_{\{R2\}}(G)$ is NP-complete even when restricted to bipartite graphs. They posed the following open problem.

Problem 1 (Rahmouni and Chellali [12]). Can you design a linear algorithm for computing the value of $i_{\{R2\}}(T)$ for any tree T ?

Chen and Lu [7] have answered Problem 1 by giving an algorithm that computes the independent Roman $\{2\}$ -domination in trees. In this paper we answer Problem 1 by a different approach.

Let f be a function on the vertex set of a graph G that assigns to each vertex a set of colors chosen from the set $\{1, 2\}$; that is $f : V(G) \rightarrow \mathcal{P}(\{1, 2\})$, where $\mathcal{P}(\{1, 2\})$ is the power set of $\{1, 2\}$. If for each vertex $v \in V(G)$ such that $f(v) = \emptyset$, we have $\bigcup_{u \in N(v)} f(u) = \{1, 2\}$, then f is called a *2-rainbow dominating function* (2RDF) of G . The weight of a 2RDF f is defined as $w(f) = \sum_{v \in V(G)} |f(v)|$. The minimum weight of a 2-rainbow dominating function is called the *2-rainbow domination number* of G , denoted by $\gamma_{r2}(G)$. We say that a function f is a $\gamma_{r2}(G)$ -function if it is a 2RDF and $w(f) = \gamma_{r2}(G)$. For a 2RDF f we let $V_1^f = \{v : f(v) = \{1\}\}$. Similarly, V_2^f, V_{12}^f and V_0^f are defined. So, we will write $f = (V_0^f, V_1^f, V_2^f, V_{12}^f)$.

A function f is called an *independent 2-rainbow dominating function* (I2RDF) of G , if f is a 2RDF and no two vertices in $V(G) - V_0^f$ are adjacent. The *independent 2-rainbow domination number*, denoted by $i_{r2}(G)$, is the minimum weight of an independent 2-rainbow dominating function of G . We say that a function f is an $i_{r2}(G)$ -function if it is an I2RDF and $w(f) = i_{r2}(G)$. The concept of rainbow domination was introduced by Brešar, Henning, and Rall [2], and further studied by several authors (see for example, [1, 3, 5, 14, 15, 17]). Chellali *et al.* [4] posed the following problem.

Problem 2 (Chellali and Jafari Rad [4]). Is there a polynomial algorithm for computing the independent 2-rainbow domination number for trees?

In this paper we study algorithmic and complexity of the independent 2-rainbow domination number as well as the independent Roman $\{2\}$ -domination number. In Section 2, we show that the decision problem for computing the independent 2-rainbow (respectively, independent Roman $\{2\}$ -domination) number is NP-complete even when restricted to chordal graphs. In Section 3, we first give a linear algorithm that computes the independent 2-rainbow domination number of a given tree, answering Problem 2. We then answer Problem 1 using the following Corollary 1 of [12]. Then we give a linear algorithm that computes the independent 2-rainbow domination number of a given unicyclic graph.

Corollary 1 (Rahmouni and Chellali [12]). *If T is a tree, then $i_{\{R2\}}(T) = i_{r2}(T)$.*

2. NP-COMPLETENESS RESULTS

Consider the following decision problems related to the optimization problems of computing the independent 2-rainbow domination number and the independent Roman $\{2\}$ -domination number of a given graph.

Independent 2-Rainbow Domination (I2RD) Problem:

Instance: A graph G and a positive integer m .

Question: Does an I2RDF f exist on G with $w(f) \leq m$?

Independent Roman $\{2\}$ -Domination (IR2D) Problem:

Instance: A graph G and a positive integer m .

Question: Is there an IR2DF f on G with $w(f) \leq m$?

We introduce a polynomial time reduction from PLANAR 3-SAT Problem to I2RD and IR2D Problems to show that I2RD and IR2D Problems are NP-complete even when restricted to planar graphs. Recall that 3-SAT is the problem of deciding whether a given boolean formula in 3-conjunctive normal form is satisfiable. Let $\Phi = \{\mathcal{C}, \mathcal{X}\}$ be an instance in 3-SAT Problem, that is, let Φ be a boolean formula in 3-conjunctive normal form. Let $\mathcal{C} = \{c_1, c_2, \dots, c_l\}$ be a set of $l \geq 1$ clauses over a set $\mathcal{X} = \{x_1, \dots, x_k\}$ of $k \geq 3$ variables. For each $1 \leq j \leq l$, the clause c_j (consisting of exactly three literals) is of the form $c_j = \{y_{1j}, y_{2j}, y_{3j}\}$, where each of y_{1j} , y_{2j} and y_{3j} is either a variable or the negative of a variable in \mathcal{X} . A natural graph associated to 3-SAT Problem is the bipartite graph $G_{\{\mathcal{C}, \mathcal{X}\}}$ that has $\mathcal{C} \cup \mathcal{X}$ as its vertex set and has an edge between the vertices x_i and c_j if c_j contains either x_i or $\neg x_i$. PLANAR 3-SAT is 3-SAT restricted to those instances $\{\mathcal{C}, \mathcal{X}\}$ for which $G_{\{\mathcal{C}, \mathcal{X}\}}$ is planar. It is well-know that PLANAR 3-SAT Problem is NP-complete [9, 11]. Let $\Phi = \{\mathcal{C}, \mathcal{X}\}$ be an instance of 3-SAT Problem such that the associated graph $G_{\{\mathcal{C}, \mathcal{X}\}}$ to Φ is planar. We construct graph G_Φ corresponding to Φ as follows.

Assume that H is a planar embedding of $G_{\{\mathcal{C}, \mathcal{X}\}}$. We replace each variable-vertex x_i of H , where $1 \leq i \leq k$, by a graph H_i as variable gadget, where H_i is obtained from a cycle graph of order $4l$ with vertices u_i^1, \dots, u_i^{4l} such that each of vertices $u_i^{4j-3}, u_i^{4j-2}, u_i^{4j-1}, u_i^{4j}$ is adjacent to a new vertex v_i^j for each $1 \leq j \leq l$. We replace each clause-vertex c_j of H , where $1 \leq j \leq l$, by a new vertex z_j . In the rest we fix indices i and j , where $1 \leq i \leq k$ and $1 \leq j \leq l$. Assume that $c_{i_1}, c_{i_2}, \dots, c_{i_m}$, where $1 \leq m \leq l$, is the sequence of all clause-vertices adjacent to x_i in H in clockwise direction starting from an arbitrary clause-vertex in $N_H(x_i)$, that is, $x_i c_{j'}$ is an edge of H for each $j' \in \{i_1, i_2, \dots, i_m\}$. If $x_i \in c_{i_r}$ (respectively, $\neg x_i \in c_{j_r}$), where $1 \leq r \leq m$, then we replace $x_i c_{j_r}$ by new edges $u_i^{4r-2} z_{j_r}$ and $u_i^{4r} z_{j_r}$ (respectively, $u_i^{4r-3} z_{j_r}$ and $u_i^{4r-1} z_{j_r}$). Let H_Φ be the resulting graph. See Figure 1. It is easy to see that H_Φ is a planar graph.

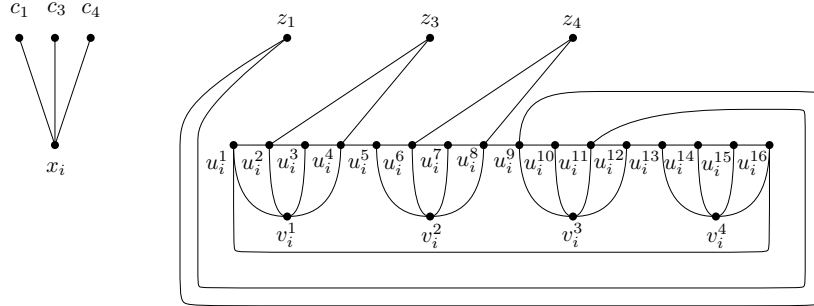


Figure 1. Illustration of replacing clause-vertex c_j by z_j for each $j \in \{1, 3, 4\}$, variable-vertex x_i by H_i , clause-edge $x_i c_3$ by edges $u_i^2 z_3$ and $u_i^4 z_3$, clause-edge $x_i c_4$ by edges $u_i^6 z_4$ and $u_i^8 z_4$ and clause-edge $x_i c_1$ by edges $u_i^9 z_1$ and $u_i^{11} z_1$ for which $l = 4$, $N_H(x_i) = \{c_3, c_4, c_1\}$, $x_i \in c_3$, $x_i \in c_4$ and $\neg x_i \in c_1$.

Lemma 2. *The boolean formula Φ is satisfiable if and only if there is an IR2DF (respectively, I2RDF) f on H_Φ with $w(f) \leq 2kl$.*

Proof. Assume that Φ is satisfiable. Let T be an assignment of truth values for the variables of \mathcal{X} for which Φ evaluates to *true*. We construct sets V_1, V'_1 and V'_2 on the vertex set of H_Φ as follows. If T assigns the value *true* to x_i , then we add all vertices in $\{u_i^{2j} : 1 \leq j \leq 2l\}$, $\{u_i^{4j-2} : 1 \leq j \leq l\}$ and $\{u_i^{4j} : 1 \leq j \leq l\}$ to V_1, V'_1 and V'_2 , respectively. If T assigns the value *false* to x_i , then we add all vertices in $\{u_i^{2j-1} : 1 \leq j \leq 2l\}$, $\{u_i^{4j-3} : 1 \leq j \leq l\}$ and $\{u_i^{4j-1} : 1 \leq j \leq l\}$ to V_1, V'_1 and V'_2 , respectively. It is easy to see that $f = (V(H_\Phi) - V_1, V_1, \emptyset)$ (respectively, $f = (V(H_\Phi) - (V'_1 \cup V'_2), V'_1, V'_2, \emptyset)$) is an IR2DF (respectively, I2RDF) on H_Φ with $w(f) = 2kl$.

Assume that there is an IR2DF $f = (V_0, V_1, V_2)$ (respectively, I2RDF $f = (V'_0, V'_1, V'_2, V'_2)$) on H_Φ with $w(f) \leq 2kl$. Consider values $f(v_i^j)$, $f(u_i^{4j-3})$, $f(u_i^{4j-2})$, $f(u_i^{4j-1})$, and $f(u_i^{4j})$ for each $1 \leq i \leq k$ and $1 \leq j \leq l$. Since at least two of vertices $u_i^{4j-3}, u_i^{4j-2}, u_i^{4j-1}, u_i^{4j}$ are not adjacent to vertex z_j for all $1 \leq j \leq l$, we find that $S_{ij} = f(v_i^j) + \sum_{s=0}^3 f(u_i^{4j-s}) \geq 2$ (respectively, $S_{ij} = |f(v_i^j)| + \sum_{s=0}^3 |f(u_i^{4j-s})| \geq 2$) for each $1 \leq i \leq k$ and $1 \leq j \leq l$. So, $w(f) \geq \sum_{i \in \{1, \dots, k\}, j \in \{1, \dots, l\}} S_{ij} \geq 2kl$. Since $w(f) \leq 2kl$, we have $S_{ij} = 2$ and $f(z_j) = 0$ (respectively, $|f(z_j)| = 0$) for each $1 \leq i \leq k$ and $1 \leq j \leq l$. Since $f(z_j) = 0$ (respectively, $|f(z_j)| = 0$) for each $1 \leq j \leq l$, if $f(v_i^j) \neq 0$ (respectively, $|f(v_i^j)| \neq 0$) for some $1 \leq i \leq k$ and $1 \leq j \leq l$, then $S_{ij} > 2$, a contradiction. So, $f(v_i^j) = 0$ (respectively, $|f(v_i^j)| = 0$) for each $1 \leq i \leq k$ and $1 \leq j \leq l$. Thus, either both $f(u_i^{2j-1}) = 1$ and $f(u_i^{2j}) = 0$ or both $f(u_i^{2j-1}) = 0$ and $f(u_i^{2j}) = 1$

(respectively, either $f(u_i^{4j-1}) \neq \emptyset$, $f(u_i^{4j-3}) \neq \emptyset$, $f(u_i^{4j-1}) \cup f(u_i^{4j-3}) = \{1, 2\}$ and $f(u_i^{2j}) = \emptyset$ or $f(u_i^{4j}) \neq \emptyset$, $f(u_i^{4j-2}) \neq \emptyset$, $f(u_i^{4j}) \cup f(u_i^{4j-2}) = \{1, 2\}$ and $f(u_i^{2j-1}) = \emptyset$) for each $1 \leq i \leq k$ and $1 \leq j \leq l$.

We fix indices i and j , where $1 \leq i \leq k$ and $1 \leq j \leq l$. If both $f(u_i^{2j-1}) = 0$ and $f(u_i^{2j}) = 1$ (respectively, $f(u_i^{4j-1}) \neq \emptyset$, $f(u_i^{4j-3}) \neq \emptyset$, $f(u_i^{4j-1}) \cup f(u_i^{4j-3}) = \{1, 2\}$ and $f(u_i^{2j}) = \emptyset$), then we assign the value *true* to the variable x_i and if both $f(u_i^{2j-1}) = 1$ and $f(u_i^{2j}) = 0$ (respectively, $f(u_i^{4j}) \neq \emptyset$, $f(u_i^{4j-2}) \neq \emptyset$, $f(u_i^{4j}) \cup f(u_i^{4j-2}) = \{1, 2\}$ and $f(u_i^{2j-1}) = \emptyset$), then we assign the value *false* to the variable x_i . We claim that Φ is satisfiable for this assignment.

Assume without loss of generality that $c_j = \{x_1, \neg x_2, x_6\}$. Since $f(z_j) = 0$ (respectively, $|f(z_j)| = 0$), we have $f(u_1^{4j'-2}) = f(u_1^{4j'}) = 1$, $f(u_2^{4j''-3}) = f(u_2^{4j''-1}) = 1$ or $f(u_6^{4j'''-2}) = f(u_6^{4j'''} = 1$ (respectively, $f(u_1^{4j'-2}) \cup f(u_1^{4j'}) = \{1, 2\}$, $f(u_2^{4j''-3}) \cup f(u_2^{4j''-1}) = \{1, 2\}$ or $f(u_6^{4j'''-2}) \cup f(u_6^{4j'''} = \{1, 2\}$) for some $j', j'', j''' \in \{1, 2, \dots, l\}$. Assume without loss of generality that $f(u_1^{4j'-2}) = f(u_1^{4j'}) = 1$ (respectively, $f(u_1^{4j'-2}) \cup f(u_1^{4j'}) = \{1, 2\}$). So, x_1 has the value *true*. It causes to satisfy the clause c_j , that is, the boolean formula Φ is satisfiable. This completes the proof. \blacksquare

Clearly, we can compute H_Φ in polynomial time. By Lemma 2 and the facts that H_Φ is a planar graph and both IR2D and I2RD Problems belong to NP we have the following.

Theorem 3. *Both IR2D and I2RD Problems are NP-complete even when restricted to planar graphs.*

3. LINEAR ALGORITHMS FOR TREES AND UNICYCLIC GRAPHS

In this section we first give a linear algorithm that computes the independent 2-rainbow domination number as well as the independent Roman $\{2\}$ -domination number of a given tree. Finally, using the above algorithm we give a linear algorithm that computes the independent 2-rainbow domination number of a given unicyclic graph.

3.1. Trees

In this section, we give a linear algorithm (Algorithm 3.1) that computes the independent 2-rainbow domination number of trees. We say that a rooted tree T with the vertex set $V = \{v_1, v_2, \dots, v_n\}$ has Property 1 if $j < i$, where v_j is the parent of $v_i \in V$. Let $G = (V, E)$ be a graph such that $u \in V$ and a vertex $v \notin V$. We define the following.

- $i_{r_2}(G, u = 0) = \min\{w(f) : f \text{ is an I2RDF on } G \text{ with } f(u) = \emptyset\},$
- $i_{r_2}(G, u = 1) = \min\{w(f) : f \text{ is an I2RDF on } G \text{ with } f(u) = \{1\}\},$
- $i_{r_2}(G, u = 2) = \min\{w(f) : f \text{ is an I2RDF on } G \text{ with } f(u) = \{2\}\},$
- $i_{r_2}(G, u = 12) = \min\{w(f) : f \text{ is an I2RDF on } G \text{ with } f(u) = \{1, 2\}\},$
- $i'_{r_2}(G, u, v = 1) = \min\{w(f) : f \text{ is an I2RDF on } G + uv \text{ with } f(u) = \emptyset \text{ and } f(v) = \{1\}\},$
- $i'_{r_2}(G, u, v = 2) = \min\{w(f) : f \text{ is an I2RDF on } G + uv \text{ with } f(u) = \emptyset \text{ and } f(v) = \{2\}\},$
- $i'_{r_2}(G, u, v = 12) = \min\{w(f) : f \text{ is an I2RDF on } G + uv \text{ with } f(u) = \emptyset \text{ and } f(v) = \{1, 2\}\}.$

Lemma 4. *Let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be two graphs with $V_1 \cap V_2 = \emptyset$ such that $u \in V_1, v \in V_2$ and a vertex $w \notin V_1 \cup V_2$. Let $G = (V_1 \cup V_2, E_1 \cup E_2 \cup \{uw\})$. Then,*

- (i) $i_{r_2}(G, u = 0) = \min\{i_{r_2}(H_1, u = 0) + i_{r_2}(H_2, v = 0), i'_{r_2}(H_1, u, w = 1) + i_{r_2}(H_2, v = 1) - 1, i'_{r_2}(H_1, u, w = 2) + i_{r_2}(H_2, v = 2) - 1, i'_{r_2}(H_1, u, w = 12) + i_{r_2}(H_2, v = 12) - 2\},$
- (ii) $i_{r_2}(G, u = 1) = i_{r_2}(H_1, u = 1) + i'_{r_2}(H_2, v, w = 1) - 1,$
- (iii) $i_{r_2}(G, u = 2) = i_{r_2}(H_1, u = 2) + i'_{r_2}(H_2, v, w = 2) - 1,$
- (iv) $i_{r_2}(G, u = 12) = i_{r_2}(H_1, u = 12) + i'_{r_2}(H_2, v, w = 12) - 2,$
- (v) $i'_{r_2}(G, u, w = 1) = \min\{i'_{r_2}(H_1, u, w = 1) + i_{r_2}(H_2, v = 0), i'_{r_2}(H_1, u, w = 1) + i_{r_2}(H_2, v = 1), i'_{r_2}(H_1, u, w = 12) + i_{r_2}(H_2, v = 2) - 1, i'_{r_2}(H_1, u, w = 12) + i_{r_2}(H_2, v = 12) - 1\},$
- (vi) $i'_{r_2}(G, u, w = 2) = \min\{i'_{r_2}(H_1, u, w = 2) + i_{r_2}(H_2, v = 0), i'_{r_2}(H_1, u, w = 12) + i_{r_2}(H_2, v = 1) - 1, i'_{r_2}(H_1, u, w = 2) + i_{r_2}(H_2, v = 2), i'_{r_2}(H_1, u, w = 12) + i_{r_2}(H_2, v = 12) - 1\},$
- (vii) $i'_{r_2}(G, u, w = 12) = \min\{i'_{r_2}(H_1, u, w = 12) + i_{r_2}(H_2, v = 0), i'_{r_2}(H_1, u, w = 12) + i_{r_2}(H_2, v = 1), i'_{r_2}(H_1, u, w = 12) + i_{r_2}(H_2, v = 2), i'_{r_2}(H_1, u, w = 12) + i_{r_2}(H_2, v = 12)\}.$

Proof. Let f be a 2RDF on G and let f_1 and f_2 be restrictions of f to H_1 and H_2 , respectively. Let $f' = g(w) \cup f, f'_1 = g(w) \cup f_1$ and $f'_2 = g(w) \cup f_2$, where $g(w) \in \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}.$

Algorithm 3.1: I2RDNT(T)

Input: A connected rooted tree T with $V(T) = \{v_1, \dots, v_n\}$, Property 1 and a vertex $w \notin V(T)$.

Output: $(i_{r_2}(T, v_1 = 0), i_{r_2}(T, v_1 = 1), i_{r_2}(T, v_1 = 2), i_{r_2}(T, v_1 = 12), i'_{r_2}(T, v_1, w = 1), i'_{r_2}(T, v_1, w = 2), i'_{r_2}(T, v_1, w = 12))$.

- 1 **for** $i = 1$ **to** n **do**
- 2 $i_{r_2}(v_i = 0) = \infty; i_{r_2}(v_i = 1) = 1; i_{r_2}(v_i = 2) = 1; i_{r_2}(v_i = 12) = 2;$
 $i'_{r_2}(v_i, w = 1) = \infty; i'_{r_2}(v_i, w = 2) = \infty; i'_{r_2}(v_i, w = 12) = 2;$
- 3 **for** $i = n$ **to** 2 **do**
- 4 Let v_j be the parent of v_i ;
- 5 $i_{r_2}(v_j = 0) = \min\{i_{r_2}(v_j = 0) + i_{r_2}(v_i = 0), i'_{r_2}(v_j, w = 1) + i_{r_2}(v_i = 1)$
 $- 1, i'_{r_2}(v_j, w = 2) + i_{r_2}(v_i = 2) - 1, i'_{r_2}(v_j, w = 12) + i_{r_2}(v_i = 12) - 2\};$
- 6 $i_{r_2}(v_j = 1) = i_{r_2}(v_j = 1) + i'_{r_2}(v_i, w = 1) - 1;$
- 7 $i_{r_2}(v_j = 2) = i_{r_2}(v_j = 2) + i'_{r_2}(v_i, w = 2) - 1;$
- 8 $i_{r_2}(v_j = 12) = i_{r_2}(v_j = 12) + i'_{r_2}(v_i, w = 12) - 2;$
- 9 $i'_{r_2}(v_j, w = 1) = \min\{i'_{r_2}(v_j, w = 1) + i_{r_2}(v_i = 0), i'_{r_2}(v_j, w = 1)$
 $+ i_{r_2}(v_i = 1), i'_{r_2}(v_j, w = 12) + i_{r_2}(v_i = 2) - 1, i'_{r_2}(v_j, w = 12)$
 $+ i_{r_2}(v_i = 12) - 1\};$
- 10 $i'_{r_2}(v_j, w = 2) = \min\{i'_{r_2}(v_j, w = 2) + i_{r_2}(v_i = 0), i'_{r_2}(v_j, w = 12)$
 $+ i_{r_2}(v_i = 1) - 1, i'_{r_2}(v_j, w = 2) + i_{r_2}(v_i = 2), i'_{r_2}(v_j, w = 12)$
 $+ i_{r_2}(v_i = 12) - 1\};$
- 11 $i'_{r_2}(v_j, w = 12) = i'_{r_2}(v_j, w = 12) + \min\{i_{r_2}(v_i = 0), i_{r_2}(v_i = 1),$
 $i_{r_2}(v_i = 2), i_{r_2}(v_i = 12)\};$
- 12 **return** $(i_{r_2}(v_1 = 0), i_{r_2}(v_1 = 1), i_{r_2}(v_1 = 2), i_{r_2}(v_1 = 12), i'_{r_2}(v_1, w = 1),$
 $i'_{r_2}(v_1, w = 2), i'_{r_2}(v_1, w = 12));$

Clearly, $f(u) \in \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ and $f(v) \in \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. Clearly, $f(u) = A$ if and only if $f(u) = A$ and $f(v) = \emptyset$, $f(u) = A$ and $f(v) = \{1\}$, $f(u) = A$ and $f(v) = \{2\}$ or $f(u) = A$ and $f(v) = \{1, 2\}$ for each $A \in \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$.

Let $f(u) = \emptyset$. Hence, f is an I2RDF on G with $f(u) = \emptyset$ if and only if f_1 is an I2RDF on H_1 with $f_1(u) = \emptyset$ and f_2 is an I2RDF on H_2 with $f_2(v) = \emptyset$, f'_1 is an I2RDF on $H_1 + uw$ with both $f'_1(u) = \emptyset$ and $f'_1(w) = \{1\}$ and f_2 is an I2RDF on H_2 with $f_2(v) = \{1\}$, f'_1 is an I2RDF on $H_1 + uw$ with both $f'_1(u) = \emptyset$ and $f'_1(w) = \{2\}$ and f_2 is an I2RDF on H_2 with $f_2(v) = \{2\}$ or f'_1 is an I2RDF on $H_1 + uw$ with both $f'_1(u) = \emptyset$ and $f'_1(w) = \{1, 2\}$ and f_2 is an I2RDF on H_2 with $f_2(v) = \{1, 2\}$. This completes the proof of part (i).

Let $f(u) = \{1\}$. Hence, f is an I2RDF on G with $f(u) = \{1\}$ if and only if f_1 is an I2RDF on H_1 with $f_1(u) = \{1\}$ and f'_2 is an I2RDF on $H_2 + vw$ with both $f'_2(v) = \emptyset$ and $f'_2(w) = \{1\}$. This completes the proof of part (ii).

Let $f(u) = \{2\}$. Hence, f is an I2RDF on G with $f(u) = \{2\}$ if and only if f_1 is an I2RDF on H_1 with $f_1(u) = \{2\}$ and f'_2 is an I2RDF on $H_2 + vw$ with both $f'_2(v) = \emptyset$ and $f'_2(w) = \{2\}$. This completes the proof of part (iii).

Let $f(u) = \{1, 2\}$. Hence, f is an I2RDF on G with $f(u) = \{1, 2\}$ if and only if f_1 is an I2RDF on H_1 with $f_1(u) = \{1, 2\}$ and f'_2 is an I2RDF on $H_2 + vw$ with both $f'_2(v) = \emptyset$ and $f'_2(w) = \{1, 2\}$. This completes the proof of part (iv).

Let $f'(u) = \emptyset$ and $f'(w) = \{1\}$. Hence, f' is an I2RDF on $G + uw$ with both $f'(u) = \emptyset$ and $f'(w) = \{1\}$ if and only if f'_1 is an I2RDF on $H_1 + uw$ with both $f'_1(u) = \emptyset$ and $f'_1(w) = \{1\}$ and f_2 is an I2RDF on H_2 with $f_2(v) = \emptyset$, f'_1 is an I2RDF on $H_1 + uw$ with both $f'_1(u) = \emptyset$ and $f'_1(w) = \{1\}$ and f_2 is an I2RDF on H_2 with $f_2(v) = \{1\}$, f'_1 is an I2RDF on $H_1 + uw$ with both $f'_1(u) = \emptyset$ and $f'_1(w) = \{1, 2\}$ and f_2 is an I2RDF on H_2 with $f_2(v) = \{2\}$, f'_1 is an I2RDF on $H_1 + uw$ with both $f'_1(u) = \emptyset$ and $f'_1(w) = \{1, 2\}$ and f_2 is an I2RDF on H_2 with $f_2(v) = \{1, 2\}$. This completes the proof of part (v).

Let $f'(u) = \emptyset$ and $f'(w) = \{2\}$. Hence, f' is an I2RDF on $G + uw$ with both $f'(u) = \emptyset$ and $f'(w) = \{2\}$ if and only if f'_1 is an I2RDF on $H_1 + uw$ with both $f'_1(u) = \emptyset$ and $f'_1(w) = \{2\}$ and f_2 is an I2RDF on H_2 with $f_2(v) = \emptyset$, f'_1 is an I2RDF on $H_1 + uw$ with both $f'_1(u) = \emptyset$ and $f'_1(w) = \{1, 2\}$ and f_2 is an I2RDF on H_2 with $f_2(v) = \{1\}$, f'_1 is an I2RDF on $H_1 + uw$ with both $f'_1(u) = \emptyset$ and $f'_1(w) = \{2\}$ and f_2 is an I2RDF on H_2 with $f_2(v) = \{2\}$, f'_1 is an I2RDF on $H_1 + uw$ with both $f'_1(u) = \emptyset$ and $f'_1(w) = \{1, 2\}$ and f_2 is an I2RDF on H_2 with $f_2(v) = \{1, 2\}$. This completes the proof of part (vi).

Let $f'(u) = \emptyset$ and $f'(w) = \{1, 2\}$. Hence, f' is an I2RDF on $G + uw$ with both $f'(u) = \emptyset$ and $f'(w) = \{1, 2\}$ if and only if f'_1 is an I2RDF on $H_1 + uw$ with both $f'_1(u) = \emptyset$ and $f'_1(w) = \{1, 2\}$ and f_2 is an I2RDF on H_2 with $f_2(v) = \emptyset$, f'_1 is an I2RDF on $H_1 + uw$ with both $f'_1(u) = \emptyset$ and $f'_1(w) = \{1, 2\}$ and f_2 is an I2RDF on H_2 with $f_2(v) = \{1\}$, f'_1 is an I2RDF on $H_1 + uw$ with both $f'_1(u) = \emptyset$ and $f'_1(w) = \{1, 2\}$ and f_2 is an I2RDF on H_2 with $f_2(v) = \{2\}$, f'_1 is an I2RDF on $H_1 + uw$ with both $f'_1(u) = \emptyset$ and $f'_1(w) = \{1, 2\}$ and f_2 is an I2RDF on H_2 with $f_2(v) = \{1, 2\}$. This completes the proof of part (vii). ■

Lemma 5. *Let T be a tree with $u \in V(T)$ and a vertex $w \notin V(T)$. Algorithm 3.1 computes values $i_{r_2}(T, u = a)$ and $i'_{r_2}(T, u, w = b)$ in linear time for each $a \in \{0, 1, 2, 12\}$ and $b \in \{1, 2, 12\}$.*

Proof. We can compute a rooted tree T' with Property 1 and the root u for T in linear time. Clearly, $i_{r_2}(T, u = a) = i_{r_2}(T', u = a)$ and $i'_{r_2}(T, u, w = b) = i'_{r_2}(T', u, w = b)$ for each $a \in \{0, 1, 2, 12\}$ and $b \in \{1, 2, 12\}$. By Lemma 4, Algorithm I2RDNT(T') computes these values. The running time of each iteration of the for loops of Algorithm I2RDNT(T') is $\mathcal{O}(1)$ and so the running time of Algorithm 3.1 is linear. This completes the proof. ■

The following is clear.

Corollary 6. *Let $G = (V, E)$ be a graph such that $u \in V$. Then, $i_{r_2}(G) = \min\{i_{r_2}(G, u = 0), i_{r_2}(G, u = 1), i_{r_2}(G, u = 2), i_{r_2}(G, u = 12)\}$.*

By Corollaries 1 and 6 and Lemma 5 we have the following.

Theorem 7. *There is a linear algorithm that computes the independent 2-rainbow domination number and the independent Roman $\{2\}$ -domination number of a given tree.*

Note that Theorem 7 provides answers to Problems 1 and 2.

3.2. Computing independent 2-rainbow domination number of unicyclic graphs

In this section using Algorithm 3.1 we give a linear algorithm that computes the independent 2-rainbow domination number of a given unicyclic graph. Recall that a connected unicyclic graph is a connected graph with a unique cycle. Let $A, C \in \{\{1\}, \{2\}, \{1, 2\}\}$ and $B \in \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$, let $G = (V, E)$ be a graph with $u, v \in V$ and vertices $w, z \notin V$. We define the following.

- $i_{r_2}(G, u = \emptyset, v = B) = \min\{w(f) : f \text{ is an I2RDF on } G \text{ with } f(u) = \emptyset \text{ and } f(v) = B\}$,
- $i_{r_2}(G, u = B, v = \emptyset) = \min\{w(f) : f \text{ is an I2RDF on } G \text{ with } f(u) = B \text{ and } f(v) = \emptyset\}$,
- $i'_{r_2}(G, u, w = A, v = B) = \min\{w(f) : f \text{ is an I2RDF on } G + uw \text{ with } f(u) = \emptyset, f(w) = A \text{ and } f(v) = B\}$,
- $i''_{r_2}(G, u, w = A, v, z = C) = \min\{w(f) : f \text{ is an I2RDF on } G + uw \text{ with } f(u) = f(v) = \emptyset, f(w) = A \text{ and } f(z) = C\}$,

Let U be a connected unicyclic graph with the unique cycle $C = v_0, \dots, v_{k-1}, v_0$, where $k \geq 3$. Define $T(v_0, R) = U - v_0v_1$. Clearly, $T(v_0, R)$ is a tree with the vertex set $V(U)$.

Lemma 8. *Let U be a connected unicyclic graph with the unique cycle v_0, \dots, v_{k-1}, v_0 ($k > 2$) with a vertex $w \notin V(U)$ and let $A \in \{\{1\}, \{2\}, \{1, 2\}\}$. Then, $i_{r_2}(U) = \min\{i_{r_2}(T(v_0, R), v_0 = \emptyset, v_1 = \emptyset), i'_{r_2}(T(v_0, R), v_0, w = A, v_1 = A) - |A|, i'_{r_2}(T(v_0, R), v_1, w = A, v_0 = A) - |A|\}$.*

Proof. Assume that $i_{r_2} = \min\{i_{r_2}(T(v_0, R), v_0 = \emptyset, v_1 = \emptyset), i'_{r_2}(T(v_0, R), v_0, w = A, v_1 = A) - |A|, i'_{r_2}(T(v_0, R), v_1, w = A, v_0 = A) - |A|\}$, where $A \in \{\{1\}, \{2\}, \{1, 2\}\}$.

Let f be an I2RDF on $T(v_0, R)$ with $f(v_0) = f(v_1) = \emptyset$ and $w(f) = i_{r_2}(T(v_0, R), v_0 = \emptyset, v_1 = \emptyset)$. Then, f is an I2RDF on U and so $i_{r_2}(U) \leq i_{r_2}(T(v_0, R), v_0 = \emptyset, v_1 = \emptyset)$.

Let f be an I2RDF on $T(v_0, R) + v_0w$ with $f(v_0) = \emptyset, f(v_1) = f(w) = A$ and $w(f) = i'_{r_2}(T(v_0, R), v_0, w = A, v_1 = A)$. Then, the restriction of f to $V(T)$ is an I2RDF on U and so $i_{r_2}(U) \leq i'_{r_2}(T(v_0, R), v_0, w = A, v_1 = A) - |A|$.

Similarly, $i_{r_2}(U) \leq i'_{r_2}(T(v_0, R), v_1, w = A, v_0 = A) - |A|$. So, $i_{r_2}(U) \leq i_{r_2}$.

Let f be an I2RDF on U with $w(f) = i_{r_2}(U)$. We have either both $f(v_0) = \emptyset$ and $f(v_1) \in \{\{1\}, \{2\}, \{1, 2\}\}$ or both $f(v_1) = \emptyset$ and $f(v_0) \in \{\{1\}, \{2\}, \{1, 2\}\}$. In the following we consider these cases.

- Let $f(v_0) = f(v_1) = \emptyset$. Then f is an I2RDF on $T(v_0, R)$ with $f(v_0) = f(v_1) = \emptyset$ and so $i_{r_2}(T(v_0, R), v_0 = \emptyset, v_1 = \emptyset) \leq i_{r_2}(U)$.
- Let $f(v_0) = \emptyset$ and $f(v_1) = g(w) = A$, where $A \in \{\{1\}, \{2\}, \{1, 2\}\}$. Then, $h = f \cup g$ is an I2RDF on $T(v_0, R) + v_0w$ with $h(v_0) = \emptyset$ and $h(v_1) = h(w) = A$ and so $i'_{r_2}(T(v_0, R), v_0, v_1 = A, w = A) - |A| \leq i_{r_2}(U)$.
- Similar to the previous case, if $f(v_1) = \emptyset$ and $f(v_0) = g(w) = A$, where $A \in \{\{1\}, \{2\}, \{1, 2\}\}$, then $i'_{r_2}(T(v_0, R), v_1, v_0 = A, w = A) - |A| \leq i_{r_2}(U)$.

So, $i_{r_2} \leq i_{r_2}(U)$. This completes the proof. ■

By Lemma 8 for computing the independent 2-rainbow domination number of a given unicyclic graph we need to compute $i_{r_2}(T, u = \emptyset, v = \emptyset)$ and $i'_{r_2}(T, u, w = A, v = A)$, where $A \in \{\{1\}, \{2\}, \{1, 2\}\}$ and T is a tree with $u, v \in V(T)$ and a vertex $w \notin V(T)$. We claim that Algorithms 3.2, 3.3, 3.4 and 3.5 compute these values. We define $P_T(v, u)$ as the shortest path between v and u in T .

Lemma 9. *Let T be a rooted tree with the root $u, v \in V(T)$ and a vertex $w \notin V(T)$. Let (a_0, \dots, a_6) be the output of Algorithm I2RDNT0(T, u, v). Then,*

- $a_0 = i_{r_2}(T, v = \emptyset, u = \emptyset)$,
- $a_1 = i_{r_2}(T, v = \emptyset, u = \{1\})$,
- $a_2 = i_{r_2}(T, v = \emptyset, u = \{2\})$,
- $a_3 = i_{r_2}(T, v = \emptyset, u = \{1, 2\})$,
- $a_4 = i'_{r_2}(T, u, w = \{1\}, v = \emptyset)$,
- $a_5 = i'_{r_2}(T, u, w = \{2\}, v = \emptyset)$,
- $a_6 = i'_{r_2}(T, u, w = \{1, 2\}, v = \emptyset)$.

Proof. Let $P_T(v, u) = w_0(= v), \dots, w_k(= u)$, where $k > 0$. The proof is by induction on $k = |P_T(v, u)|$. Let $k = 1$. So, u is the parent of v . Let $T' = T_u - T_v$. So,

- $i_{r_2}(T, v = \emptyset, u = \emptyset) = i_{r_2}(T_v, v = 0) + i_{r_2}(T', u = 0)$,
- $i_{r_2}(T, v = \emptyset, u = \{1\}) = i'_{r_2}(T_v, v, w = 1) + i_{r_2}(T', u = 1) - 1$,
- $i_{r_2}(T, v = \emptyset, u = \{2\}) = i'_{r_2}(T_v, v, w = 2) + i_{r_2}(T', u = 2) - 1$,
- $i_{r_2}(T, v = \emptyset, u = \{1, 2\}) = i'_{r_2}(T_v, v, w = 12) + i_{r_2}(T', u = 12) - 2$,

Algorithm 3.2: I2RDNT0(T, u, v)

Input: A rooted tree T with the root $u, v \in V(T)$ and a vertex $w \notin V(T)$.

Output: $\{i_{r2}(T, v = \emptyset, u = A), i'_{r2}(T, u, w = B, v = \emptyset) : A \in \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}, B \in \{\{1\}, \{2\}, \{1, 2\}\}\}$.

- 1 Let $P_T(u, v) = w_0(= v), \dots, w_k(= u)$.
- 2 $T' = T_{w_1} - T_{w_0}$;
- 3 $i_{00} = i_{r2}(T_{w_0}, w_0 = 0) + i_{r2}(T', w_1 = 0)$;
- 4 $i_{01} = i'_{r2}(T_{w_0}, w_0, w = 1) + i_{r2}(T', w_1 = 1) - 1$;
- 5 $i_{02} = i'_{r2}(T_{w_0}, w_0, w = 2) + i_{r2}(T', w_1 = 2) - 1$;
- 6 $i_{012} = i'_{r2}(T_{w_0}, w_0, w = 12) + i_{r2}(T', w_1 = 12) - 2$;
- 7 $i'_1 = i_{r2}(T_{w_0}, w_0 = 0) + i'_{r2}(T', w_1, w = 1)$;
- 8 $i'_2 = i_{r2}(T_{w_0}, w_0 = 0) + i'_{r2}(T', w_1, w = 2)$;
- 9 $i'_{12} = i_{r2}(T_{w_0}, w_0 = 0) + i'_{r2}(T', w_1, w = 12)$;
- 10 **for** $i = 2$ **to** k **do**
- 11 $T' = T_{w_i} - T_{w_{i-1}}$;
- 12 $\alpha_{00} = \min\{i_{r2}(T', w_i = 0) + i_{00}, i'_{r2}(T', w_i, w = 1) + i_{01} - 1, i'_{r2}(T', w_i, w = 2) + i_{02} - 1, i'_{r2}(T', w_i, w = 12) + i_{012} - 2\}$;
- 13 $\alpha_{01} = i_{r2}(T', w_i = 1) + i'_1 - 1$;
- 14 $\alpha_{02} = i_{r2}(T', w_i = 2) + i'_2 - 1$;
- 15 $\alpha_{012} = i_{r2}(T', w_i = 12) + i'_{12} - 2$;
- 16 $i'_1 = \min\{i'_{r2}(T', w, w_i = 1) + i_{00}, i'_{r2}(T', w, w_i = 1) + i_{01}, i'_{r2}(T', w, w_i = 12) + i_{02} - 1, i'_{r2}(T', w, w_i = 12) + i_{012} - 1\}$;
- 17 $i'_2 = \min\{i'_{r2}(T', w, w_i = 2) + i_{00}, i'_{r2}(T', w, w_i = 12) + i_{01} - 1, i'_{r2}(T', w, w_i = 2) + i_{02}, i'_{r2}(T', w, w_i = 12) + i_{012} - 1\}$;
- 18 $i'_{12} = i'_{r2}(T', w, w_i = 12) + \min\{i_{00}, i_{01}, i_{02}, i_{012}\}$;
- 19 $i_{00} = \alpha_{00}; i_{01} = \alpha_{01}; i_{02} = \alpha_{02}; i_{012} = \alpha_{012}$;
- 20 **return** $(i_{00}, i_{01}, i_{02}, i_{012}, i'_1, i'_2, i'_{12})$;

- $i'_{r2}(T, u, w = \{1\}, v = \emptyset) = i_{r2}(T_v, v = 0) + i'_{r2}(T', u, w = 1)$,
- $i'_{r2}(T, u, w = \{2\}, v = \emptyset) = i_{r2}(T_v, v = 0) + i'_{r2}(T', u, w = 2)$,
- $i'_{r2}(T, u, w = \{1, 2\}, v = \emptyset) = i_{r2}(T_v, v = 0) + i'_{r2}(T', u, w = 12)$.

Since $k = 1$, the for loop of Algorithm I2RDNT0(T, u, v) does not execute. This proves the base case of the induction. Assume that the result is true for any rooted tree T' with the root $u, v \in V(T')$, a vertex $w \notin V(T')$ and $|P_{T'}(v, u)| \leq m$, where $m \geq 1$. Let T be a rooted tree with the root $u, v \in V(T)$, a vertex $w \notin V(T)$, $|P_T(v, u)| = m + 1$ and $P_T(v, u) = w_0(= v), \dots, w_m, w_{m+1}(= u)$. Let T_{w_m} be the rooted subtree of T with the root w_m . Let (a_0, \dots, a_6) and (b_0, \dots, b_6) be the outputs of Algorithms I2RDNT0(T, u, v) and I2RDNT0(T_{w_m}, w_m, v), re-

Algorithm 3.3: I2RDNT1(T, u, v)

Input: A rooted tree T with root u , $v \in V(T)$ and vertices $w, z \notin V(T)$.

Output: $\{i'_{r_2}(T, v, w = \{1\}, u = A), i''_{r_2}(T, v, w = \{1\}, u, z = B) : A \in \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}, B \in \{\{1\}, \{2\}, \{1, 2\}\}\}$.

```

1 Let  $P_T(u, v) = w_0(=v), \dots, w_k(=u)$ .
2  $T' = T_{w_1} - T_{w_0}$ ;
3  $i_{10} = i'_{r_2}(T_{w_0}, w_0, w = 1) + i_{r_2}(T', w_1 = 0)$ ;
4  $i_{11} = i'_{r_2}(T_{w_0}, w_0, w = 1) + i_{r_2}(T', w_1 = 1)$ ;
5  $i_{12} = i'_{r_2}(T_{w_0}, w_0, w = 12) + i_{r_2}(T', w_1 = 2) - 1$ ;
6  $i_{112} = i'_{r_2}(T_{w_0}, w_0, w = 12) + i_{r_2}(T', w_1 = 12) - 1$ ;
7  $i'_{11} = i'_{r_2}(T_{w_0}, w_0, w = 1) + i'_{r_2}(T', w_1, w = 1)$ ;
8  $i'_{12} = i'_{r_2}(T_{w_0}, w_0, w = 1) + i'_{r_2}(T', w_1, w = 2)$ ;
9  $i'_{112} = i'_{r_2}(T_{w_0}, w_0, w = 1) + i'_{r_2}(T', w_1, w = 12)$ ;
10 for  $i = 2$  to  $k$  do
11      $T' = T_{w_i} - T_{w_{i-1}}$ ;
12      $\alpha_{10} = \min\{i_{r_2}(T', w_i = 0) + i_{10}, i'_{r_2}(T', w_i, w = 1)$ 
13          $+ i_{11} - 1, i'_{r_2}(T', w_i, w = 2) + i_{12} - 1, i'_{r_2}(T', w_i, w = 12) + i_{112} - 2\}$ ;
14      $\alpha_{11} = i_{r_2}(T', w_i = 1) + i'_{11} - 1$ ;
15      $\alpha_{12} = i_{r_2}(T', w_i = 2) + i'_{12} - 1$ ;
16      $\alpha_{112} = i_{r_2}(T', w_i = 12) + i'_{112} - 2$ ;
17      $i'_{11} = \min\{i'_{r_2}(T', w_i, w = 1) + i_{10}, i'_{r_2}(T', w_i, w = 1)$ 
18          $+ i_{11}, i'_{r_2}(T', w_i, w = 12) + i_{12} - 1, i'_{r_2}(T', w_i, w = 12) + i_{112} - 1\}$ ;
19      $i'_{12} = \min\{i'_{r_2}(T', w_i, w = 2) + i_{10}, i'_{r_2}(T', w_i, w = 12)$ 
20          $+ i_{11} - 1, i'_{r_2}(T', w_i, w = 2) + i_{12}, i'_{r_2}(T', w_i, w = 12) + i_{112} - 1\}$ ;
21      $i'_{112} = i'_{r_2}(T', w_i, w = 12) + \min\{i_{10}, i_{11}, i_{12}, i_{112}\}$ ;
22      $i_{10} = \alpha_{10}; i_{11} = \alpha_{11}; i_{12} = \alpha_{12}; i_{112} = \alpha_{112}$ ;
23 return  $(i_{10}, i_{11}, i_{12}, i_{112}, i'_{11}, i'_{12}, i'_{112})$ ;
    
```

spectively. By the induction hypothesis,

- $b_0 = i_{r_2}(T_{w_m}, v = \emptyset, w_m = \emptyset)$,
- $b_1 = i_{r_2}(T_{w_m}, v = \emptyset, w_m = \{1\})$,
- $b_2 = i_{r_2}(T_{w_m}, v = \emptyset, w_m = \{2\})$,
- $b_3 = i_{r_2}(T_{w_m}, v = \emptyset, w_m = \{1, 2\})$,
- $b_4 = i'_{r_2}(T_{w_m}, w_m, w = \{1\}, v = \emptyset)$,
- $b_5 = i'_{r_2}(T_{w_m}, w_m, w = \{2\}, v = \emptyset)$,
- $b_6 = i'_{r_2}(T_{w_m}, w_m, w = \{1, 2\}, v = \emptyset)$.

Let (o_0^k, \dots, o_6^k) be values of variables $(i_{00}, i_{01}, i_{02}, i_{012}, i'_1, i'_2, i'_3)$ of Algorithm I2RDNT0(T, u, v), respectively, after the iteration of the for loop for each $2 \leq k \leq$

Algorithm 3.4: I2RDNT2(T, u, v)

Input: A rooted tree T with root $u, v \in V(T)$ and vertices $w, z \notin V(T)$.
Output: $\{i'_{r_2}(T, v, w = \{2\}, u = A), i''_{r_2}(T, v, w = \{2\}, u, z = B) : A \in \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}, B \in \{\{1\}, \{2\}, \{1, 2\}\}\}$.

- 1 Let $P_T(u, v) = w_0(= v), \dots, w_k(= u)$.
- 2 $T' = T_{w_1} - T_{w_0}$;
- 3 $i_{20} = i'_{r_2}(T_{w_0}, w_0, w = 2) + i_{r_2}(T', w_1 = 0)$;
- 4 $i_{21} = i'_{r_2}(T_{w_0}, w_0, w = 12) + i_{r_2}(T', w_1 = 1) - 1$;
- 5 $i_{22} = i'_{r_2}(T_{w_0}, w_0, w = 2) + i_{r_2}(T', w_1 = 2)$;
- 6 $i_{212} = i'_{r_2}(T_{w_0}, w_0, w = 12) + i_{r_2}(T', w_1 = 12) - 1$;
- 7 $i'_{21} = i'_{r_2}(T_{w_0}, w_0, w = 2) + i'_{r_2}(T', w_1, w = 1)$;
- 8 $i'_{22} = i'_{r_2}(T_{w_0}, w_0, w = 2) + i'_{r_2}(T', w_1, w = 2)$;
- 9 $i'_{212} = i'_{r_2}(T_{w_0}, w_0, w = 2) + i'_{r_2}(T', w_1, w = 12)$;
- 10 **for** $i = 2$ **to** k **do**
- 11 $T' = T_{w_i} - T_{w_{i-1}}$;
- 12 $\alpha_{20} = \min\{i_{r_2}(T', w_i = 0) + i_{20}, i'_{r_2}(T', w_i, w = 1) + i_{21} - 1, i'_{r_2}(T', w_i, w = 2) + i_{22} - 1, i'_{r_2}(T', w_i, w = 12) + i_{212} - 2\}$;
- 13 $\alpha_{21} = i_{r_2}(T', w_i = 1) + i'_{21} - 1$;
- 14 $\alpha_{22} = i_{r_2}(T', w_i = 2) + i'_{22} - 1$;
- 15 $\alpha_{212} = i_{r_2}(T', w_i = 12) + i'_{212} - 2$;
- 16 $i'_{21} = \min\{i'_{r_2}(T', w_i, w = 1) + i_{20}, i'_{r_2}(T', w_i, w = 1) + i_{21}, i'_{r_2}(T', w_i, w = 12) + i_{22} - 1, i'_{r_2}(T', w_i, w = 12) + i_{212} - 1\}$;
- 17 $i'_{22} = \min\{i'_{r_2}(T', w_i, w = 2) + i_{20}, i'_{r_2}(T', w_i, w = 12) + i_{21} - 1, i'_{r_2}(T', w_i, w = 2) + i_{22}, i'_{r_2}(T', w_i, w = 12) + i_{212} - 1\}$;
- 18 $i'_{112} = i'_{r_2}(T', w_i, w = 12) + \min\{i_{20}, i_{21}, i_{22}, i_{212}\}$;
- 19 $i_{20} = \alpha_{20}; i_{21} = \alpha_{21}; i_{22} = \alpha_{22}; i_{212} = \alpha_{212}$;
- 20 **return** $(i_{20}, i_{21}, i_{22}, i_{212}, i'_{21}, i'_{22}, i'_{212})$;

$m + 1$. Clearly, $(b_0, \dots, b_6) = (o_0^m, \dots, o_6^m)$ and $(a_0, \dots, a_6) = (o_0^{m+1}, \dots, o_6^{m+1})$. Let $T' = T - T_{w_m}$. Since u is the parent of $w_m (\neq v)$ in T , we have

- $i_{r_2}(T, v = \emptyset, u = \emptyset) = \min\{i_{r_2}(T', w_i = 0) + b_0, i'_{r_2}(T', w_i, w = 1) + b_1 - 1, i'_{r_2}(T', w_i, w = 2) + b_2 - 1, i'_{r_2}(T', w_i, w = 12) + b_3 - 2\}$,
- $i_{r_2}(T, v = \emptyset, u = \{1\}) = i_{r_2}(T', w_i = 1) + b_4 - 1$,
- $i_{r_2}(T, v = \emptyset, u = \{2\}) = i_{r_2}(T', w_i = 2) + b_5 - 1$,
- $i_{r_2}(T, v = \emptyset, u = \{1, 2\}) = i_{r_2}(T', w_i = 12) + b_6 - 2$,
- $i'_{r_2}(T, u, w = \{1\}, v = \emptyset) = \min\{i'_{r_2}(T', w, w_i = 1) + b_0, i'_{r_2}(T', w, w_i = 1) + b_1, i'_{r_2}(T', w, w_i = 12) + b_2 - 1, i'_{r_2}(T', w, w_i = 12) + b_3 - 1\}$,
- $i'_{r_2}(T, u, w = \{2\}, v = \emptyset) = \min\{i'_{r_2}(T', w, w_i = 2) + b_0, i'_{r_2}(T', w, w_i = 12) + b_3 - 1, i'_{r_2}(T', w, w_i = 12) + b_3 - 1\}$,

Algorithm 3.5: I2RDNT12(T, u, v)

Input: A rooted tree T with root u , $v \in V(T)$ and vertices $w, z \notin V(T)$.

Output: $\{i'_{r_2}(T, v, w = \{1, 2\}, u = A), i''_{r_2}(T, v, w = \{1, 2\}, u, z = B) :$

$A \in \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}, B \in \{\{1\}, \{2\}, \{1, 2\}\}$.

- 1 Let $P_T(u, v) = w_0(= v), \dots, w_k(= u)$.
- 2 $T' = T_{w_1} - T_{w_0}$;
- 3 $i_{120} = i'_{r_2}(T_{w_0}, w_0, w = 12) + i_{r_2}(T', w_1 = 0)$;
- 4 $i_{121} = i'_{r_2}(T_{w_0}, w_0, w = 12) + i_{r_2}(T', w_1 = 1)$;
- 5 $i_{122} = i'_{r_2}(T_{w_0}, w_0, w = 12) + i_{r_2}(T', w_1 = 2)$;
- 6 $i_{1212} = i'_{r_2}(T_{w_0}, w_0, w = 12) + i_{r_2}(T', w_1 = 12)$;
- 7 $i'_{121} = i'_{r_2}(T_{w_0}, w_0, w = 12) + i'_{r_2}(T', w_1, w = 1)$;
- 8 $i'_{122} = i'_{r_2}(T_{w_0}, w_0, w = 12) + i'_{r_2}(T', w_1, w = 2)$;
- 9 $i'_{1212} = i'_{r_2}(T_{w_0}, w_0, w = 12) + i'_{r_2}(T', w_1, w = 12)$;
- 10 **for** $i = 2$ **to** k **do**
- 11 $T' = T_{w_i} - T_{w_{i-1}}$;
- 12 $\alpha_{120} = \min\{i_{r_2}(T', w_i = 0) + i_{120}, i'_{r_2}(T', w_i, w = 1)$
 $+ i_{121} - 1, i'_{r_2}(T', w_i, w = 2) + i_{122} - 1, i'_{r_2}(T', w_i, w = 12) + i_{1212} - 2\}$;
- 13 $\alpha_{121} = i_{r_2}(T', w_i = 1) + i'_{121} - 1$;
- 14 $\alpha_{122} = i_{r_2}(T', w_i = 2) + i'_{122} - 1$;
- 15 $\alpha_{1212} = i_{r_2}(T', w_i = 12) + i'_{1212} - 2$;
- 16 $i'_{121} = \min\{i'_{r_2}(T', w_i, w = 1) + i_{120}, i'_{r_2}(T', w_i, w = 1)$
 $+ i_{121}, i'_{r_2}(T', w_i, w = 12) + i_{122} - 1, i'_{r_2}(T', w_i, w = 12) + i_{1212} - 1\}$;
- 17 $i'_{122} = \min\{i'_{r_2}(T', w_i, w = 2) + i_{120}, i'_{r_2}(T', w_i, w = 12)$
 $+ i_{121} - 1, i'_{r_2}(T', w_i, w = 2) + i_{122}, i'_{r_2}(T', w_i, w = 12) + i_{1212} - 1\}$;
- 18 $i'_{1212} = i'_{r_2}(T', w_i, w = 12) + \min\{i_{120}, i_{121}, i_{122}, i_{1212}\}$;
- 19 $i_{120} = \alpha_{120}; i_{121} = \alpha_{121}; i_{122} = \alpha_{122}; i_{1212} = \alpha_{1212}$;
- 20 **return** $(i_{120}, i_{121}, i_{122}, i_{1212}, i'_{121}, i'_{122}, i'_{1212})$;

$$b_1 - 1, i'_{r_2}(T', w, w_i = 2) + b_2, i'_{r_2}(T', w, w_i = 12) + b_3 - 1\},$$

$$\bullet i'_{r_2}(T, u, w = \{1, 2\}, v = \emptyset) = i'_{r_2}(T', w, w_i = 12) + \min\{b_0, b_1, b_2, b_3\}.$$

This completes the proof. ■

Similar to Lemma 9 we have the following.

Lemma 10. *Let T be a rooted tree with the root u , $v \in V(T)$ and vertices $w, z \notin V(T)$. Let (a_0, \dots, a_6) be the output of Algorithm I2RDNT1(T, u, v). Then,*

- $a_0 = i'_{r_2}(T, v, w = \{1\}, u = \emptyset)$,
- $a_1 = i'_{r_2}(T, v, w = \{1\}, u = \{1\})$,
- $a_2 = i'_{r_2}(T, v, w = \{1\}, u = \{2\})$,

- $a_3 = i'_{r_2}(T, v, w = \{1\}, u = \{1, 2\})$,
- $a_4 = i''_{r_2}(T, v, w = \{1\}, u, z = \{1\})$,
- $a_5 = i''_{r_2}(T, v, w = \{1\}, u, z = \{2\})$,
- $a_6 = i''_{r_2}(T, v, w = \{1\}, u, z = \{1, 2\})$.

Lemma 11. *Let T be a rooted tree with the root u , $v \in V(T)$ and a vertex $w \notin V(T)$. Let (a_0, \dots, a_6) be the output of Algorithm I2RDNT2(T, u, v). Then,*

- $a_0 = i'_{r_2}(T, v, w = \{2\}, u = \emptyset)$,
- $a_1 = i'_{r_2}(T, v, w = \{2\}, u = \{1\})$,
- $a_2 = i'_{r_2}(T, v, w = \{2\}, u = \{2\})$,
- $a_3 = i'_{r_2}(T, v, w = \{2\}, u = \{1, 2\})$,
- $a_4 = i''_{r_2}(T, v, w = \{2\}, u, z = \{1\})$,
- $a_5 = i''_{r_2}(T, v, w = \{2\}, u, z = \{2\})$,
- $a_6 = i''_{r_2}(T, v, w = \{2\}, u, z = \{1, 2\})$.

Lemma 12. *Let T be a rooted tree with the root u , $v \in V(T)$ and a vertex $w \notin V(T)$. Let (a_0, \dots, a_6) be the output of Algorithm I2RDNT12(T, u, v). Then,*

- $a_0 = i'_{r_2}(T, v, w = \{1, 2\}, u = \emptyset)$,
- $a_1 = i'_{r_2}(T, v, w = \{1, 2\}, u = \{1\})$,
- $a_2 = i'_{r_2}(T, v, w = \{1, 2\}, u = \{2\})$,
- $a_3 = i'_{r_2}(T, v, w = \{1, 2\}, u = \{1, 2\})$,
- $a_4 = i''_{r_2}(T, v, w = \{1, 2\}, u, z = \{1\})$,
- $a_5 = i''_{r_2}(T, v, w = \{1, 2\}, u, z = \{2\})$,
- $a_6 = i''_{r_2}(T, v, w = \{1, 2\}, u, z = \{1, 2\})$.

Theorem 13. *There is a linear algorithm that computes the independent 2-rainbow domination number of a given unicyclic graph.*

Proof. Let U be a connected unicyclic graph with the unique cycle v_0, \dots, v_{k-1}, v_0 such that a vertex $w \notin V(U)$. By Lemma 8, $i_{r_2}(U) = \min\{i_{r_2}(T(v_0, R), v_0 = \emptyset, v_1 = \emptyset), i'_{r_2}(T(v_0, R), v_0, w = A, v_1 = A) - |A|, i'_{r_2}(T(v_0, R), v_1, w = A, v_0 = A) - |A|\}$, where $A \in \{\{1\}, \{2\}, \{1, 2\}\}$. Let T_u be a rooted tree of $T(v_0, R)$ with the root u and let $v \in V(T_u)$. Clearly, $i_{r_2}(T(v_0, R), v = A, u = A) = i_{r_2}(T_u, v = A, u = A)$ and $i'_{r_2}(T(v_0, R), v, w = A, u = A) = i'_{r_2}(T_u, v, w = A, u = A)$, where $A \in \{\{1\}, \{2\}, \{1, 2\}\}$. It follows from Lemmas 9, 10, 11 and 12 that Algorithms 3.2, 3.3, 3.4 and 3.5 compute $i_{r_2}(U)$.

It remains to compute the running time of these algorithms. Let $P_T(v, u) = w_0(= v), \dots, w_k(= u)$, where $k > 0$. Clearly, we can compute T_u and $P(T, v, u)$ in linear time. Let T_m be the value of variable T' of Algorithm I2RDNT0(T, u, v)

after the iteration of the for loop for each $2 \leq m \leq k$. By Lemma 5, the running time of lines 2–9 of Algorithm I2RDNT0(T, u, v) is $\mathcal{O}(V(T_1))$ and the running time of the iteration of the for loop of Algorithm I2RDNT0(T, u, v) for $2 \leq m \leq k$ is $\mathcal{O}(V(T_m))$. Clearly, $V(T_i) \cap V(T_j) = \emptyset$ for each $2 \leq i < j \leq k$. So, the running time of Algorithm I2RDNT0(T, u, v) is equal to $\sum_{i=2}^k \mathcal{O}(V(T_m)) = \mathcal{O}(V(T))$ and so the running time of Algorithm 3.2 is linear. Similarly, the running times of Algorithms of 3.3, 3.4 and 3.5 are linear. This completes the proof. ■

REFERENCES

- [1] J. Amjadi, M. Falahat, N. Jafari Rad and S.M. Sheikholeslami, *Strong equality between the 2-rainbow domination and independent 2-rainbow domination numbers in trees*, Bull. Malays. Math. Sci. Soc. **39** (2016) 205–218.
<https://doi.org/10.1007/s40840-015-0284-0>
- [2] B. Brešar, M.A. Henning and D.F. Rall, *Rainbow domination in graphs*, Taiwanese J. Math. **12** (2008) 213–225.
<https://doi.org/10.11650/twjm/1500602498>
- [3] B. Brešar and T.K. Šumenjak, *On the 2-rainbow domination in graphs*, Discrete Appl. Math. **155** (2007) 2394–2400.
<https://doi.org/10.1016/j.dam.2007.07.018>
- [4] M. Chellali and N. Jafari Rad, *Independent 2-rainbow domination in graphs*, J. Combin. Math. Combin. Comput. **94** (2015) 133–148.
- [5] M. Chellali and N. Jafari Rad, *On 2-rainbow domination and Roman domination in graphs*, Australas. J. Combin. **56** (2013) 85–93.
- [6] M. Chellali, T.W. Haynes, S.T. Hedetniemi and A. MacRae, *Roman {2}-domination*, Discrete Appl. Math. **204** (2016) 22–28..
- [7] H. Chen and Ch. Lu, *A note on Roman {2}-domination problem in graphs*.
arXiv:1804.09338.pdf
- [8] E.J. Cockayne, P.M. Dreyer Jr., S.M. Hedetniemi and S.T. Hedetniemi, *Roman domination in graphs*, Discrete Math. **278** (2004) 11–22.
<https://doi.org/10.1016/j.disc.2003.06.004>
- [9] M.R. Garey and D.S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness* (W.H. Freeman, New York, 1979).
- [10] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Fundamentals of Domination in Graphs* (Marcel Dekker, Inc., New York, 1998).
- [11] D. Lichtenstein, *Planar formulae and their uses*, SIAM J. Comput. **11** (1982) 329–343.
<https://doi.org/10.1137/0211025>

- [12] A. Rahmouni and M. Chellali, *Independent Roman $\{2\}$ -domination in graphs*, *Discrete Appl. Math.* **236** (2018) 408–414.
<https://doi.org/10.1016/j.dam.2017.10.028>
- [13] C.S. ReVelle and K.E. Rosing, *Defendens imperium Romanum: A classical problem in military strategy*, *Amer. Math. Monthly* **107** (2000) 585–594.
<https://doi.org/10.1080/00029890.2000.12005243>
- [14] Z. Shao, Z. Li, A. Peperko, J. Wan and J. Žerovnik, *Independent rainbow domination of graphs*, *Bull. Malays. Math. Sci. Soc.* **42** (2019) 417–435.
<https://doi.org/10.1007/s40840-017-0488-6>
- [15] Z. Shao, M. Liang, C. Yin, X. Xu, P. Pavlič and J. Žerovnik, *On rainbow domination numbers of graphs*, *Inform. Sci.* **254** (2014) 225–234.
<https://doi.org/10.1016/j.ins.2013.07.020>
- [16] I. Stewart, *Defend the Roman empire!*, *Sci. Amer.* **281** (1999) 136–139.
<https://doi.org/10.1038/scientificamerican1299-136>
- [17] Y. Wu and N. Jafari Rad, *Bounds on the 2-rainbow domination number of graphs*, *Graphs Combin.* **29** (2013) 1125–1133.
<https://doi.org/10.1007/s00373-012-1158-y>

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