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CORRIGENDUM TO: INDEPENDENT TRANSVERSAL DOMINATION IN GRAPHS [DISCUSS. MATH. GRAPH THEORY 32 (2012) 5–17]

Emma Guzman-Garcia

and

Rocío Sánchez-López [1]

Facultad de Ciencias, Universidad Nacional Autónoma de México
Círcuito Exterior s/n, Coyoacán,
Ciudad Universitaria, 04510
Ciudad de México, CDMX

e-mail: guzmanemma.604@gmail.com
usagitsukinomx@yahoo.com.mx

Abstract

In [*Independent transversal domination in graphs*, Discuss. Math. Graph Theory 32 (2012) 5–17], Hamid claims that if G is a connected bipartite graph with bipartition $\{X, Y\}$ such that $|X| \leq |Y|$ and $|X| = \gamma(G)$, then $\gamma_{it}(G) = \gamma(G) + 1$ if and only if every vertex x in X is adjacent to at least two pendant vertices. In this corrigendum, we give a counterexample for the sufficient condition of this sentence and we provide a right characterization. On the other hand, we show an example that disproves a construction which is given in the same paper.

Keywords: domination, independent, transversal, covering, matching.

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1. Introduction

Among the results that Hamid shows in [4] we find the following.

Theorem 1.1 [4]**.** *Let G be a connected bipartite graph with bipartition $\{X, Y\}$ such that $|X| \leq |Y|$ and $|X| = \gamma(G)$. Then $\gamma_{it}(G) = \gamma(G) + 1$ if and only if every vertex x in X is adjacent to at least two pendant vertices.*

[1] Corresponding author.

We find a connected bipartite graph G with bipartition $\{X, Y\}$ such that $|X| \leq |Y|$, $|X| = \gamma(G)$ and $\gamma_{it}(G) = \gamma(G) + 1$. But there exists a vertex in X which is not adjacent to at least two pendant vertices.

A problem that arises with Theorem 1.1 is that it is used in [1] in order to prove the following result.

Corollary 1.2 [1]**.** *Let T be a tree with bipartition $\{X, Y\}$ such that $1 \leq |X| \leq |Y|$ and $\gamma(T) = |X|$. Then, $\gamma_{it}(T) = \gamma(T)$ if and only if there is a vertex in X which is adjacent to at most one pendant vertex.*

In this corrigendum, we provide a right characterization for bipartite graphs G with bipartition $\{X, Y\}$, $|X| \leq |Y|$ and $|X| = \gamma(G)$, such that $\gamma_{it}(G) = \gamma(G) + 1$. As a consequence of the main result, we show the corrected version of Corollary 1.2.

Other result showed in [4] is the following.

Theorem 1.3 [4]**.** *Let a and b be two positive integers with $b \geq 2a - 1$. Then there exists a connected graph G on b vertices such that $\gamma_{it}(G) = a$.*

In order to prove Theorem 1.3, Hamid proposes the following construction: set $b = 2a + r$, with $r \geq -1$, and let H be any connected graph on a vertices. Let $\mathrm{V}(H) = \{v_1, v_2, \ldots, v_a\}$ be the vertex set of H and let G be the graph obtained from H by attaching $r + 1$ pendant edges at v_1 and one pendant edge at each v_i, for $i \geq 2$. Let u_i be the pendant vertex in G adjacent to v_i, for $i \geq 2$.

Hamid claims that $\gamma_{it}(G) = a$ and $S = \{v_1, u_2, u_3, \ldots, u_a\}$ is a $\gamma_{it}(G)$-set. Further, every maximum independent set of G intersects S and hence $\gamma_{it}(G) = a$.

We find that, in some cases for H, G holds $\gamma_{it}(G) \neq a$ and there exists an $\alpha(G)$-set which does not intersect S.

In this corrigendum we provide a correct proof of Theorem 1.3 for $b \geq 2a$.

2. Definitions and Known Results

We use [2] and [3] for terminology and notation not defined here and consider finite and simple graphs only. For introductory notation, let G be a graph. $n(G)$ denotes $|V(G)|$. Let v be a vertex of G, the *open neighborhood of* v in G, denoted by $N(v)$, is defined as the set $\{u \in V(G) : uv \in E(G)\}$. The *degree* of a vertex v, denoted by $\delta(v)$, is the number $|N(v)|$. We say that a vertex u is a *pendant vertex* if $\delta(u) = 1$. For a graph G, the number $\min\{\delta(u) : u \in V(G)\}$ is denoted by $\delta(G)$. An edge of a graph is said to be a *pendant edge* if one of its vertices is a pendant vertex. A *complete graph* is a graph with n vertices and an edge between every two vertices, denoted by K_n. A subset I of $V(G)$ is said to be *independent* if every two vertices of I are non-adjacent. We say that a graph G is *bipartite* if there exists a partition $\{X, Y\}$ of $V(G)$ such that X and Y are independent sets (we call that partition a *bipartition*). If G contains every edge joining X and Y, then G is a *complete bipartite graph*, denoted by $K_{m,n}$ with $|X| = m$ and $|Y| = n$. The complete bipartite graph $K_{1,n}$ is called a *star*.

A subset D of $V(G)$ is said to be *dominating* if for every u in $V(G) - D$ it holds $N(u) \cap D \neq \emptyset$. The cardinality of a smallest dominating set is the *domination number*, denoted by $\gamma(G)$, and we refer to such a set as a $\gamma(G)$*-set*. The cardinality of a largest independent set in G is the *independence number*, denoted by $\alpha(G)$, and an independent set having cardinality $\alpha(G)$ is called a *maximum independent set*. We refer to such a set as an $\alpha(G)$*-set*. A subset M of $E(G)$ is a *matching* if every two edges of M are non-adjacent. A *maximum matching* is one of largest cardinality in G. The number of edges in a maximum matching of a graph G is called the *matching number* of G, denoted by $\alpha'(G)$. A subset K of $V(G)$ such that every edge of G has at least one end in K is called a *covering* of G. The number of vertices in a minimum covering of G is the *covering number* of G, denoted by $\beta(G)$. An *independent transversal dominating set* in G is a dominating set that intersects every maximum independent set in G. The *independent transversal domination number*, denoted by $\gamma_{it}(G)$, is the smallest cardinality of an independent transversal dominating set of G. An independent transversal dominating set of cardinality $\gamma_{it}(G)$ is called a *minimum independent transversal dominating set*. We refer to such a set as a $\gamma_{it}(G)$*-set*.

We need the following results.

Theorem 2.1 [5]. *For any tree T, $\gamma(T) = n(T) - \Delta(T)$ if and only if T is a wounded spider.*

Proposition 2.1 ([4], Example 3.1). $\gamma_{it}(K_{m,n}) = 2$.

Theorem 2.2 [4]. *For any graph G, we have $\gamma(G) \leq \gamma_{it}(G) \leq \gamma(G) + \delta(G)$.*

Lemma 2.3 ([2], page 74). *Let M be a matching and K a covering such that $|M| = |K|$. Then M is a maximum matching and K is a minimum covering.*

Lemma 2.4 ([2], page 101). *Let G be a graph. Then $\alpha(G) + \beta(G) = n(G)$.*

3. A COUNTEREXAMPLE FOR THEOREM 1.1

Consider the graph G in Figure 1. Since $M = \{x_1y_2, x_2y_5, x_3y_4, x_4y_6\}$ is a matching and X is a covering such that $|M| = |X|$, it follows from Lemmas 2.3 and 2.4 that $\alpha(G) = 7$; also it is straightforward to see that $\gamma(G) = 4$. On the other hand, notice that X and $(X - \{x_4\}) \cup \{y_6\}$ are the only one $\gamma(G)$-sets. Therefore, since Y and $(Y - \{y_6\}) \cup \{x_4\}$ are $\alpha(G)$-sets such that $X \cap Y = \emptyset$ and $((Y - \{y_6\}) \cup \{x_4\})$

$\cap\ ((X - \{x_4\}) \cup \{y_6\}) = \emptyset$, we get from Theorem 2.2 that $\gamma_{it}(G) = \gamma(G) + 1$ (because $\delta(G) = 1$). As x_4 is not adjacent to at least two pendant vertices, we obtain a counterexample for Theorem 1.1.

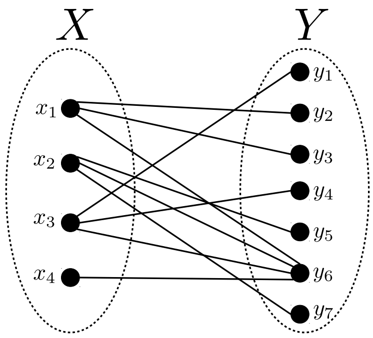

Figure 1. $N(x_4)$ has no pendant vertices.

4. Right Characterization for Bipartite Graphs G Such That $|X| \le |Y|$, $|X| = \gamma(G)$ and $\gamma_{it}(G) = \gamma(G) + 1$

We need the following results.

Corollary 4.1 [4]**.** *If G has an isolated vertex, then $\gamma_{it}(G) = \gamma(G)$.*

Theorem 4.2 shows the right version of Theorem 1.1. Moreover, Theorem 4.2 allows disconnected graphs.

Theorem 4.2. *Let G be a bipartite graph with bipartition $\{X, Y\}$ such that $|X| \le |Y|$ and $|X| = \gamma(G)$. Then $\gamma_{it}(G) = \gamma(G) + 1$ if and only if*

1. *every vertex x in X, such that $\delta(x) \neq 1$, is adjacent to at least two pendant vertices,*
2. *Y has no isolated vertices.*

Proof. If $|V(G)| = 2$, hypothesis $|X| = \gamma(G)$ implies that $G = K_2$ and therefore G satisfies Theorem 4.2. Assume that $|V(G)| \ge 3$.

Suppose that $\gamma_{it}(G) = \gamma(G) + 1$. It follows from Corollary 4.1 that G has no isolated vertices, which implies that $\delta(G) \ge 1$. Therefore, in particular Y has no isolated vertices. Thus, it remains to prove that every vertex x in X, such that $\delta(x) \neq 1$, is adjacent to at least two pendant vertices. Suppose that there exists a vertex w in X such that $\delta(w) \ge 2$.

Notice that X is a $\gamma(G)$-set (because for every u in $(V(G) - X) = Y$, $\delta(u) \ge 1$, and $|X| = \gamma(G)$).

Consider the following claims.

Claim 1. $\alpha(G) = |Y|$.

Given that Y is an independent set in G, we get that $\alpha(G) \geq |Y|$. On the other hand, the hypotheses $\gamma_{it}(G) = \gamma(G) + 1$ and $|X| = \gamma(G)$ imply that there exists an $\alpha(G)$-set S such that $X \cap S = \emptyset$. Since $S \subseteq Y$, then $\alpha(G) = |S| \leq |Y|$. Therefore, $\alpha(G) = |Y|$.

Claim 2. $\delta(G) = 1$.

Proceeding by contradiction, suppose that $\delta(G) \geq 2$. Let u and v be two vertices in G such that $u \in X$ and $v \in N(u)$. Set $S = (X - \{u\}) \cup \{v\}$.

Claim 2.1. *S is a dominating set in G.*

Since $\delta(w) \geq 2$ for every w in $Y - \{v\}$, there exists x_w in $X - \{u\}$ such that $wx_w \in E(G)$. Therefore, S is a dominating set in G (consider the choice of v).

Claim 2.2. *$S \cap J \neq \emptyset$ for every $\alpha(G)$-set J.*

Let J be an $\alpha(G)$-set. If $v \in J$, then $S \cap J \neq \emptyset$. Suppose that $v \notin J$. Given that $|J| = \alpha(G) = |Y|$ (by Claim 1) and $v \notin J$, it follows that $X \cap J \neq \emptyset$. If $u \notin J$, we get $(X - \{u\}) \cap J \neq \emptyset$ (because $X \cap J \neq \emptyset$), which implies that $S \cap J \neq \emptyset$. Thus, suppose that $u \in J$. Since $\delta(u) \geq 2$, there exists z in $Y - \{v\}$ such that $uz \in E(G)$, which implies that $|J \cap Y| \leq |Y| - 2$ (because $u \in J$, $\{uv, uz\} \subseteq E(G)$ and J is an independent set). Therefore, $2 \leq |X \cap J|$, which implies that $(X - \{u\}) \cap J \neq \emptyset$. Thus, $S \cap J \neq \emptyset$.

We get from Claims 2.1, 2.2, the definition of S and the hypothesis that $\gamma_{it}(G) \leq |S| = |X| = \gamma(G)$, a contradiction with $\gamma_{it}(G) = \gamma(G) + 1$. Therefore, $\delta(G) = 1$.

Let u be a vertex in X such that $\delta(u) \geq 2$. We will prove that u is adjacent to at least two pendant vertices. Proceeding by contradiction, suppose that $N(u)$ contains at most one pendant vertex. If $N(u)$ contains a pendant vertex v, choose v, otherwise let v be any vertex in $N(u)$. Set $S = (X - \{u\}) \cup \{v\}$.

Claim 3. *S is a dominating set in G.*

Given that $\delta(w) \geq 1$ for every w in $Y - N(u)$, it follows that there exists x_w in $X - \{u\}$ such that $wx_w \in E(G)$. On the other hand, since for every z in $N(u) - \{v\}$ it holds that $\delta(z) \geq 2$, then there exists x_z in $X - \{u\}$ such that $zx_z \in E(G)$. Therefore, S is a dominating set in G.

Claim 4. *If J is an $\alpha(G)$-set, then $S \cap J \neq \emptyset$.*

The proof is the same as the proof of Claim 2.2.

We get from Claims 3, 4, the definition of S and the hypothesis that $\gamma_{it}(G) \leq |S| = |X| = \gamma(G)$, a contradiction with $\gamma_{it}(G) = \gamma(G) + 1$. Hence, u is adjacent to at least two pendant vertices.

Therefore, every vertex x in X, such that $\delta(x) \neq 1$, is adjacent to at least two pendant vertices.

Suppose that for every vertex w in X, such that $\delta(w) \neq 1$, $N(w)$ contains at least two pendant vertices and Y has no isolated vertices. Notice that it follows from the hypothesis that $\delta(G) \geq 1$. Consider the following claims.

Claim A. $\alpha(G) = |Y|$.

Given that Y is an independent set, we get that $\alpha(G) \geq |Y|$. Proceeding by contradiction, suppose that $\alpha(G) > |Y|$ and let J be an $\alpha(G)$-set.

Since $\alpha(G) > |Y|$ and $|X| \leq |Y|$, we get that $J \cap X \neq \emptyset$ and $J \cap Y \neq \emptyset$. Set $X' = J \cap X$, $Y' = J \cap Y$, $X_1 = \{x \in X' : \delta(x) \geq 2\}$ and $X_2 = \{x \in X' : \delta(x) = 1\}$.

Claim A.1. $|X_1| \geq 1$.

As $|Y| = |Y'| + |Y - Y'|$, $|J| = |X'| + |Y'|$ and $|J| > |Y|$, it follows that $|X'| > |Y - Y'|$, which implies that there exist two vertices in X', say u_1 and u_2, and there exists a vertex y in $Y - Y'$ such that $\{u_1y, u_2y\} \subseteq E(G)$.

Proceeding by contradiction, suppose that $X_1 = \emptyset$. Since $\delta(u_1) = 1$ and $\delta(u_2) = 1$, then for every z in $Y - (Y' \cup \{y\})$ there exists x_z in $X - \{u_1, u_2\}$ such that $zx_z \in E(G)$ (recall that $\delta(G) \geq 1$). On the other hand, given that J is an independent set, we get that for every w in Y' there exists x_w in $X - X'$ such that $wx_w \in E(G)$. Hence, $(X - \{u_1, u_2\}) \cup \{y\}$ is a dominating set, a contradiction with $|X| = \gamma(G)$. Therefore, $|X_1| \geq 1$.

Since $N(X') \subseteq Y - Y'$ and every vertex of X_1 is adjacent to at least two pendant vertices, we get from the definition of X_2 that $|Y - Y'| \geq 2|X_1| + |X_2|$; that is, $|Y - Y'| \geq |X'| + |X_1|$, which implies that $|X_1| + |X'| + |Y'| \leq |Y|$. Hence, since $|X_1| + |J| \leq |Y|$, $1 \leq |X_1|$ (by Claim A.1) and $|Y| < |J|$, we get a contradiction.

Therefore, $\alpha(G) = |Y|$.

Claim B. *If D is a $\gamma(G)$-set, then $V(G) - D$ is an $\alpha(G)$-set.*

Let D be a $\gamma(G)$-set. Since $|D| = \gamma(G) = |X|$, then $|V(G) - D| = (|V(G)| - |X|) = |Y| = \alpha(G)$ (by Claim A). It remains to prove that $V(G) - D$ is an independent set. It is clear that $V(G) - D$ is an independent set if either $(V(G) - D) \subseteq X$ or $(V(G) - D) \subseteq Y$. Hence, suppose that $(V(G) - D) \cap X \neq \emptyset$ and $(V(G) - D) \cap Y \neq \emptyset$. Let u and v be two vertices in $V(G) - D$; we will prove that $uv \notin E(G)$. Suppose that $u \in (V(G) - D) \cap X$ and $v \in (V(G) - D) \cap Y$.

Claim B.1. $\delta(u) = 1$.

Proceeding by contradiction, suppose that $\delta(u) \geq 2$. It follows from the hypothesis that $N(u)$ has at least two pendant vertices, say w and z. Since $u \notin D$, we get that $\{w, z\} \subseteq D$ (because D is a dominating set).

We will see that $S = (D - \{w, z\}) \cup \{u\}$ is a dominating set. Notice that $V(G) - S = (((V(G) - D) \cap X) - \{u\}) \cup (((V(G) - D) \cap Y) \cup \{w, z\})$, $D = (D \cap X) \cup (D \cap Y)$ and $S = (D \cap X) \cup ((D \cap Y) - \{w, z\}) \cup \{u\}$. Given that D is a dominating set, we get that for every y in $(V(G) - D) \cap Y$ there exists x_y in $D \cap X$ such that $yx_y \in E(G)$. In the same way for every x in $((V(G) - D) \cap X) - \{u\}$ there exists y_x in $D \cap Y$ such that $xy_x \in E(G)$ ($y_x \notin \{w, z\}$ because w and z are pendant vertices which are adjacent to u). Hence, we conclude that S is a dominating set. Since $|S| = |X| - 1$, we get a contradiction with $|X| = \gamma(G)$. Therefore, $\delta(u) = 1$.

Given that $\delta(u) = 1$, $u \notin D$ and D is a dominating set, it follows that $N(u) \subseteq D$, which implies that $uv \notin E(G)$ (because $v \notin D$).

Therefore, $V(G) - D$ is an independent set. Hence, $V(G) - D$ is an $\alpha(G)$-set.

Claim C. $\delta(G) = 1$.

Recall that $\delta(G) \geq 1$. If X has a pendant vertex, then we are done; otherwise, it follows from the hypothesis that for u in X there exists a pendant vertex in $N(u)$. Therefore, $\delta(G) = 1$.

It follows from Claim B that $\gamma_{it}(G) \neq \gamma(G)$. Therefore, we get from Claim C and Theorem 2.2 that $\gamma_{it}(G) = \gamma(G) + 1$. ■

5. Some Consequences of Theorem 4.2

A *subdivision* of an edge uv is obtained by replacing the edge uv with a path (u, w, v), where w is a new vertex. For a positive integer t, a *wounded spider* is a star $K_{1,t}$ with at most $t - 1$ of its edges subdivided. Similarly, for an integer $t \geq 2$, a *healthy spider* is a star $K_{1,t}$ with all of its edges subdivided.

Remark 5.1. It is straightforward to see that if G is a healthy spider, then $\gamma(G) = \Delta(G)$. On the other hand, if G is a healthy spider, it follows from Theorem 4.2 that $\gamma_{it}(G) = \gamma(G)$.

Remark 5.2. Let G be a wounded spider which is not a star. Suppose that G is obtained from $K_{1,t}$ by subdividing r of its edges, with $1 \leq r \leq t - 1$ and $t \geq 2$.

1. If $r \leq t - 2$, then $\gamma_{it}(G) = \gamma(G) + 1 = r + 2$.
2. If $r = t - 1$, then $\gamma_{it}(G) = \gamma(G) = t$.

Proof. Suppose that $V(G) = \{u_1, v_2, \ldots, v_t, v_{t+1}\} \cup \{u_2, \ldots, u_r, u_{r+1}\}$, $E(G) = \{u_1v_j : j \in \{2, \ldots, t + 1\}\} \cup \{u_iv_i : i \in \{2, \ldots, r + 1\}\}$. Set $X = \{u_1, u_2, \ldots, u_r, u_{r+1}\}$ and $Y = \{v_2, \ldots, v_t, v_{t+1}\}$.

1. Suppose that $r \leq t-2$. It follows from Theorem 2.1 that $\gamma(G) = ((t+1)+r) - t = r+1$ which implies that $|X| = \gamma(G)$. Therefore, we get from Theorem 4.2 that $\gamma_{it}(G) = \gamma(G) + 1 = (r+1) + 1$.

2. Suppose that $r = t-1$. It follows from Theorem 2.1 that $\gamma(G) = t$. Since $|X| = \gamma(G)$ and u_1 is not adjacent to at least two pendant vertices in G, it follows from Theorem 4.2 that $\gamma_{it}(G) \neq \gamma(G) + 1$. Therefore, given that $\delta(G) = 1$, we get from Theorem 2.2 that $\gamma_{it}(G) = \gamma(G)$. Hence, $\gamma_{it}(G) = t$. ■

Corollary 5.1. *Let T be a tree with bipartition $\{X, Y\}$ such that $1 \leq |X| \leq |Y|$ and $\gamma(T) = |X|$. Then, $\gamma_{it}(T) = \gamma(T)$ if and only if there is a vertex x in X, with $\delta(x) \neq 1$, which is adjacent to at most one pendant vertex.*

6. Example Disproving Construction in Theorem 1.3

Recall that, in order to prove Theorem 1.3, Hamid proposes the following construction: set $b = 2a + r$, with $r \geq -1$, and let H be any connected graph on a vertices. Let $\mathrm{V}(H) = \{v_1, v_2, \ldots, v_a\}$ be the vertex set of H and let G be the graph obtained from H by attaching $r+1$ pendant edges at v_1 and one pendant edge at each v_i, for $i \geq 2$. Let u_i ($i \geq 2$) be the pendant vertex in G adjacent to v_i.

Hamid claims that $\gamma_{it}(G) = a$ and $S = \{v_1, u_2, u_3, \ldots, u_a\}$ is a $\gamma_{it}(G)$-set. Further, every maximum independent set of G intersects S and hence $\gamma_{it}(G) = a$.

• We find that, when $r = -1$ and $a \geq 3$, for the graph $H = K_a$, the associated graph G does not hold the conclusion of Theorem 1.3, see Figure 2.

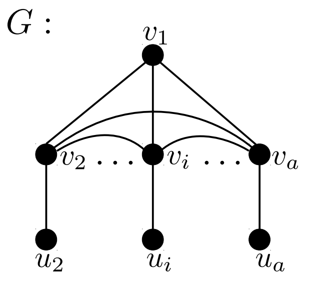

Figure 2

In this case, since $K = (V(H) - \{v_1\})$ is a covering and $M = \{v_iu_i : i \in \{2, \ldots, a\}\}$ is a matching such that $|K| = a - 1 = |M|$, we get from Lemma 2.3 that $|K| = \beta(G)$. Thus, it follows from Lemma 2.4 that $\alpha(G) = 2a-1-(a-1) = a$. Hence $(V(G) - K) = \{u_2, \ldots, u_a, v_1\}$ is the only one independent set in G such that $|V(G) - K| = \alpha(G)$. Therefore, $V(G) - ((V(H) - \{v_a\}) \cup \{u_a\})$ is an independent transversal dominating set in G, which implies that $\gamma_{it}(G) \leq a-1$. On the other hand, let S be a $\gamma_{it}(G)$-set. Given that S is a dominating set, then $\{v_i, u_i\} \cap S \neq \emptyset$ for every i in $\{2, \ldots, a\}$, which implies that $a-1 \leq |S|$. Therefore, $\gamma_{it}(G) = a-1$

• We find that, when $r > 0$ and $a \geq 2$, for the graph $H = K_{1,a-1}$, the associated graph G is a wounded spider and this does not hold the conclusion of Theorem 1.3, see Figure 3.

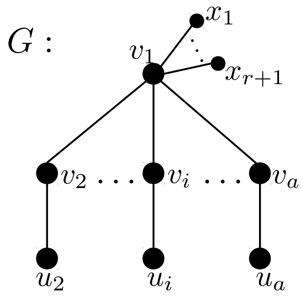

Figure 3

Notice that G is also obtained from $K_{1,a+r}$ by subdividing exactly $a-1$ of its edges, where $a-1 \leq (a+r)-2$. Therefore, it follows from Remark 5.2 that $\gamma_{it}(G) = \gamma(G)+1 = (a-1)+2 = a+1$.

• When $r > 0$ and $a = 1$ we have that $G = K_{1,r+1}$ and in this case we get from Proposition 2.1 that $\gamma_{it}(G) = 2 = a+1$, see Figure 4.

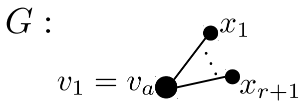

Figure 4

• When $H = K_{1,a-1}$, for $r \geq 0$ and $a \geq 2$, there exists an $\alpha(G)$-set in G which does not intersect $S = \{v_1, u_2, u_3, \ldots, u_a\}$.

For every i in $\{1, \ldots, r+1\}$ let x_i be the pendant vertex adjacent to v_1. Since $M = \{v_1x_1, v_2u_2, \ldots, v_au_a\}$ is a matching and $K = V(H)$ is a covering such that $|M| = |K|$, then we get from Lemma 2.3 that K is a minimum covering. On the other hand, it follows from Lemma 2.4 that $2a + r = |V(G)| = \alpha(G) + \beta(G) = \alpha(G) + a$, which implies that $\alpha(G) = a + r$.

Therefore $(V(H) - \{v_1\}) \cup \{x_1, \ldots, x_{r+1}\}$ is an $\alpha(G)$-set in G which does not intersect S.

For $b \geq 2a$ we proceed to prove the following.

Theorem 6.1. *Let a and b be two positive integers with $b \geq 2a$. Then there exists a connected graph G on b vertices such that $\gamma_{it}(G) = a$.*

Proof. Suppose that $b = 2a + r$, for some r in $\mathbb{N}$. Let H be a connected graph of order a, such that $H \ncong K_{1,a-1}$, with vertex set $V(H) = \{v_1, \ldots, v_a\}$. Let $\{x_1, \ldots, x_{r+1}\}$ and $\{u_2, \ldots, u_a\}$ be two sets such that $\{x_1, \ldots, x_{r+1}\} \cap \{u_2, \ldots, u_a\} = \emptyset$, $\{x_1, \ldots, x_{r+1}\} \cap V(H) = \emptyset$ and $V(H) \cap \{u_2, \ldots, u_a\} = \emptyset$. Let G be the graph with $V(G) = V(H) \cup \{x_1, \ldots, x_{r+1}\} \cup \{u_2, \ldots, u_a\}$ and $E(G) = E(H) \cup \{v_i u_i : i \in \{2, \ldots, a\}\} \cup \{v_1 x_i : i \in \{1, \ldots, r+1\}\}$.

Claim 1. $a \leq \gamma_{it}(G)$.

We will prove that $\gamma(G) = a$. Since $V(H)$ is a dominating set in G, then $\gamma(G) \leq a$. On the other hand, let S be a $\gamma(G)$-set. Given that $\{u_i, v_i\} \cap S \neq \emptyset$ (because S is a dominating set) for every i in $\{2, \ldots, a\}$ and $r + 1 \geq 1$ we get that $|S| \geq a$. Hence, $\gamma(G) = a$. Therefore, it follows from Theorem 2.2 that $a \leq \gamma_{it}(G)$.

Claim 2. $\alpha(G) = r + a$.

Since $K = V(H)$ is a covering and $M = (\{v_i u_i : i \in \{2, \ldots, a\}\} \cup \{v_1 x_1\})$ is a matching such that $|K| = a = |M|$, it follows from Lemma 2.3 that $|K| = \beta(G)$. Hence, we get from Lemma 2.4 that $\alpha(G) = r + a$.

Claim 3. *$S = \{v_1, u_2, \ldots, u_a\}$ is an independent transversal dominating set in G.*

Given that S is a dominating set, it remains to prove that S intersects every maximum independent set in G. Since $H \ncong K_{1,a-1}$ and H is connected, we get that $V(H) - \{v_1\}$ is not an independent set in G, which implies that $(V(H) - \{v_1\}) \cup \{x_1, \ldots, x_{r+1}\}$ is not an independent set in G. Since $|(V(H) - \{v_1\}) \cup \{x_1, \ldots, x_{r+1}\}| = a + r$, it follows that S intersects every maximum independent set in G.

Therefore, we get from Claims 1 and 3 that $a \leq \gamma_{it}(G) \leq a$. ∎

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