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A CLASSIFICATION OF CACTUS GRAPHS ACCORDING TO THEIR DOMINATION NUMBER

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Abstract

A set S of vertices in a graph G is a dominating set of G if every vertex not in S is adjacent to some vertex in S. The domination number, $\gamma(G)$, of G is the minimum cardinality of a dominating set of G. The authors proved in [A new lower bound on the domination number of a graph, J. Comb. Optim. 38 (2019) 721–738] that if G is a connected graph of order $n \ge 2$ with $k \ge 0$ cycles and ℓ leaves, then $\gamma(G) \ge \lceil (n - \ell + 2 - 2k)/3 \rceil$. As a consequence of the above bound, $\gamma(G) = (n - \ell + 2(1 - k) + m)/3$ for some integer $m \ge 0$. In this paper, we characterize the class of cactus graphs achieving equality here, thereby providing a classification of all cactus graphs according to their domination number.

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1. INTRODUCTION

A dominating set of a graph G is a set S of vertices of G such that every vertex not in S has a neighbor in S, where two vertices are neighbors in G if they are adjacent. The minimum cardinality of a dominating set is the domination number of G, denoted by $\gamma(G)$. A dominating set of cardinality $\gamma(G)$ is called a γ -set of G. As remarked in [5], the notion of domination and its variations in graphs has been studied a great deal; a rough estimate says that it occurs in more than 6000 papers to date. For fundamentals of domination theory in graphs we refer the reader to the so-called domination books by Haynes, Hedetniemi, and Slater [6, 7]. An updated glossary of domination parameters can be found in [4].

Two vertices u and v in a graph G are *connected* if there exists a (u, v)-path in G. The graph G is *connected* if every two vertices in G are connected. A *block* of G is a maximal connected subgraph of G which has no cut-vertex of its own. A *cactus* is a connected graph in which every edge belongs to at most one cycle. Equivalently, a (nontrivial) cactus is a connected graph in which every block is an edge or a cycle. The *distance* between two vertices u and v in a connected graph G is the minimum length of a (u, v)-path in G. The *diameter*, diam(G), of G is the maximum distance among pairs of vertices in G.

For notation and graph theory terminology we generally follow [8]. In particular, the order of a graph G with vertex set V(G) and edge set E(G) is given by n(G) = |V(G)| and its size by m(G) = |E(G)|. A neighbor of a vertex v in G is a vertex adjacent to v, and the open neighborhood of v is the set of neighbors of v, denoted $N_G(v)$. The closed neighborhood of v is the set $N_G[v] = N_G(v) \cup \{v\}$. The degree of a vertex v in G is given by $d_G(v) = |N_G(v)|$.

For a set S of vertices in a graph G, the subgraph induced by S is denoted by G[S]. Further, the subgraph obtained from G by deleting all vertices in S and all edges incident with vertices in S is denoted by G - S. If $S = \{v\}$, we simply denote $G - \{v\}$ by G - v. A *leaf* of a graph G is a vertex of degree 1 in G, and its unique neighbor is called a *support vertex*. The set of all leaves of G is denoted by L(G), and we let $\ell(G) = |L(G)|$ be the number of leaves in G. We denote the set of support vertices of G by S(G). We call a vertex of degree at least 2 a *non-leaf*.

Following our notation in [5], we denote the path and cycle on n vertices by P_n and C_n , respectively. A complete graph on n vertices is denoted by K_n , while a complete bipartite graph with partite sets of size n and m is denoted by $K_{n,m}$. A star is the graph $K_{1,k}$, where $k \ge 1$. Further if k > 1, the vertex of degree k is called the *center* vertex of the star, while if k = 1, arbitrarily designate either vertex of P_2 as the center. A double star is a tree with exactly two (adjacent) non-leaf vertices.

A rooted tree T distinguishes one vertex r called the root. For each vertex

 $v \neq r$ of T, the *parent* of v is the neighbor of v on the unique (r, v)-path, while a *child* of v is any other neighbor of v. A *descendant* of v is a vertex $u \neq v$ such that the unique (r, u)-path contains v. In particular, every child of v is a descendant of v. We let D(v) denote the set of descendants of v, and we define $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at v is the subtree of T induced by D[v], and is denoted by T_v . We use the standard notation $[k] = \{1, \ldots, k\}$.

2. Main Result

Our aim in this paper is to provide a classification of all cactus graphs according to their domination number. For this purpose, we shall use a result of the authors in [5] (which we present in Section 4) that establishes a lower bound on the domination number of a graph in terms of its order, number of vertices of degree 1, and number of cycles. From this result, we prove our desired characterization below, where \mathcal{G}_k^m is a family of graphs defined in Section 3.

Theorem 1. Let $m \ge 0$ be an integer. If G is a cactus graph of order $n \ge 2$ with $k \ge 0$ cycles and ℓ leaves, then $\gamma(G) = \frac{1}{3}(n - \ell + 2(1 - k) + m)$, if and only if $G \in \mathcal{G}_k^m$.

We proceed as follows. In Section 3 we define the families \mathcal{G}_k^m of graphs for each integer $k \geq 0$ and $m \geq 0$. Known results on the domination number are given in Section 4. In Section 5 we present a proof of our main result.

3. The Families \mathcal{G}_k^m for $m \ge 0$ and $k \ge 0$

In this section, we define the families \mathcal{G}_k^m of graphs for each integer $k \geq 0$ and $m \geq 0$. The families \mathcal{G}_k^0 , \mathcal{G}_k^1 , \mathcal{G}_k^2 , $\mathcal{T}_0^{1,1}$, $\mathcal{T}_0^{2,1}$ of graphs were defined by the authors in [5]. For completeness, we include these definitions in Sections 3.1 and 3.2. We first define the families \mathcal{G}_k^0 , \mathcal{G}_k^1 and \mathcal{G}_k^2 of graphs in the special case when k = 0.

3.1. The families \mathcal{G}_0^0 , \mathcal{G}_0^1 and \mathcal{G}_0^2

Hajian *et al.* [5] defined the class of trees \mathcal{G}_0^0 , \mathcal{G}_0^1 and \mathcal{G}_0^2 as follows.

• Let \mathcal{G}_0^0 be the class of all trees T that can be obtained from a sequence T_1, \ldots, T_k of trees where $k \geq 1$ such that T_1 is a star with at least three vertices, $T = T_k$, and, if $k \geq 2$, then the tree T_{i+1} can be obtained from the tree T_i by applying Operation \mathcal{O} defined below for all $i \in [k-1]$.

Operation \mathcal{O} . Add a vertex disjoint copy of a star Q_i with at least three vertices to the tree T_i and add an edge joining a leaf of Q_i and a leaf of T_i .

• Let $\mathcal{T}_0^{1,1}$ be the class of all trees T that can be obtained from a tree $T' \in \mathcal{G}_0^0$ by adding a vertex disjoint copy of a star with at least three vertices and adding an edge from a leaf of the added star to a non-leaf in T'. Now, let \mathcal{G}_0^1 be the class of all trees T that can be obtained from a sequence T_1, \ldots, T_k of trees where $k \ge 1$ such that $T_1 \in \mathcal{T}_0^{1,1} \cup \{P_2\}, T = T_k$, and, if $k \ge 2$, then the tree T_{i+1} can be obtained from the tree T_i by applying Operation \mathcal{O} for all $i \in [k-1]$.

• Let $\mathcal{T}_0^{2,1}$ be the class of all trees T that can be obtained from a tree $T' \in \mathcal{G}_0^0$ by adding a vertex disjoint copy of a star (with at least two vertices) and adding an edge from the center of the added star to a non-leaf in T'. Let $\mathcal{T}_0^{2,2}$ be the class of all trees T that can be obtained from a tree $T' \in \mathcal{G}_0^1$ by adding a vertex disjoint copy of a star with at least three vertices and adding an edge from a leaf of the added star to a non-leaf in T'. Now, let \mathcal{G}_0^2 be the class of all trees Tthat can be obtained from a sequence T_1, \ldots, T_k of trees, where $k \ge 1$, such that $T_1 \in \mathcal{T}_0^{2,1} \cup \mathcal{T}_0^{2,2} \cup \{P_4\}, T = T_k$, and, if $k \ge 2$, then the tree T_{i+1} can be obtained from the tree T_i by applying Operation \mathcal{O} for all $i \in [k-1]$.

3.2. The families \mathcal{G}_k^0 , \mathcal{G}_k^1 and \mathcal{G}_k^2 when $k \geq 1$

For $k \geq 1$, Hajian *et al.* [5] defined the families of graphs \mathcal{G}_k^0 , \mathcal{G}_k^1 and \mathcal{G}_k^2 as follows.

• For $k \ge 1$, they recursively defined the family \mathcal{G}_i^0 of graphs for each $i \in [k]$ by the following procedure.

Procedure A. For $i \in [k]$, a graph G_i belongs to the family \mathcal{G}_i^0 if it contains an edge e = xy such that the graph $G_i - e$ belongs to the family \mathcal{G}_{i-1}^0 and the vertices x and y are leaves in $G_i - e$ that are connected by a unique path in $G_i - e$.

• For $k \ge 1$, they recursively defined the family \mathcal{G}_i^1 of graphs for each $i \in [k]$ by the following two procedures.

Procedure B. For $i \in [k]$, a graph G_i belongs to the family \mathcal{G}_i^1 if it contains an edge e = xy such that the graph $G_i - e$ belongs to the family \mathcal{G}_{i-1}^1 and the vertices x and y are leaves in $G_i - e$ that are connected by a unique path in $G_i - e$.

Procedure C. For $i \in [k]$, a graph G_i belongs to the family \mathcal{G}_i^1 if it contains an edge e = xy such that the graph $G_i - e$ belongs to the family \mathcal{G}_{i-1}^0 and the vertices x and y are connected by a unique path in $G_i - e$. Further, exactly one of x and y is a leaf in $G_i - e$.

• For $k \ge 1$, they recursively defined the family \mathcal{G}_i^2 of graphs for each $i \in [k]$ by the following four procedures.

Procedure D. For $i \in [k]$, a graph G_i belongs to the family \mathcal{G}_i^2 if it contains an edge e = xy such that the graph $G_i - e$ belongs to the family \mathcal{G}_{i-1}^2 and the vertices x and y are leaves in $G_i - e$ that are connected by a unique path in $G_i - e$.

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Procedure E. For $i \in [k]$, a graph G_i belongs to the family \mathcal{G}_i^2 if it contains an edge e = xy such that the graph $G_i - e$ belongs to the family \mathcal{G}_{i-1}^1 and the vertices x and y are connected by a unique path in $G_i - e$. Further, exactly one of x and y is a leaf in $G_i - e$.

Procedure F. For $i \in [k]$, a graph G_i belongs to the family \mathcal{G}_i^2 if it contains an edge e = xy such that the graph $G_i - e$ belongs to the family \mathcal{G}_{i-1}^0 and the vertices x and y are connected by a unique path in $G_i - e$. Further, both x and y are non-leaves in $G_i - e$.

Procedure G. For $2 \leq i \in [k]$, a graph G_i belongs to the family \mathcal{G}_i^2 if it contains an edge e = xy such that the graph $G_i - e$ belongs to the family \mathcal{G}_{i-2}^0 and the vertices x and y are connected by exactly two paths in $G_i - e$. Further, both xand y are leaves in $G_i - e$.

3.3. The family \mathcal{G}_0^m when $m \geq 3$

In this section, we define a family of graphs \mathcal{G}_0^m for each integer $m \geq 3$ as follows. We call a non-leaf x in a tree T a special vertex if $\gamma(T-x) \geq \gamma(T)$. For $m \geq 3$, we first recursively define the class $\mathcal{T}_0^{m,1}$ and $\mathcal{T}_0^{m,2}$ of trees as follows.

• Let $\mathcal{T}_0^{m,1}$ be the class of all trees T that can be obtained from a tree $T' \in \mathcal{G}_0^{m-2}$ by adding a vertex disjoint copy of a star Q and joining the center of Q to a special vertex in T'.

• Let $\mathcal{T}_0^{m,2}$ be the class of all trees T that can be obtained from a tree $T' \in \mathcal{G}_0^{m-1}$ by adding a vertex disjoint copy of a star Q with at least three vertices and joining a leaf of Q to a non-leaf in T'.

For $m \geq 3$, we next recursively define the family \mathcal{G}_0^m of graphs constructed from the families \mathcal{G}_0^{m-1} and \mathcal{G}_0^{m-2} as follows.

• Let \mathcal{G}_0^m be the class of all trees T that can be obtained from a sequence T_1, \ldots, T_q of trees, where $q \ge 1$ and where the tree $T_1 \in \mathcal{T}_0^{m,1} \cup \mathcal{T}_0^{m,2}$ and the tree $T = T_q$. Further, if $q \ge 2$, then for each $i \in [q] \setminus \{1\}$, the tree T_i can be obtained from the tree T_{i-1} by applying the Operation \mathcal{O} defined in Section 3.1.

Operation \mathcal{O} . Add a vertex disjoint copy of a star Q_i with at least three vertices to the tree T_i and add an edge joining a leaf of Q_i and a leaf of T_i .

3.4. The family \mathcal{G}_k^m when $m \geq 3$ and $k \geq 1$

For $m \geq 3$ and $k \geq 1$, we construct the family \mathcal{G}_k^m from \mathcal{G}_{k-1}^{m-2} , \mathcal{G}_{k-1}^{m-1} and \mathcal{G}_{k-1}^m , recursively, as follows.

Procedure H. For $i \in [k]$, a graph G_i belongs to the family \mathcal{G}_i^m if it contains an edge e = xy such that the graph $G_i - e$ belongs to the family G_{i-1}^m and the vertices x and y are connected by a unique path in $G_i - e$ and $\gamma(G_i) = \gamma(G_i - e)$. Further, both x and y are leaves in $G_i - e$. **Procedure I.** For $i \in [k]$, a graph G_i belongs to the family \mathcal{G}_i^m if it contains an edge e = xy such that the graph $G_i - e$ belongs to the family G_{i-1}^{m-1} and the vertices x and y are connected by a unique path in $G_i - e$ and $\gamma(G_i) = \gamma(G_i - e)$. Further, exactly one of x and y is a leaf in $G_i - e$.

Procedure J. For $i \in [k]$, a graph G_i belongs to the family \mathcal{G}_i^m if it contains an edge e = xy such that the graph $G_i - e$ belongs to the family G_{i-1}^{m-2} and the vertices x and y are connected by a unique path in $G_i - e$ and $\gamma(G_i) = \gamma(G_i - e)$. Further, both x and y are non-leaves in $G_i - e$.

4. KNOWN RESULTS

In this section, we present some preliminary observations and known results. We begin with the following properties of graphs that belong to the families \mathcal{G}_k^0 , \mathcal{G}_k^1 and \mathcal{G}_k^2 for $k \ge 0$.

Observation 1. The following properties hold in a graph $G \in \mathcal{G}_k^0 \cup \mathcal{G}_k^1 \cup \mathcal{G}_k^2$, where $k \geq 0$.

- (a) The graph G contains exactly k cycles.
- (b) The graph $G \in \mathcal{G}_k^0 \cup \mathcal{G}_k^1$ is a cactus graph.

We shall also need the following elementary property of a dominating set in a graph.

Observation 2. If G is connected graph of order at least 3, then there exists a γ -set of G that contains no leaf of G.

The following lemma is established in [5].

Lemma 2 [5]. If G is a connected graph and C is an arbitrary cycle in G, then there is an edge e of C such that $\gamma(G - e) = \gamma(G)$.

Several authors obtained bounds on the domination number in terms of different variants of graphs, see for example [1, 2, 3, 6, 9]. Let \mathcal{R} be the family of all trees in which the distance between any two distinct leaves is congruent to 2 modulo 3. Lemańska [9] established the following lower bound on the domination number of a tree in terms of its order and number of leaves.

Theorem 3 [9]. If T is a tree of order $n \ge 2$ with ℓ leaves, then $\gamma(T) \ge (n - \ell + 2)/3$, with equality if and only if $T \in \mathcal{R}$.

Hajian *et al.* [5] showed that the family \mathcal{R} is precisely the family \mathcal{G}_0^0 ; that is, $\mathcal{R} = \mathcal{G}_0^0$.

As a consequence of Theorem 3, we have the following result.

Corollary 4 [9]. If T is a tree of order $n \ge 2$ with ℓ leaves, then $\gamma(T) = \frac{1}{3}(n-\ell+2+m)$ for some integer $m \ge 0$.

Hajian et al. [5] strengthened the result in Theorem 3 as follows.

Theorem 5 [5]. If T is a tree of order $n \ge 2$ with ℓ leaves, then the following holds.

- (a) $\gamma(T) \geq \frac{1}{3}(n-\ell+2)$, with equality if and only if $T \in \mathcal{G}_0^0$.
- (b) $\gamma(T) = \frac{1}{3}(n-\ell+3)$ if and only if $T \in \mathcal{G}_0^1$.
- (c) $\gamma(T) = \frac{1}{3}(n-\ell+4)$ if and only if $T \in \mathcal{G}_0^2$.

The result of Theorem 5 was generalized in [5] to connected graphs as follows.

Theorem 6 [5]. If G is a connected graph of order $n \ge 2$ with $k \ge 0$ cycles and ℓ leaves, then the following holds.

- (a) $\gamma(G) \geq \frac{1}{3}(n-\ell+2(1-k))$, with equality if and only if $G \in \mathcal{G}_k^0$.
- (b) $\gamma(G) = \frac{1}{3}(n-\ell+3-2k)$ if and only if $G \in \mathcal{G}_k^1$.
- (c) $\gamma(G) = \frac{1}{3}(n-\ell+4-2k)$ if and only if $G \in \mathcal{G}_k^2$.

As a consequence of Theorem 6(a), we have the following.

Corollary 7 [5]. If G is a connected graph of order $n \ge 2$ with $k \ge 0$ cycles and ℓ leaves, then $\gamma(G) = \frac{1}{3}(n - \ell + 2(1 - k) + m)$ for some integer $m \ge 0$.

5. Proof of Main Result

In this section, we present a proof of our main result, namely Theorem 1. For this purpose, we first prove Theorem 1 in the special case when k = 0, that is, when the cactus is a tree.

Theorem 8. Let $m \ge 0$ be an integer. If T is a tree of order $n \ge 2$ with ℓ leaves, then $\gamma(T) = \frac{1}{3}(n - \ell + 2 + m)$ if and only if $T \in \mathcal{G}_0^m$.

Proof. Let T be a tree of order $n \ge 2$ with ℓ leaves. We proceed by induction on $m \ge 0$, namely first-induction, to show that $\gamma(T) = \frac{1}{3}(n - \ell + 2 + m)$, if and only if $T \in \mathcal{G}_0^m$. For the base step of the first-induction let $m \le 2$. If m = 0, then the result follows by Theorem 5(a). If m = 1, then the result follows by Theorem 5(b). If m = 2, then the result follows by Theorem 5(c). This establishes the base step of the induction. Let $m \ge 3$ and assume that the result holds for all trees T_0 of order n_0 with ℓ_0 leaves, for $m_0 < m$. Let T be a tree of order n and with ℓ leaves. We will show that $\gamma(T) = \frac{1}{3}(n - \ell + 2 + m)$, if and only if $T \in \mathcal{G}_0^m$.

 (\Longrightarrow) Assume that $\gamma(T) = \frac{1}{3}(n-\ell+2+m)$ (where we recall that here $m \ge 3$). We show that $T \in \mathcal{G}_0^m$. If $T = P_2$, then by the definition of the family

 \mathcal{G}_0^1 , we have $T \in \mathcal{G}_0^1$. Then by Theorem 5(b), $\gamma(T) = \frac{1}{3}(n-\ell+2+1)$, and so m = 1, a contradiction. Hence we may assume that diam $(T) \ge 2$, for otherwise the desired result follows. If diam(T) = 2, then T is a star, and by the definition of the family G_0^0 , we have $T \in G_0^0$. Thus by Theorem 5(a), $\gamma(T) = \frac{1}{3}(n-\ell+2+1)$, and so m = 0, a contradiction. If diam(T) = 2, then T is a double star, and by definition of the family \mathcal{G}_0^2 we have $T \in T_0^{2,1} \subseteq \mathcal{G}_0^2$. Thus by Theorem 5(c), $\gamma(T) = \frac{1}{3}(n-\ell+2+2)$, and so m = 2, a contradiction. Hence, diam $(T) \ge 4$ and $n \ge 5$.

We now root the tree T at a vertex r at the end of a longest path P in T. Let u be a vertex at maximum distance from r, and so $d_T(u, r) = \operatorname{diam}(T)$. Necessarily, r and u are leaves. Let v be the parent of u, let w be the parent of v, let x be the parent of w, and let y be the parent of x. Possibly, y = r. Since u is a vertex at maximum distance from the root r, every child of v is a leaf. By Observation 2, there exists a γ -set, say S, of T that contains no leaf of T; that is, $L(T) \cap S = \emptyset$. In particular, we note that $|S| = \gamma(T) = \frac{1}{3}(n - \ell + 2 + m)$. In order to dominate the vertex u, we note therefore that $v \in S$. Let $d_T(v) = t$. We note that $t \geq 2$.

Claim 1. If $d_T(w) \geq 3$, then $T \in \mathcal{G}_0^m$.

Proof. Suppose that $d_T(w) \geq 3$. In this case, we consider the tree $T' = T - V(T_v)$, where T_v is the maximal subtree at v. Let T' have order n' and let T' have ℓ' leaves. We note that n' = n - t. Since w is not a leaf in T', we have $\ell' = \ell - (t - 1) = \ell - t + 1$. By Corollary 4, $\gamma(T') = \frac{1}{3}(n' - \ell' + 2 + m')$ for some integer $m' \geq 0$. If a child of w is a leaf in T', then since the dominating set S contains no leaves, we have that $w \in S$. If no child of w is a leaf in T, then every child of w is a support vertex and therefore belongs to the set S. In both cases, we note that the set $S \setminus \{v\}$ is a dominating set of T', implying that $\gamma(T') \leq |S| - 1 = \gamma(T) - 1$. Every γ -set of T' can be extended to a dominating set of T by adding to it the vertex v, implying that $\gamma(T) \leq \gamma(T') + 1$. Consequently, $\gamma(T') = \gamma(T) - 1$. Thus,

$$\begin{split} \gamma(T') &= \gamma(T) - 1 \\ &= \frac{1}{3}(n - \ell + 2 + m) - 1 \\ &= \frac{1}{3}(n - \ell + m - 1) \\ &= \frac{1}{3}((n' + t) - (\ell' + t - 1) + m - 1) \\ &= \frac{1}{3}(n' - \ell' + m). \end{split}$$

As observed earlier, $\gamma(T') = \frac{1}{3}(n' - \ell' + 2 + m')$ for some integer $m' \ge 0$. Thus, m' = m - 2. Applying the inductive hypothesis to the tree T', we have $T' \in \mathcal{G}_0^{m-2}$. Let v' be a child of w different from v. We note that the tree $T_{v'}$ is a component of T' - w and this component is dominated by the vertex v'. We can therefore choose a γ -set of T' - w to contain the vertex v'. Such a γ -set of T' - w is also a dominating set of T', implying that $\gamma(T') \leq \gamma(T' - w)$; that is, the vertex w is a special vertex of T'. Thus, the tree T is obtained from the tree $T' \in \mathcal{G}_0^{m-2}$ by adding a vertex disjoint copy of a star T_v and joining the center v of T_v to a special vertex w in T'. Thus $T \in \mathcal{T}_0^{m,1}$. Consequently, $T \in \mathcal{G}_0^m$. This completes the proof of Claim 1.

By Claim 1, we may assume that $d_T(w) = 2$, for otherwise $T \in \mathcal{G}_0^m$ as desired. We now consider the tree $T' = T - V(T_w)$, where T_w is the maximal subtree at w. Let T' have order n' and let T' have ℓ' leaves. We note that n' = n - t - 1. By Corollary 4, $\gamma(T') = \frac{1}{3}(n' - \ell' + 2 + m')$ for some integer $m' \ge 0$.

As observed earlier, the vertex v belongs to the dominating set S. If $w \in S$, then we can replace w in S with the vertex x to produce a new γ -set of T that contains no leaf of T. Hence we may assume that $w \notin S$, implying that the set $S \setminus \{v\}$ is a dominating set of T' and therefore $\gamma(T') \leq |S| - 1 = \gamma(T) - 1$. Every γ -set of T' can be extended to a dominating set of T by adding to it the vertex v, implying that $\gamma(T) \leq \gamma(T') + 1$. Consequently, $\gamma(T') = \gamma(T) - 1$.

Claim 2. If $d_T(x) \ge 3$, then $T \in \mathcal{G}_0^m$.

Proof. Suppose that $d_T(x) \ge 3$. In this case, the vertex x is not a leaf of T', implying that $\ell' = \ell - (t-1) = \ell - t + 1$. Thus,

$$\begin{aligned} \gamma(T') &= \gamma(T) - 1 \\ &= \frac{1}{3}(n - \ell + m - 1) \\ &= \frac{1}{3}((n' + t + 1) - (\ell' + t - 1) + m - 1) \\ &= \frac{1}{3}(n' - \ell' + m + 1). \end{aligned}$$

As observed earlier, $\gamma(T') = \frac{1}{3}(n' - \ell' + 2 + m')$ for some integer $m' \ge 0$. Thus, m' = m - 1. Applying the inductive hypothesis to the tree T', we have $T' \in \mathcal{G}_0^{m-1}$. Thus, the tree T is obtained from the tree $T' \in \mathcal{G}_0^{m-1}$ by adding a vertex disjoint copy of a star T_v with at least three vertices and joining a leaf of the star T_v to the non-leaf x of T'. Thus $T \in \mathcal{T}_0^{m,2}$. Consequently, $T \in \mathcal{G}_0^m$. \Box

By Claim 2, we may assume that $d_T(x) = 2$, for otherwise $T \in \mathcal{G}_0^m$ as desired. In this case, the vertex x is a leaf of T', implying that $\ell' = \ell - (t-1) + 1 = \ell - t + 2$. Thus,

$$\frac{1}{3}(n'-\ell'+2+m') = \gamma(T') = \gamma(T) - 1$$

= $\frac{1}{3}(n-\ell+m-1)$
= $\frac{1}{3}((n'+t+1)-(\ell'+t-2)+m-1)$
= $\frac{1}{3}(n'-\ell'+m+2),$

and so m = m'. Applying the inductive hypothesis to the tree T', we have $T' \in \mathcal{G}_0^m$. Thus, the tree T is obtained from the tree $T' \in \mathcal{G}_0^m$ by adding a vertex disjoint copy of a star T_v with at least three vertices and adding the edge xw joining a leaf w of T_v and a leaf x of T'; that is, T is obtained from T' by Operation \mathcal{O} . Hence, by definition of the family \mathcal{G}_0^m , we have $T \in \mathcal{G}_0^m$, as desired. This completes the necessity part of the proof of Theorem 8.

(\Leftarrow) Conversely, assume that $T \in \mathcal{G}_0^m$, where $m \ge 0$. Recall that T is a tree of order $n \ge 2$ with ℓ leaves. Thus, T is obtained from a sequence T_1, \ldots, T_q of trees, where $q \ge 1$ and where the tree $T_1 \in \mathcal{T}_0^{m,1} \cup \mathcal{T}_0^{m,2}$, and the tree $T = T_q$. Further, if $q \ge 2$, then for each $i \in [q] \setminus \{1\}$, the tree T_i can be obtained from the tree T_{i-1} by applying the following Operation \mathcal{O} . We proceed by induction on $q \ge 1$, namely *second-induction*, to show that $\gamma_t(T) = \frac{1}{3}(n - \ell + 2 + m)$.

Claim 3. If q = 1, then $\gamma_t(T) = \gamma(T) = \frac{1}{3}(n - \ell + 2 + m)$.

Proof. Suppose that q = 1. Thus, $T_1 \in \mathcal{T}_0^{m,1} \cup \mathcal{T}_0^{m,2}$. We consider the two possibilities in turn, and in both cases we will show that the tree $T \in \mathcal{G}_0^m$ satisfies $\gamma(T) = \frac{1}{3}(n - \ell + 2 + m)$.

Claim 3.1. If $T \in \mathcal{T}_0^{m,1}$, then $\gamma_t(T) = \frac{1}{3}(n - \ell + 2 + m)$.

Proof. Suppose that $T \in \mathcal{T}_0^{m,1}$. Thus, T is obtained from a tree $T' \in \mathcal{G}_0^{m-2}$ by adding a vertex disjoint copy of a star Q with $t \ge 2$ vertices and joining the center of Q, say y, to a special vertex x in T'. Let T' have order n', and so n' = n - t. Further, let T' have ℓ' leaves. Since x is a non-leaf of T', we have $\ell' = \ell - (t - 1)$. Applying the first-induction hypothesis to the tree $T' \in \mathcal{G}_0^{m-2}$, we have $\gamma_t(T') = \frac{1}{3}(n' - \ell' + 2 + (m - 2)) = \frac{1}{3}(n' - \ell' + m)$.

We show next that $\gamma(T) = \gamma(T') + 1$. Since x is a special vertex of T', we note that $\gamma(T'-x) \geq \gamma(T')$. Every γ -set of T' can be extended to a dominating set of T by adding to it the vertex y, implying that $\gamma(T) \leq \gamma(T') + 1$. Conversely, we can choose a γ -set, say D, of T to contain the vertex y which dominates the star Q. If $x \in D$, then $D \setminus \{y\}$ is a dominating set of T', and so $\gamma(T') \leq |D| - 1$. If $x \notin D$, then $D \setminus \{y\}$ is a dominating set of T' - x, and so $\gamma(T') \leq \gamma(T' - x) \leq |D| - 1$. In both cases, $\gamma(T') \leq |D| - 1 = \gamma(T) - 1$. Consequently, $\gamma(T) = \gamma(T') + 1$. Thus,

$$\begin{split} \gamma(T) &= \gamma(T') + 1 \\ &= \frac{1}{3}(n' - \ell' + m) + 1 \\ &= \frac{1}{3}((n-t) - (\ell - t + 1) + m) + 1 \\ &= \frac{1}{3}(n - \ell + 2 + m). \end{split}$$

This completes the proof of Claim 3.1.

Claim 3.2. If $T \in \mathcal{T}_0^{m,2}$, then $\gamma_t(T) = \frac{1}{3}(n - \ell + 2 + m)$.

Proof. Suppose that $T \in \mathcal{T}_0^{m,2}$. Thus, T is obtained from a tree $T' \in \mathcal{G}_0^{m-1}$ by adding a vertex disjoint copy of a star Q with $t \geq 3$ vertices and joining a leaf, say v, of Q to a non-leaf, say w, in T'. Let u be the center of the star Q. Let T' have order n', and so n' = n - t. Further, let T' have ℓ' leaves. Since w is a non-leaf of T', we have $\ell' = \ell - (t-2)$. Applying the first-induction hypothesis to the tree $T' \in \mathcal{G}_0^{m-1}$, we have $\gamma_t(T') = \frac{1}{3}(n' - \ell' + 2 + (m-1)) = \frac{1}{3}(n' - \ell' + m + 1)$.

We show next that $\gamma(T) = \gamma_t(T') + 1$. Every γ -set of T' can be extended to a dominating of T by adding to it the vertex u, implying that $\gamma(T) \leq \gamma(T') + 1$. By Observation 2, there exists a γ -set D of T that contains no leaf of G. Thus, $u \in D$. If $v \in D$, then we can replace v in D with the vertex w. Hence we may assume that $v \notin D$, implying that $D \setminus \{u\}$ is a dominating set of T', and so $\gamma(T') \leq |D| - 1 = \gamma(T) - 1$. Consequently, $\gamma(T) = \gamma(T') + 1$. Thus,

$$\begin{split} \gamma(T) &= \gamma(T') + 1 \\ &= \frac{1}{3}(n' - \ell' + m + 1) + 1 \\ &= \frac{1}{3}((n-t) - (\ell - t + 2) + m + 1) + 1 \\ &= \frac{1}{3}(n - \ell + 2 + m). \end{split}$$

This completes the proof of Claim 3.2.

By Claims 3.1 and 3.2, if $T \in \mathcal{T}_0^{m,1} \cup \mathcal{T}_0^{m,2}$, then $\gamma(T) = \frac{1}{3}(n-\ell+2+m)$. This completes the proof of Claim 3.

By Claim 3, if q = 1, then $\gamma(T) = \frac{1}{3}(n - \ell + 2 + m)$. This establishes the base step of the second-induction. Let $q \ge 2$ and assume that if q' is an integer where $1 \le q' < q$ and if $T' \in \mathcal{G}_0^m$ is a tree of order $n' \ge 2$ with ℓ' leaves obtained from a sequence of q' trees, then $\gamma(T) = \frac{1}{3}(n' - \ell' + 2 + m)$. Recall that T is obtained from a sequence T_1, \ldots, T_q of trees, where $q \ge 1$ and where the tree $T_1 \in \mathcal{T}_0^{m,1} \cup \mathcal{T}_0^{m,2}$, and the tree $T = T_q$. Further for each $i \in [q] \setminus \{1\}$, the tree T_i can be obtained from the tree T_{i-1} by applying the Operation \mathcal{O} .

We now consider the tree $T' = T_{q-1}$. Thus, the tree $T \in \mathcal{G}_0^m$ is obtained from the tree T' by adding a vertex disjoint copy of a star Q with $t \ge 3$ vertices and adding an edge joining a leaf of Q to a leaf of T'. Let T' have order n' and let T' have ℓ' leaves. We note that n' = n - t and $\ell' = \ell - (t - 2) + 1 = \ell - t + 3$. Applying the second-induction hypothesis to the tree $T' \in \mathcal{G}_0^m$, we have $\gamma(T') = \frac{1}{3}(n' - \ell' + 2 + m)$. Analogous arguments as before show that $\gamma(T) = \gamma_t(T') + 1$. Thus,

$$\gamma(T) = \gamma(T') + 1$$

= $\frac{1}{3}(n' - \ell' + 2 + m) + 1$
= $\frac{1}{3}((n - t) - (\ell - t + 3) + 2 + m) + 1$
= $\frac{1}{3}(n - \ell + 2 + m).$

Hence we have shown that if $T \in \mathcal{G}_0^m$, where $m \ge 0$ and where T has order $n \ge 2$ with ℓ leaves, then $\gamma(T) = \frac{1}{3}(n - \ell + 2 + m)$. This completes the proof of Theorem 8.

We are now in a position to prove our main result, namely Theorem 1. Recall its statement.

Theorem 1. Let $m \ge 0$ be an integer. If G is a cactus graph of order $n \ge 2$ with $k \ge 0$ cycles and ℓ leaves, then $\gamma(G) = \frac{1}{3}(n - \ell + 2(1 - k) + m)$, if and only if $G \in \mathcal{G}_k^m$.

Proof. Let $m \ge 0$ be an integer, and let G be a cactus graph of order $n \ge 2$ with $k \ge 0$ cycles and ℓ leaves. We proceed by induction on k to show that $\gamma(G) = \frac{1}{3}(n - \ell + 2(1 - k) + m)$ if and only if $G \in \mathcal{G}_k^m$. If k = 0, then the result follows from Theorem 8. This establishes the base case. Let $k \ge 1$ and assume that if G' is a cactus graph of order $n' \ge 2$ with k' cycles and ℓ' leaves where $0 \le k' < k$, then $\gamma(G) = \frac{1}{3}(n' - \ell' + 2(1 - k') + m')$ if and only if $G \in \mathcal{G}_{k'}^{m'}$. Let G be a cactus graph of order $n \ge 2$ with $k \ge 0$ cycles and ℓ leaves. We will show that $\gamma(G) = \frac{1}{3}(n - \ell + 2(1 - k) + m)$, if and only if $G \in \mathcal{G}_k^m$. If m = 0, then the result follows by Theorem 6(a). If m = 1, then the result follows by Theorem 6(b). If m = 2, then the result follows by Theorem 6(c). Thus, we may assume that $m \ge 3$, for otherwise the desired result follows.

 (\Longrightarrow) Assume that $\gamma(G) = \frac{1}{3}(n-\ell+2+m-2k)$ (where we recall that here $m \geq 3$). We will show that $T \in \mathcal{G}_k^m$. By Lemma 2, the graph G contains a cycle edge e such that $\gamma(G-e) = \gamma(G)$. Let e = uv, and consider the graph G' = G - e. Let G' have order n' with $k' \geq 0$ cycles and ℓ' leaves. We note that n' = n. Further, since G is a cactus graph, k' = k - 1. Removing the cycle edge e from G produces at most two new leaves, namely the ends of the edge e, implying that $\ell' - 2 \leq \ell \leq \ell'$. By Corollary 7, we have $\gamma(G') = \frac{1}{3}(n' - \ell' + 2 + m' - 2k')$ for some integer $m' \geq 0$. Applying the inductive hypothesis to the cactus graph G', we have that $G' \in \mathcal{G}_{k'}^{m'} = \mathcal{G}_{k-1}^{m'}$. Our earlier observations imply that

$$\begin{aligned} \frac{1}{3}(n-\ell+2+m-2k) &= \gamma(G) = \gamma(G') \\ &= \frac{1}{3}(n'-\ell'+2+m'-2k') \\ &= \frac{1}{3}(n-\ell'+2+m'-2(k-1)), \end{aligned}$$

and so $m - \ell = m' - \ell' + 2$. Since G is a cactus, the vertices u and v are connected in G' = G - e by a unique path. As observed earlier, $\ell' - 2 \le \ell \le \ell'$.

Suppose that $\ell = \ell'$. In this case, neither u nor v is a leaf of G', implying that both u and v have degree at least 2 in G'. Further, the equation $m-\ell = m'-\ell'+2$ simplifies to m' = m-2. Thus, $G' \in \mathcal{G}_{k-1}^{m-2}$. Hence, the graph G is obtained from G' by Procedure J and therefore $G \in \mathcal{G}_k^m$.

Suppose that $\ell = \ell' - 1$. In this case, exactly one of u and v is a leaf of G'. Further, the equation $m - \ell = m' - \ell' + 2$ simplifies to m' = m - 1. Thus, $G' \in \mathcal{G}_{k-1}^{m-1}$. Hence, the graph G is obtained from G' by Procedure I, and therefore $G \in \mathcal{G}_k^m$.

Suppose that $\ell = \ell' - 2$. In this case, both u and v are leaves in G'. Further, the equation $m - \ell = m' - \ell' + 2$ simplifies to m' = m. Thus, $G' \in \mathcal{G}_{k-1}^m$. Hence, the graph G is obtained from G' by Procedure H, and therefore $G \in \mathcal{G}_k^m$. This completes the necessity part of the proof of Theorem 1.

(\Leftarrow) Conversely, assume that $G \in \mathcal{G}_k^m$. Recall that by our earlier assumptions, $m \geq 3$ and $k \geq 1$. Thus, the graph G is obtained from either a graph $G' \in \mathcal{G}_{k-1}^m$ by Procedure H or from a graph $G' \in \mathcal{G}_{k-1}^{m-1}$ by Procedure I or from a graph $G' \in \mathcal{G}_{k-1}^{m-2}$ by Procedure J. In all three cases, let G' have order n' with $k' \geq 0$ cycles and ℓ' leaves. Further, in all cases we note that n' = n and k' = k-1. We consider the three possibilities in turn.

Suppose firstly that G is obtained from a graph $G' \in \mathcal{G}_{k-1}^m$ by Procedure H. In this case, $\ell = \ell' - 2$ and $\gamma(G) = \gamma(G')$. Applying the inductive hypothesis to the graph $G' \in \mathcal{G}_{k-1}^m$, we have $\gamma(G) = \gamma(G') = \frac{1}{3}(n' - \ell' + 2 + m - 2(k-1)) = \frac{1}{3}(n - (\ell+2) + 4 + m - 2k) = \frac{1}{3}(n - \ell + 2 + m - 2k).$

Suppose next that G is obtained from a graph $G' \in \mathcal{G}_{k-1}^{m-1}$ by Procedure I. In this case, $\ell = \ell' - 1$ and $\gamma(G) = \gamma(G')$. Applying the inductive hypothesis to the graph $G' \in \mathcal{G}_{k-1}^{m-1}$, we have $\gamma(G) = \gamma(G') = \frac{1}{3}(n'-\ell'+2+(m-1)-2(k-1)) = \frac{1}{3}(n-(\ell+1)+3+m-2k) = \frac{1}{3}(n-\ell+2+m-2k).$

Suppose finally that G is obtained from a graph $G' \in \mathcal{G}_{k-1}^{m-2}$ by Procedure J. In this case, $\ell = \ell'$ and $\gamma(G) = \gamma(G')$. Applying the inductive hypothesis to the graph $G' \in \mathcal{G}_{k-1}^{m-2}$, we have $\gamma(G) = \gamma(G') = \frac{1}{3}(n'-\ell'+2+(m-2)-2(k-1)) = \frac{1}{3}(n-\ell+2+m-2k)$. In all three cases, $\gamma(G) = \frac{1}{3}(n-\ell+2+m-2k)$. This completes the proof of Theorem 1.

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