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A CLASSIFICATION OF CACTUS GRAPHS ACCORDING TO THEIR DOMINATION NUMBER

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Abstract

A set S of vertices in a graph G is a dominating set of G if every vertex not in S is adjacent to some vertex in S. The domination number, $\gamma(G)$, of G is the minimum cardinality of a dominating set of G. The authors proved in [*A new lower bound on the domination number of a graph*, J. Comb. Optim. 38 (2019) 721–738] that if G is a connected graph of order $n \geq 2$ with $k \geq 0$ cycles and ℓ leaves, then $\gamma(G) \geq \lceil (n - \ell + 2 - 2k)/3 \rceil$. As a consequence of the above bound, $\gamma(G) = (n - \ell + 2(1-k) + m)/3$ for some integer $m \geq 0$. In this paper, we characterize the class of cactus graphs achieving equality here, thereby providing a classification of all cactus graphs according to their domination number.

Keywords: domination number, lower bounds, cycles, cactus graphs.

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1. Introduction

A *dominating set* of a graph G is a set S of vertices of G such that every vertex not in S has a neighbor in S, where two vertices are neighbors in G if they are adjacent. The minimum cardinality of a dominating set is the *domination number* of G, denoted by $\gamma(G)$. A dominating set of cardinality $\gamma(G)$ is called a γ*-set* of G. As remarked in [5], the notion of domination and its variations in graphs has been studied a great deal; a rough estimate says that it occurs in more than 6000 papers to date. For fundamentals of domination theory in graphs we refer the reader to the so-called domination books by Haynes, Hedetniemi, and Slater [6, 7]. An updated glossary of domination parameters can be found in [4].

Two vertices u and v in a graph G are *connected* if there exists a (u, v)-path in G. The graph G is *connected* if every two vertices in G are connected. A *block* of G is a maximal connected subgraph of G which has no cut-vertex of its own. A *cactus* is a connected graph in which every edge belongs to at most one cycle. Equivalently, a (nontrivial) cactus is a connected graph in which every block is an edge or a cycle. The *distance* between two vertices u and v in a connected graph G is the minimum length of a (u, v)-path in G. The *diameter*, $\mathrm{diam}(G)$, of G is the maximum distance among pairs of vertices in G.

For notation and graph theory terminology we generally follow [8]. In particular, the *order* of a graph G with vertex set $V(G)$ and edge set $E(G)$ is given by $n(G) = |V(G)|$ and its *size* by $m(G) = |E(G)|$. A *neighbor* of a vertex v in G is a vertex adjacent to v, and the *open neighborhood* of v is the set of neighbors of v, denoted $N_G(v)$. The *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of a vertex v in G is given by $d_G(v) = |N_G(v)|$.

For a set S of vertices in a graph G, the subgraph induced by S is denoted by $G[S]$. Further, the subgraph obtained from G by deleting all vertices in S and all edges incident with vertices in S is denoted by $G - S$. If $S = \{v\}$, we simply denote $G - \{v\}$ by $G - v$. A *leaf* of a graph G is a vertex of degree 1 in G, and its unique neighbor is called a *support vertex*. The set of all leaves of G is denoted by $L(G)$, and we let $\ell(G) = |L(G)|$ be the number of leaves in G. We denote the set of support vertices of G by $S(G)$. We call a vertex of degree at least 2 a *non-leaf*.

Following our notation in [5], we denote the path and cycle on n vertices by P_n and C_n, respectively. A *complete graph* on n vertices is denoted by K_n, while a *complete bipartite graph* with partite sets of size n and m is denoted by $K_{n,m}$. A *star* is the graph $K_{1,k}$, where $k \geq 1$. Further if $k > 1$, the vertex of degree k is called the *center* vertex of the star, while if $k = 1$, arbitrarily designate either vertex of P_2 as the center. A *double star* is a tree with exactly two (adjacent) non-leaf vertices.

A *rooted tree* T distinguishes one vertex r called the *root*. For each vertex

$v \neq r$ of T, the *parent* of v is the neighbor of v on the unique (r, v)-path, while a *child* of v is any other neighbor of v. A *descendant* of v is a vertex $u \neq v$ such that the unique (r, u)-path contains v. In particular, every child of v is a descendant of v. We let $D(v)$ denote the set of descendants of v, and we define $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v. We use the standard notation $[k] = \{1, \ldots, k\}$.

2. Main Result

Our aim in this paper is to provide a classification of all cactus graphs according to their domination number. For this purpose, we shall use a result of the authors in [5] (which we present in Section 4) that establishes a lower bound on the domination number of a graph in terms of its order, number of vertices of degree 1, and number of cycles. From this result, we prove our desired characterization below, where $\mathcal{G}_k^m$ is a family of graphs defined in Section 3.

Theorem 1. *Let $m \geq 0$ be an integer. If G is a cactus graph of order $n \geq 2$ with $k \geq 0$ cycles and ℓ leaves, then $\gamma(G) = \frac{1}{3}(n - \ell + 2(1 - k) + m)$, if and only if $G \in \mathcal{G}_k^m$.*

We proceed as follows. In Section 3 we define the families $\mathcal{G}_k^m$ of graphs for each integer $k \geq 0$ and $m \geq 0$. Known results on the domination number are given in Section 4. In Section 5 we present a proof of our main result.

3. The Families $\mathcal{G}_k^m$ for $m \geq 0$ and $k \geq 0$

In this section, we define the families $\mathcal{G}_k^m$ of graphs for each integer $k \geq 0$ and $m \geq 0$. The families $\mathcal{G}_k^0$, $\mathcal{G}_k^1$, $\mathcal{G}_k^2$, $\mathcal{T}_0^{1,1}$, $\mathcal{T}_0^{2,1}$ of graphs were defined by the authors in [5]. For completeness, we include these definitions in Sections 3.1 and 3.2. We first define the families $\mathcal{G}_k^0$, $\mathcal{G}_k^1$ and $\mathcal{G}_k^2$ of graphs in the special case when $k = 0$.

3.1. The families $\mathcal{G}_0^0$, $\mathcal{G}_0^1$ and $\mathcal{G}_0^2$

Hajian *et al.* [5] defined the class of trees $\mathcal{G}_0^0$, $\mathcal{G}_0^1$ and $\mathcal{G}_0^2$ as follows.

• Let $\mathcal{G}_0^0$ be the class of all trees T that can be obtained from a sequence $T_1, \ldots, T_k$ of trees where $k \geq 1$ such that T_1 is a star with at least three vertices, $T = T_k$, and, if $k \geq 2$, then the tree T_{i+1} can be obtained from the tree T_i by applying Operation $\mathcal{O}$ defined below for all $i \in [k - 1]$.

Operation $\mathcal{O}$. Add a vertex disjoint copy of a star Q_i with at least three vertices to the tree T_i and add an edge joining a leaf of Q_i and a leaf of T_i.

• Let $\mathcal{T}_0^{1,1}$ be the class of all trees T that can be obtained from a tree $T' \in \mathcal{G}_0^0$ by adding a vertex disjoint copy of a star with at least three vertices and adding an edge from a leaf of the added star to a non-leaf in T'. Now, let $\mathcal{G}_0^1$ be the class of all trees T that can be obtained from a sequence $T_1, \ldots, T_k$ of trees where $k \geq 1$ such that $T_1 \in \mathcal{T}_0^{1,1} \cup \{P_2\}$, $T = T_k$, and, if $k \geq 2$, then the tree T_{i+1} can be obtained from the tree T_i by applying Operation $\mathcal{O}$ for all $i \in [k-1]$.

• Let $\mathcal{T}_0^{2,1}$ be the class of all trees T that can be obtained from a tree $T' \in \mathcal{G}_0^0$ by adding a vertex disjoint copy of a star (with at least two vertices) and adding an edge from the center of the added star to a non-leaf in T'. Let $\mathcal{T}_0^{2,2}$ be the class of all trees T that can be obtained from a tree $T' \in \mathcal{G}_0^1$ by adding a vertex disjoint copy of a star with at least three vertices and adding an edge from a leaf of the added star to a non-leaf in T'. Now, let $\mathcal{G}_0^2$ be the class of all trees T that can be obtained from a sequence $T_1, \ldots, T_k$ of trees, where $k \geq 1$, such that $T_1 \in \mathcal{T}_0^{2,1} \cup \mathcal{T}_0^{2,2} \cup \{P_4\}$, $T = T_k$, and, if $k \geq 2$, then the tree T_{i+1} can be obtained from the tree T_i by applying Operation $\mathcal{O}$ for all $i \in [k-1]$.

3.2. The families $\mathcal{G}_k^0$, $\mathcal{G}_k^1$ and $\mathcal{G}_k^2$ when $k \geq 1$

For $k \geq 1$, Hajian *et al.* [5] defined the families of graphs $\mathcal{G}_k^0$, $\mathcal{G}_k^1$ and $\mathcal{G}_k^2$ as follows.

• For $k \geq 1$, they recursively defined the family $\mathcal{G}_i^0$ of graphs for each $i \in [k]$ by the following procedure.

Procedure A. For $i \in [k]$, a graph G_i belongs to the family $\mathcal{G}_i^0$ if it contains an edge $e = xy$ such that the graph $G_i - e$ belongs to the family $\mathcal{G}_{i-1}^0$ and the vertices x and y are leaves in $G_i - e$ that are connected by a unique path in $G_i - e$.

• For $k \geq 1$, they recursively defined the family $\mathcal{G}_i^1$ of graphs for each $i \in [k]$ by the following two procedures.

Procedure B. For $i \in [k]$, a graph G_i belongs to the family $\mathcal{G}_i^1$ if it contains an edge $e = xy$ such that the graph $G_i - e$ belongs to the family $\mathcal{G}_{i-1}^1$ and the vertices x and y are leaves in $G_i - e$ that are connected by a unique path in $G_i - e$.

Procedure C. For $i \in [k]$, a graph G_i belongs to the family $\mathcal{G}_i^1$ if it contains an edge $e = xy$ such that the graph $G_i - e$ belongs to the family $\mathcal{G}_{i-1}^0$ and the vertices x and y are connected by a unique path in $G_i - e$. Further, exactly one of x and y is a leaf in $G_i - e$.

• For $k \geq 1$, they recursively defined the family $\mathcal{G}_i^2$ of graphs for each $i \in [k]$ by the following four procedures.

Procedure D. For $i \in [k]$, a graph G_i belongs to the family $\mathcal{G}_i^2$ if it contains an edge $e = xy$ such that the graph $G_i - e$ belongs to the family $\mathcal{G}_{i-1}^2$ and the vertices x and y are leaves in $G_i - e$ that are connected by a unique path in $G_i - e$.

Procedure E. For $i \in [k]$, a graph G_i belongs to the family $\mathcal{G}_i^2$ if it contains an edge $e = xy$ such that the graph $G_i - e$ belongs to the family $\mathcal{G}_{i-1}^1$ and the vertices x and y are connected by a unique path in $G_i - e$. Further, exactly one of x and y is a leaf in $G_i - e$.

Procedure F. For $i \in [k]$, a graph G_i belongs to the family $\mathcal{G}_i^2$ if it contains an edge $e = xy$ such that the graph $G_i - e$ belongs to the family $\mathcal{G}_{i-1}^0$ and the vertices x and y are connected by a unique path in $G_i - e$. Further, both x and y are non-leaves in $G_i - e$.

Procedure G. For $2 \leq i \in [k]$, a graph G_i belongs to the family $\mathcal{G}_i^2$ if it contains an edge $e = xy$ such that the graph $G_i - e$ belongs to the family $\mathcal{G}_{i-2}^0$ and the vertices x and y are connected by exactly two paths in $G_i - e$. Further, both x and y are leaves in $G_i - e$.

3.3. The family $\mathcal{G}_0^m$ when $m \geq 3$

In this section, we define a family of graphs $\mathcal{G}_0^m$ for each integer $m \geq 3$ as follows. We call a non-leaf x in a tree T a *special vertex* if $\gamma(T - x) \geq \gamma(T)$. For $m \geq 3$, we first recursively define the class $\mathcal{T}_0^{m,1}$ and $\mathcal{T}_0^{m,2}$ of trees as follows.

• Let $\mathcal{T}_0^{m,1}$ be the class of all trees T that can be obtained from a tree $T' \in \mathcal{G}_0^{m-2}$ by adding a vertex disjoint copy of a star Q and joining the center of Q to a special vertex in T'.

• Let $\mathcal{T}_0^{m,2}$ be the class of all trees T that can be obtained from a tree $T' \in \mathcal{G}_0^{m-1}$ by adding a vertex disjoint copy of a star Q with at least three vertices and joining a leaf of Q to a non-leaf in T'.

For $m \geq 3$, we next recursively define the family $\mathcal{G}_0^m$ of graphs constructed from the families $\mathcal{G}_0^{m-1}$ and $\mathcal{G}_0^{m-2}$ as follows.

• Let $\mathcal{G}_0^m$ be the class of all trees T that can be obtained from a sequence $T_1, \ldots, T_q$ of trees, where $q \geq 1$ and where the tree $T_1 \in \mathcal{T}_0^{m,1} \cup \mathcal{T}_0^{m,2}$ and the tree $T = T_q$. Further, if $q \geq 2$, then for each $i \in [q] \setminus \{1\}$, the tree T_i can be obtained from the tree T_{i-1} by applying the Operation $\mathcal{O}$ defined in Section 3.1.

Operation $\mathcal{O}$. Add a vertex disjoint copy of a star Q_i with at least three vertices to the tree T_i and add an edge joining a leaf of Q_i and a leaf of T_i.

3.4. The family $\mathcal{G}_k^m$ when $m \geq 3$ and $k \geq 1$

For $m \geq 3$ and $k \geq 1$, we construct the family $\mathcal{G}_k^m$ from $\mathcal{G}_{k-1}^{m-2}$, $\mathcal{G}_{k-1}^{m-1}$ and $\mathcal{G}_{k-1}^m$, recursively, as follows.

Procedure H. For $i \in [k]$, a graph G_i belongs to the family $\mathcal{G}_i^m$ if it contains an edge $e = xy$ such that the graph $G_i - e$ belongs to the family G_{i-1}^m and the vertices x and y are connected by a unique path in $G_i - e$ and $\gamma(G_i) = \gamma(G_i - e)$. Further, both x and y are leaves in $G_i - e$.

Procedure I. For $i \in [k]$, a graph G_i belongs to the family $\mathcal{G}_i^m$ if it contains an edge $e = xy$ such that the graph $G_i - e$ belongs to the family G_{i-1}^{m-1} and the vertices x and y are connected by a unique path in $G_i - e$ and $\gamma(G_i) = \gamma(G_i - e)$. Further, exactly one of x and y is a leaf in $G_i - e$.

Procedure J. For $i \in [k]$, a graph G_i belongs to the family $\mathcal{G}_i^m$ if it contains an edge $e = xy$ such that the graph $G_i - e$ belongs to the family G_{i-1}^{m-2} and the vertices x and y are connected by a unique path in $G_i - e$ and $\gamma(G_i) = \gamma(G_i - e)$. Further, both x and y are non-leaves in $G_i - e$.

4. Known Results

In this section, we present some preliminary observations and known results. We begin with the following properties of graphs that belong to the families $\mathcal{G}_k^0$, $\mathcal{G}_k^1$ and $\mathcal{G}_k^2$ for $k \geq 0$.

Observation 1. *The following properties hold in a graph $G \in \mathcal{G}_k^0 \cup \mathcal{G}_k^1 \cup \mathcal{G}_k^2$, where $k \geq 0$.*

(a) *The graph G contains exactly k cycles.*

(b) *The graph $G \in \mathcal{G}_k^0 \cup \mathcal{G}_k^1$ is a cactus graph.*

We shall also need the following elementary property of a dominating set in a graph.

Observation 2. *If G is connected graph of order at least 3, then there exists a γ-set of G that contains no leaf of G.*

The following lemma is established in [5].

Lemma 2 [5]**.** *If G is a connected graph and C is an arbitrary cycle in G, then there is an edge e of C such that $\gamma(G - e) = \gamma(G)$.*

Several authors obtained bounds on the domination number in terms of different variants of graphs, see for example [1, 2, 3, 6, 9]. Let $\mathcal{R}$ be the family of all trees in which the distance between any two distinct leaves is congruent to 2 modulo 3. Lemańska [9] established the following lower bound on the domination number of a tree in terms of its order and number of leaves.

Theorem 3 [9]**.** *If T is a tree of order $n \geq 2$ with ℓ leaves, then $\gamma(T) \geq (n - \ell + 2)/3$, with equality if and only if $T \in \mathcal{R}$.*

Hajian *et al.* [5] showed that the family $\mathcal{R}$ is precisely the family $\mathcal{G}_0^0$; that is, $\mathcal{R} = \mathcal{G}_0^0$.

As a consequence of Theorem 3, we have the following result.

Corollary 4 [9]**.** *If T is a tree of order $n \geq 2$ with ℓ leaves, then $\gamma(T) = \frac{1}{3}(n - \ell + 2 + m)$ for some integer $m \geq 0$.*

Hajian *et al.* [5] strengthened the result in Theorem 3 as follows.

Theorem 5 [5]**.** *If T is a tree of order $n \geq 2$ with ℓ leaves, then the following holds.*

(a) *$\gamma(T) \geq \frac{1}{3}(n - \ell + 2)$, with equality if and only if $T \in \mathcal{G}_0^0$.*

(b) *$\gamma(T) = \frac{1}{3}(n - \ell + 3)$ if and only if $T \in \mathcal{G}_0^1$.*

(c) *$\gamma(T) = \frac{1}{3}(n - \ell + 4)$ if and only if $T \in \mathcal{G}_0^2$.*

The result of Theorem 5 was generalized in [5] to connected graphs as follows.

Theorem 6 [5]**.** *If G is a connected graph of order $n \geq 2$ with $k \geq 0$ cycles and ℓ leaves, then the following holds.*

(a) *$\gamma(G) \geq \frac{1}{3}(n - \ell + 2(1 - k))$, with equality if and only if $G \in \mathcal{G}_k^0$.*

(b) *$\gamma(G) = \frac{1}{3}(n - \ell + 3 - 2k)$ if and only if $G \in \mathcal{G}_k^1$.*

(c) *$\gamma(G) = \frac{1}{3}(n - \ell + 4 - 2k)$ if and only if $G \in \mathcal{G}_k^2$.*

As a consequence of Theorem 6(a), we have the following.

Corollary 7 [5]**.** *If G is a connected graph of order $n \geq 2$ with $k \geq 0$ cycles and ℓ leaves, then $\gamma(G) = \frac{1}{3}(n - \ell + 2(1 - k) + m)$ for some integer $m \geq 0$.*

5. Proof of Main Result

In this section, we present a proof of our main result, namely Theorem 1. For this purpose, we first prove Theorem 1 in the special case when $k = 0$, that is, when the cactus is a tree.

Theorem 8. *Let $m \geq 0$ be an integer. If T is a tree of order $n \geq 2$ with ℓ leaves, then $\gamma(T) = \frac{1}{3}(n - \ell + 2 + m)$ if and only if $T \in \mathcal{G}_0^m$.*

Proof. Let T be a tree of order $n \geq 2$ with ℓ leaves. We proceed by induction on $m \geq 0$, namely *first-induction*, to show that $\gamma(T) = \frac{1}{3}(n - \ell + 2 + m)$, if and only if $T \in \mathcal{G}_0^m$. For the base step of the first-induction let $m \leq 2$. If $m = 0$, then the result follows by Theorem 5(a). If $m = 1$, then the result follows by Theorem 5(b). If $m = 2$, then the result follows by Theorem 5(c). This establishes the base step of the induction. Let $m \geq 3$ and assume that the result holds for all trees T_0 of order n_0 with ℓ_0 leaves, for $m_0 < m$. Let T be a tree of order n and with ℓ leaves. We will show that $\gamma(T) = \frac{1}{3}(n - \ell + 2 + m)$, if and only if $T \in \mathcal{G}_0^m$.

($\Longrightarrow$) Assume that $\gamma(T) = \frac{1}{3}(n - \ell + 2 + m)$ (where we recall that here $m \geq 3$). We show that $T \in \mathcal{G}_0^m$. If $T = P_2$, then by the definition of the family

$\mathcal{G}_0^1$, we have $T \in \mathcal{G}_0^1$. Then by Theorem 5(b), $\gamma(T) = \frac{1}{3}(n - \ell + 2 + 1)$, and so $m = 1$, a contradiction. Hence we may assume that $\mathrm{diam}(T) \geq 2$, for otherwise the desired result follows. If $\mathrm{diam}(T) = 2$, then T is a star, and by the definition of the family G_0^0, we have $T \in G_0^0$. Thus by Theorem 5(a), $\gamma(T) = \frac{1}{3}(n-\ell+2+1)$, and so $m = 0$, a contradiction. If $\mathrm{diam}(T) = 2$, then T is a double star, and by definition of the family $\mathcal{G}_0^2$ we have $T \in T_0^{2,1} \subseteq \mathcal{G}_0^2$. Thus by Theorem 5(c), $\gamma(T) = \frac{1}{3}(n - \ell + 2 + 2)$, and so $m = 2$, a contradiction. Hence, $\mathrm{diam}(T) \geq 4$ and $n \geq 5$.

We now root the tree T at a vertex r at the end of a longest path P in T. Let u be a vertex at maximum distance from r, and so $d_T(u, r) = \mathrm{diam}(T)$. Necessarily, r and u are leaves. Let v be the parent of u, let w be the parent of v, let x be the parent of w, and let y be the parent of x. Possibly, $y = r$. Since u is a vertex at maximum distance from the root r, every child of v is a leaf. By Observation 2, there exists a γ-set, say S, of T that contains no leaf of T; that is, $L(T) \cap S = \emptyset$. In particular, we note that $|S| = \gamma(T) = \frac{1}{3}(n - \ell + 2 + m)$. In order to dominate the vertex u, we note therefore that $v \in S$. Let $d_T(v) = t$. We note that $t \geq 2$.

Claim 1. *If $d_T(w) \geq 3$, then $T \in \mathcal{G}_0^m$.*

Proof. Suppose that $d_T(w) \geq 3$. In this case, we consider the tree $T' = T - V(T_v)$, where T_v is the maximal subtree at v. Let T' have order n' and let T' have ℓ' leaves. We note that $n' = n - t$. Since w is not a leaf in T', we have $\ell' = \ell - (t - 1) = \ell - t + 1$. By Corollary 4, $\gamma(T') = \frac{1}{3}(n' - \ell' + 2 + m')$ for some integer $m' \geq 0$. If a child of w is a leaf in T', then since the dominating set S contains no leaves, we have that $w \in S$. If no child of w is a leaf in T, then every child of w is a support vertex and therefore belongs to the set S. In both cases, we note that the set $S \setminus \{v\}$ is a dominating set of T', implying that $\gamma(T') \leq |S| - 1 = \gamma(T) - 1$. Every γ-set of T' can be extended to a dominating set of T by adding to it the vertex v, implying that $\gamma(T) \leq \gamma(T') + 1$. Consequently, $\gamma(T') = \gamma(T) - 1$. Thus,

$$\begin{aligned}
\gamma(T') &= \gamma(T) - 1 \\
&= \tfrac{1}{3}(n - \ell + 2 + m) - 1 \\
&= \tfrac{1}{3}(n - \ell + m - 1) \\
&= \tfrac{1}{3}((n' + t) - (\ell' + t - 1) + m - 1) \\
&= \tfrac{1}{3}(n' - \ell' + m).
\end{aligned}$$

As observed earlier, $\gamma(T') = \frac{1}{3}(n' - \ell' + 2 + m')$ for some integer $m' \geq 0$. Thus, $m' = m - 2$. Applying the inductive hypothesis to the tree T', we have $T' \in \mathcal{G}_0^{m-2}$. Let v' be a child of w different from v. We note that the tree $T_{v'}$ is a component of $T' - w$ and this component is dominated by the vertex v'. We can therefore choose a γ-set of $T' - w$ to contain the vertex v'. Such a γ-set of $T' - w$ is also a dominating set of T', implying that $\gamma(T') \leq \gamma(T' - w)$; that is, the vertex w is a special vertex of T'. Thus, the tree T is obtained from the tree $T' \in \mathcal{G}_0^{m-2}$ by adding a vertex disjoint copy of a star T_v and joining the center v of T_v to a special vertex w in T'. Thus $T \in \mathcal{T}_0^{m,1}$. Consequently, $T \in \mathcal{G}_0^m$. This completes the proof of Claim 1. □

By Claim 1, we may assume that $d_T(w) = 2$, for otherwise $T \in \mathcal{G}_0^m$ as desired. We now consider the tree $T' = T - V(T_w)$, where T_w is the maximal subtree at w. Let T' have order n' and let T' have ℓ' leaves. We note that $n' = n - t - 1$. By Corollary 4, $\gamma(T') = \frac{1}{3}(n' - \ell' + 2 + m')$ for some integer $m' \geq 0$.

As observed earlier, the vertex v belongs to the dominating set S. If $w \in S$, then we can replace w in S with the vertex x to produce a new γ-set of T that contains no leaf of T. Hence we may assume that $w \notin S$, implying that the set $S \setminus \{v\}$ is a dominating set of T' and therefore $\gamma(T') \leq |S| - 1 = \gamma(T) - 1$. Every γ-set of T' can be extended to a dominating set of T by adding to it the vertex v, implying that $\gamma(T) \leq \gamma(T') + 1$. Consequently, $\gamma(T') = \gamma(T) - 1$.

Claim 2. *If $d_T(x) \geq 3$, then $T \in \mathcal{G}_0^m$.*

Proof. Suppose that $d_T(x) \geq 3$. In this case, the vertex x is not a leaf of T', implying that $\ell' = \ell - (t - 1) = \ell - t + 1$. Thus,

$$\begin{aligned}\gamma(T') &= \gamma(T) - 1\\ &= \tfrac{1}{3}(n - \ell + m - 1)\\ &= \tfrac{1}{3}((n' + t + 1) - (\ell' + t - 1) + m - 1)\\ &= \tfrac{1}{3}(n' - \ell' + m + 1).\end{aligned}$$

As observed earlier, $\gamma(T') = \frac{1}{3}(n' - \ell' + 2 + m')$ for some integer $m' \geq 0$. Thus, $m' = m - 1$. Applying the inductive hypothesis to the tree T', we have $T' \in \mathcal{G}_0^{m-1}$. Thus, the tree T is obtained from the tree $T' \in \mathcal{G}_0^{m-1}$ by adding a vertex disjoint copy of a star T_v with at least three vertices and joining a leaf of the star T_v to the non-leaf x of T'. Thus $T \in \mathcal{T}_0^{m,2}$. Consequently, $T \in \mathcal{G}_0^m$. □

By Claim 2, we may assume that $d_T(x) = 2$, for otherwise $T \in \mathcal{G}_0^m$ as desired. In this case, the vertex x is a leaf of T', implying that $\ell' = \ell - (t-1) + 1 = \ell - t + 2$. Thus,

$$\begin{aligned}\tfrac{1}{3}(n' - \ell' + 2 + m') = \gamma(T') &= \gamma(T) - 1\\ &= \tfrac{1}{3}(n - \ell + m - 1)\\ &= \tfrac{1}{3}((n' + t + 1) - (\ell' + t - 2) + m - 1)\\ &= \tfrac{1}{3}(n' - \ell' + m + 2),\end{aligned}$$

and so $m = m'$. Applying the inductive hypothesis to the tree T', we have $T' \in \mathcal{G}_0^m$. Thus, the tree T is obtained from the tree $T' \in \mathcal{G}_0^m$ by adding a vertex disjoint copy of a star T_v with at least three vertices and adding the edge xw joining a leaf w of T_v and a leaf x of T'; that is, T is obtained from T' by Operation $\mathcal{O}$. Hence, by definition of the family $\mathcal{G}_0^m$, we have $T \in \mathcal{G}_0^m$, as desired. This completes the necessity part of the proof of Theorem 8.

($\Longleftarrow$) Conversely, assume that $T \in \mathcal{G}_0^m$, where $m \geq 0$. Recall that T is a tree of order $n \geq 2$ with ℓ leaves. Thus, T is obtained from a sequence $T_1, \ldots, T_q$ of trees, where $q \geq 1$ and where the tree $T_1 \in \mathcal{T}_0^{m,1} \cup \mathcal{T}_0^{m,2}$, and the tree $T = T_q$. Further, if $q \geq 2$, then for each $i \in [q] \setminus \{1\}$, the tree T_i can be obtained from the tree T_{i-1} by applying the following Operation $\mathcal{O}$. We proceed by induction on $q \geq 1$, namely *second-induction*, to show that $\gamma_t(T) = \frac{1}{3}(n - \ell + 2 + m)$.

Claim 3. *If $q = 1$, then $\gamma_t(T) = \gamma(T) = \frac{1}{3}(n - \ell + 2 + m)$.*

Proof. Suppose that $q = 1$. Thus, $T_1 \in \mathcal{T}_0^{m,1} \cup \mathcal{T}_0^{m,2}$. We consider the two possibilities in turn, and in both cases we will show that the tree $T \in \mathcal{G}_0^m$ satisfies $\gamma(T) = \frac{1}{3}(n - \ell + 2 + m)$.

Claim 3.1. *If $T \in \mathcal{T}_0^{m,1}$, then $\gamma_t(T) = \frac{1}{3}(n - \ell + 2 + m)$.*

Proof. Suppose that $T \in \mathcal{T}_0^{m,1}$. Thus, T is obtained from a tree $T' \in \mathcal{G}_0^{m-2}$ by adding a vertex disjoint copy of a star Q with $t \geq 2$ vertices and joining the center of Q, say y, to a special vertex x in T'. Let T' have order n', and so $n' = n - t$. Further, let T' have ℓ' leaves. Since x is a non-leaf of T', we have $\ell' = \ell - (t - 1)$. Applying the first-induction hypothesis to the tree $T' \in \mathcal{G}_0^{m-2}$, we have $\gamma_t(T') = \frac{1}{3}(n' - \ell' + 2 + (m - 2)) = \frac{1}{3}(n' - \ell' + m)$.

We show next that $\gamma(T) = \gamma(T') + 1$. Since x is a special vertex of T', we note that $\gamma(T' - x) \geq \gamma(T')$. Every γ-set of T' can be extended to a dominating set of T by adding to it the vertex y, implying that $\gamma(T) \leq \gamma(T') + 1$. Conversely, we can choose a γ-set, say D, of T to contain the vertex y which dominates the star Q. If $x \in D$, then $D \setminus \{y\}$ is a dominating set of T', and so $\gamma(T') \leq |D| - 1$. If $x \notin D$, then $D \setminus \{y\}$ is a dominating set of $T' - x$, and so $\gamma(T') \leq \gamma(T' - x) \leq |D| - 1$. In both cases, $\gamma(T') \leq |D| - 1 = \gamma(T) - 1$. Consequently, $\gamma(T) = \gamma(T') + 1$. Thus,

$$\begin{aligned} \gamma(T) &= \gamma(T') + 1 \\ &= \tfrac{1}{3}(n' - \ell' + m) + 1 \\ &= \tfrac{1}{3}((n - t) - (\ell - t + 1) + m) + 1 \\ &= \tfrac{1}{3}(n - \ell + 2 + m). \end{aligned}$$

This completes the proof of Claim 3.1. □

Claim 3.2. *If* $T \in \mathcal{T}_0^{m,2}$, *then* $\gamma_t(T) = \frac{1}{3}(n - \ell + 2 + m)$.

Proof. Suppose that $T \in \mathcal{T}_0^{m,2}$. Thus, T is obtained from a tree $T' \in \mathcal{G}_0^{m-1}$ by adding a vertex disjoint copy of a star Q with $t \geq 3$ vertices and joining a leaf, say v, of Q to a non-leaf, say w, in T'. Let u be the center of the star Q. Let T' have order n', and so $n' = n - t$. Further, let T' have ℓ' leaves. Since w is a non-leaf of T', we have $\ell' = \ell - (t-2)$. Applying the first-induction hypothesis to the tree $T' \in \mathcal{G}_0^{m-1}$, we have $\gamma_t(T') = \frac{1}{3}(n' - \ell' + 2 + (m-1)) = \frac{1}{3}(n' - \ell' + m + 1)$.

We show next that $\gamma(T) = \gamma_t(T') + 1$. Every γ-set of T' can be extended to a dominating of T by adding to it the vertex u, implying that $\gamma(T) \leq \gamma(T') + 1$. By Observation 2, there exists a γ-set D of T that contains no leaf of G. Thus, $u \in D$. If $v \in D$, then we can replace v in D with the vertex w. Hence we may assume that $v \notin D$, implying that $D \setminus \{u\}$ is a dominating set of T', and so $\gamma(T') \leq |D| - 1 = \gamma(T) - 1$. Consequently, $\gamma(T) = \gamma(T') + 1$. Thus,

$$\begin{aligned}
\gamma(T) &= \gamma(T') + 1 \\
&= \tfrac{1}{3}(n' - \ell' + m + 1) + 1 \\
&= \tfrac{1}{3}((n - t) - (\ell - t + 2) + m + 1) + 1 \\
&= \tfrac{1}{3}(n - \ell + 2 + m).
\end{aligned}$$

This completes the proof of Claim 3.2. □

By Claims 3.1 and 3.2, if $T \in \mathcal{T}_0^{m,1} \cup \mathcal{T}_0^{m,2}$, then $\gamma(T) = \frac{1}{3}(n - \ell + 2 + m)$. This completes the proof of Claim 3. □

By Claim 3, if $q = 1$, then $\gamma(T) = \frac{1}{3}(n - \ell + 2 + m)$. This establishes the base step of the second-induction. Let $q \geq 2$ and assume that if q' is an integer where $1 \leq q' < q$ and if $T' \in \mathcal{G}_0^m$ is a tree of order $n' \geq 2$ with ℓ' leaves obtained from a sequence of q' trees, then $\gamma(T) = \frac{1}{3}(n' - \ell' + 2 + m)$. Recall that T is obtained from a sequence $T_1, \ldots, T_q$ of trees, where $q \geq 1$ and where the tree $T_1 \in \mathcal{T}_0^{m,1} \cup \mathcal{T}_0^{m,2}$, and the tree $T = T_q$. Further for each $i \in [q] \setminus \{1\}$, the tree T_i can be obtained from the tree T_{i-1} by applying the Operation $\mathcal{O}$.

We now consider the tree $T' = T_{q-1}$. Thus, the tree $T \in \mathcal{G}_0^m$ is obtained from the tree T' by adding a vertex disjoint copy of a star Q with $t \geq 3$ vertices and adding an edge joining a leaf of Q to a leaf of T'. Let T' have order n' and let T' have ℓ' leaves. We note that $n' = n - t$ and $\ell' = \ell - (t-2) + 1 = \ell - t + 3$. Applying the second-induction hypothesis to the tree $T' \in \mathcal{G}_0^m$, we have $\gamma(T') = \frac{1}{3}(n' - \ell' + 2 + m)$. Analogous arguments as before show that $\gamma(T) = \gamma_t(T') + 1$. Thus,

$$\begin{aligned}
\gamma(T) &= \gamma(T') + 1 \\
&= \tfrac{1}{3}(n' - \ell' + 2 + m) + 1 \\
&= \tfrac{1}{3}((n - t) - (\ell - t + 3) + 2 + m) + 1 \\
&= \tfrac{1}{3}(n - \ell + 2 + m).
\end{aligned}$$

Hence we have shown that if $T \in \mathcal{G}_0^m$, where $m \geq 0$ and where T has order $n \geq 2$ with ℓ leaves, then $\gamma(T) = \frac{1}{3}(n - \ell + 2 + m)$. This completes the proof of Theorem 8. ■

We are now in a position to prove our main result, namely Theorem 1. Recall its statement.

Theorem 1. *Let $m \geq 0$ be an integer. If G is a cactus graph of order $n \geq 2$ with $k \geq 0$ cycles and ℓ leaves, then $\gamma(G) = \frac{1}{3}(n - \ell + 2(1-k) + m)$, if and only if $G \in \mathcal{G}_k^m$.*

Proof. Let $m \geq 0$ be an integer, and let G be a cactus graph of order $n \geq 2$ with $k \geq 0$ cycles and ℓ leaves. We proceed by induction on k to show that $\gamma(G) = \frac{1}{3}(n - \ell + 2(1-k) + m)$ if and only if $G \in \mathcal{G}_k^m$. If $k = 0$, then the result follows from Theorem 8. This establishes the base case. Let $k \geq 1$ and assume that if G' is a cactus graph of order $n' \geq 2$ with k' cycles and ℓ' leaves where $0 \leq k' < k$, then $\gamma(G) = \frac{1}{3}(n' - \ell' + 2(1-k') + m')$ if and only if $G \in \mathcal{G}_{k'}^{m'}$. Let G be a cactus graph of order $n \geq 2$ with $k \geq 0$ cycles and ℓ leaves. We will show that $\gamma(G) = \frac{1}{3}(n - \ell + 2(1-k) + m)$, if and only if $G \in \mathcal{G}_k^m$. If $m = 0$, then the result follows by Theorem 6(a). If $m = 1$, then the result follows by Theorem 6(b). If $m = 2$, then the result follows by Theorem 6(c). Thus, we may assume that $m \geq 3$, for otherwise the desired result follows.

($\Longrightarrow$) Assume that $\gamma(G) = \frac{1}{3}(n - \ell + 2 + m - 2k)$ (where we recall that here $m \geq 3$). We will show that $T \in \mathcal{G}_k^m$. By Lemma 2, the graph G contains a cycle edge e such that $\gamma(G - e) = \gamma(G)$. Let $e = uv$, and consider the graph $G' = G - e$. Let G' have order n' with $k' \geq 0$ cycles and ℓ' leaves. We note that $n' = n$. Further, since G is a cactus graph, $k' = k - 1$. Removing the cycle edge e from G produces at most two new leaves, namely the ends of the edge e, implying that $\ell' - 2 \leq \ell \leq \ell'$. By Corollary 7, we have $\gamma(G') = \frac{1}{3}(n' - \ell' + 2 + m' - 2k')$ for some integer $m' \geq 0$. Applying the inductive hypothesis to the cactus graph G', we have that $G' \in \mathcal{G}_{k'}^{m'} = \mathcal{G}_{k-1}^{m'}$. Our earlier observations imply that

$$\begin{aligned} \tfrac{1}{3}(n - \ell + 2 + m - 2k) &= \gamma(G) = \gamma(G') \\ &= \tfrac{1}{3}(n' - \ell' + 2 + m' - 2k') \\ &= \tfrac{1}{3}(n - \ell' + 2 + m' - 2(k-1)), \end{aligned}$$

and so $m - \ell = m' - \ell' + 2$. Since G is a cactus, the vertices u and v are connected in $G' = G - e$ by a unique path. As observed earlier, $\ell' - 2 \leq \ell \leq \ell'$.

Suppose that $\ell = \ell'$. In this case, neither u nor v is a leaf of G', implying that both u and v have degree at least 2 in G'. Further, the equation $m - \ell = m' - \ell' + 2$ simplifies to $m' = m - 2$. Thus, $G' \in \mathcal{G}_{k-1}^{m-2}$. Hence, the graph G is obtained from G' by Procedure J and therefore $G \in \mathcal{G}_k^m$.

Suppose that $\ell = \ell' - 1$. In this case, exactly one of u and v is a leaf of G'. Further, the equation $m - \ell = m' - \ell' + 2$ simplifies to $m' = m - 1$. Thus, $G' \in \mathcal{G}_{k-1}^{m-1}$. Hence, the graph G is obtained from G' by Procedure I, and therefore $G \in \mathcal{G}_k^m$.

Suppose that $\ell = \ell' - 2$. In this case, both u and v are leaves in G'. Further, the equation $m - \ell = m' - \ell' + 2$ simplifies to $m' = m$. Thus, $G' \in \mathcal{G}_{k-1}^m$. Hence, the graph G is obtained from G' by Procedure H, and therefore $G \in \mathcal{G}_k^m$. This completes the necessity part of the proof of Theorem 1.

($\Longleftarrow$) Conversely, assume that $G \in \mathcal{G}_k^m$. Recall that by our earlier assumptions, $m \geq 3$ and $k \geq 1$. Thus, the graph G is obtained from either a graph $G' \in \mathcal{G}_{k-1}^m$ by Procedure H or from a graph $G' \in \mathcal{G}_{k-1}^{m-1}$ by Procedure I or from a graph $G' \in \mathcal{G}_{k-1}^{m-2}$ by Procedure J. In all three cases, let G' have order n' with $k' \geq 0$ cycles and ℓ' leaves. Further, in all cases we note that $n' = n$ and $k' = k-1$. We consider the three possibilities in turn.

Suppose firstly that G is obtained from a graph $G' \in \mathcal{G}_{k-1}^m$ by Procedure H. In this case, $\ell = \ell' - 2$ and $\gamma(G) = \gamma(G')$. Applying the inductive hypothesis to the graph $G' \in \mathcal{G}_{k-1}^m$, we have $\gamma(G) = \gamma(G') = \frac{1}{3}(n' - \ell' + 2 + m - 2(k-1)) = \frac{1}{3}(n - (\ell + 2) + 4 + m - 2k) = \frac{1}{3}(n - \ell + 2 + m - 2k)$.

Suppose next that G is obtained from a graph $G' \in \mathcal{G}_{k-1}^{m-1}$ by Procedure I. In this case, $\ell = \ell' - 1$ and $\gamma(G) = \gamma(G')$. Applying the inductive hypothesis to the graph $G' \in \mathcal{G}_{k-1}^{m-1}$, we have $\gamma(G) = \gamma(G') = \frac{1}{3}(n' - \ell' + 2 + (m-1) - 2(k-1)) = \frac{1}{3}(n - (\ell + 1) + 3 + m - 2k) = \frac{1}{3}(n - \ell + 2 + m - 2k)$.

Suppose finally that G is obtained from a graph $G' \in \mathcal{G}_{k-1}^{m-2}$ by Procedure J. In this case, $\ell = \ell'$ and $\gamma(G) = \gamma(G')$. Applying the inductive hypothesis to the graph $G' \in \mathcal{G}_{k-1}^{m-2}$, we have $\gamma(G) = \gamma(G') = \frac{1}{3}(n' - \ell' + 2 + (m-2) - 2(k-1)) = \frac{1}{3}(n - \ell + 2 + m - 2k)$. In all three cases, $\gamma(G) = \frac{1}{3}(n - \ell + 2 + m - 2k)$. This completes the proof of Theorem 1. ■

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