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ON FACE IRREGULAR EVALUATIONS OF PLANE GRAPHS

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Abstract

We investigate face irregular labelings of plane graphs and we introduce new graph characteristics, namely face irregularity strength of type (α, β, γ). We obtain some estimation on these parameters and determine the precise values for certain families of plane graphs that prove the sharpness of the lower bounds.

Keywords: plane graphs, irregular assignment, irregularity strength, face irregular labeling, face irregularity strength.

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1. Introduction

All graphs, $G = (V, E, F)$ considered in this paper are simple, finite, undirected and planar. A *plane* graph is a particular drawing of a planar graph on the Euclidean plane. In all cases, a k-labeling (k-evaluation) will refer to a mapping from the set of graph elements into a set of positive integers $\{1, 2, \ldots, k\}$. If the domain is the vertex set or the edge set or the face set, the k-labelings are called, respectively, *vertex labelings* or *edge labelings* or *face labelings* and are denoted, respectively, as k-labelings of type $(1, 0, 0)$ or $(0, 1, 0)$ or $(0, 0, 1)$. If the domain is $V(G) \cup E(G)$ or $V(G) \cup E(G) \cup F(G)$ or $V(G) \cup F(G)$ or $E(G) \cup F(G)$, then we call the k-labelings, respectively, as *total k-labeling* or *entire k-labeling* or *vertex-face k-labeling* or *edge-face k-labeling* and we denote these k-labelings, respectively, as labelings of type $(1, 1, 0)$ or $(1, 1, 1)$ or $(1, 0, 1)$ or $(0, 1, 1)$.

Suppose that $G = (V, E, F)$ is a plane graph. The weight of a face f under a k-labeling is the sum of labels (if present) carried by that face and all the edges and vertices surrounding it. In general, for a k-labeling φ of type (α, β, γ), where $\alpha, \beta, \gamma \in \{0, 1\}$, the associated *face-weight* of a face $f \in F(G)$ is defined as

$$wt_{\varphi_{(\alpha,\beta,\gamma)}}(f) = \alpha \sum_{v \sim f} \varphi(v) + \beta \sum_{e \sim f} \varphi(e) + \gamma\, \varphi(f), \tag{1}$$

where the sums are taken over all vertices and all edges adjacent to the face f, respectively. Note that the trivial case $(\alpha, \beta, \gamma) = (0, 0, 0)$ is not allowed.

A k-labeling φ of type (α, β, γ) of the plane graph G is defined to be a *face irregular k-labeling of type* (α, β, γ) if for every two different faces f and g of G there is

$$wt_{\varphi_{(\alpha,\beta,\gamma)}}(f) \neq wt_{\varphi_{(\alpha,\beta,\gamma)}}(g).$$

The *face irregularity strength of type* (α, β, γ) of a plane graph G, denoted by $\mathrm{fs}_{(\alpha,\beta,\gamma)}(G)$, is the smallest integer k such that G admits a face irregular k-labeling of type (α, β, γ). Note that for some classes of graphs and some values of parameters α, β and γ the corresponding graph invariant is infinite. We will discuss these cases later.

The irregular labelings of plane graphs with restrictions placed on the weights of faces were introduced in [8] and there is defined the face irregularity strength of type $(1, 1, 1)$ as *entire face irregularity strength*, denoted by efs(G). Some partial results on irregularity strength of type $(1, 1, 0)$ under the notation total face irregularity strength can be found in [20].

The face irregular k-labeling of type (α, β, γ) of the plane graphs is a modification of the well-known irregular assignments and vertex (edge) irregular total labelings. The irregular assignments were introduced by Chartrand *et al.* in [12] as edge k-labeling of a graph G, where the sums of the labels of edges incident with every two distinct vertices in G are distinct. An *irregularity strength* s(G)

of a graph G is known as the minimum k for which G has an irregular assignment using labels at most k. This parameter has attracted much attention, see [13, 16, 18].

A total k-labeling λ is a *vertex irregular k-labeling* if vertex-weights

$$wt_\lambda(x) = \lambda(x) + \sum_{xy \in E(G)} \lambda(xy)$$

are distinct for all couples of different vertices in G and it is an *edge irregular total k-labeling* if edge-weights

$$wt_\lambda(xy) = \lambda(x) + \lambda(xy) + \lambda(y)$$

are distinct for all couples of different edges in G. The *total vertex (edge) irregularity strength* is defined as the minimum k for which G admits a vertex (edge) irregular total k-labeling.

The total vertex (edge) irregularity strengths were introduced in [10], where are determined several bounds and exact values for different families of graphs. These results were then improved by Przybyło [22], Anholcer *et al.* [3] and Nurdin *et al.* [21]. Other interesting results on total vertex (edge) irregularity strengths can be found in [1, 2, 4, 11, 14, 15, 17, 19].

In this paper, we generalize the concept of the face irregular labelings of plane graphs and we obtain some estimations on the face irregularity strength of type (α, β, γ). We also determine the precise values of the corresponding graph invariants for certain families of plane graphs that prove the sharpness of the lower bounds.

2. Lower Bounds for Face Irregularity Strength

At the beginning of this section we determine the face irregularity strength of type (α, β, γ) for a cycle C_n, $n \geq 3$.

Theorem 1. *Let $\alpha, \beta, \gamma \in \{0, 1\}$. If $n \geq 3$, then*

$$\mathrm{fs}_{(\alpha,\beta,\gamma)}(C_n) = \begin{cases} 2, & \text{if } \gamma = 1, \\ \infty, & \text{if } \gamma = 0. \end{cases}$$

Proof. The cycle C_n contains exactly two n-sided faces, say f_n^1 and f_n^2. Evidently, every vertex and every edge of C_n lays on the boundary of both these faces. For the weight of a face f_n^i, $i = 1, 2$, under a labeling φ of type (α, β, γ) we get

$$wt_{\varphi_{(\alpha,\beta,\gamma)}}(f_n^i) = \alpha \sum_{v \in V(C_n)} \varphi(v) + \beta \sum_{e \in E(C_n)} \varphi(e) + \gamma\, \varphi(f_n^i).$$

This implies that the faces have distinct weights if and only if distinct labels are assigned to both faces. Trivially, if $\gamma = 0$, this is not possible and thus $\mathrm{fs}_{(\alpha,\beta,\gamma)}(C_n) = \infty$. If $\gamma = 1$, then $\mathrm{fs}_{(\alpha,\beta,\gamma)}(C_n) = 2$. ■

The next theorem gives a lower bound of the face irregularity strength of type (α, β, γ).

Theorem 2. *Let $G = (V, E, F)$ be a 2-connected plane graph with n_i i-sided faces, $i \geq 3$. Let $\alpha, \beta, \gamma \in \{0, 1\}$, $a = \min\{i : n_i \neq 0\}$ and $b = \max\{i : n_i \neq 0\}$. Then the face irregularity strength of type (α, β, γ) of G is*

$$\mathrm{fs}_{(\alpha,\beta,\gamma)}(G) \geq \left\lceil \frac{(\alpha+\beta)a + \gamma + |F(G)| - 1}{(\alpha+\beta)b + \gamma} \right\rceil.$$

Proof. Assume that φ is a face irregular k-labeling of type (α, β, γ) of a 2-connected plane graph G with $\mathrm{fs}_{(\alpha,\beta,\gamma)}(G) = k$.

The smallest face-weight under the face irregular k-labeling φ admits the value at least $(\alpha+\beta)a + \gamma$. Since $|F(G)| = \sum_{i=3}^{b} n_i$, it follows that the largest face-weight attains the value at least $(\alpha+\beta)a + \gamma + |F(G)| - 1$ and at most $((\alpha+\beta)b + \gamma)\, k$.

Thus,

$$(\alpha+\beta)a + \gamma + |F(G)| - 1 \leq ((\alpha+\beta)b + \gamma)\, k$$

and

$$k \geq \left\lceil \frac{(\alpha+\beta)a + \gamma + |F(G)| - 1}{(\alpha+\beta)b + \gamma} \right\rceil.$$ ■

We use the previous result to find the face irregularity strength of type (α, β, γ) for graphs with all faces of different sizes.

Theorem 3. *Let $\alpha, \beta, \gamma \in \{0, 1\}$ and let $G = (V, E, F)$ be a 2-connected plane graph in which no different faces have the same size. Then*

$$\mathrm{fs}_{(\alpha,\beta,\gamma)}(G) = \begin{cases} 1, & \text{if } (\alpha, \beta) \neq (0, 0), \\ |F(G)|, & \text{if } (\alpha, \beta) = (0, 0). \end{cases}$$

Proof. Let $G = (V, E, F)$ be a 2-connected plane graph containing exactly one s_i-sided face, $i = 1, 2, \ldots, |F(G)|$. Let us denote the s_i-sided face by f_i. Let $a = \min\{s_i : i = 1, 2, \ldots, |F(G)|\}$ and $b = \max\{s_i : i = 1, 2, \ldots, |F(G)|\}$. Evidently, $b \geq a + |F(G)| - 1$. Then

$$\begin{aligned} (\alpha+\beta)b + \gamma &\geq (\alpha+\beta)(a + |F(G)| - 1) + \gamma \\ &= (\alpha+\beta)a + (\alpha+\beta)|F(G)| - (\alpha+\beta) + \gamma. \end{aligned}$$

For $(\alpha,\beta) \neq (0,0)$ we get

$$(\alpha+\beta)b+\gamma \geq (\alpha+\beta)a+(\alpha+\beta)|F(G)|-1+\gamma$$

and thus

$$\frac{(\alpha+\beta)a+(\alpha+\beta)|F(G)|-1+\gamma}{(\alpha+\beta)b+\gamma} \leq 1.$$

Trivially, according to Theorem 2 we get that for $(\alpha,\beta) \neq (0,0)$ is $\mathrm{fs}_{(\alpha,\beta,\gamma)}(G) \geq 1$. To prove that $\mathrm{fs}_{(\alpha,\beta,\gamma)}(G) \leq 1$ in this case we consider a 1-labeling φ of (α,β,γ) of G defined in the following way.

$$\varphi(x) = \begin{cases} \alpha, & \text{if } x \in V(G), \\ \beta, & \text{if } x \in E(G), \\ \gamma, & \text{if } x \in F(G). \end{cases}$$

Then the weight of a face f_i, $i = 1,2,\ldots,|F(G)|$, under the labeling φ is

$$\begin{aligned} wt_{\varphi_{(\alpha,\beta,\gamma)}}(f_i) &= \alpha \sum_{v\sim f_i} \varphi(v) + \beta \sum_{e\sim f_i} \varphi(e) + \gamma\,\varphi(f_i) = \alpha \sum_{v\sim f_i} \alpha + \beta \sum_{e\sim f_i} \beta + \gamma\cdot\gamma \\ &= \alpha^2\cdot s_i + \beta^2\cdot s_i + \gamma^2 = (\alpha+\beta)s_i + \gamma. \end{aligned}$$

Evidently, if $(\alpha,\beta) \neq (0,0)$, the weights of all faces in G are distinct.

In the case when $(\alpha,\beta) = (0,0)$ the weight of a face f_i, $i = 1,2,\ldots,|F(G)|$, under any k-labeling ψ of type $(0,0,1)$ is

$$wt_{\psi_{(0,0,1)}}(f_i) = \psi(f_i).$$

As the weights of all faces must be distinct, evidently the labels of all faces must be distinct. Thus $k \geq |F(G)|$. This concludes the proof. ∎

Note that if a plane graph contains at least two faces of the same size, then $\mathrm{fs}_{(\alpha,\beta,\gamma)}(G) \geq 2$ for all feasible (α,β,γ).

Immediately from Theorem 2 and using the similar arguments as in the proof of the previous theorem we get the following result.

Lemma 4. *For every 2-connected plane graph $G = (V,E,F)$ is*

$$\mathrm{fs}_{(0,0,1)}(G) = |F(G)|.$$

Immediately from Theorem 2 we get a lower bound for the entire face irregularity strength of a 2-connected plane graph.

$$\mathrm{fs}_{(1,1,1)}(G) = \mathrm{efs}(G) \geq \left\lceil \frac{2a+n_3+n_4+\cdots+n_b}{2b+1} \right\rceil. \tag{2}$$

This result was proved by Bača *et al.* [8]. Furthermore, the existence of a face irregular entire 2-labeling of octahedron was shown in [9] and it proves the sharpness of the lower bound (2).

In the case when a 2-connected plane graph G contains only one face of the largest size, i.e., $n_b = 1$ and $c = \max\{i : n_i \neq 0,\ i < b\}$, then by using similar procedure as in the proof of Theorem 2 we can improve a lower bound for the face irregularity strength of type (α, β, γ).

Theorem 5. *Let $G = (V, E, F)$ be a 2-connected plane graph with n_i i-sided faces, $i \geq 3$. Let $\alpha, \beta, \gamma \in \{0, 1\}$, $a = \min\{i : n_i \neq 0\}$, $b = \max\{i : n_i \neq 0\}$, $n_b = 1$ and $c = \max\{i : n_i \neq 0,\ i < b\}$. Then the face irregularity strength of type (α, β, γ) of G is*

$$\mathrm{fs}_{(\alpha,\beta,\gamma)}(G) \geq \left\lceil \frac{(\alpha+\beta)a + \gamma + |F(G)| - 2}{(\alpha+\beta)c + \gamma} \right\rceil.$$

From this result we get that if a 2-connected plane graph contains only one largest face, then a lower bound on the face irregularity strength of type $(1, 1, 1)$ is estimated as follows

$$\mathrm{fs}_{(1,1,1)}(G) = \mathrm{efs}(G) \geq \left\lceil \frac{2a + |F(G)| - 1}{2c + 1} \right\rceil. \tag{3}$$

This bound was presented in [8]. Moreover, the exact value of the entire face irregularity strength for ladder $L_n \simeq P_n \Box P_2$, $n \geq 3$, was determined in [8] and it proves the sharpness of the lower bound (3).

Let us consider a simple graph $G = (V, E)$ such that every edge in $E(G)$ belongs at least to one subgraph of G isomorphic to a given graph H. We say that G admits an H-covering in this case. The graph G admitting H-covering admits an *H-irregular total k-labeling* $f : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ if distinct subgraphs in G isomorphic to H have different H-weights, where the associated H-weight is defined such that

$$wt_f(H) = \sum_{v \in V(H)} f(v) + \sum_{e \in E(H)} f(e).$$

The minimum k for which the graph G has an H-irregular total k-labeling is called the *total H-irregularity strength* of the graph G and is denoted by $\mathrm{ths}(G, H)$. Analogously, we can define the vertex H-irregularity strength $\mathrm{vhs}(G, H)$ and the edge H-irregularity strength $\mathrm{ehs}(G, H)$. These concepts were defined in [6] and [5], respectively.

Next theorem provides the exact values of the face irregularity strength of type (α, β, γ) for ladder $L_n \simeq P_n \Box P_2$, $n \geq 3$.

Theorem 6. *Let $L_n \simeq P_n \Box P_2$, $n \geq 3$, be a ladder and let $\alpha, \beta, \gamma \in \{0,1\}$. Then*

$$\mathrm{fs}_{(\alpha,\beta,\gamma)}(L_n) = \begin{cases} n, & \text{if } (\alpha,\beta,\gamma) = (0,0,1), \\ \left\lceil \frac{n+2}{4} \right\rceil, & \text{if } (\alpha,\beta,\gamma) = (1,0,0);\ (0,1,0), \\ \left\lceil \frac{n+3}{5} \right\rceil, & \text{if } (\alpha,\beta,\gamma) = (1,0,1);\ (0,1,1), \\ \left\lceil \frac{n+6}{8} \right\rceil, & \text{if } (\alpha,\beta,\gamma) = (1,1,0), \\ \left\lceil \frac{n+7}{9} \right\rceil, & \text{if } (\alpha,\beta,\gamma) = (1,1,1). \end{cases}$$

Proof. Let $L_n \cong P_n \Box P_2$, $n \geq 3$, be a ladder with the vertex set $V(L_n) = \{v_i, u_i : i = 1,2,\ldots,n\}$ and the edge set $E(L_n) = \{v_i v_{i+1}, u_i u_{i+1} : i = 1,2,\ldots,n-1\} \cup \{v_i u_i : i = 1,2,\ldots,n\}$. We denote the 4-sided faces f_i, $i = 1,2,\ldots,n-1$, such that the face f_i is surrounded by vertices u_i, u_{i+1}, v_i, v_{i+1} and edges $u_i u_{i+1}$, $v_i v_{i+1}$, $u_i v_i$, $u_{i+1} v_{i+1}$. The ladder contains also one external $2n$-sided face f_{ext}.

The result for $\mathrm{fs}_{(0,0,1)}(L_n)$ follows from Lemma 4.

Using a connection between the face irregularity strength and the entire face irregularity strength of ladders $\mathrm{efs}(L_n)$ and the total/vertex/edge C_4-irregularity strength of ladders, denoted by $\mathrm{ths}(L_n, C_4)/\mathrm{vhs}(L_n, C_4)/\mathrm{ehs}(L_n, C_4)$, we get that for $n \geq 3$

$$\begin{aligned} \mathrm{fs}_{(1,1,1)}(L_n) &= \mathrm{efs}(L_n) = \left\lceil \tfrac{n+7}{9} \right\rceil, && \text{see [8]}, \\ \mathrm{fs}_{(1,1,0)}(L_n) &= \mathrm{ths}(L_n, C_4) = \left\lceil \tfrac{n+6}{8} \right\rceil, && \text{see [6]}, \\ \mathrm{fs}_{(1,0,0)}(L_n) &= \mathrm{vhs}(L_n, C_4) = \left\lceil \tfrac{n+2}{4} \right\rceil, && \text{see [5]}, \\ \mathrm{fs}_{(0,1,0)}(L_n) &= \mathrm{ehs}(L_n, C_4) = \left\lceil \tfrac{n+2}{4} \right\rceil, && \text{see [5]}. \end{aligned}$$

Now consider two remaining cases.

Case 1. When $(\alpha,\beta,\gamma) = (1,0,1)$. According to Theorem 5 we obtain $\mathrm{fs}_{(1,0,1)}(L_n) \geq \lceil (n+3)/5 \rceil$. Put $k = \lceil (n+3)/5 \rceil$. To show that k is an upper bound for the face irregularity strength of type $(1,0,1)$ of L_n we define a k-labeling of type $(1,0,1)$ $\varphi : V(L_n) \cup F(L_n) \to \{1,2,\ldots,k\}$ in the following way

$$\begin{aligned} \varphi(v_i) &= \left\lceil \tfrac{i+1}{5} \right\rceil, && \text{for } i = 1,2,\ldots,n, \\ \varphi(u_i) &= \left\lceil \tfrac{i+3}{5} \right\rceil, && \text{for } i = 1,2,\ldots,n, \\ \varphi(f_i) &= \left\lceil \tfrac{i}{5} \right\rceil, && \text{for } i = 1,2,\ldots,n-1, \\ \varphi(f_{ext}) &= k. \end{aligned}$$

It is a routine matter to verify that under the labeling φ all vertex and face labels are at most k. For the face-weight of the face f_i, $i = 1,2,\ldots,n-1$, under the labeling φ of type $(1,0,1)$ we get

$$wt_{\varphi_{(1,0,1)}}(f_i) = \varphi(u_i) + \varphi(u_{i+1}) + \varphi(v_i) + \varphi(v_{i+1}) + \varphi(f_i).$$

Thus, for $i = 1, 2, \ldots, n-2$ we obtain

$$wt_{\varphi_{(1,0,1)}}(f_{i+1}) - wt_{\varphi_{(1,0,1)}}(f_i) = \varphi(u_{i+2}) + \varphi(v_{i+2}) + \varphi(f_{i+1}) - \varphi(u_i) - \varphi(v_i) - \varphi(f_i) = \left\lceil \tfrac{i+5}{5} \right\rceil + \left\lceil \tfrac{i+3}{5} \right\rceil + \left\lceil \tfrac{i+1}{5} \right\rceil - \left\lceil \tfrac{i+3}{5} \right\rceil - \left\lceil \tfrac{i+1}{5} \right\rceil - \left\lceil \tfrac{i}{5} \right\rceil = 1.$$

This means that all weights of 4-sided faces are different. Moreover, the weight of the external face is

$$wt_{\varphi_{(1,0,1)}}(f_{ext}) = \sum_{i=1}^{n} \varphi(u_i) + \sum_{i=1}^{n} \varphi(v_i) + \varphi(f_{ext}) > \sum_{i=n-1}^{n} \varphi(u_i) + \sum_{i=n-1}^{n} \varphi(v_i) + k \geq wt_{\varphi_{(1,0,1)}}(f_{n-1}).$$

Case 2. When $(\alpha, \beta, \gamma) = (0, 1, 1)$. By Theorem 5 we get $\mathrm{fs}_{(0,1,1)}(L_n) \geq \lceil (n+3)/5 \rceil$. Put $k = \lceil (n+3)/5 \rceil$. Consider a k-labeling of type $(0, 1, 1)$ $\psi : E(L_n) \cup F(L_n) \to \{1, 2, \ldots, k\}$ defined such that

$$\begin{aligned}
\psi(u_i u_{i+1}) &= \left\lceil \tfrac{i+2}{5} \right\rceil, && \text{for } i = 1, 2, \ldots, n-1,\\
\psi(v_i v_{i+1}) &= \left\lceil \tfrac{i+1}{5} \right\rceil, && \text{for } i = 1, 2, \ldots, n-1,\\
\psi(u_i v_i) &= \left\lceil \tfrac{i+3}{5} \right\rceil, && \text{for } i = 1, 2, \ldots, n,\\
\psi(f_i) &= \left\lceil \tfrac{i}{5} \right\rceil, && \text{for } i = 1, 2, \ldots, n-1,\\
\psi(f_{ext}) &= k.
\end{aligned}$$

Evidently, under the labeling ψ all edge and face labels are at most k. The face-weight of the face f_i, $i = 1, 2, \ldots, n-1$, under the labeling ψ of type $(0, 1, 1)$ is

$$wt_{\psi(0,1,1)}(f_i) = \psi(u_i u_{i+1}) + \psi(v_i v_{i+1}) + \psi(u_i v_i) + \psi(u_{i+1} v_{i+1}) + \psi(f_i).$$

For $i = 1, 2, \ldots, n-2$ we obtain

$$\begin{aligned}
wt_{\psi(0,1,1)}(f_{i+1}) - wt_{\psi(0,1,1)}(f_i) &= \psi(u_{i+1}u_{i+2}) + \psi(v_{i+1}v_{i+2}) + \psi(u_{i+2}v_{i+2})\\
&\quad + \psi(f_{i+1}) - \psi(u_i u_{i+1}) - \psi(v_i v_{i+1}) - \psi(u_i v_i) - \psi(f_i)\\
&= \left\lceil \tfrac{i+3}{5} \right\rceil + \left\lceil \tfrac{i+2}{5} \right\rceil + \left\lceil \tfrac{i+5}{5} \right\rceil + \left\lceil \tfrac{i+1}{5} \right\rceil - \left\lceil \tfrac{i+2}{5} \right\rceil - \left\lceil \tfrac{i+1}{5} \right\rceil - \left\lceil \tfrac{i+3}{5} \right\rceil - \left\lceil \tfrac{i}{5} \right\rceil\\
&= 1.
\end{aligned}$$

Thus the weights of all 4-sided faces are different. For the weight of the external face we get

$$\begin{aligned} wt_{\psi(0,1,1)}(f_{ext}) &= \sum_{i=1}^{n-1}\psi(u_iu_{i+1}) + \sum_{i=1}^{n-1}\psi(v_iv_{i+1}) + \psi(u_1v_1) + \psi(u_nv_n) + \psi(f_{ext}) \\ &> \sum_{i=n-1}^{n-1}\psi(u_iu_{i+1}) + \sum_{i=n-1}^{n-1}\psi(v_iv_{i+1}) + \psi(u_{n-1}v_{n-1}) + \psi(u_nv_n) + k \\ &\geq\ wt_{\psi(0,1,1)}(f_{n-1}). \end{aligned}$$

This concludes the proof. ■

In some cases the lower bound of the face irregularity strength of type $(1,\beta,\gamma)$ can be improved when we consider the maximum degree in a 2-connected plane graph.

Theorem 7. *Let $G=(V,E,F)$ be a 2-connected plane graph with the maximum degree $\Delta(G)$. Let $\beta,\gamma\in\{0,1\}$ and x be a vertex of degree $\Delta(G)$. Let the smallest face and the biggest face incident with x be an $\hat{a}$-sided face and a $\hat{b}$-sided face, respectively. Then the face irregularity strength of type $(1,\beta,\gamma)$ of G is*

$$\mathrm{fs}_{(1,\beta,\gamma)}(G) \geq \left\lceil \frac{\hat{a}(\beta+1)+\gamma+\Delta(G)-2}{\hat{b}(\beta+1)+\gamma-1} \right\rceil.$$

Proof. Consider a 2-connected plane graph with the maximum degree $\Delta(G)$ and with a face irregular k-labeling φ of type $(1,\beta,\gamma)$, where $\mathrm{fs}_{(1,\beta,\gamma)}(G)=k$. Suppose that $f_1,f_2,\ldots,f_{\Delta(G)}$ are the faces incident with a fixed vertex x of the maximum degree $\Delta(G)$. Let the smallest face and the biggest face incident with x be an $\hat{a}$-sided face and a $\hat{b}$-sided face, respectively.

Clearly, the face-weights $wt_{\varphi_{(1,\beta,\gamma)}}(f_i)$, $i=1,2,\ldots,\Delta(G)$, are all distinct and each of them contains the value $\varphi(x)$.

The smallest face-weight under the face irregular k-labeling φ admits the value at least $\varphi(x)+\hat{a}(\beta+1)+\gamma-1$. Since there are $\Delta(G)$ faces incident with x, it follows that the largest face-weight between them attains the value at least $\varphi(x)+\hat{a}(\beta+1)+\gamma+\Delta(G)-2$ and at most $\varphi(x)+\left(\hat{b}(\beta+1)+\gamma-1\right)k$. Then

$$\varphi(x)+\hat{a}(\beta+1)+\gamma+\Delta(G)-2 \leq \varphi(x)+\left(\hat{b}(\beta+1)+\gamma-1\right)k$$

and

$$k \geq \left\lceil \frac{\hat{a}(\beta+1)+\gamma+\Delta(G)-2}{\hat{b}(\beta+1)+\gamma-1} \right\rceil.$$

■

In [8] is proved that

$$\mathrm{fs}_{(1,1,1)}(G) = \mathrm{efs}(G) \geq \left\lceil \frac{2\hat{a} + \Delta(G) - 1}{2\hat{b}} \right\rceil . \tag{4}$$

Sharpness of the lower bound in (4) is proved for wheels W_n on $n+1$ vertices, $n \geq 3$.

If many faces of the same size are incident with a fixed vertex, then for some plane graphs it is possible to get even better lower bound.

Theorem 8. *Let $G = (V, E, F)$ be a 2-connected plane graph with the biggest face of size b. Let $n_a(x)$ denote the number of a-sided faces incident with a vertex x in G, where $a = 3, 4, \ldots, b$. Let $\beta, \gamma \in \{0, 1\}$. Then the face irregularity strength of type $(1, \beta, \gamma)$ of G is*

$$\mathrm{fs}_{(1,\beta,\gamma)}(G) \geq \max\left\{ \max\left\{ \left\lceil \frac{a(\beta+1) + \gamma + n_a(x) - 2}{a(\beta+1) + \gamma - 1} \right\rceil : a = 3, 4, \ldots, b \right\} : x \in V(G) \right\}.$$

Proof. Consider a 2-connected plane graph with the biggest face of size b. Let $n_a(x)$ be the number of a-sided faces incident with a vertex x in G, $a = 3, 4, \ldots, b$. Note, that $\sum_{a=3}^{b} n_a(x) = \deg(x)$.

Let φ be a face irregular k-labeling of type $(1, \beta, \gamma)$, where $\mathrm{fs}_{(1,\beta,\gamma)}(G) = k$. Suppose that $f_{a,1}(x), f_{a,2}(x), \ldots, f_{a,n_a(x)}(x)$ are the a-sided faces incident with a fixed vertex x, where $a = 3, 4, \ldots, b$. Clearly, the face-weights $wt_{\varphi(1,\beta,\gamma)}(f_{a,i}(x))$, $i = 1, 2, \ldots, n_a(x)$, are all distinct, each of them contains the value $\varphi(x)$ and the smallest of them admits the value at least $\varphi(x) + a(\beta+1) + \gamma - 1$. Since there are $n_a(x)$ a-sided faces incident with the vertex x, it follows that the corresponding largest face-weight attains the value at least $\varphi(x) + a(\beta+1) + \gamma + n_a(x) - 2$ and at most $\varphi(x) + (a(\beta+1) + \gamma - 1)\, k$. Then

$$\varphi(x) + a(\beta+1) + \gamma + n_a(x) - 2 \leq \varphi(x) + (a(\beta+1) + \gamma - 1)\, k$$

and

$$k \geq \left\lceil \frac{a(\beta+1) + \gamma + n_a(x) - 2}{a(\beta+1) + \gamma - 1} \right\rceil .$$

Moreover, this inequality must hold for every a, $a = 3, 4, \ldots, b$ and every vertex x in G. ∎

A *fan* F_n, $n \geq 2$, is a graph obtained by joining all vertices of path P_n to a further vertex. The next theorem gives the exact value of the face irregularity strength of type (α, β, γ) for a fan.

Theorem 9. *Let F_n, $n \geq 3$, be a fan and let $\alpha, \beta, \gamma \in \{0,1\}$. Then*

$$\mathrm{fs}_{(\alpha,\beta,\gamma)}(F_n) = \begin{cases} n, & \text{if } (\alpha,\beta,\gamma) = (0,0,1), \\ \left\lceil \frac{n}{2} \right\rceil, & \text{if } (\alpha,\beta,\gamma) = (1,0,0), \\ \left\lceil \frac{n+1}{3} \right\rceil, & \text{if } (\alpha,\beta,\gamma) = (1,0,1);\ (0,1,0), \\ \left\lceil \frac{n+2}{4} \right\rceil, & \text{if } (\alpha,\beta,\gamma) = (0,1,1), \\ \left\lceil \frac{n+3}{5} \right\rceil, & \text{if } (\alpha,\beta,\gamma) = (1,1,0), \\ \left\lceil \frac{n+4}{6} \right\rceil, & \text{if } (\alpha,\beta,\gamma) = (1,1,1). \end{cases}$$

Proof. The fan F_n contains $n+1$ vertices, say, $w, v_1, v_2, \ldots, v_n$ and $2n-1$ edges wv_i, $i = 1, 2, \ldots, n$, and v_iv_{i+1}, $i = 1, 2, \ldots, n-1$. We denote its 3-sided faces by the symbol f_i, $i = 1, 2, \ldots, n-1$, such that the face f_i is surrounded by vertices v_i, v_{i+1}, w and edges v_iv_{i+1}, v_iw, $v_{i+1}w$. The external $(n+1)$-sided face we denote by f_{ext}.

The value for the parameter $\mathrm{fs}_{(0,0,1)}(F_n)$ follows from Lemma 4.

Using a connection between the face irregularity strength and the total / vertex / edge C_3-irregularity strength of fan graphs, denoted by $\mathrm{ths}(F_n, C_3)$ / $\mathrm{vhs}(F_n, C_3)$ / $\mathrm{ehs}(F_n, C_3)$, we get that for $n \geq 3$

$$\begin{aligned} \mathrm{fs}_{(1,1,0)}(F_n) &= \mathrm{ths}(F_n, C_3) = \left\lceil \tfrac{n+3}{5} \right\rceil, && \text{see [6]}, \\ \mathrm{fs}_{(1,0,0)}(F_n) &= \mathrm{vhs}(F_n, C_3) = \left\lceil \tfrac{n}{2} \right\rceil, && \text{see [5]}, \\ \mathrm{fs}_{(0,1,0)}(F_n) &= \mathrm{ehs}(F_n, C_3) = \left\lceil \tfrac{n+1}{3} \right\rceil, && \text{see [7]}. \end{aligned}$$

Next consider three remaining cases.

Case 1. When $(\alpha, \beta, \gamma) = (1,1,1)$. According to Theorem 8 we have $\mathrm{fs}_{(1,1,1)}(F_n) \geq \lceil (n+4)/6 \rceil$. Put $k = \lceil (n+4)/6 \rceil$. We define a k-labeling of type $(1,1,1)$ $\varphi : V(F_n) \cup E(F_n) \cup F(F_n) \to \{1, 2, \ldots, k\}$ such that

$$\begin{aligned} \varphi(v_i) &= \left\lceil \tfrac{i+4}{6} \right\rceil, && \text{for } i = 1, 2, \ldots, n, \\ \varphi(w) &= 1, \\ \varphi(v_iw) &= \left\lceil \tfrac{i+1}{6} \right\rceil, && \text{for } i = 1, 2, \ldots, n, \\ \varphi(v_iv_{i+1}) &= \left\lceil \tfrac{i+3}{6} \right\rceil, && \text{for } i = 1, 2, \ldots, n-1, \\ \varphi(f_i) &= \left\lceil \tfrac{i}{6} \right\rceil, && \text{for } i = 1, 2, \ldots, n-1, \\ \varphi(f_{ext}) &= k. \end{aligned}$$

It is easy to see that all labels are at most k. The face-weight of the face f_i, $i = 1, 2, \ldots, n-1$, under the labeling φ of type $(1,1,1)$ is

$$wt_{\varphi(1,1,1)}(f_i) = \varphi(v_i) + \varphi(v_iv_{i+1}) + \varphi(v_{i+1}) + \varphi(w) + \varphi(v_iw) + \varphi(v_{i+1}w) + \varphi(f_i).$$

Thus, for $i = 1, 2, \ldots, n-2$ we obtain

$$\begin{aligned} wt_{\varphi_{(1,1,1)}}(f_{i+1}) - wt_{\varphi_{(1,1,1)}}(f_i) &= \varphi(v_{i+2}) + \varphi(v_{i+1}v_{i+2}) + \varphi(v_{i+2}w) + \varphi(f_{i+1}) \\ &\quad - \varphi(v_i) - \varphi(v_iv_{i+1}) - \varphi(v_iw) - \varphi(f_i) \\ &= \left\lceil \tfrac{i+6}{6} \right\rceil + \left\lceil \tfrac{i+4}{6} \right\rceil + \left\lceil \tfrac{i+3}{6} \right\rceil + \left\lceil \tfrac{i+1}{6} \right\rceil - \left\lceil \tfrac{i+4}{6} \right\rceil - \left\lceil \tfrac{i+3}{6} \right\rceil \\ &\quad - \left\lceil \tfrac{i+1}{6} \right\rceil - \left\lceil \tfrac{i}{6} \right\rceil = 1. \end{aligned}$$

This means that all weights of 3-sided faces are different. The weight of the external $(n+1)$-sided face is

$$\begin{aligned} wt_{\varphi_{(1,1,1)}}(f_{ext}) &= \sum_{i=1}^{n} \varphi(v_i) + \psi(w) + \sum_{i=1}^{n-1} \varphi(v_iv_{i+1}) + \varphi(v_1w) + \varphi(v_nw) + \varphi(f_{ext}) \\ &> \sum_{i=n-1}^{n} \varphi(v_i) + \psi(w) + \sum_{i=n-1}^{n-1} \varphi(v_iv_{i+1}) + \varphi(v_{n-1}w) + \varphi(v_nw) + k \\ &\geq wt_{\varphi_{(1,1,1)}}(f_{n-1}). \end{aligned}$$

Case 2. When $(\alpha, \beta, \gamma) = (1, 0, 1)$. Using Theorem 8 we get the lower bound, i.e., $\mathrm{fs}_{(1,0,1)}(F_n) \geq \lceil (n+1)/3 \rceil$. Put $k = \lceil (n+1)/3 \rceil$. We define a k-labeling of type $(1, 0, 1)$ $\psi : V(F_n) \cup F(F_n) \to \{1, 2, \ldots, k\}$ such that

$$\begin{aligned} \psi(v_i) &= \left\lceil \tfrac{i+1}{3} \right\rceil, && \text{for } i = 1, 2, \ldots, n, \\ \psi(w) &= 1, \\ \psi(f_i) &= \left\lceil \tfrac{i}{3} \right\rceil, && \text{for } i = 1, 2, \ldots, n-1, \\ \psi(f_{ext}) &= k. \end{aligned}$$

For $i = 1, 2, \ldots, n-1$ we get

$$wt_{\psi_{(1,0,1)}}(f_i) = \psi(v_i) + \psi(v_{i+1}) + \psi(w) + \psi(f_i)$$

and for $i = 1, 2, \ldots, n-2$ we obtain

$$\begin{aligned} wt_{\psi_{(1,0,1)}}(f_{i+1}) - wt_{\psi_{(1,0,1)}}(f_i) &= \psi(v_{i+2}) + \psi(f_{i+1}) - \psi(v_i) - \psi(f_i) \\ &= \left\lceil \tfrac{i+3}{3} \right\rceil + \left\lceil \tfrac{i+1}{3} \right\rceil - \left\lceil \tfrac{i+1}{3} \right\rceil - \left\lceil \tfrac{i}{3} \right\rceil = 1. \end{aligned}$$

Thus the 3-sided face-weights are distinct. Moreover,

$$\begin{aligned} wt_{\psi_{(1,0,1)}}(f_{ext}) &= \sum_{i=1}^{n} \psi(v_i) + \psi(w) + \psi(f_{ext}) > \sum_{i=n-1}^{n} \psi(v_i) + \psi(w) + k \\ &\geq wt_{\psi_{(1,0,1)}}(f_{n-1}). \end{aligned}$$

Case 3. When $(\alpha,\beta,\gamma) = (0,1,1)$. By Theorem 5 we get $\mathrm{fs}_{(0,1,1)}(F_n) \geq \lceil (n+2)/4 \rceil$. Put $k = \lceil (n+2)/4 \rceil$. We define a k-labeling of type $(0,1,1)$ $\xi : E(F_n) \cup F(F_n) \to \{1,2,\ldots,k\}$ such that

$$\begin{aligned} \xi(v_i w) &= \left\lceil \tfrac{i+2}{4} \right\rceil, && \text{for } i = 1,2,\ldots,n, \\ \xi(v_i v_{i+1}) &= \left\lceil \tfrac{i+1}{4} \right\rceil, && \text{for } i = 1,2,\ldots,n-1, \\ \xi(f_i) &= \left\lceil \tfrac{i}{4} \right\rceil, && \text{for } i = 1,2,\ldots,n-1, \\ \xi(f_{ext}) &= k. \end{aligned}$$

The face-weight of the face f_i, $i = 1,2,\ldots,n-1$, under the labeling ξ of type $(0,1,1)$ is

$$wt_{\xi_{(0,1,1)}}(f_i) = \xi(v_i v_{i+1}) + \xi(v_i w) + \xi(v_{i+1} w) + \xi(f_i)$$

and for $i = 1,2,\ldots,n-2$ we get

$$\begin{aligned} wt_{\xi_{(0,1,1)}}(f_{i+1}) - wt_{\xi_{(0,1,1)}}(f_i) = {} & \xi(v_{i+1}v_{i+2}) + \xi(v_{i+2}w) + \xi(f_{i+1}) - \xi(v_i v_{i+1}) \\ & - \xi(v_i w) - \xi(f_i) = \left\lceil \tfrac{i+2}{4} \right\rceil + \left\lceil \tfrac{i+4}{4} \right\rceil + \left\lceil \tfrac{i+1}{4} \right\rceil \\ & - \left\lceil \tfrac{i+1}{4} \right\rceil - \left\lceil \tfrac{i+2}{4} \right\rceil - \left\lceil \tfrac{i}{4} \right\rceil = 1. \end{aligned}$$

Thus all weights of 3-sided faces are different. To finish the proof we need to show that the weight of the external $(n+1)$-sided face is not equal to the weight of any 3-sided face. However, this follows from the following inequality.

$$\begin{aligned} wt_{\xi_{(0,1,1)}}(f_{ext}) &= \sum_{i=1}^{n-1} \xi(v_i v_{i+1}) + \xi(v_1 w) + \xi(v_n w) + \xi(f_{ext}) \\ &> \sum_{i=n-1}^{n-1} \xi(v_i v_{i+1}) + \xi(v_{n-1} w) + \xi(v_n w) + k \geq wt_{\xi_{(0,1,1)}}(f_{n-1}). \end{aligned}$$

■

Sometimes two or more faces can have not just one, but two vertices in common. The following two theorems present the lower bounds if more faces share two common vertices. The first theorem deals with the case when we consider all such faces, the second theorem describes the situation when we consider only faces of the same size.

Theorem 10. *Let $G = (V,E,F)$ be a 2-connected plane graph. Let $\beta,\gamma \in \{0,1\}$. Let $n(x,y)$ be the number of faces incident with both vertices x and y in G. Let the smallest face and the biggest face incident with x and y be an $a_{x,y}$-sided face and a $b_{x,y}$-sided face, respectively. Then the face irregularity strength of type $(1,\beta,\gamma)$ of G is*

$$\mathrm{fs}_{(1,\beta,\gamma)}(G) \geq \max\left\{ \left\lceil \frac{a_{x,y}(\beta+1) + \gamma + n(x,y) - 3}{b_{x,y}(\beta+1) + \gamma - 2} \right\rceil : x,y \in V(G) \right\}.$$

Proof. Consider a 2-connected plane graph. Let $n(x,y)$ denote the number of faces incident with both vertices x and y in G. Let the smallest face and the biggest face incident with x and y be an $a_{x,y}$-sided face and a $b_{x,y}$-sided face, respectively.

Let φ be a face irregular k-labeling of type $(1,\beta,\gamma)$, where $\mathrm{fs}_{(1,\beta,\gamma)}(G)=k$. Suppose that $f_1(x,y), f_2(x,y),\ldots,f_{n(x,y)}(x,y)$ are the faces incident with both fixed vertices x and y in G. Clearly, the face-weights $wt_{\varphi_{(1,\beta,\gamma)}}(f_i(x,y))$, $i=1,2,\ldots,n(x,y)$, are all distinct and each of them contains the values $\varphi(x)$ and $\varphi(y)$. The corresponding smallest face-weight under the face irregular k-labeling φ admits the value at least $\varphi(x)+\varphi(y)+a_{x,y}(\beta+1)+\gamma-2$. Since there are $n(x,y)$ faces incident with both vertices x,y, it follows that the corresponding largest face-weight attains the value at least $\varphi(x)+\varphi(y)+a_{x,y}(\beta+1)+\gamma+n(x,y)-3$ and at most $\varphi(x)+\varphi(y)+(b_{x,y}(\beta+1)+\gamma-2)\,k$. Then

$$\varphi(x)+\varphi(y)+a_{x,y}(\beta+1)+\gamma+n(x,y)-3\le\varphi(x)+\varphi(y)+(b_{x,y}(\beta+1)+\gamma-2)\,k$$

and

$$k\ge\left\lceil\frac{a_{x,y}(\beta+1)+\gamma+n(x,y)-3}{b_{x,y}(\beta+1)+\gamma-2}\right\rceil.$$

Moreover, this inequality must hold for all couples of vertices x,y in G. ■

Theorem 11. *Let $G=(V,E,F)$ be a 2-connected plane graph with the biggest face of size b. Let $\beta,\gamma\in\{0,1\}$. Let $n_a(x,y)$ denote the number of a-sided faces incident with both vertices x and y in G, where $a=3,4,\ldots,b$. Then the face irregularity strength of type $(1,\beta,\gamma)$ of G is*

$$\mathrm{fs}_{(1,\beta,\gamma)}(G)\ge\max\left\{\max\left\{\left\lceil\tfrac{a(\beta+1)+\gamma+n_a(x,y)-3}{a(\beta+1)+\gamma-2}\right\rceil:a=3,4,\ldots,b\right\}:x,y\in V(G)\right\}.$$

Proof. Consider a 2-connected plane graph with the biggest face of size b. Let $n_a(x,y)$ denote the number of a-sided faces incident with both vertices x and y in G, where $a=3,4,\ldots,b$.

Let φ be a face irregular k-labeling of type $(1,\beta,\gamma)$, where $\mathrm{fs}_{(1,\beta,\gamma)}(G)=k$. Suppose that $f_{a,1}(x,y), f_{a,2}(x,y),\ldots,f_{a,n_a(x,y)}(x,y)$ are the a-sided faces incident with both fixed vertices x and y in G, where $a=3,4,\ldots,b$. Clearly, the face-weights $wt_{\varphi_{(1,\beta,\gamma)}}(f_{a,i}(x,y))$, $i=1,2,\ldots,n_a(x,y)$, are all distinct and each of them contains the values $\varphi(x)$ and $\varphi(y)$. The corresponding smallest face-weight under the face irregular k-labeling φ admits the value at least $\varphi(x)+\varphi(y)+a(\beta+1)+\gamma-2$. Since there are $n_a(x,y)$ a-sided faces incident with both vertices x,y, it follows that the corresponding largest face-weight attains the value at least $\varphi(x)+\varphi(y)+a(\beta+1)+\gamma+n_a(x,y)-3$ and at most $\varphi(x)+\varphi(y)+(a(\beta+1)+\gamma-2)\,k$. Then

$$\varphi(x)+\varphi(y)+a(\beta+1)+\gamma+n_a(x,y)-3\le\varphi(x)+\varphi(y)+(a(\beta+1)+\gamma-2)\,k$$

and

$$k \geq \left\lceil \frac{a(\beta+1)+\gamma+n_a(x,y)-3}{a(\beta+1)+\gamma-2} \right\rceil .$$

Moreover, this inequality must hold for all couples of vertices x, y in G and for every a, $a = 3, 4, \ldots, b$. This concludes the proof. ■

3. Upper Bounds for Face Irregularity Strength

In Theorem 1 we proved that $\mathrm{fs}_{(\alpha,\beta,0)}(C_n) = \infty$. In the following section of the paper we show that except this case the face irregularity strength of type (α, β, γ) of any 2-connected plane graph G is always finite.

Theorem 12. *Let $\alpha, \beta \in \{0, 1\}$ and let $G = (V, E, F)$ be a 2-connected plane graph containing faces of D different sizes. Then*

$$\mathrm{fs}_{(\alpha,\beta,1)}(G) \leq \begin{cases} |F(G)|, & \text{if } (\alpha,\beta) = (0,0), \\ |F(G)| + 1 - D, & \text{if } (\alpha,\beta) \neq (0,0). \end{cases}$$

Proof. Let $\alpha, \beta \in \{0, 1\}$ and let $G = (V, E, F)$ be a 2-connected plane graph containing faces of D different sizes. Let us denote the faces of G arbitrarily by the symbols f_i such that if s_i is the size of a face f_i, $i = 1, 2, \ldots, |F(G)|$, then for every $i = 1, 2, \ldots, |F(G)| - 1$

$$s_i \leq s_{i+1}. \tag{5}$$

We define a labeling φ of type $(\alpha, \beta, 1)$ of G in the following way.

$$\varphi(x) = \begin{cases} \alpha, & \text{if } x \in V(G), \\ \beta, & \text{if } x \in E(G), \end{cases}$$

$$\varphi(f_i) = \begin{cases} 1, & \text{if } i = 1, \\ \varphi(f_{i-1}), & \text{if } (\alpha+\beta)s_i \neq (\alpha+\beta)s_{i-1},\ i = 2, 3, \ldots, |F(G)|, \\ \varphi(f_{i-1}) + 1, & \text{if } (\alpha+\beta)s_i = (\alpha+\beta)s_{i-1},\ i = 2, 3, \ldots, |F(G)|. \end{cases}$$

For the weights of faces f_i, $i = 1, 2, \ldots, |F(G)|$, under the labeling φ we get

$$\begin{aligned} wt_{\varphi_{(\alpha,\beta,1)}}(f_i) &= \alpha \sum_{v \sim f_i} \varphi(v) + \beta \sum_{e \sim f_i} \varphi(e) + \varphi(f_i) = \alpha s_i + \beta s_i + \varphi(f_i) \\ &= (\alpha+\beta)s_i + \varphi(f_i). \end{aligned}$$

Thus by the definition of the labeling φ and according to (5) we get that for every $i = 1, 2, \ldots, |F(G)| - 1$

$$wt_{\varphi_{(\alpha,\beta,1)}}(f_i) = (\alpha+\beta)s_i + \varphi(f_i) < (\alpha+\beta)s_{i+1} + \varphi(f_{i+1}) = wt_{\varphi_{(\alpha,\beta,1)}}(f_{i+1}).$$

It means that all face weights are distinct and thus φ is a face irregular labeling of type $(\alpha, \beta, 1)$ of G. This implies that

$$\mathrm{fs}_{(\alpha,\beta,1)}(G) \leq \varphi(f_{|F(G)|}) = \begin{cases} |F(G)|, & \text{if } (\alpha,\beta) = (0,0), \\ |F(G)| + 1 - D, & \text{if } (\alpha,\beta) \neq (0,0). \end{cases}$$ ■

Now we will deal with the case when $\gamma = 0$ and we prove that the face irregularity strength of type $(1,0,0)$ of any 2-connected graph G different from a cycle is again finite.

Theorem 13. *For every 2-connected plane graph $G = (V, E, F)$ different from a cycle we have*

$$\mathrm{fs}_{(1,0,0)}(G) \leq 2^{|V(G)|-1}.$$

Proof. Let $G = (V, E, F)$ be a 2-connected plane graph different from a cycle.

Let us denote the faces in G arbitrarily by the symbols $f_1, f_2, \ldots, f_{|F(G)|}$ and let us denote the vertices in G arbitrarily by the symbols $v_1, v_2, \ldots, v_{|V(G)|}$. We define a $2^{|V(G)|-1}$-labeling φ of type $(1,0,0)$ of G such that

$$\varphi(v_i) = 2^{i-1}, \quad \text{for } i = 1, 2, \ldots, |V(G)|.$$

Consider a labeling θ defined such that

$$\theta_{i,j} = \begin{cases} 1, & \text{if } v_i \sim f_j, \\ 0, & \text{if } v_i \not\sim f_j, \end{cases}$$

where $i = 1, 2, \ldots, |V(G)|$, $j = 1, 2, \ldots, |F(G)|$.

Every face-weight under a labeling of type $(1,0,0)$ is the sum of all labels of vertices surrounding this face. Thus, for $j = 1, 2, \ldots, |F(G)|$ we have

$$wt_{\varphi_{(1,0,0)}}(f_j) = \sum_{v \sim f_j} \varphi(v) = \sum_{v_i \sim f_j} 2^{i-1} = \sum_{i=1}^{|V(G)|} \theta_{i,j} \cdot 2^{i-1}. \tag{6}$$

To prove that the face-weights are all distinct it is enough to show that the sums $\sum_{i=1}^{|V(G)|} \theta_{i,j} \cdot 2^{i-1}$ are distinct for every $j = 1, 2, \ldots, |F(G)|$. However, this is evident if we consider that the ordered $|V(G)|$-tuple $(\theta_{|V(G)|,j}\theta_{|V(G)|-1,j} \ldots \theta_{2,j}\theta_{1,j})$ corresponds to the binary code representation of the sum (6). As G is not isomorphic to a cycle we get that different faces cannot have the same vertex sets and we immediately get that the $|V(G)|$-tuples are different for different faces. ■

In a similar way we can prove the following results.

Theorem 14. *For every 2-connected plane graph $G = (V, E, F)$ different from a cycle we have*

$$\mathrm{fs}_{(0,1,0)}(G) \leq 2^{|E(G)|-1}.$$

Theorem 15. *For every 2-connected plane graph $G = (V, E, F)$ different from a cycle we have*

$$\mathrm{fs}_{(1,1,0)}(G) \leq 2^{|V(G)|+|E(G)|-1}.$$

4. Modification of the Problem

In the previous sections we discussed the case when the face-weights of different faces are distinct. However, in the light of a face-antimagic labeling we can consider a modification of the foregoing concept that we will require that just the weights of the faces of the same size must be distinct. This means that the faces of the different size can have the same weights.

A k-labeling φ of type (α, β, γ) of the plane graph G is called a *same-face irregular k-labeling* if for every number s the face-weights are different for all pairs of distinct s-sided faces. Thus for every couple of s-sided faces f and g in G there is

$$wt_{\varphi_{(\alpha,\beta,\gamma)}}(f) \neq wt_{\varphi_{(\alpha,\beta,\gamma)}}(g).$$

The *same-face irregularity strength of type* (α, β, γ) of a plane graph G, denoted by $\mathrm{sfs}_{(\alpha,\beta,\gamma)}(G)$, is the smallest integer k such that G admits a same-face irregular k-labeling of type (α, β, γ).

Trivially, if all face-weights are distinct, then also weights of faces of the same size are distinct.

Theorem 16. *If G is a 2-connected plane graph and $\alpha, \beta, \gamma \in \{0, 1\}$, then*

$$\mathrm{sfs}_{(\alpha,\beta,\gamma)}(G) \leq \mathrm{fs}_{(\alpha,\beta,\gamma)}(G).$$

In some cases the equality holds.

Theorem 17. *If $n \geq 3$ and $\alpha, \beta, \gamma \in \{0, 1\}$, then*

$$\begin{aligned} \mathrm{sfs}_{(\alpha,\beta,\gamma)}(C_n) &= \mathrm{fs}_{(\alpha,\beta,\gamma)}(C_n), \\ \mathrm{sfs}_{(\alpha,\beta,\gamma)}(L_n) &= \mathrm{fs}_{(\alpha,\beta,\gamma)}(L_n), \\ \mathrm{sfs}_{(\alpha,\beta,\gamma)}(F_n) &= \mathrm{fs}_{(\alpha,\beta,\gamma)}(F_n). \end{aligned}$$

Using similar arguments as in the previous parts we can prove the following results.

Theorem 18. *Let $\alpha, \beta, \gamma \in \{0, 1\}$ and let $G = (V, E, F)$ be a 2-connected plane graph in which no different faces have the same size. Then*

$$\mathrm{sfs}_{(\alpha,\beta,\gamma)}(G) = 1.$$

Theorem 19. *Let $G = (V, E, F)$ be a 2-connected plane graph with the biggest face of size b. Let $n_a(G)$, $a = 3, 4, \ldots, b$, denote the number of a-sided faces in G. Then*

$$\mathrm{sfs}_{(0,0,1)}(G) = \max\{n_a(G) : a = 3, 4, \ldots, b\}.$$

Theorem 20. *Let $G = (V, E, F)$ be a 2-connected plane graph with the biggest face of size b. Let $n_a(x)$ denote the number of a-sided faces incident with a vertex x in G, where $a = 3, 4, \ldots, b$. Let $\beta, \gamma \in \{0, 1\}$. Then the same-face irregularity strength of type $(1, \beta, \gamma)$ of G is*

$$\mathrm{sfs}_{(1,\beta,\gamma)}(G) \geq \max\left\{\max\left\{\left\lceil \frac{a(\beta+1)+\gamma+n_a(x)-2}{a(\beta+1)+\gamma-1} \right\rceil : a = 3, 4, \ldots, b\right\} : x \in V(G)\right\}.$$

Theorem 21. *Let $G = (V, E, F)$ be a 2-connected plane graph with the biggest face of size b. Let $\beta, \gamma \in \{0, 1\}$. Let $n_a(x, y)$ denote the number of a-sided faces incident with both vertices x and y in G, where $a = 3, 4, \ldots, b$. Then the same-face irregularity strength of type $(1, \beta, \gamma)$ of G is*

$$\mathrm{sfs}_{(1,\beta,\gamma)}(G) \geq \max\left\{\max\left\{\left\lceil \frac{a(\beta+1)+\gamma+n_a(x,y)-3}{a(\beta+1)+\gamma-2} \right\rceil : a = 3, 4, \ldots, b\right\} : x, y \in V(G)\right\}.$$

We conclude this section with an upper bound for the same-face irregularity strength of type $(\alpha, \beta, 1)$.

Theorem 22. *Let $\alpha, \beta \in \{0, 1\}$. Let $G = (V, E, F)$ be a 2-connected plane graph with the biggest face of size b. Let $n_a(G)$, $a = 3, 4, \ldots, b$, denote the number of a-sided faces in G. Then*

$$\mathrm{sfs}_{(\alpha,\beta,1)}(G) \leq \max\{n_a(G) : a = 3, 4, \ldots, b\}.$$

5. Conclusion

In the paper, we estimated the lower bounds and the upper bounds of the face irregularity strength of type (α, β, γ) for 2-connected plane graphs and determined the precise values of these parameters for certain families of plane graphs, namely ladders and fan graphs to prove the sharpness of the lower bounds. We also conjecture that if the studied parameter is finite it is equal to the maximum of the presented lower bounds.

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