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# ON LIST EQUITABLE TOTAL COLORINGS OF THE GENERALIZED THETA GRAPH

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**Abstract**

In 2003, Kostochka, Pelsmajer, and West introduced a list analogue of equitable coloring called equitable choosability. A $k$-assignment, $L$, for a graph $G$ assigns a list, $L(v)$, of $k$ available colors to each $v \in V(G)$, and an equitable $L$-coloring of $G$ is a proper coloring, $f$, of $G$ such that $f(v) \in L(v)$ for each $v \in V(G)$ and each color class of $f$ has size at most $\lceil |V(G)|/k \rceil$. Graph $G$ is equitably $k$-choosable if $G$ is equitably $L$-colorable whenever $L$ is a $k$-assignment for $G$. In 2018, Kaul, Mudrock, and Pelsmajer subsequently introduced the List Equitable Total Coloring Conjecture which states that if $T$ is a total graph of some simple graph, then $T$ is equitably $k$-choosable for each $k \geq \max\{\chi_\ell(T), \Delta(T)/2 + 2\}$ where $\Delta(T)$ is the maximum degree of a vertex in $T$ and $\chi_\ell(T)$ is the list chromatic number of $T$. In this paper, we verify the List Equitable Total Coloring Conjecture for subdivisions of stars and the generalized theta graph.

**Keywords:** graph coloring, total coloring, equitable coloring, list coloring, equitable choosability.

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## 1. Introduction

In this paper, all graphs are nonempty, finite, simple graphs unless otherwise noted. Generally speaking we follow West [37] for terminology and notation. The set of natural numbers is $\mathbb{N} = \{1, 2, 3, \ldots\}$. For $m \in \mathbb{N}$, we write $[m]$ for the set $\{1, \ldots, m\}$. For sets $A$ and $B$, we write $A - B$ for the set of all elements of $A$ that are not elements of $B$. If $G$ is a graph and $S \subseteq V(G)$, we use $G[S]$ for the subgraph of $G$ induced by $S$. For $v \in V(G)$, we write $d_G(v)$ for the degree of vertex $v$ in the graph $G$ and $\Delta(G)$ for the maximum degree of a vertex in $G$, and we write $N_G(v)$ for the neighborhood of vertex $v$ in the graph $G$. If $e, w \in E(G)$ and $v \in V(G)$, we say $e$ and $v$ are incident if $v$ is an endpoint of $e$, and we say $e$ and $w$ are adjacent if $e$ and $w$ share an endpoint. Also, $G^k$ denotes the $k^{th}$ power of graph $G$ (i.e. $G^k$ has the same vertex set as $G$ and edges between any two vertices within distance $k$ in $G$).

### 1.1. Total coloring, equitable coloring, and list coloring

In this paper, we study a conjecture that combines different types of colorings, namely total coloring, equitable coloring, and list coloring. So, we begin by briefly reviewing these three notions.

Given a graph $G$, in the classic vertex coloring problem we wish to color the elements of $V(G)$ with colors from the set $[m]$ so that adjacent vertices receive different colors, a so-called *proper $m$-coloring.* The *chromatic number* of $G$, denoted $\chi(G)$, is the smallest $k$ such that a proper $k$-coloring of $G$ exists.

#### 1.1.1. Total coloring

A *total $m$-coloring* of $G$ is a labeling $f : V(G) \cup E(G) \to [m]$ where $f(u) \neq f(v)$ whenever $u$ and $v$ are adjacent or incident in $G$. The *total chromatic number* of a graph $G$, denoted $\chi''(G)$, is the smallest $k$ such that $G$ has a total $k$-coloring. Clearly, for any graph $G$, $\chi''(G) \geq \Delta(G) + 1$. A famous open problem is the Total Coloring Conjecture (see [1, 34]) which states that for any graph $G$, $\chi''(G) \leq \Delta(G) + 2$. See [22] for some other applications of total coloring.

It is possible to rephrase total coloring in terms of classic vertex coloring. Specifically, the *total graph* of graph $G$, $T(G)$, is the graph with vertex set $V(G) \cup E(G)$ and vertices are adjacent in $T(G)$ if and only if the corresponding elements are adjacent or incident in $G$. Then $G$ has a total $k$-coloring if and only if $T(G)$ has a proper $k$-coloring. It follows that $\chi''(G) = \chi(T(G))$.

Given a graph $G$, one can construct $T(G)$ in two steps: first subdivide every edge of $G$ to get a new graph $H$, then take its square (i.e. $T(G) = H^2$). For example, $T(P_m) = P^2_{2m-1}$.

#### 1.1.2. Equitable coloring

The study of equitable coloring began in 1964 with a conjecture of Erdős [9], but it was formally introduced by Meyer in the 1970's [27]. An *equitable $k$-coloring* of a graph $G$ is a proper $k$-coloring of $G$, $f$, such that the sizes of the color classes differ by at most one (where a proper $k$-coloring has exactly $k$ color classes). In an equitable $k$-coloring, the color classes associated with the coloring are each of size $\lceil |V(G)|/k \rceil$ or $\lfloor |V(G)|/k \rfloor$. We say that a graph $G$ is *equitably $k$-colorable* if there exists an equitable $k$-coloring of $G$. Many applications of equitable coloring exist, see for example [16, 17, 31, 33].

Unlike classic vertex coloring, increasing the number of colors can make equitable coloring more difficult in certain cases. Indeed for any $m \in \mathbb{N}$, $K_{2m+1,2m+1}$ is equitably $2m$-colorable, but it is not equitably $(2m+1)$-colorable. In 1970, Hajnál and Szemerédi [15] proved the 1964 conjecture of Erdős: every graph $G$ has an equitable $k$-coloring when $k \geq \Delta(G) + 1$.

In 1994, Chen, Lih, and Wu [5] conjectured that the result of Hajnál and Szemerédi can be improved by 1 for most connected graphs. Their conjecture is known as the $\Delta$-Equitable Coloring Conjecture, and it is still open. Formally, the $\Delta$-Equitable Coloring Conjecture states: a connected graph $G$ is equitably $\Delta(G)$-colorable if it is different from $K_m$, $C_{2m+1}$, and $K_{2m+1,2m+1}$. The $\Delta$-Equitable Coloring Conjecture has been proven true for interval graphs, bipartite graphs, outerplanar graphs, subcubic graphs, certain planar graphs, and several other classes of graphs (see [5, 7, 8, 25, 26, 38]).

#### 1.1.3. List coloring

List coloring is another variation on classic vertex coloring that was introduced independently by Vizing [35] and Erdős, Rubin, and Taylor [10] in the 1970's. In list coloring, we associate with graph $G$ a *list assignment*, $L$, that assigns to each vertex $v \in V(G)$ a list, $L(v)$, of available colors. Graph $G$ is said to be *$L$-colorable* if there exists a proper coloring $f$ of $G$ such that $f(v) \in L(v)$ for each $v \in V(G)$ (we refer to $f$ as a *proper $L$-coloring* of $G$). A list assignment $L$ is called a *$k$-assignment* for $G$ if $|L(v)| = k$ for each $v \in V(G)$. We say $G$ is *$k$-choosable* if $G$ is $L$-colorable whenever $L$ is a $k$-assignment for $G$. The *list chromatic number* of $G$, denoted $\chi_\ell(G)$, is the smallest $k$ for which $G$ is $k$-choosable. Since a $k$-assignment can assign the same $k$ colors to every vertex of a graph, $\chi(G) \leq \chi_\ell(G)$.

### 1.2. Equitable total coloring and list equitable coloring

We now briefly review work that has been done on equitably coloring total graphs, and we review a list analogue of equitable coloring. We will then spend the remainder of the paper focused upon list equitable total coloring.

#### 1.2.1. Equitable total coloring

The study of equitable total coloring was initiated by Fu in 1994 [11]. Specifically, Fu introduced the Equitable Total Coloring Conjecture which we now state.

**Conjecture 1** (Equitable Total Coloring Conjecture [11]). *For every graph $G$, $T(G)$ has an equitable $k$-coloring for each $k \geq \max\{\chi(T(G)), \Delta(G)+2\}$.*

The "$\Delta(G)+2$" is required because Fu [11] found an infinite family of graphs $G$ with $\chi''(G) = \Delta(G)+1$ but $T(G)$ is not equitably $(\Delta(G)+1)$-colorable (cf. Proposition 2.10 in [11]). Note that if the Total Coloring Conjecture is true, we would have $\max\{\chi''(G), \Delta(G)+2\} = \Delta(G)+2$.

Fu [11] showed that Conjecture 1 holds for complete bipartite graphs, complete $t$-partite graphs of odd order, trees, and certain split graphs. Equitable total coloring has also been studied for graphs with maximum degree 3 [36], joins of certain graphs [13, 14, 39], the Cartesian product of cycles [6], and the corona product of cubic graphs [12].

#### 1.2.2. List equitable coloring

In 2003, Kostochka, Pelsmajer, and West introduced a list analogue of equitable coloring called equitable choosability [20]. Suppose that $L$ is a $k$-assignment for graph $G$. An *equitable $L$-coloring* of $G$ is a proper $L$-coloring, $f$, of $G$ such that $f$ uses no color more than $\lceil |V(G)|/k \rceil$ times[1]. When an equitable $L$-coloring of $G$ exists we say that $G$ is *equitably $L$-colorable.* Graph $G$ is *equitably $k$-choosable* if $G$ is equitably $L$-colorable whenever $L$ is a $k$-assignment for $G$.

We now mention a convention used in this paper. Suppose that $H$ is a subgraph of $G$, and suppose that $L$ is a $k$-assignment for $G$. When there is an equitable $L'$-coloring of $H$ where $L'$ is the $k$-assignment for $H$ defined by $L'(v) = L(v)$ for each $v \in V(H)$, we say $H$ has an equitable $L$-coloring. Notice an equitable $L$-coloring of $H$ requires color classes of size at most $\lceil |V(H)|/k \rceil$ which may be more restrictive than the bound required for an equitable $L$-coloring of $G$.

It is important to note that, similar to equitable coloring, making the lists larger may make equitable list coloring more difficult in certain cases. Indeed $K_{1,9}$ is equitably 4-choosable, but it is not equitably 5-choosable. Also, equitable $k$-choosability does not imply equitable $k$-colorability unless $k = 2$. Indeed $K_{1,6}$ is equitably 3-choosable, but it is not equitably 3-colorable (see [29]). In [20], there are (perhaps surprising) conjectures that are list analogues of Hajnál and Szemerédi's result and the $\Delta$-Equitable Coloring Conjecture.

**Conjecture 2** [20]. *Every graph $G$ is equitably $k$-choosable when $k \geq \Delta(G)+1$.*

**Conjecture 3** [20]. *A connected graph $G$ is equitably $k$-choosable for each $k \geq \Delta(G)$ if it is different from $K_m$, $C_{2m+1}$, and $K_{2m+1,2m+1}$.*

In [20], it is shown that Conjectures 2 and 3 hold for forests, connected interval graphs, and 2-degenerate graphs with maximum degree at least 5. It

[1]When it comes to equitable choosability, the word equitable indicates that no color is used excessively often.

was also shown in [20] that if $G$ is a graph and $k \geq \max\{\Delta(G), |V(G)|/2\}$, then $G$ is equitably $k$-choosable unless $G$ contains $K_{k+1}$ or is $K_{k,k}$ with $k$ odd in the latter case. Thus, Conjecture 3 is true for small graphs (at most $2k$ vertices). Conjectures 2 and 3 have also been verified for outerplanar graphs [42], powers of paths and cycles [18], series-parallel graphs [41], and certain planar graphs (see [23, 40, 43]). In 2013, Kierstead and Kostochka made substantial progress on Conjecture 2, as follows.

**Theorem 4** [19]**.** *If $G$ is any graph, then $G$ is equitably $k$-choosable whenever*

$$k \geq \begin{cases} \Delta(G)+1 & \text{if } \Delta(G) \leq 7, \\ \Delta(G)+\frac{\Delta(G)+6}{7} & \text{if } 8 \leq \Delta(G) \leq 30, \\ \Delta(G)+\frac{\Delta(G)}{6} & \text{if } \Delta(G) \geq 31. \end{cases}$$

### 1.3. List equitable total coloring

In 2018, Kaul, Pelsmajer, and the first author, began studying the equitable choosability of total graphs [18] which was originally suggested by Nakprasit [30]. Motivated by the Equitable Total Coloring Conjecture (Conjecture 1), they introduced the List Equitable Total Coloring Conjecture (LETCC for short).

**Conjecture 5** (List Equitable Total Coloring Conjecture [18])**.** *For every graph $G$, $T(G)$ is equitably $k$-choosable for each $k \geq \max\{\chi_\ell(T(G)), \Delta(G)+2\}$.*

Note that since $\Delta(T(G)) = 2\Delta(G)$, the LETCC is saying something stronger about total graphs than Conjectures 2 and 3 when $\Delta(G) > 2$. Also, Fu's infinite family of graphs $G$ with $\chi''(G) = \Delta(G)+1$ and $T(G)$ is not equitably $\Delta(G)+1$-colorable also has the property that $\chi_\ell(T(G)) = \Delta(G)+1$ and $T(G)$ is not equitably $(\Delta(G)+1)$-choosable. So the LETCC would be sharp if true. The LETCC has been verified for all graphs $G$ with $\Delta(G) \leq 2$, stars, double stars, and trees of maximum degree 3 (see [18, 28]).

### 1.4. Outline of results and an open question

In this paper, we study list equitable total coloring of generalized theta graphs. Suppose that $m \in \mathbb{N}$ and $l_1, \ldots, l_m \in \mathbb{N}$ satisfy $l_1 \leq \cdots \leq l_m$. Then, the *generalized theta graph* $\Theta(l_1, \ldots, l_m)$ is the equivalence class of graphs consisting of two vertices joined by internally disjoint paths of lengths $l_1, \ldots, l_m$. We will assume that $l_2 \geq 2$ when $m \geq 2$ since we will only be considering simple graphs in this paper.[2] Studying list equitable total coloring of generalized theta graphs is quite natural as theta graphs and generalized theta graphs have many interesting

[2]For the remainder of this paper, whenever we see $\Theta(l_1, \ldots, l_m)$, we will always assume $m, l_1, \ldots, l_m \in \mathbb{N}$ with $l_1 \leq \cdots \leq l_m$ and $l_2 \geq 2$ when $m \geq 2$.

properties that have been studied by many researchers (see [2, 3, 4, 10, 21, 24, 32]). For this paper, we prove a positive answer to the question: Does the LETCC hold for generalized theta graphs?

In order to build up to the generalized theta graph, in Section 2 we study list equitable total coloring of subdivisions of stars. We say that $H$ is a *subdivision of* $G$ if $H$ is a graph obtained from $G$ by replacing the edges of $G$ with internally disjoint paths. In Section 2, we prove the following theorem.

**Theorem 6.** *Suppose $G$ is a subdivision of $K_{1,m}$. If $m = 1$, then $T(G)$ is equitably $k$-choosable whenever $k \geq 3$. Otherwise $T(G)$ is equitably $k$-choosable whenever $k \geq m+1$.*

Suppose $G$ is a subdivision of $K_{1,m}$. It is worth noting that Theorem 6 is the best result possible since $\chi_\ell(T(G)) \geq \max\{3, m+1\}$. Also, the result of Theorem 6 is saying something stronger than: the LETCC holds for subdivisions of stars. This is because the LETCC only says that $T(G)$ should be equitably $k$-choosable for each $k \geq m+2$.

Finally, in Section 3 we prove the following.

**Theorem 7.** *Suppose $G = \Theta(l_1, \ldots, l_m)$, then $T(G)$ is equitably $k$-choosable whenever $k \geq m+2$.*

So, the LETCC holds for generalized theta graphs. Notice that in Theorem 7, $T(G)$ is a path square and cycle square when $m$ is 1 and 2 respectively. Since path squares with at least 3 vertices are not equitably 2-choosable, and all cycle squares with order not divisible by 3 are not equitably 3-choosable (see [18] for further details), one can see that in the case of $m = 1, 2$, $T(G)$ may not be equitably $(m+1)$-choosable. The question of whether $T(G)$ is equitably $(m+1)$-choosable when $m \geq 3$ is open.

**Question 8.** *Suppose $G = \Theta(l_1, \ldots, l_m)$. If $m \geq 3$, does it follow that $T(G)$ is equitably $(m+1)$-choosable?*

## 2. Subdivisions of Stars

In this section, we prove Theorem 6. Suppose that $m \in \mathbb{N}$ and $l_1, \ldots, l_m \in \mathbb{N}$ satisfy $l_1 \leq \cdots \leq l_m$. Then, we use $B(l_1, \ldots, l_m)$ to denote the equivalence class of subdivisions of $K_{1,m}$ where the edges of $K_{1,m}$ have been replaced with internally disjoint paths of lengths: $l_1, \ldots, l_m$.[3]

For the remainder of this paper when $G = B(l_1, \ldots, l_m)$, we assume that $V(G) = \{u\} \cup \{v_{i,j} : i \in [m], j \in [l_i]\}$, and the edges of $G$ are drawn so that for

[3]For the remainder of this paper, whenever we see $B(l_1, \ldots, l_m)$, we will always assume $m, l_1, \ldots, l_m \in \mathbb{N}$ with $l_1 \leq \cdots \leq l_m$.

each $i \in [m]$ vertices are adjacent if and only if they appear consecutively in the ordering: $u, v_{i,1}, \ldots, v_{i,l_i}$.

Note that when $G = B(l_1, \ldots, l_m)$, $T(G)$ is isomorphic to a copy of $[B(2l_1, \ldots, 2l_m)]^2$. So, in order to prove Theorem 6, we begin by proving the following result which will imply Theorem 6 for each $m \geq 3$.

**Theorem 9.** *For $m \geq 3$, $(B(l_1, \ldots, l_m))^2$ is equitably $k$-choosable for each $k \geq m+1$.*

Notice that for $m \geq 3$ Theorem 9 is saying something stronger than Theorem 6 since we are allowing any of the natural numbers $l_1, \ldots, l_m$ to be odd. We now prove a Lemma that is closely related to a Lemma appearing in [20]; we use this Lemma frequently to prove our results.

**Lemma 10.** *Let $G$ be a graph and let $L$ be a $k$-assignment for $G$. Suppose that $|V(G)| = kq + r$ where $1 \leq r \leq k$. Suppose $t$ satisfies $r \leq t \leq k$. Let $S = \{x_1, \ldots, x_t\}$ be a set of $t$ distinct vertices in $G$. If $G - S$ has an equitable $L$-coloring and*

$$|N_G(x_i) - S| \leq k - i$$

*for each $i \in [t]$, then $G$ has an equitable $L$-coloring.*

***Proof.*** Suppose that $f$ is an equitable $L$-coloring of $G - S$ (notice $G - S$ could be the empty graph). Note that no color is used more than $q$ times by $f$. In an equitable $L$-coloring of $G$, we must use no color more than $\lceil (kq + r)/k \rceil = q + 1$ times. Let $L'(x_i) = L(x_i) - \{f(v) : v \in N_G(x_i) - S\}$ for each $i \in [t]$. Since $|N_G(x_i) - S| \leq k - i$, we know that $|L'(x_i)| \geq i$. So, there is a proper $L'$-coloring of $G[S]$ that uses $t$ distinct colors. Such a coloring along with $f$ completes an equitable $L$-coloring of $G$. ■

We will now prove five lemmas that will imply Theorem 9. The first two of the five lemmas will take care of the case where $k \geq m + 2$, and the last three of the five lemmas will deal with $k = m + 1$.

**Lemma 11.** *Suppose $m \geq 3$. If $H = B(l_1, l_2, \ldots, l_m)$ and $G = H^2$, then $G$ is equitably $k$-choosable whenever $k \geq m + 3$.*

***Proof.*** The result is obvious when $k \geq |V(G)| = 1 + \sum_{i=1}^{m} l_i$. So, we may assume that $L$ is an arbitrary $k$-assignment for $G$ such that $m + 3 \leq k < 1 + \sum_{i=1}^{m} l_i$. We will show that $G$ is equitably $L$-colorable.

Since $1 + \sum_{i=1}^{m} l_i > m + 3$, $l_m > 1$. Let $S_0 = \{v_{i,1} : i \in [m]\} \cup A$ where $A = \{u, v_{m,2}\}$ if $v_{m,3} \notin V(G)$ (i.e. $l_m = 2$) and $A = \{u, v_{m,2}, v_{m,3}\}$ otherwise. Note $m + 2 \leq |S_0| \leq m + 3$. Let $d = k - |S_0|$. Let $S_1$ be an arbitrary subset of $V(G) - S_0$ of size $d$, and let $S = S_0 \cup S_1$. Note that $G - S$ is a graph with maximum degree at most 4. By Theorem 4, there is an equitable $L$-coloring of $G - S$.

Now, let $x_1 = u$, $x_{k-3} = v_{m-2,1}$, $x_{k-2} = v_{m-1,1}$, $x_{k-1} = v_{m,2}$, and $x_k = v_{m,1}$. We then arbitrarily name the remaining vertices in $S$: $x_2, \dots, x_{k-4}$ in an injective fashion. By the way $S$ is constructed, $|N_G(x_{k-i}) - S| \leq 2$ for $i = 2, 3$, $|N_G(x_{k-1}) - S| \leq 1$, and $|N_G(x_k) - S| = 0$. Moreover, $|N_G(x_1) - S| \leq m - 1 \leq (m + 3) - 1 \leq k - 1$. Finally, for $2 \leq i \leq k - 4$, $|N_G(x_i) - S| \leq 4 \leq k - i$. So, Lemma 10 implies that $G$ is equitably $L$-colorable. ■

**Lemma 12.** *Suppose $m \geq 3$. If $H = B(l_1, l_2, \dots, l_m)$ and $G = H^2$, then $G$ is equitably $(m + 2)$-choosable.*

***Proof.*** Note that the result is obvious when $l_m = 1$ since $G$ is a copy of $K_{m+1}$ when $l_m = 1$. So, we assume that $l_m > 1$. Let $L$ be an arbitrary $(m + 2)$-assignment for $G$. We will show that $G$ is equitably $L$-colorable in the following cases: $l_m = 2$, $l_m = 3$, and $l_m \geq 4$.

In the case $l_m = 2$, let $S = \{v_{i,1} : i \in [m]\} \cup \{u, v_{m,2}\}$. We name the vertices of $S$ as $x_1, x_2, \dots, x_{m+2}$ where $x_1 = u$, $x_{m+2} = v_{m,2}$, and for each $j \in [m]$, $x_{j+1} = v_{j,1}$. By Theorem 4, $G - S$ is equitably $L$-colorable. It is easy to see that $|N_G(x_1) - S| \leq m - 1 \leq (m + 2) - 1$, $|N_G(x_{m+2}) - S| = 0$, $|N_G(x_{m+1}) - S| = 0$, and for each $2 \leq j \leq m$, $|N_G(x_j) - S| \leq 1 \leq m + 2 - j$. Thus, $G$ is equitably $L$-colorable by Lemma 10.

In the case $l_m = 3$, let $S = \{v_{i,1} : 2 \leq i \leq m\} \cup \{u, v_{m,2}, v_{m,3}\}$. We name the vertices of $S$ as $x_1, x_2, \dots, x_{m+2}$ where $x_1 = u$, $x_{m+2} = v_{m,3}$, $x_{m+1} = v_{m,2}$, and for each $2 \leq i \leq m$, $x_i = v_{i,1}$. By Theorem 4, $G - S$ is equitably $L$-colorable. It is easy to see that $|N_G(x_1) - S| \leq m \leq (m + 2) - 1$, $|N_G(x_{m+2}) - S| = 0$, $|N_G(x_{m+1}) - S| = 0$, $|N_G(x_m) - S| = 1$, and for each $2 \leq j \leq m-1$, $|N_G(x_j) - S| \leq 3 \leq m + 2 - j$. Thus, $G$ is equitably $L$-colorable by Lemma 10.

In the case $l_m \geq 4$, let $S = \{v_{i,1} : 3 \leq i \leq m\} \cup \{u, v_{m,2}, v_{m,3}, v_{m,4}\}$. We name the vertices of $S$ as $x_1, x_2, \dots, x_{m+2}$ where $x_1 = u$, $x_{m+2} = v_{m,2}$, $x_{m+1} = v_{m,3}$, $x_m = v_{m,4}$, and $x_{m-1} = v_{m,1}$. Finally, if $m \geq 4$, then for each $2 \leq j \leq m - 2$ we let $x_j = v_{j+1,1}$. Notice $G - S$ has maximum degree at most 4. So, by Theorem 4, $G - S$ is equitably $L$-colorable. It is easy to see that $|N_G(x_1) - S| \leq m + 1 \leq (m + 2) - 1$, $|N_G(x_{m+2}) - S| = 0$, $|N_G(x_{m+1}) - S| \leq 1$, $|N_G(x_m) - S| \leq 2$, and $|N_G(x_{m-1}) - S| = 2$. Finally, if $m \geq 4$ then for each $2 \leq j \leq m - 2$, $|N_G(x_j) - S| \leq 4 \leq m + 2 - j$. Thus, $G$ is equitably $L$-colorable by Lemma 10. ■

We now turn our attention to the case of $k = m + 1$. Notice that in the case when $m = 3$, when we try to use Lemma 10, we will no longer be able to use Theorem 4 to show $G - S$ is equitably $L$-colorable. So, we need a result from [18].

**Proposition 13** [18]. *For $p, n \in \mathbb{N}$, $P_n^p$ is equitably $k$-choosable whenever $k \geq p+1$.*

Notice that Proposition 13 immediately implies that if $G$ is a spanning subgraph of a path square, then $G$ is equitably $k$-choosable whenever $k \geq 3$.

**Lemma 14.** *Suppose $m \geq 3$. If $H = B(l_1, l_2, \ldots, l_m)$, $G = H^2$, and $l_1 \leq l_2 \leq 2$, then $G$ is equitably $(m+1)$-choosable.*

***Proof.*** We may assume that $|V(G)| > m+1$. Suppose $L$ is an arbitrary $(m+1)$-assignment for $G$. In the case that $l_2 = 1$, let $S = \{u, v_{1,1}, v_{2,1}, \ldots, v_{m,1}\}$. Clearly $G - S$ has an equitable $L$-coloring by Proposition 13. Let $x_1 = u$, and let $x_{i+1} = v_{m+1-i,1}$ for each $i \in [m]$. Note that $|N_G(x_1) - S| \leq m-2 \leq (m+1)-1$, $|N_G(x_i) - S| \leq 2 \leq m+1-i$ for all $2 \leq i \leq m-1$, and $|N_G(x_i) - S| = 0 \leq m+1-i$ when $i = m, m+1$. By Lemma 10, we know that $G$ has an equitable $L$-coloring.

In the case that $l_2 = 2$, let $S = \{u, v_{1,1}, v_{2,1}, \ldots, v_{m-1,1}, v_{2,2}\}$. Clearly $G - S$ has an equitable $L$-coloring by Proposition 13. Let $x_1 = u$, $x_{m-1} = v_{1,1}$, $x_m = v_{2,1}$, and $x_{m+1} = v_{2,2}$. If $m \geq 4$, let $x_{i+1} = v_{m-i,1}$ for each $i \in [m-3]$. Note: $|N_G(x_1) - S| \leq m+1-1$, $|N_G(x_{m-1}) - S| \leq 2$, $|N_G(x_m) - S| = 1$, and $|N_G(x_{m+1}) - S| = 0$. Furthermore, if $m \geq 4$, for each $2 \leq i \leq m-2$, $|N_G(x_i) - S| \leq 3 \leq m+1-i$. By Lemma 10, we know that $G$ has an equitable $L$-coloring. ■

We now would like to prove for $m \geq 3$ that if $H = B(l_1, l_2, \ldots, l_m)$ and $G = H^2$, then $G$ is equitably $(m+1)$-choosable by induction on $\sum_{i=1}^m l_i$. Lemma 14 takes care of the base case. The next lemma takes care of a small issue in the inductive step when $m = 3$.

**Lemma 15.** *If $H = B(1,3,3)$ and $G = H^2$, then $G$ is equitably 4-choosable.*

***Proof.*** Suppose that $L$ is an arbitrary 4-assignment for $G$. Let $S = \{v_{2,3}, v_{3,3}, v_{3,2}, v_{3,1}\}$. Note that $G - S$ is the square of a path. So, by Proposition 13 we know that $G - S$ has an equitable $L$-coloring. We then let $x_1 = v_{3,1}$, $x_2 = v_{2,3}$, $x_3 = v_{3,2}$, and $x_4 = v_{3,3}$. Note that $|N_G(x_i) - S| = 4 - i$ for all $i \in [4]$. So, by Lemma 10, we know that $G$ has an equitable $L$-coloring. ■

We are finally ready to complete our proof of Theorem 9.

**Lemma 16.** *Suppose that $m \geq 3$. If $H = B(l_1, l_2, \ldots, l_m)$ and $G = H^2$, then $G$ is equitably $(m+1)$-choosable.*

***Proof.*** We will prove the desired by induction on $\sum_{i=1}^m l_i = |V(G)| - 1$. Let $C = \sum_{i=1}^m l_i$. Let $L$ be an arbitrary $(m+1)$-assignment for $G$. We will show an equitable $L$-coloring of $G$ exists for each $C \geq m$. For the base case suppose that

$m \leq C \leq 3m-3$. Since $C \leq 3m-3$, we know that $l_1 \leq l_2 \leq 2$. So, the desired result holds by Lemma 14.

For the inductive step suppose that $C \geq 3m-2$ and assume that the desired result holds for all natural numbers less than $C$ and at least $m$. Note that when $l_2 \leq 2$ the result holds by Lemma 14; so, we may assume that $l_2 \geq 3$.

If $l_m \geq 4$ we let $S = \{v_{j,l_j} : 2 \leq j \leq m-1\} \cup \{v_{m,l_m-2}, v_{m,l_m-1}, v_{m,l_m}\}$. By the inductive hypothesis, $G - S$ is equitably $L$-colorable. Now let $x_i = v_{i+1,l_{i+1}}$ for all $i \in [m-2]$, $x_{m-1} = v_{m,l_m-2}$, $x_m = v_{m,l_m-1}$, and $x_{m+1} = v_{m,l_m}$. Note that $|N_G(x_i) - S| = 2 \leq (m+1) - i$ for all $i \in [m-1]$, $|N_G(x_m) - S| = 1$, and $|N_G(x_{m+1}) - S| = 0$. Thus, Lemma 10 implies that $G$ is equitably $L$-colorable.

Now, assume that $l_m \leq 3$ (note that this means $l_m = 3$ since $l_2 \geq 3$). This implies that $3m-2 \leq C \leq 3m$. Let $d = C-(2m+1)$. In the case that $C = 3m-2$, we may assume that $m \geq 4$ by Lemma 15. Note that $m-3 \leq d \leq m-1$ and that $|V(G)| - d = C + 1 - d = 2m + 2$. We let $S = \{v_{2,l_2}, \ldots, v_{1+d,l_{1+d}}\}$. By the inductive hypothesis, we know that $G - S$ has an equitable $L$-coloring. Let $x_i = v_{i+1,l_{i+1}}$ for all $i \in [d]$. Note that $|N_G(x_i) - S| = 2 \leq (m+1) - i$ for all $i \in [d]$. Lemma 10 then implies that $G$ has an equitable $L$-coloring. The induction step is now complete. ■

Finally, notice the result of Theorem 6 is implied by Theorem 9 when $m \geq 3$, and the result of Theorem 6 is implied by Proposition 13 when $m = 1, 2$.

## 3. Generalized Theta Graphs

In this Section, we will prove Theorem 7. Throughout this section if $G = \Theta(l_1, \ldots, l_m)$, we will assume that the vertices that are the common endpoints in $V(G)$ are $u$ and $w$. We also let the vertices of the $i$th path be[4]:

$$u, v_{i,1}, \ldots, v_{i,l_i-1}, w.$$

When it comes to proving Theorem 7, it is crucial to note that if $G = \Theta(l_1, \ldots, l_m)$, then $T(G)$ is a copy of $[\Theta(2l_1, \ldots, 2l_m)]^2$ where $2l_2 \geq 4$ whenever $m \geq 2$. So, the results in this Section will focus upon the equitable choosability of the squares of generalized theta graphs with sufficiently long paths. Notice that if $G = \Theta(l_1, \ldots, l_m)$, $T(G)$ is a path square and cycle square on at least 6 vertices when $m$ is 1 and 2 respectively. So, when $m = 1, 2$ the result of Theorem 7 is implied by Proposition 13 and the following result.

**Proposition 17** [18]. *Suppose that $p, n \in \mathbb{N}$ with $p \geq 2$ and $n \geq 2p+2$. Then, $C_n^p$ is equitably $k$-choosable for each $k \geq 2p$.*

[4]Notice that if $l_1 = 1$, the first path has no internal vertices.

So, to complete the proof of Theorem 7, we may focus our attention on the case where $m \geq 3$. We begin by proving the following result.

**Theorem 18.** *For $m \geq 3$, $l_1 \geq 2$, and $l_2 \geq 4$, $[\Theta(l_1, l_2, \ldots, l_m)]^2$ is equitably $k$-choosable whenever $k \geq m + 3$.*

We will establish two lemmas that will immediately imply Theorem 18.

**Lemma 19.** *Suppose that $m \geq 3$. Suppose $H = \Theta(l_1, l_2, \ldots, l_m)$ where $l_1 \geq 2$ and $l_2 \geq 4$. If $G = H^2$, then $G$ is equitably $k$-choosable whenever $k \geq 2m + 2$.*

***Proof.*** The result is obvious when $k \geq |V(G)|$; thus, we will assume that $2m + 2 \leq k < |V(G)| = 2 + \sum_{i=1}^{m}(l_i - 1)$. Suppose $L$ is an arbitrary $k$-assignment for $G$. Let

$$S_0 = \{u, w, v_{m,2}, v_{m,3}\} \cup \{v_{i,1} : 1 \leq i \leq m\} \cup \{v_{i,l_i-1} : 3 \leq i \leq m\}.$$

Note that $G - S_0$ is a spanning subgraph of a disjoint union of path squares. Let $d = k - |S_0|$. Let $S_1$ be an arbitrary subset of $V(G) - S_0$ of size $d$. Then let $S = S_0 \cup S_1$. Note that $G - S$ has maximum degree at most 4. By Theorem 4, there is an equitable $L$-coloring of $G - S$. We will name the vertices of $S$: $x_1, x_2, \ldots, x_k$ such that $x_1 = w$, $x_2 = u$, $x_{k-3} = v_{2,1}$, $x_{k-2} = v_{m,3}$, $x_{k-1} = v_{m,2}$, $x_k = v_{m,1}$. We then arbitrarily name the remaining vertices in $S$: $x_3, x_4, \ldots, x_{k-4}$ in an injective fashion. For $i \in \{k-3, k-2, k-1, k\}$ we have $|N_G(x_i) - S| \leq k - i$. For $3 \leq i \leq k-4$ we have that $|N_G(x_i) - S| \leq 4 \leq k - i$. Finally, $|N_G(x_2) - S| \leq m - 1 \leq k - 2$ and $|N_G(x_1) - S| \leq m + 2 \leq k - 1$. By Lemma 10, there is an equitable $L$-coloring for $G$. ∎

**Lemma 20.** *Suppose that $m \geq 3$. Suppose $H = \Theta(l_1, l_2, \ldots, l_m)$ where $l_1 \geq 2$ and $l_2 \geq 4$. If $G = H^2$, then $G$ is equitably $k$-choosable whenever $m + 3 \leq k \leq 2m + 1$.*

***Proof.*** Suppose that $L$ is an arbitrary $k$-assignment for $G$ such that $m + 3 \leq k \leq 2m + 1$. We let $S = \{u, v_{m,3}\} \cup \{v_{i,1} : i \in [m]\} \cup \{v_{i,2} : 2m + 3 - k \leq i \leq m\}$. Note that $|S| = k$. Note that there is an $r \in \{m-2, m-1, m\}$ and natural numbers $a_1, \ldots, a_r$ with $a_1 \leq \cdots \leq a_r$ such that $G - S$ is isomorphic to $[B(a_1, a_2, \ldots, a_r)]^2$. When $r \leq 2$, we know that $G - S$ has an equitable $L$-coloring by Theorem 4 since $\Delta(G - S) \leq 4$. When $r \geq 3$, we know that $G - S$ has an equitable $L$-coloring by Lemma 11. Let $x_1 = v_{m,3}$, $x_2 = v_{1,1}$, $x_3 = u$, $x_{k-1} = v_{m,2}$, and $x_k = v_{m,1}$. Finally, we arbitrarily name the remaining vertices in $S$: $x_3, x_4, \ldots, x_{k-2}$ in an injective fashion. Note that $|N_G(x_1) - S| \leq m \leq k - 1$, $|N_G(x_2) - S| \leq m \leq k - 2$, $|N_G(x_3) - S| \leq m - 1 \leq k - 3$, $|N_G(x_{k-1}) - S| = 1$, and $|N_G(x_k) - S| = 0$. Finally note that $|N_G(x_i) - S| \leq 2 \leq k - i$ for all $4 \leq i \leq k - 2$. So by Lemma 10 we know that $G$ has an equitable $L$-coloring. ∎

We are now finished with the proof of Theorem 18. To complete the proof of Theorem 7 we must show that if $G = \Theta(l_1, l_2, \ldots, l_m)$, then $T(G)$ is equitably $k$-choosable when $m \geq 3$ and $k = m + 2$. Recall for $m \geq 3$ that if $G = \Theta(l_1, l_2, \ldots, l_m)$, then $T(G)$ is a copy of $[\Theta(2l_1, 2l_2, \ldots, 2l_m)]^2$ where $2l_2 \geq 4$. So, we begin working on the case of $k = m + 2$ by dealing with $[\Theta(2, 4, \ldots, 4)]^2$ and $[\Theta(4, \ldots, 4)]^2$. We will then finish the case of $k = m + 2$ by considering $[\Theta(l_1, l_2, \ldots, l_m)]^2$ with $l_1 \geq 2$, $l_2 \geq 4$, and $l_m \geq 6$.

We begin with two specific cases to which our general arguments do not apply.

**Lemma 21.** *Suppose $H = \Theta(2, 4, 4)$ and $G = H^2$. Then, $G$ is equitably 5-choosable.*

***Proof.*** Suppose $L$ is an arbitrary 5-assignment for $G$. We will show that $G$ has an equitable $L$-coloring in two cases: (1) $L(v)$ is the same list for each $v \in V(G)$ and (2) there exist $x, y \in V(G)$ such that $L(x) \neq L(y)$. In case (1) we may suppose $L(v) = \{1, 2, 3, 4, 5\}$ for each $v \in V(G)$. We let $f$ be the proper $L$-coloring of $G$ defined as follows: $f(u) = 1$, $f(v_{1,1}) = 2$, $f(v_{2,1}) = 3$, $f(v_{3,1}) = 4$, $f(v_{2,2}) = 2$, $f(v_{3,2}) = 3$, $f(v_{2,3}) = 4$, $f(v_{3,3}) = 1$, and $f(w) = 5$. Clearly, $f$ is an equitable $L$-coloring of $G$.

We now turn our attention to case (2). Let $S_1 = \{v_{2,1}, v_{2,2}, v_{2,3}\}$ and $S_2 = \{v_{3,1}, v_{3,2}, v_{3,3}\}$. Note that $(V(G) - S_1) \cup (V(G) - S_2) = V(G)$ and $(V(G) - S_1) \cap (V(G) - S_2) \neq \emptyset$. So, it must be the case that there exists at least two lists in either $\{L(v) : v \in V(G) - S_1\}$ or $\{L(v) : v \in V(G) - S_2\}$. We assume without loss of generality that there are at least two lists in $\{L(v) : v \in V(G) - S_1\}$. It is then possible to find a proper $L$-coloring, $f$, of $G - S_1$ that uses six distinct colors. Let $L'(v_{2,i}) = L(v_{2,i}) - \{f(v) : v \in N_G(v_{3,i}) - S_1\}$ for each $i \in [3]$. Note that $|L'(v_{2,1})| \geq 2$, $|L'(v_{2,2})| \geq 3$, and $|L'(v_{2,3})| \geq 2$. So, there is a proper $L'$-coloring of $G[S_1]$. Such a coloring completes an equitable $L$-coloring of $G$. ■

**Lemma 22.** *Suppose $H = \Theta(2, 4, 4, 4)$ and $G = H^2$. Then, $G$ is equitably 6-choosable.*

***Proof.*** Suppose $L$ is an arbitrary 6-assignment for $G$. Note an equitable $L$-coloring of $G$ uses no color more than twice. We will show that $G$ has an equitable $L$-coloring in two cases: (1) $L(v)$ is the same list for each $v \in V(G) - \{u, w\}$ and (2) there exist $x, y \in V(G) - \{u, w\}$ such that $L(x) \neq L(y)$. In case (1) we may suppose $L(v) = \{1, 2, 3, 4, 5, 6\}$ for each $v \in V(G) - \{u, w\}$. Let $f$ be the proper $L$-coloring for $G - \{u, w\}$ given by: $f(v_{1,1}) = 2$, $f(v_{2,1}) = 3$, $f(v_{3,1}) = 4$, $f(v_{4,1}) = 5$, $f(v_{2,2}) = 2$, $f(v_{3,2}) = 6$, $f(v_{4,2}) = 6$, $f(v_{2,3}) = 1$, $f(v_{3,3}) = 5$, and $f(v_{4,3}) = 4$. Then, let $L'(u) = L(u) - \{2, 3, 4, 5, 6\}$ and $L'(w) = L(w) - \{1, 2, 4, 5, 6\}$. As long as $L'(u)$ and $L'(w)$ are not the same set of size 1, we can find a proper $L'$-coloring of $G[\{u, w\}]$ which along with $f$ completes an equitable $L$-coloring of $G$.

So, we assume that $L'(w) = L'(u) = \{7\}$. This means $L(u) = \{2, 3, 4, 5, 6, 7\}$ and $L(w) = \{1, 2, 4, 5, 6, 7\}$. An equitable $L$ coloring, $g$, of $G$ can then be constructed as follows: $g(u) = 7$, $g(v_{1,1}) = 2$, $g(v_{2,1}) = 3$, $g(v_{3,1}) = 4$, $g(v_{4,1}) = 5$, $g(v_{2,2}) = 2$, $g(v_{3,2}) = 3$, $g(v_{4,2}) = 6$, $g(v_{2,3}) = 6$, $g(v_{3,3}) = 5$, $g(v_{4,3}) = 4$, and $g(w) = 1$.

For case (2), let $S_1 = \{u, w, v_{4,1}, v_{4,2}, v_{3,2}\}$, $S_2 = \{u, w, v_{3,1}, v_{3,2}, v_{2,2}\}$, and $S_3 = \{u, w, v_{2,1}, v_{2,2}, v_{4,2}\}$. Note that $(V(G) - S_1) \cup (V(G) - S_2) \cup (V(G) - S_3) = V(G) - \{u, w\}$ and $\bigcap_{i=1}^{3} (V(G) - S_i) \neq \emptyset$. Thus, there must exist at least two lists in $\{L(v) : v \in V(G) - S_1\}$, $\{L(v) : v \in V(G) - S_2\}$, or $\{L(v) : v \in V(G) - S_3\}$. Assume without loss of generality that $\{L(v) : v \in V(G) - S_1\}$ contains at least two lists. It is possible to find a proper $L$-coloring, $h$, of $G - S_1$ that uses seven distinct colors. Let $L'(v) = L(v) - \{h(x) : x \in N_G(v) - S_1\}$ for each $v \in S_1$. Note that $|L'(u)| \geq 2$, $|L'(w)| \geq 1$, $|L'(v_{4,1})| \geq 2$, $|L'(v_{3,2})| \geq 4$, $|L'(v_{4,2})| \geq 5$. Now, we greedily construct a proper $L'$-coloring of $G[S_1]$, $h'$, that uses at least 4 distinct colors by coloring the vertices of $S_1$ in the order: $w, u, v_{4,1}, v_{3,2}, v_{4,2}$ (this is possible since $v_{4,1}$ is not adjacent to $w$ in $G$). In the case $|h'(S_1)| = 5$, $h$ together with $h'$ forms an equitable $L$-coloring of $G$. So, we suppose that $|h'(S_1)| = 4$. By the way $h'$ is constructed, it must be that $h'(w) = h'(v_{4,1}) = c$.

Now, for the sake of contradiction, we suppose that $h$ together with $h'$ is not an equitable $L$-coloring of $G$. This means that $c$ was used by $h$. Note that $V(G) - S_1 \subseteq N_G(w) \cup N_G(v_{4,1})$. So, by the definition of $L'$, $c \notin L'(w)$ or $c \notin L'(v_{4,1})$ which is a contradiction. Thus $h$ along with $h'$ forms an equitable $L$-coloring of $G$. ■

We are now ready to prove two Lemmas that will complete the case of $k = m + 2$ for $[\Theta(2, 4, \ldots, 4)]^2$ and $[\Theta(4, \ldots, 4)]^2$.

**Lemma 23.** *Suppose $m \geq 5$. Suppose $H = \Theta(l_1, l_2, \ldots, l_m)$ where $l_1 = 2$, and $l_2 = l_m = 4$. If $G = H^2$, then $G$ is equitably $(m + 2)$-choosable.*

***Proof.*** Suppose that $L$ is an arbitrary $(m + 2)$-assignment for $G$. We will construct an equitable $L$-coloring of $G$. Since $m \geq 5$, in an equitable $L$-coloring of $G$, no color can be used more than $\lceil 3m/(m + 2) \rceil = 3$ times. Let $S_1 = \{u\} \cup \{v_{i,1} : i \in [m]\}$, $S_2 = \{w\} \cup \{v_{i,3} : 2 \leq i \leq m\}$, and $S_3 = \{v_{i,2} : 2 \leq i \leq m\}$. Note that $|S_1| = m + 1$, $|S_2| = m$, and $|S_3| = m - 1$. We begin by coloring the vertices in $S_1$ with $m + 1$ distinct colors. For each $v \in S_2$ suppose that $L'(v)$ is obtained from $L(v)$ by deleting the colors used on the neighbors of $v$ in the coloring of the vertices in $S_1$. Note that $|L'(v)| \geq m$ for each $v \in S_2$. So, we can color each $v \in S_2$ with a color $c \in L'(v)$ such that the vertices in $S_2$ are colored with $m$ distinct colors.

Now, for each $v \in S_1 \cup S_2$ let $f(v)$ be the color used to color $v$. Note that $f$ uses at most $m$ colors twice, and $f$ uses no color more than twice. For each $v \in S_3$ let $L''(v) = L(v) - \{f(x) : x \in (S_1 \cup S_2) \cap N_G(v)\}$. Since each vertex in $S_3$ has degree 4 in $G$, $|L''(v)| \geq m - 2$ for each $v \in S_3$.

If it is not the case that: $|L''(v)| = m-2$ for all $v \in S_3$ and $L''(v_{2,2}) = L''(v_{3,2}) = \cdots = L''(v_{m,2})$, then it is clear that there exists a proper $L''$-coloring of $G[S_3]$ that uses $m-1$ distinct colors which completes an equitable $L$-coloring of $G$. So, we assume that $|L''(v)| = m-2$ for all $v \in S_3$ and $L''(v_{2,2}) = L''(v_{3,2}) = \cdots = L''(v_{m,2})$. Let $A = L''(v_{2,2})$. For the sake of contradiction, we assume that all colors in $A$ are used twice by $f$. Note that $f^{-1}(A) \subseteq S_1 \cup S_2$ and $|f^{-1}(A)| = 2|A| = 2m-4$. Since $|S_1 \cup S_2| = 2m+1$, there are at most 5 vertices in $(S_1 \cup S_2) - f^{-1}(A)$. Note that

$$\left| \bigcup_{i=2}^{m} N_G(v_{i,2}) \right| = 2m \geq 10 > 5.$$

So, there must be a $z \in S_1 \cup S_2$ that is in $f^{-1}(A) \cap \bigcup_{i=2}^{m} N_G(v_{i,2})$. This however implies that for some $2 \leq i \leq m$, $f(z)$ was deleted from $L(v_{i,2})$ when forming $L''(v_{i,2})$ which implies $f(z) \notin L''(v_{i,2})$ which is a contradiction.

Thus, there must exist an element $a \in A$ that was not used twice by $f$. So, we can complete an equitable $L$-coloring of $G$ by coloring $v_{2,2}$ and $v_{3,2}$ with $a$ and coloring the remaining vertices in $S_3$ with the $m-3$ distinct colors in $A - \{a\}$. ■

**Lemma 24.** *Suppose $m \geq 3$. Suppose $H = \Theta(l_1, l_2, \ldots, l_m)$ where $l_1 = l_m = 4$. If $G = H^2$, then $G$ is equitably $(m+2)$-choosable.*

***Proof.*** Suppose that $L$ is an arbitrary $(m+2)$-assignment for $G$. We will construct an equitable $L$-coloring of $G$. Since $m \geq 3$, in an equitable $L$-coloring of $G$, no color can be used more than $\lceil (3m+2)/(m+2) \rceil = 3$ times. Let $S_1 = \{u, v_{1,2}\} \cup \{v_{i,1} : i \in [m]\}$, $S_2 = \{w\} \cup \{v_{i,3} : i \in [m]\}$, and $S_3 = \{v_{i,2} : 2 \leq i \leq m\}$. Note that $|S_1| = m+2$, $|S_2| = m+1$, and $|S_3| = m-1$. We begin by coloring the vertices in $S_1$ with $m+2$ distinct colors. For each $v \in S_2$ suppose that $L'(v)$ is obtained from $L(v)$ by deleting the colors used on the neighbors of $v$ in the coloring of the vertices in $S_1$. Note that $|L'(w)| \geq m+1$ and $|L'(v_{i,3})| \geq m$ for all $i \in [m]$. So, we can color each $v \in S_2$ with a color $c \in L'(v)$ such that the vertices in $S_2$ are colored with $m+1$ distinct colors.

Now, for each $v \in S_1 \cup S_2$ let $f(v)$ be the color used to color $v$. Note that $f$ uses at most $m+1$ colors twice, and $f$ uses no color more than twice. For each $v \in S_3$ suppose that $L''(v) = L(v) - \{f(x) : x \in (S_1 \cup S_2) \cap N_G(v)\}$. Since each vertex in $S_3$ has degree 4 in $G$, $|L''(v)| \geq m-2$ for each $v \in S_3$.

If it is not the case that: $|L''(v)| = m-2$ for all $v \in S_3$ and $L''(v_{2,2}) = L''(v_{3,2}) = \cdots = L''(v_{m,2})$, then it is clear that there exists a proper $L''$-coloring of $G[S_3]$ that uses $m-1$ distinct colors which completes an equitable $L$-coloring of $G$. So, we assume that $|L''(v)| = m-2$ for all $v \in S_3$ and $L''(v_{2,2}) = L''(v_{3,2}) = \cdots = L''(v_{m,2})$. Let $A = L''(v_{2,2})$. For the sake of contradiction, we assume that all colors in $A$ are used twice by $f$. Note that $f^{-1}(A) \subseteq S_1 \cup S_2$ and

$|f^{-1}(A)| = 2|A| = 2m - 4$. Since $|S_1 \cup S_2| = 2m + 3$, there are at most 7 vertices in $(S_1 \cup S_2) - f^{-1}(A)$. Note that

$$\left| \bigcup_{i=2}^{m} N_G(v_{i,2}) \right| = 2m.$$

So, when $m \geq 4$, there must be a $z \in S_1 \cup S_2$ that is in $f^{-1}(A) \cap \bigcup_{i=2}^{m} N_G(v_{i,2})$, and we reach a contradiction as we did in the proof of Lemma 23. When $m = 3$ note that since $\{v_{1,1},\, v_{1,2},\, v_{1,3}\}$ is a clique in $G$, the single element in $A$ must be in $f\big(\bigcup_{i=2}^{3} N_G(v_{i,2})\big)$. So, when $m = 3$ there must also be a $z \in S_1 \cup S_2$ that is in $f^{-1}(A) \cap \bigcup_{i=2}^{m} N_G(v_{i,2})$, and we reach a contradiction as we did in the proof of Lemma 23.

Finally, we can complete an equitable $L$-coloring of $G$ as we did in the proof of Lemma 23. ■

We now complete the proof of Theorem 7 with two lemmas. The next lemma will be important for proving the final lemma which will address all remaining cases needed for Theorem 7.

**Lemma 25.** *Suppose $m \geq 3$. Suppose $H = B(l_1, l_2, \ldots, l_m)$, where $l_m \geq 3$. Suppose $G' = H^2$. Let $G$ be the graph obtained from $G'$ by adding an extra edge between the vertices $v_{a,l_a}$ and $v_{b,l_b}$ where $a$ and $b$ are chosen so that $1 \leq a < b \leq m$. Then, $G$ is equitably $(m+2)$-choosable.*

Note that this lemma can be extended to $l_m = 2$, but this is not necessary to complete the proof of Theorem 7.

***Proof.*** Let $L$ be an arbitrary $(m+2)$-assignment for $G$. We will show that $G$ has an equitable $L$-coloring in the following cases: (1) $l_m = 3$ and (2) $l_m \geq 4$.

In the case $l_m = 3$, let $S = \{u\} \cup \{v_{i,1} : 2 \leq i \leq m\} \cup \{v_{m,2}, v_{m,3}\}$. Then, we name the vertices of $S$: $x_1, x_2, \ldots, x_{m+2}$ where $x_1 = u$, $x_{m+2} = v_{m,2}$, $x_{m+1} = v_{m,3}$, and for each $2 \leq i \leq m$, $x_i = v_{i,1}$. Note $\Delta(G - S) \leq 3$. So, by Theorem 4, $G - S$ is equitably $L$-colorable. It is easy to see that $|N_G(x_1) - S| = m \leq m+2-1$, $|N_G(x_{m+2}) - S| = 0$, $|N_G(x_{m+1}) - S| \leq 1$, $|N_G(x_m) - S| = 1$, and for each $2 \leq j \leq m-1$, $|N_G(x_j) - S| \leq 3 \leq m + 2 - j$. Thus $G$ is equitably $L$-colorable by Lemma 10.

For case (2), we will show that $G$ has an equitable $L$-coloring in the following subcases: (a) $m \geq 4$ and (b) $m = 3$. When $m \geq 4$, choose a $t$ such that $t \in [m]$, $t \neq a$, $t \neq b$, and $t \neq m$. Let

$$S = \{u, v_{m,2}, v_{m,3}, v_{m,4}\} \cup (\{v_{i,1} : i \in [m]\} - \{v_{a,1}, v_{t,1}\}).$$

Then we name the vertices of $S$: $x_1, x_2, \ldots, x_{m+2}$ where $x_1 = u$, $x_{m+2} = v_{m,2}$, $x_{m+1} = v_{m,3}$, $x_m = v_{m,4}$, and $x_{m-1} = v_{m,1}$. Finally, we arbitrarily name the remaining vertices in $S$: $x_2, \ldots, x_{m-2}$ in an injective fashion. Note $\Delta(G-S) \leq 4$, and so by Theorem 4, $G - S$ is equitably $L$-colorable. It is easy to see that $|N_G(x_1) - S| = m + 1 \leq m + 2 - 1$, $|N_G(x_{m+2}) - S| = 0$, $|N_G(x_{m+1}) - S| \leq 1$, $|N_G(x_m) - S| \leq 2$, and $|N_G(x_{m-1}) - S| = 2$. Finally, for each $2 \leq j \leq m - 2$, $|N_G(x_j) - S| \leq 4 \leq m + 2 - j$. Thus, $G$ is equitably $L$-colorable by Lemma 10.

For subcase (b), $m = 3$, and we let $S = \{u, v_{3,1}, v_{3,2}, v_{3,3}, v_{3,4}\}$. Note that $\Delta(G-S) \leq 4$. So, by Theorem 4 we know that $G-S$ has an equitable $L$-coloring. We name the vertices of $S$ as follows: $x_1 = u$, $x_2 = v_{3,1}$, $x_3 = v_{3,4}$, $x_4 = v_{3,3}$, and $x_5 = v_{3,2}$. Note $|N_G(x_1) - S| \leq 4$, $|N_G(x_2) - S| = 2$, $|N_G(x_3) - S| \leq 2$, $|N_G(x_4) - S| \leq 1$, and $|N_G(x_5) - S| = 0$. So, by Lemma 10 we know that $G$ has an equitable $L$-coloring. ■

**Lemma 26.** *Suppose $m \geq 3$. Suppose $H = \Theta(l_1, l_2, \ldots, l_m)$ where $l_1 \geq 2$, $l_2 \geq 4$, and $l_m \geq 6$. If $G = H^2$, then $G$ is equitably $(m+2)$-choosable.*

***Proof.*** Let $L$ be an arbitrary $(m+2)$-assignment for $G$. We will show that $G$ has an equitable $L$-coloring. We let

$$S = \{u, v_{m,2}, v_{m,3}, v_{m,4}\} \cup \{v_{i,1} : 3 \leq i \leq m\}.$$

There exist natural numbers $a_1, \ldots, a_m$ satisfying $a_1 \leq \cdots \leq a_m$ and $a_m \geq 3$ such that the following holds. If $H = [B(a_1, \ldots, a_m)]^2$, then there are $i, j \in [m]$ satisfying $1 \leq i < j \leq m$ such that $G - S$ is $H$ plus an edge between the vertices $v_{i,a_i}$ and $v_{j,a_j}$. Note that we know such an $i$ and $j$ exist since $v_{1,1}$ and $v_{2,1}$ are adjacent in $G - S$. Also note that we know $a_m \geq 3$ since $v_{2,1}$, $v_{2,2}$, and $v_{2,3}$ are in $G - S$. By Lemma 25 we know that $G - S$ has an equitable $L$-coloring. We name the vertices of $S$: $x_1, x_2, \ldots, x_{m+2}$ where $x_1 = u$, $x_{m-1} = v_{m,4}$, $x_m = v_{m,1}$, $x_{m+1} = v_{m,3}$ and $x_{m+2} = v_{m,2}$. Finally, if $m \geq 4$, we arbitrarily name the remaining vertices in $S$: $x_2, \ldots, x_{m-2}$ in an injective fashion. Note that $|N_G(x_1)-S| = m+1$, $|N_G(x_{m-1})-S| = 2$, $|N_G(x_m)-S| = 1$, $|N_G(x_{m+1})-S| = 1$, $|N_G(x_{m+2}) - S| = 0$, and $|N_G(x_i) - S| \leq 4 \leq m + 2 - i$ for all $2 \leq i \leq m - 2$. Thus, $G$ has an equitable $L$-coloring by Lemma 10. ■

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