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HIGH GIRTH HYPERGRAPHS WITH UNAVOIDABLE MONOCHROMATIC OR RAINBOW EDGES

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Abstract

A classical result of Erdős and Hajnal claims that for any integers $k, r, g \geq 2$ there is an r-uniform hypergraph of girth at least g with chromatic number at least k. This implies that there are sparse hypergraphs such that in any coloring of their vertices with at most $k-1$ colors there is a monochromatic hyperedge. When there is no restriction on the number of the colors used, one can easily avoid monochromatic hyperedges. Then, however, so-called rainbow or multicolored hyperedges might appear. Nešetřil and Rödl [19] called hypergraphs such that in any vertex-coloring there is either a monochromatic or a rainbow hyperedge, *selective*. They showed an existence of selective r-uniform hypergraphs of girth g for any integers $r, g \geq 2$ using probabilistic and explicit constructions. In this paper, we provide a slightly different construction of such hypergraphs and summarize the probabilistic approaches. The main building block of the construction, a part-rainbow-forced hypergraph, is of independent interest. This is an r-uniform r-partite hypergraph with a given girth such that in any vertex-coloring that is rainbow on each part, there is a rainbow hyperedge. We give a simple construction of such a hypergraph that does not use iterative amalgamation.

Keywords: hypergraph, chromatic number, mixed hypergraph, bihypergraphs, monochromatic, rainbow, girth, selective.

2010 Mathematics Subject Classification: 05C15, 05C35, 05C65.

1. Introduction

A classical result of Erdős and Hajnal [5], Corollary 13.4, claims that for any integers $k, r, g \geq 2$ there is an r-uniform hypergraph of girth at least g with chromatic number at least k. This implies that there are sparse hypergraphs such that in any coloring of their vertices with at most $k-1$ colors there is a monochromatic hyperedge. The original proof was probabilistic. Other probabilistic constructions were given by Nešetřil and Rödl [20], Duffus *et al.* [3], Kostochka and Rödl [11], and, in case of graphs only, by Erdős [4]. Several explicit constructions were found later, see Lovász [15], Erdős and Lovász [6], Nešetřil and Rödl [20], Duffus *et al.* [3], Alon *et al.* [1], Kříž [12], Kostochka and Nešetřil [10]. Nešetřil [18] as well as Raigorodskii and Shabanov [21] gave surveys on the topic. Some interesting generalizations and applications were treated by Feder and Vardi [8], Kun [13], Müller [16, 17], Kupavskii and Shabanov [14], as well as by Nešetřil [18].

When the number of colors used on the vertices of a hypergraph is not restricted, the monochromatic hyperedges could easily be avoided by simply using a lot of different colors. Then, however, so-called rainbow (totally multicolored) hyperedges could appear. The notion of a proper coloring when both rainbow and monochromatic hyperedges are forbidden was introduced by Erdős, Nešetřil and Rödl [7], see also [19]. In these papers, the authors called hypergraphs for which any vertex-coloring results in either monochromatic or rainbow hyperedge as *selective hypergraph.* Various aspects of these hypergraphs were addressed in these papers. Caro *et al.* [2] used the notion of selective hypergraph coloring in a different context, when the color distribution on the hyperedges satisfy a given partition pattern. Voloshin called hypergraphs that allow for a vertex-coloring in which every edge is neither monochromatic nor rainbow, *bihypergraphs*, [23], he also introduced a more general notion of *mixed hypergraphs*, see also Karrer [9].

One of the results of Nešetřil and Rödl [7] is that there are selective hypergraphs of arbitrarily high girth. Surprisingly, these fundamental results of Erdős, Nešetřil and Rödl [7, 19] do not have a single citation in Mathscinet. Here we provide explicit and probabilistic constructions quite similar to the ones of Nešetřil and Rödl. One of the key components of the explicit construction are so-called part-rainbow-forced hypergraphs. We give two new constructions of those, one of them is significantly smaller than in the original construction. An observation we make is that one could build such a hypergraph without iterative amalgamation simply from a hypergraph of high girth and the number of edges exceeding the number of vertices.

A cycle of length g in a hypergraph is a subhypergraph consisting of $g \geq 2$ distinct hyperedges $E_0, \dots, E_{g-1}$ and containing distinct vertices $x_0, \dots, x_{g-1}$, such that $x_i \in E_i \cap E_{i+1}$, $i = 0, \dots, g-1$, addition of indices modulo g. The *girth* of a hypergraph is the length of a shortest cycle if such exists, and infinity otherwise. Next is our main result, that is also a result from [19].

Theorem 1. *For any integers $r, g \geq 2$ there is an r-uniform hypergraph of girth at least g such that in any coloring of its vertices there is either a monochromatic or a rainbow* (*totally multicolored*) *edge.*

To shorten the presentation, we shall say that a hypergraph is *rm-unavoidable* if any coloring of its vertices has either a rainbow or a monochromatic edge. We give an explicit construction and use it to prove the main theorem in Section 2. The probabilistic proof is given in Section 3. The proofs of a few standard results we use are presented in Appendix.

2. Explicit Construction of rm-Unavoidable Hypergraphs

The goal of this section is to construct, for each $r \geq 2$ and $g \geq 2$, an rm-unavoidable hypergraph, that we shall call $H(r,g)$, of uniformity r and girth g. The three main concepts we use are amalgamation, special partite hypergraphs forcing rainbow edges, and so-called complete partite factors. All of these notions are defined for partite hypergraphs. A hypergraph is *a-partite* if its vertex set can be partitioned in at most a parts such that each hyperedge contains at most one vertex from each part. We shall first define a part-rainbow-forced hypergraph as a hypergraph having some special coloring properties and give an explicit construction of such a hypergraph $PR(r,g)$. Then we incorporate this hypergraph into a more involved construction of an rm-unavoidable hypergraph $H(r,g)$. Both of these constructions use amalgamation.

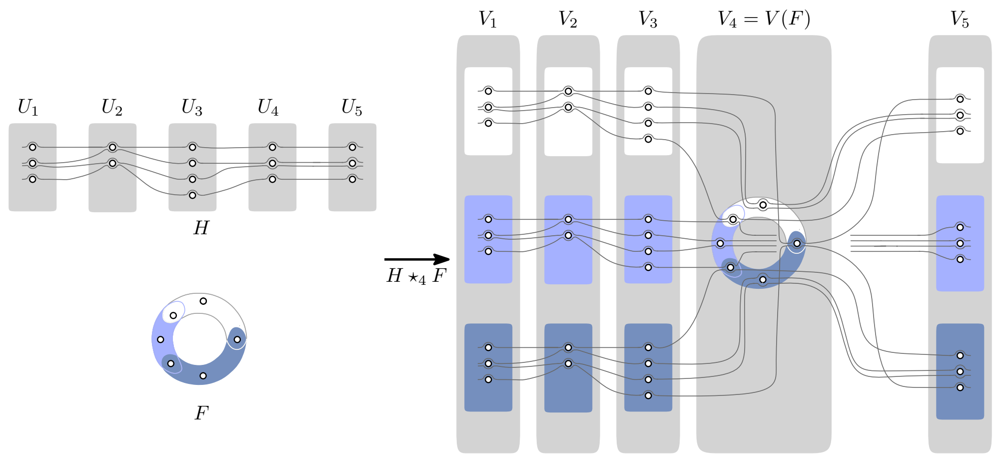

Figure 1. Amalgamation of F and H along the 4^{th} part. Here F is a 3-uniform cycle on 3 edges, H is 5-uniform, 5-partite with 4 edges. The resulting graph is 5-partite, 5-uniform, with curves indicating hyperedges and colors indicating distinct copies of H, corresponding to the edges of F.

Amalgamation

Given an a-partite hypergraph H with the i^{th} part of size r_i and given an r_i-uniform hypergraph $F = (V, \mathcal{E})$, an *amalgamation* of H and F along the i^{th} part, denoted by $H \star_i F$, is an a-partite hypergraph obtained by taking $|\mathcal{E}|$ vertex-disjoint copies of H and identifying the ith part of each such copy with a hyperedge of F such that distinct copies get identified with distinct hyperedges. Moreover, the j^{th} part of $H \star_i F$ is a pairwise disjoint union of the j^{th} parts from the copies of H, for $j \in \{1, \ldots, a\} \setminus \{i\}$, see Figure 1. We shall sometimes say that $H \star_i F$ is obtained by amalgamating copies of H along the part i using F.

Partite factor

Let F be an r-uniform r-partite hypergraph. A *complete a-partite F-factor* is an a-partite r-uniform hypergraph G that is a union of pairwise vertex-disjoint copies $F_1, \ldots, F_{\binom{a}{r}}$ of F, such that each part of F_i is contained in some part of G, $i = 1, \ldots, \binom{a}{r}$, and such that the union of any r parts of G contains the vertex set of F_i, for some $i = 1, \ldots, \binom{a}{r}$, see Figure 2.

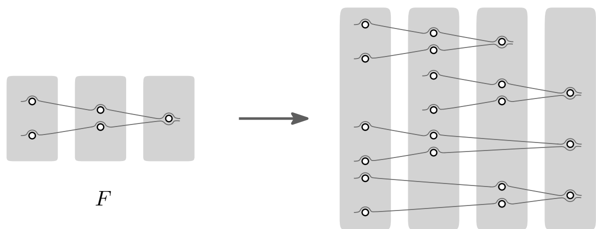

Figure 2. An example of a complete 4-partite F-factor, where F is a 3-partite 3-uniform hypergraph with two edges.

Part-rainbow-forced hypergraph

A vertex coloring of an a-partite hypergraph with parts $X_1, \ldots, X_a$ that assign $|X_i|$ colors to part i, for each $i = 1, \ldots, a$, is called *part rainbow*. We say that an a-partite hypergraph is *part-rainbow-forced* if in any part-rainbow coloring there is a rainbow edge.

The following constructions give part-rainbow-forced hypergraphs.

Construction of a hypergraph $PR(r, g)$

Let $r, g \geq 2$, $g \geq 2$ be fixed. Let $g \geq 2$, let $PR(2, g)$ be a bipartite graph on vertices x, y, z and edges xy, yz.

Assume now that $H_r = PR(r, g)$ has been constructed and it is an r-uniform, r-partite hypergraph, $r \geq 2$. Let F' be an ℓ-uniform hypergraph of girth at least g and minimum degree $\ell(r + 1)$, where $\ell = |E(H_r)|$. For completeness, we show the existence of F' in Appendix. It was shown by Sauer [22], see also a survey on related topics by Nešetřil [18].

For an r-uniform r-partite hypergraph H, let $\widetilde{H}$ be an $(r+1)$-partite $(r+1)$-uniform hypergraph that is obtained from H by expanding each of its edges by a vertex in a new, $(r + 1)^{\text{st}}$ part, such that each edge is extended by an own vertex, i.e., the size of the $(r + 1)^{\text{st}}$ part is equal to the number of edges in H, see Figure 3.

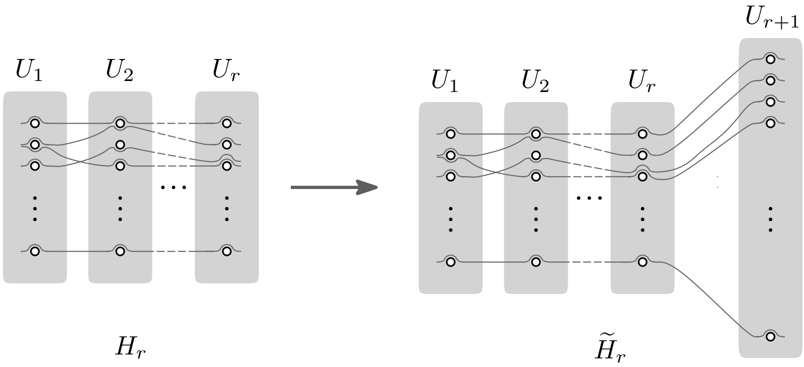

Figure 3. Extension of an r-partite r-uniform hypergraph H_r to an $(r+1)$-partite $(r+1)$-uniform hypergraph $\widetilde{H}_r$.

Let $PR(r+1,g) = \widetilde{PR(r,g)} \star_{r+1} F'$, i.e., it is an amalgamation of copies of $\widetilde{PR(r,g)}$ along the $(r+1)^{\text{st}}$ part using F', see Figure 4.

Next, we give an alternative construction.

Construction of a hypergraph $\boldsymbol{PR*(r,g)}$

Let $r,g \geq 2$, $g \geq 2$ be fixed. Let $g \geq 2$, let $PR(2,g)$ be a bipartite graph on vertices x,y,z and edges xy, yz.

Let $r=3$, $g \geq 2$ be fixed. Let $PR^*(3,g)$ have parts U_1,U_2,U_3. First consider G, a bipartite graph with parts U_1 and U_2 that is a union of two cycles of length at least g each, sharing a unique edge and no other vertices except for that edge's. We see that G has girth at least g and it has more edges than vertices. Let $PR^*(3,g)$ be obtained from G by extending each edge of G to a hyperedge with three vertices where the third vertex is in U_3 and has degree one in the resulting hypergraph. More formally, let $U_3 = \{v_e : e \in E(G)\}$, $|U_3| = |E(G)|$, and the set of hyperedges of $PR^*(3,g)$ is $\{e \cup v_e : e \in E(G)\}$.

In general, for $r \geq 3$, let G be an $(r-1)$-uniform hypergraph that is partite with parts $U_1,\dots,U_{r-1}$, having girth g, and with set of edges E, $|E| > |V(G)|$. Let $PR^*(r,g)$ be an r-partite r-uniform hypergraph with parts $U_1,\dots,U_r$, $|U_r| = |E|$, $U_r = \{v_e : e \in E\}$ with the edge set $\{e \cup v_e : e \in E\}$, i.e., H restricted to the parts $U_1,\dots,U_{r-1}$ coincides with G and each edge of H is extended to a distinct vertex of U_r.

Lemma 2. *For any integers r,g, $g > r \geq 2$, $PR(r,g)$ and $PR^*(r,g)$ are part-rainbow-forced r-uniform hypergraphs of girth at least g.*

Proof. By construction, $PR(r,g)$ is an r-uniform r-partite hypergraph, $r \geq 2$. We shall prove by induction on r that $PR(r,g)$ is part-rainbow-forced hypergraph of girth at least g.

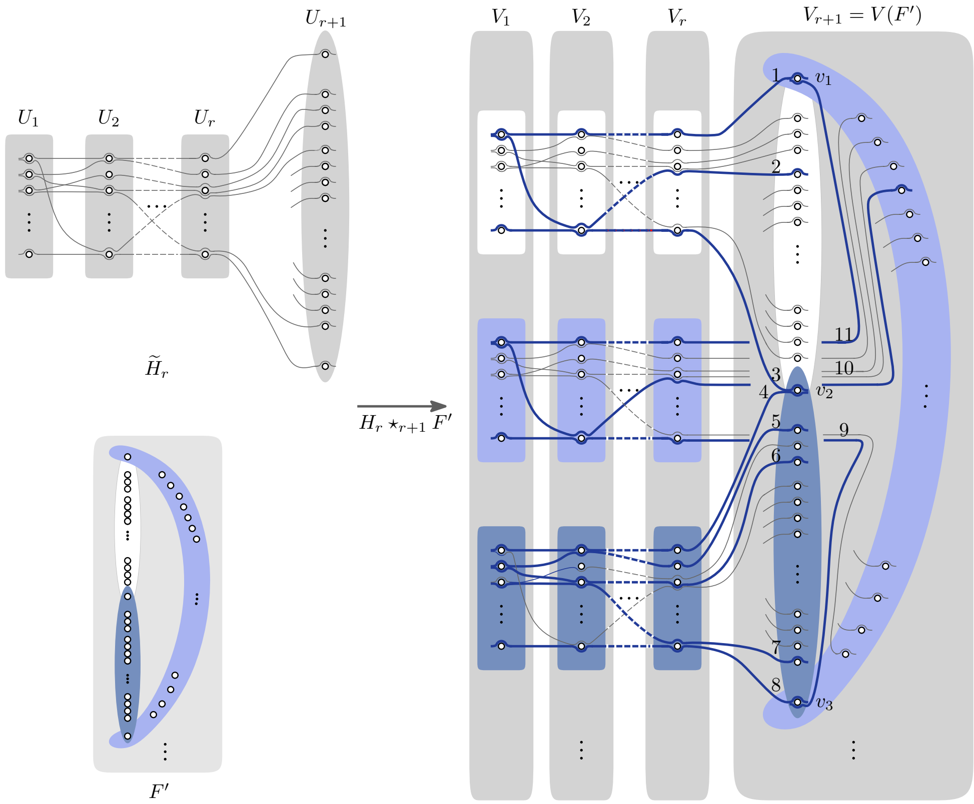

Figure 4. Illustration of a part-rainbow-forced $(r+1)$-uniform hypergraph $PR(r+1, g)$ and a cycle of length 3 in the amalgamated hypergraph F'. The bold hyperedges form a cycle of length 11 in the resulted hypergraph.

When $r = 2$, we see that a part-rainbow coloring assigns distinct colors to x and z. Thus, no matter how y is colored, xy or yz is rainbow. Moreover this graph is acyclic, so it has infinite girth.

Assume that $PR(r, g)$ is part-rainbow-forced hypergraph of girth at least g. Let us prove that $H_{r+1} = PR(r+1, g)$ is also part-rainbow-forced hypergraph of girth at least g. Let $H_r = PR(r, g)$. Recall that H_{r+1} is an amalgamation of copies $\widetilde{H}_r^1$, $\widetilde{H}_r^1, \ldots, \widetilde{H}_r^{e'}$ of $\widetilde{H_r}$ along the $(r+1)^{\text{st}}$ part using F', where F' is an ℓ-uniform hypergraph of girth at least g, minimum degree $\ell(r+1)$, $\ell = |E(H_r)|$, and $e' = |E(F')|$. Recall further, that $\widetilde{H}_r^i$ is obtained by an extension operation tilde from H_r^i, a copy of H_r.

First we shall verify that any part-rainbow coloring c of H_{r+1} results in a rainbow edge. For any $i = 1, \ldots, e'$, consider a restriction of c to the vertex set of H_r^i. Since it is a copy of $H_r = PR(r, g)$, it is again part-rainbow, so there is a rainbow edge E_i' in that copy. Let $E_i' \cup \{v_i\}$ be a corresponding uniquely defined edge of $\widetilde{H}_r^i$. The vertices $v_1, \ldots, v_{e'}$ are vertices of F'. Since the minimum degree of F' is at least $\ell(r+1)$, then $e' = |E(F')| \geq |V(F')|\ell(r+1)/\ell = |V(F')|(r+1)$. Thus there are at least $r+1$ repeated vertices in the list $v_1, \ldots, v_{e'}$, i.e., without loss of generality, $v = v_1 = \cdots = v_{r+1}$. Thus v extends rainbow edges $E'_1, E'_2, \ldots, E'_{r+1}$ in $H_r^1, H_r^2, \ldots, H_r^{r+1}$. We claim that at least one of the extended edges $E'_1 \cup \{v\}, E'_2 \cup \{v\}, \ldots, E'_{r+1} \cup \{v\}$ is rainbow. Assume not, then $c(v)$ is present in each of $E'_1, E'_2, \ldots, E'_{r+1}$. However, there are at most r vertices of each given color in the first r parts. Since $E'_1, E'_2, \ldots, E'_{r+1}$ are pairwise disjoint, we have a contradiction.

To see that the girth of H_{r+1} is at least g, consider a cycle C in H_{r+1}, see bold edges in Figure 4. If the edges of C come from one copy of $\widetilde{H}_r$, then the length of C is at least g as the girth of $\widetilde{H}_r$ is the same as girth of H_r. If the edges of C come from at least two distinct copies of $\widetilde{H}_r$, then C is a union of hyperpaths $P_0, P_1, \ldots, P_{m-1}$ from different copies of $\widetilde{H}_r$, such that the consecutive paths share a vertex in the last $(r+1)^{\text{st}}$ part, i.e., $V(P_i) \cap V(P_{i+1}) = \{u_i\}$, $u_0, \ldots, u_{m-1}$ are distinct vertices from V_{r+1}, addition modulo m. Thus u_i and u_{i+1} belong to the same copy of $\widetilde{H}_r$ and thus the same edge of F', $i = 0, \ldots, m-1$, addition modulo m. We see that these edges of F' form a cycle in F' of length at most the length of C. On the other hand, we know that any cycle in F' has length at least g, implying that C has length at least g. This concludes the proof that $PR(r+1, g)$ is part-rainbow-forced of girth at least g.

Now, we shall argue that $PR^*(r, g)$ is a part-rainbow forced r-uniform hypergraph of girth at least g, for $g > r$. The fact that it has girth at least g follows from the fact that G has girth at least g. Consider a part-rainbow coloring of $PR^*(r, g)$. Assume that there is no rainbow hyperedge. Since $|U_r| = |E(G)| > |V(G)|$, there is a vertex in U_r having a color not present in $U_1 \cup \cdots \cup U_{r-1}$. Let U'_r be the set of all such vertices, let $x = |U'_r|$. Each $v \in U'_r$ belongs to a unique hyperedge e_v of H. This hyperedge is not rainbow, there are some two vertices in $e_v \cap (U_1 \cup \cdots \cup U_{r-1})$ that have the same color. We select exactly one such pair, denote its vertices u_v, u'_v. Consider a fixed color, say 1, and consider all selected vertices u_v, u'_v of color 1. We see that the total number of such vertices is at most $r-1$ since each U_i, $i = 1, \ldots, r-1$ is rainbow colored. Moreover, the respective edges e_v do not form a cycle since $g > r$. Thus the pairs $u_v u'_v$ correspond to a forest, thus having more vertices than edges. This implies that the number of vertices of color 1 is at least the number of respective hyperedges e_v plus one. Thus the total number of distinct colors used on $U_1 \cup \cdots \cup U_{r-1}$ is at most $|V(G)| - x$. On the other hand, x is the number of vertices from U_r of colors not used on $U_1 \cup \cdots \cup U_{r-1}$, so $x \geq |U_r| - (|V(G)| - x) = |E(G)| - |V(G)| + x \geq 1 + x$, a contradiction. ∎

Now we construct an rm-unavoidable hypergraph $H(r, g)$ of uniformity r and girth at least g.

Construction of a hypergraph $H(r, g)$

For $g = 2$ and any $r \geq 2$, let $H(r, 2)$ be a complete r-uniform hypergraph on $(r-1)^2+1$ vertices. Assume that for any $r \geq 2$, $H(r, g-1)$ has been constructed. Let $F = PR(r, g)$ be as given in the previous construction. Let $a = (r-1)^2 + r$ and let $\mathcal{M}_1$ be a complete a-partite F-factor. For any partite hypergraph G, let $|G|_i$ denote the size of the i^{th} part of G.

Let $\mathcal{M}_2 = \mathcal{M}_1 \star_1 \mathcal{H}_1$, where $\mathcal{H}_1 = H(|\mathcal{M}_1|_1, g-1)$. Let $\mathcal{M}_3 = \mathcal{M}_2 \star_2 \mathcal{H}_2$, where $\mathcal{H}_2 = H(|\mathcal{M}_2|_2, g-1)$. In general, let $\mathcal{M}_{j+1} = \mathcal{M}_j \star_j \mathcal{H}_j$, where $\mathcal{H}_j = H(|\mathcal{M}_j|_j, g-1)$. We see that the j^{th} part of $\mathcal{M}_{j+1}$ corresponds to the vertex set of $\mathcal{H}_j$. Let $H(r, g) = \mathcal{M}_{a+1}$.

Now, we shall prove that this construction gives an rm-unavoidable hypergraph that is r-uniform and has girth g. This will give a proof of Theorem 1.

***Proof of Theorem* 1.** We shall show that $H(r, g)$ is an rm-unavoidable hypergraph of girth at least g, by induction on g. When $g = 2$, $H(r, 2)$ is a compete r-uniform hypergraph on $(r-1)^2 + 1$ vertices. It has girth 2 and in any vertex coloring there are either r vertices of the same color, forming a monochromatic edge, or r vertices of distinct colors, forming a rainbow edge. Assume that for any $r \geq 2$, $H(r, g-1)$ is an rm-unavoidable hypergraph of girth at least $g-1$.

Consider $H(r, g) = \mathcal{M} = \mathcal{M}_{a+1}$ given in the construction. Let c be a vertex coloring of $\mathcal{M}$. Consider the a^{th} part of $\mathcal{M} = \mathcal{M}_{a+1}$. This part corresponds to the vertex set of $\mathcal{H}_a = H(|\mathcal{M}_a|_a, g-1)$, an rm-unavoidable hypergraph. Thus, there is a monochromatic or rainbow subset X_a in the a^{th} part of $\mathcal{M}$ of size equal to the uniformity of $\mathcal{H}_a$, i.e., of size $|\mathcal{M}_a|_a$. Since $X_a \in \mathcal{E}(\mathcal{H}_a)$, X_a is the a^{th} part of a copy of $\mathcal{M}_a$.

Consider $(a-1)^{\text{st}}$ part of this copy of $\mathcal{M}_a$. Similarly to the above, there is a monochromatic or rainbow subset X_{a-1} of this part of size equal to the uniformity of $\mathcal{H}_{a-1} = H(|\mathcal{M}_{a-1}|_{a-1}, g-1)$, i.e., of size $|\mathcal{M}_{a-1}|_{a-1}$. Since $X_{a-1} \in \mathcal{E}(\mathcal{H}_{a-1})$, X_{a-1} is the $(a-1)^{\text{st}}$ part of a copy of $\mathcal{M}_{a-1}$ such that the a^{th} part of this copy is a subset of X_a.

Continuing in this manner we see that there is a monochromatic or a rainbow subset X_j of j^{th} part of $\mathcal{M}_{j+1}$ of size equal to the uniformity of $\mathcal{H}_j$, i.e., of size $|\mathcal{M}_j|_j$. We have that X_j is the j^{th} part of a copy of $\mathcal{M}_j$ such that the $(j+t)^{\text{th}}$ part of this copy is a subset of X_{j+t}, $j + t \in \{j+1, j+2, \ldots, a\}$.

Thus $X_1, X_2, \ldots, X_a$ form parts of an a-uniform sub-hypergraph of $\mathcal{M}$ containing a copy of $\mathcal{M}_1$. Recall that $\mathcal{M}_1$ is a complete a-partite F-factor. Each of these parts is monochromatic or rainbow. Since $a = (r-1)^2 + r$, there are either at least r parts that are rainbow or at least $(r-1)^2 + 1$ parts that are monochromatic. If there are r rainbow parts, the copy of F on these parts contains a rainbow edge as F is part-rainbow-forced. So, assume that there are at least $(r-1)^2 + 1$ monochromatic parts. If there are r of those that are of the same color, any edge in a copy of F on these parts is monochromatic. Otherwise there are at most $(r-1)$ parts of each given color, so there are r monochromatic parts of distinct colors. These r parts in turn contain an edge of F, and since an edge has at most one vertex from each part, this edge is rainbow.

Now, we verify that the girth of $\mathcal{M}$ is at least g by an argument similar to one of Lemma 2. To do that, we shall prove by induction on j that $\mathcal{M}_j$ has girth at least g, $j = 1, \ldots, a$. Since $\mathcal{M}_1$ is a complete a-partite F factor, it has girth equal to the girth of F, that is at least g. Assume that $\mathcal{M}_j$ has girth at least g. Let us prove that $\mathcal{M}_{j+1}$ has girth at least g. Recall that $\mathcal{M}_{j+1} = \mathcal{M}_j \star_j \mathcal{H}_j$, i.e., $\mathcal{M}_{j+1}$ is obtained by amalgamating copies of $\mathcal{M}_j$ along $\mathcal{H}_j = H(|\mathcal{M}_j|_j, g-1)$. Let X be the j^{th} part of $\mathcal{M}_{j+1}$, i.e., the vertex set of $\mathcal{H}_j$. Consider a shortest cycle C in $\mathcal{M}_{j+1}$. If C is a subgraph of one of these copies of $\mathcal{M}_j$, then by induction C has length at least g. If the edges of C come from at least two distinct copies of $\mathcal{M}_j$, then C is an edge-disjoint union of hyperpaths $P_0, P_1, \ldots, P_{m-1}$, each with at least 2 edges, from different copies of $\mathcal{M}_j$, such that the consecutive paths share a vertex in X, i.e., $V(P_i) \cap V(P_{i+1}) = \{u_i\}$, $i = 0, \ldots, m-1$, and $u_0, \ldots, u_{m-1}$ are distinct vertices from X, addition modulo m. Thus u_i and u_{i+1} belong to the same copy of $\mathcal{M}_j$ and thus correspond to the vertices from the same edge of $\mathcal{H}_j$, $i = 0, \ldots, m-1$, addition modulo m. We see that these edges of $\mathcal{H}_j$ form a cycle in $\mathcal{H}_j$ of length at most half the length of C. On the other hand, we know that any cycle in $\mathcal{H}_j$ has length at least $g-1$, implying that C has length at least $2(g-1) \geq g$. This concludes the proof of Theorem 1 using an explicit construction. ■

3. Proof of Theorem 1 — Probabilistic Construction

This proof is just a slight generalization of the probabilistic construction for high-girth, high-chromatic-number hypergraphs by Nešetřil and Rödl. Let an ℓ-cycle be a cycle of length ℓ. Let r, g be fixed, put $R = (r-1)^2 + 1$ and consider an R-uniform hypergraph $\mathcal{H} = \mathcal{H}(n, R, g) = (X, \mathcal{E})$ with n vertices, girth at least g, and with $|\mathcal{E}| = \left\lceil n^{1+\frac{1}{g}} \right\rceil$. Such a graph exists, if n is large enough by Lemma 5, see Appendix.

Let us order the hyperedges of $\mathcal{H}$ as $E_1, E_2, \ldots, E_m$. Let $\mathcal{M}_n$ be the family of all sequences $(E'_1, \ldots, E'_m)$ such that $|E'_i| = r$ and $E'_i \subseteq E_i$, $i = 1, \ldots, m$. For a given sequence $Q \in \mathcal{M}_n$, let $\mathcal{H}_Q$ be a hypergraph whose hyperedges are elements of Q. We say that a coloring of X is *good* for Q if there are no monochromatic and no rainbow edges under this coloring of $\mathcal{H}_Q$. We say that Q is *colorable* if there is a coloring of X that is good for Q. We shall count the number of colorable sequences and shall show that it is strictly less than the number of all sequences in $\mathcal{M}_n$. This will imply that there is a non-colorable sequence corresponding to an rm-unavoidable hypergraph.

Each hypergraph $\mathcal{H}_Q$, $Q \in \mathcal{M}_n$ has girth at least g since $\mathcal{H}$ has this property. In addition $|\mathcal{M}_n| \geq a^{n^{1+\frac{1}{g}}}$, where $a = \binom{R}{r}$, since there are a ways to choose an r-element subset from an edge of $\mathcal{H}$ and $m \geq n^{1+\frac{1}{g}}$. Now we consider a coloring of X with arbitrary number of colors. Each edge E of $\mathcal{H}$ is colored with at least r or less than r colors. If E is colored with less than r colors, there are r vertices in E of the same color since E has $R = (r-1)^2 + 1$ elements and $\frac{R}{(r-1)} > (r-1)$. If E is colored with at least r colors, there are r vertices with pairwise distinct colors. Thus each edge E of $\mathcal{H}$ contains a "bad" subset that is either monochromatic or rainbow, and only at most $\binom{|E|}{r} - 1 = \binom{R}{r} - 1 = a - 1$ of all r-element subsets of E could be "good". Therefore each coloring c of X is good for at most $(a-1)^{\lceil n^{1+\frac{1}{g}} \rceil} \leq (a-1)^{1+n^{1+\frac{1}{g}}}$ members of $\mathcal{M}_n$. Since the total number of colors in X is at most n in any coloring, it is enough to consider colorings with colors $1, \ldots, n$. Since there are n^n colorings with n colors we have that

$$\begin{aligned} \big|\{Q \in \mathcal{M}_n | \; Q \text{ is colorable}\}\big| &= \left| \bigcup_{c: X \to [n]} \bigcup_{Q \in \mathcal{M}_n} \{Q \mid c \text{ is good for } Q\} \right| \\ &\leq \sum_{c: X \to [n]} \left| \bigcup_{Q \in \mathcal{M}_n} \{Q \mid c \text{ is good for } Q\} \right| \\ &\leq n^n \cdot (a-1)^{1+n^{1+\frac{1}{g}}}. \end{aligned}$$

Next we shall show that $n^n \cdot (a-1)^{1+n^{1+\frac{1}{g}}} < a^{n^{1+\frac{1}{g}}}$ for all sufficiently large n. Indeed, $n^n (a-1)^{1+n^{1+\frac{1}{g}}} < a^{n^{1+\frac{1}{g}}} \Leftrightarrow n \ln(n) + \ln(a-1) < n^{1+\frac{1}{g}} \ln\left(\frac{a}{a-1}\right)$. The last inequality holds since $\ln\left(\frac{a}{a-1}\right) > 0$. Therefore the number of colorable members from $\mathcal{M}_n$ is less than the total number of members in $\mathcal{M}_n$ and thus there is an non-colorable $Q \in \mathcal{M}_n$ that gives $\mathcal{H}_Q$, an r-uniform hypergraph of girth at least g that is rm-unavoidable. $\square$

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4. Appendix

Lemma 3. *For any $\ell, g \geq 2$, $q \geq 1$ there is an ℓ-uniform hypergraph of girth at least g and minimum degree at least q.*

Proof. To see that such a hypergraph exists, consider an ℓ-uniform hypergraph F of girth at least g and chromatic number greater than q. If F has a vertex v that belongs to at most $q-1$ edges, delete it from F. We obtain a hypergraph $F-v$ of chromatic number greater than q again because otherwise we can take a proper coloring of $F-v$ with at most q colors and extend it to a proper coloring of F. Indeed, if $E_1, \dots, E_{q'}$, $q' \leq q-1$ are the edges incident to v, choose a color for v that is not a color of monochromatic $E_i - v$ under the proper coloring of $F-v$, $i = 1, \dots, q'$, if such a monochromatic edge exists. Since only at most $q-1$ colors are forbidden for v, one color is still available. Continue this deletion process until possible. The process must stop with a non-empty graph of chromatic number greater than q and minimum degree at least q. Since it is a sub-hypergraph of the original hypergraph, it has girth at least g. ∎

Lemma 4 [20]**.** *Let $C(r,\ell,n)$ be the number of ℓ-cycles in the r-uniform complete hypergraph on n vertices, $r \geq 3$. Then $C(r,\ell,n) \leq c(r,\ell)\binom{n}{(r-1)\ell}$, for a function $c(r,\ell)$ independent of n.*

Proof. Observe that the largest number of vertices in an ℓ-cycle C of length ℓ is $(r-1)\ell$. Indeed a cycle C of length ℓ is defined as a subhypergraph C with ℓ distinct vertices $x_0, \dots, x_{\ell-1}$, $\ell \geq 2$ and distinct hyperedges $E_0, \dots E_{\ell-1}$ such that $x_i, x_{i+1} \in E_i, i = 0, \dots, \ell-1$, addition of indices modulo ℓ. Thus, each hyperedge E_i, $i = 0, \dots, \ell-1$, has at most $r-2$ vertices not in the set $\{x_0, \dots, x_{\ell-1}\}$. Therefore the total number of vertices in C is at most $\ell(r-2) + |\{x_0, \dots, x_{\ell-1}\}| = \ell(r-2) + \ell = \ell(r-1)$. Thus, an upper bound on the number of all ℓ-cycles is $\binom{n}{\ell(r-1)} \cdot c(r,\ell)$, where $\binom{n}{\ell(r-1)}$ is the number of ways to choose a set on $\ell(r-1)$ vertices and $c(r,\ell)$ is the number of ℓ-cycles on a given set of $\ell(r-1)$ vertices. ∎

Lemma 5 [20]**.** *For any positive integers r and s, $r \geq 2, s \geq 3$ there exists an $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$, $n \in \mathbb{N}$ there exists an r-uniform hypergraph $(X, \mathcal{E})$ with girth at least s and with $|\mathcal{E}| > n^{1+\frac{1}{s}}$.*

Proof. We consider a set $\mathcal{M} = \mathcal{M}(n,r,s)$ of all r-uniform hypergraphs on vertex set $[n]$ with $m = 2\left\lceil n^{1+\frac{1}{s}} \right\rceil$ edges. Then $|\mathcal{M}| = \binom{\binom{n}{r}}{m}$. Choose a hypergraph $\mathcal{H}$ from $\mathcal{M}$ randomly and uniformly, i.e., with probability $\frac{1}{|\mathcal{M}|}$. Let K be a complete r-uniform hypergraph on vertex set $[n]$. Call cycles of length smaller than s bad. Let X_j be the number of cycles of length j in $\mathcal{H}$ and X_{bad} be the number of bad cycles. Then $\mathrm{Exp}(X_j) = \sum_C \mathrm{Prob}(C \subseteq \mathcal{H})$, where the sum is over all cycles C of length j in K. Then $\mathrm{Exp}(X_j) \leq C(r,j,n)\frac{\binom{\binom{n}{r}-j}{m-j}}{\binom{\binom{n}{r}}{m}}$, where $C(r,j,n)$ is the number of cycles of length j in K and second term is the probability of occurrence of such a cycle. Using Lemma 4, we have that $\mathrm{Exp}(X_j) \leq c(r,j)\binom{n}{(r-1)j}\frac{\binom{\binom{n}{r}-j}{m-j}}{\binom{\binom{n}{r}}{m}}$. Then, for constants $\widetilde{c}(r,j)$, $j = 2,\dots,s-2$ and $\widetilde{C}(r,s)$, we have

$$\begin{aligned}
\mathrm{Exp}(X_{bad}) &= \sum_{j=2}^{s-1} \mathrm{Exp}(X_j) \\
&\leq \sum_{j=2}^{s-1} c(r,j)\cdot\binom{n}{(r-1)j}\frac{\binom{\binom{n}{r}-j}{m-j}}{\binom{\binom{n}{r}}{m}} \\
&= \sum_{j=2}^{s-1} c(r,j)\cdot\binom{n}{(r-1)j}\frac{m\cdot(m-1)\cdots(m-j+1)}{\binom{n}{r}\cdot(\binom{n}{r}-1)\cdots(\binom{n}{r}-j+1)} \\
&\leq \sum_{j=2}^{s-1} c(r,j)\cdot\binom{n}{(r-1)j}\left(\frac{m}{\binom{n}{r}}\right)^j \\
&\leq \sum_{j=2}^{s-1} \widetilde{c}(r,j)n^{(r-1)j-rj}m^j \\
&\leq \sum_{j=2}^{s-1} \widetilde{c}(r,j)n^{(r-1)j-rj}n^{(1+1/s)j} \\
&\leq \widetilde{C}(r,s)n.
\end{aligned}$$

Since $\mathrm{Exp}(X_{bad}) \leq \widetilde{C}(r,s)n$, there is a hypergraph from $\mathcal{M}$ with at most $\widetilde{C}(r,s)n$ cycles of length at most $s-1$. Delete an edge from each such cycle and obtain a hypergraph on at least $2n^{1+1/s} - \widetilde{C}(r,s)n > n^{1+1/s}$ edges and girth at least s. ∎