LABELED PACKING OF CYCLES AND CIRCUITS

Alice Joffard

AND

Hamamache Kheddouci

Université Lyon 1

e-mail: alice.joffard@liris.cnrs.fr
hamamache.kheddouci@univ-lyon1.fr

Abstract

In 2013, Duchêne, Kheddouci, Nowakowski and Tahraoui introduced a labeled version of the graph packing problem. It led to the introduction of a new graph parameter, the k-packing label-span λ^k. This parameter corresponds, given a graph H on n vertices, to the maximum number of labels we can assign to the vertices of the graph, such that there exists a packing of k copies of H into the complete graph K_n, coherent with the labels of the vertices.

The authors intensively studied the labeled packing of cycles, and, among other results, they conjectured that for every cycle C_n of order $n = 2k + x$, with $k \geq 2$ and $1 \leq x \leq 2k - 1$, the value of $\lambda^k(C_n)$ was 2 if x is 1 and k is even, and $x + 2$ otherwise.

In this paper, we disprove this conjecture by giving a counterexample. We however prove that it gives a valid lower bound, and we give sufficient conditions for the upper bound to hold.

We then give some similar results for the labeled packing of circuits.

Keywords: packing of graphs, labeled packing, cycles, circuits.

2010 Mathematics Subject Classification: 05C70, 05C38.

1. Definitions and Notations

All along the paper, the graphs are simple graphs, with no multiple edges nor loops. For an undirected graph $G = (V(G), E(G))$, $V(G)$ represents the set of vertices of G, $E(G)$ represents the set of edges of G. For a directed graph $G = (V(G), E(G))$, $V(G)$ represents the set of vertices of G, $E(G)$ represents the

set of arcs of G. The *order* n of a graph G is its number of vertices, its *size* m is its number of edges or arcs, and $\Delta(G)$ represents its maximal degree.

The graph K_n is the complete undirected graph of order n, $K_{p,q}$ the complete bipartite graph of partition orders p and q, P_n the path of order n, C_n the cycle of order n, and S_n the star of order n. $\overleftrightarrow{K_n}$ is the *complete directed graph* of order n, meaning that its set of arcs contains all the possible arcs from one of its vertices to another, and $\overrightarrow{C_n}$ is the *circuit* of order n, that is the cyclic orientation of the cycle C_n.

The *union* of G and H is the graph $G \cup H = (V(G) \cup V(H), E(G) \cup E(H))$. The *complement* of G, is the graph $\overline{G} = (V(G), E(K_n) \setminus E(G))$. The *k-th power* G^k of an undirected graph G is the graph with vertex set $V(G)$ and where $(u,v) \in E(G^k)$ if and only if there exists a path of length at most k between u and v in G.

For p, q and r any integers, $p \bmod q$ denotes the remainder of the Euclidean division of p by q, and $p \equiv r \bmod q$ is equivalent to $p \bmod q = r \bmod q$. We denote by $[k]$ the set of integers $\{1, 2, \ldots, k\}$.

2. INTRODUCTION

The graph packing problem has been widely studied in the literature. The general idea of the problem is to find sufficient conditions on a set of graphs $(H_1, H_2, \ldots, H_k)$ or on a graph G that guarantee that $(H_1, H_2, \ldots, H_k)$ admits a packing in G, a packing being defined in the following way.

Definition. A packing of $(H_1, H_2, \ldots, H_k)$ in G is a set of injections $\alpha_i : V(H_i) \to V(G)$, for any $i \in [k]$, such that $\alpha_i^*(E(H_i)) \cap \alpha_j^*(E(H_j)) = \emptyset$ for $i \neq j$ where $\alpha_i^* : E(H_i) \to E(G)$ is induced by α_i.

If for every $i \in [k], |V(H_i)| = |V(G)| = n$, the injections are actually bijections. In the case of a *k-placement*, where the vertices of the same graph H are sent k times to the vertices of K_n, the bijections are actually permutations on the vertices of H. In this context, we call a *fixed point* any vertex $\alpha \in V(G)$ such that for every $i \in [k]$, $\sigma_i(\alpha) = \alpha$.

The first results concerning the problem of graph packing focused on the packings into the complete graph K_n. In this case, the only conditions for the injections to form a packing is that the induced images of the edges should not intersect. Spencer and Sauer [8] got first interested in the packing of two graphs, and found two sufficient conditions on the graphs for them to admit a packing in K_n.

Theorem 1 [8]**.** *If* $|E(H_1)||E(H_2)| < \binom{n}{2}$, *then* (H_1, H_2) *pack into* K_n.

Theorem 2 [8]**.** *If* $2\Delta(H_1)\Delta(H_2) < n$, *then* (H_1, H_2) *pack into* K_n.

These two theorems are quite representative of two classical approaches in graph packing theory. One consists in bounding the size of the graphs, and the other in bounding their maximal degree. Concerning the more general case of the packing of k graphs, Bollobás and Eldridge formulated the following conjecture, still focusing on small sizes graphs.

Conjecture 3 [3]. *If $|E(H_1)|, |E(H_2)|, \ldots, |E(H_k)| \leq n - k$, then $(H_1, H_2, \ldots, H_k)$ pack in K_n.*

The conjecture has been proven for $k = 2$ [8] and $k = 3$ [7]. For $k \geq 4$, the conjecture remains open and is still the subject of many researches, as it is considered to be one of the most important open problems in graph packing theory. Another very important conjecture, about the packing of two graphs in K_n is the following:

Conjecture 4 [3]. *If $(\Delta(H_1) + 1)(\Delta(H_2) + 1) \leq n + 1$, then (H_1, H_2) pack into K_n.*

Most of the results concerning graph packing focused on the conditions on the graphs H_i to pack into the complete graph K_n. However, some authors chose a different point of view, which is to fix the graphs $H_1, H_2, \ldots, H_k$ and search for a minimal size graph G that would make the packing of the H_i in G possible. A great part of these studies have focused on the cases where the H_i were trees.

Theorem 5 [5, 6]. *For any $n \geq 4$ and any tree T of order n distinct from S_n, there exists a 2-placement σ of T such that $\sigma(T) \subset T^3$.*

For more details concerning the packing of graphs, the reader is referred to the numerous surveys on the subject, e.g. [12, 13].

In 2012, Duchene, Kheddouci, Nowakowski, and Tahraoui introduced the notion of labeled packings of graphs, used in several studies since then [9, 10, 11]. They gave the following definition of a p-labeled packing of k copies of a graph H.

Definition [4, 9]. For every $p \geq 1$ and every graph H of order at most n, the pair (f, σ), where f is a mapping from $V(K_n)$ to a set of labels of cardinality p and σ is a set of injections $\sigma_i : V(H) \to V(K_n)$ for every i in $[k]$, is a p-labeled packing of k copies of H into K_n if and only if:

$$\begin{cases} \text{For every } i \neq j, \sigma_i^*(E(H)) \cap \sigma_j^*(E(H)) = \emptyset, \text{ where } \sigma_i^* : E(H) \to E(K_n) \\ \text{is induced by } \sigma_i. \\ \text{For every } v \in V(H), f(\sigma_1(v)) = f(\sigma_2(v)) = \cdots = f(\sigma_k(v)). \end{cases}$$

In other words, a p-labeled packing of k copies of H is a labeling f of the vertices of K_n with exactly p distinct labels and a set σ of k injections from the vertices

of H to the vertices of K_n such that the images of the edges of H never overlap in K_n, and for any vertex v, the k images of v by σ have the same label. It can be seen as a generalization of the problem of graph packing where all the vertices of H have the same label and can thus be sent to any other vertex.

This definition allowed the authors to introduce a new parameter for a graph H, that we call the k-packing label-span of H.

Definition [4, 9]**.** The k-packing label-span of H, denoted by $\lambda^k(H)$, is the largest p such that H admits a p-labeled packing of k copies of H.

The authors linked this new parameter to the cycle decomposition of permutations. Indeed, for $k = 2$, if (σ_1, σ_2) is a packing of two copies of H, where σ_1 is the identity and σ_2 is a permutation on the vertices decomposed into p disjoint cycles, then we know that in σ_2 the vertices of H are sent to vertices that belong to the same cycle of the permutation, so that we can always associate one label to each of the p cycles. That way, we can obtain a valid p-labeled packing of two copies of H, so that we have $\lambda^2(H) \geq p$.

The authors, among other results, analyzed the value of λ^k for cycles, and proved the following.

Theorem 6 [4, 9]**.** *Every cycle C_n of order $n \leq 2k$ does not admit any p-labeled k-packing. For every cycle C_n of order $n = 2km + x$, where $k, m \geq 2$ and $x < 2k$, we have*

$$\lambda^k(C_n) = \begin{cases} \frac{n}{2} + 1 & \text{if } m = 2, \ k > 2 \text{ and } x \equiv 2 \pmod 4, \\ \left\lfloor \frac{n}{2} \right\rfloor + m + 1 & \text{if } x = 2k - 1, \\ \left\lfloor \frac{n}{2} \right\rfloor + m & \text{otherwise.} \end{cases}$$

Concerning the left values of n, they proposed the following conjecture.

Conjecture 7 [4, 9]**.** *For every cycle C_n of order $n = 2k + x$, where $k \geq 2$ and $1 \leq x \leq 2k - 1$, we have*

$$\lambda^k(C_n) = \begin{cases} 2 & \text{if } x = 1 \text{ and } k \text{ is even,} \\ x + 2 & \text{otherwise.} \end{cases}$$

To support this conjecture, they gave the following results, that focus on some particular values of k and x.

Theorem 8 [4, 9]**.** *Let C_n be the cycle of order $n = 2k + x$, where $k \geq 2$, $2k - 3 \leq x \leq 2k - 1$ and $(k, n) \neq (2, 5)$. We have $\lambda^k(C_n) = x + 2$.*

Theorem 9 [4, 9]**.** *Let C_n be the cycle of order $n = 2k + x$, where k is a power of 2 and $x = 1$. We have $\lambda^k(C_n) = 2$.*

Theorem 10 [4, 9]**.** *Let C_n be the cycle of order $n = 2k + x$, where k is prime and $x = 1$. We have $\lambda^k(C_n) \leq x + 2$.*

Theorem 11 [4, 9]**.** *Let C_n be the cycle of order $n = 2k + x$ where k is even and $x = 2$. We have $\lambda^k(C_n) \leq x + 2$.*

In this paper, we disprove Conjecture 7 by giving a counterexample to ensure that the upper bound is not valid in general. However, we give some particular cases for which the upper bound is valid, and we prove that the lower bound always holds. Then, we study the directed version of the previous problem, that is the labeled packing of circuits.

3. LABELED PACKING OF CYCLES

Theorem 12. *Conjecture 7 is false.*

Proof. The counterexample uses the values $k = 9$, $x = 3$, and therefore $n = 21$, that is the smallest possible value of n for a counterexample, as shown later. We show that there exists a 7-labeled packing of 9 copies of C_{21}, so that $\lambda^9(C_{21}) \geq 7$, while Conjecture 7 would give $\lambda^9(C_{21}) = 3 + 2 = 5$. The 7-labeled packing of 9 copies of C_{21} is presented in Figure 1. ∎

Despite the fact that Conjecture 7 is now proven to be false, we show that it gives a good lower bound for $\lambda^k(C_n)$. The rest of this section is devoted to the proof of this statement. To complete this proof, we use the following lemma.

Lemma 13. *Le f be a mapping from $V(C_n)$ to a set of labels of cardinality p and σ be a set of permutations σ_i of $V(C_n)$ for every i in $[k]$. Let $V(C_n) = \{v_0, \ldots, v_{n-1}\}$ be the vertices of C_n, partitioned into a set F containing all the fixed points of σ, and a set V_l for each other label l of f, containing all the vertices of label l. In other words:*

$F = \{v \in V(C_n) : \text{ for every } j \in [k], \sigma_j(v) = v\}$,
$V_l = \{v \in V(C_n) : f(v) = l\}$ *for any label l.*

Let the set of edges $E(C_n)$ be partitioned into the following sets:

$E_F = \{(u, v) \in E(C_n) : u \in F \text{ or } v \in F\}$,
$E_{p,q} = \{(u, v) \in E(C_n) : u \in V_p \text{ and } v \in V_q\}$ *for any labels p and q.*

If the four following conditions hold, then (f, σ) is a labeled packing of k copies of C_n.

(i) *For each $v \notin F$ and $j \neq j'$, $f(\sigma_j(v)) = f(\sigma_{j'}(v))$.*

(ii) *For any p, q and any $(u, v), (u', v') \in E_{p,q}$, if $\sigma_j^*((u, v)) = \sigma_{j'}^*((u', v'))$, then $j = j'$.*

(iii) *For any $v \notin F$ and $j \neq j'$, $\sigma_j(v) \neq \sigma_{j'}(v)$.*

(iv) *For each pair u, v where there exists $w \in F$ such that $(u, w), (w, v) \in E(C_n)$, if $f(u) = f(v)$, then, for every $j \neq j'$, $\sigma_j(u) \neq \sigma_{j'}(v)$ and $\sigma_j(v) \neq \sigma_{j'}(u)$.*
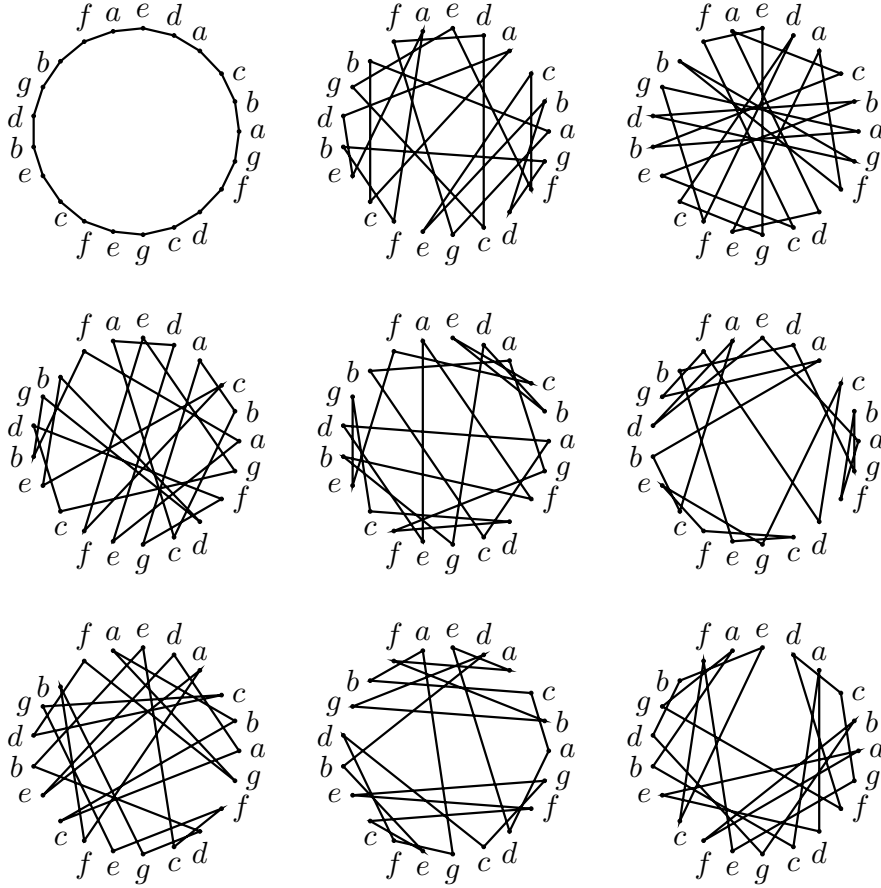

Figure 1. The counterexample used to prove Theorem 12.

Proof. By definition, for (f, σ) to be a labeled packing, we must have, for every $j \neq j', \sigma_j^*(E(C_n)) \cap \sigma_{j'}^*(E(C_n)) = \emptyset$ and for every $v \in V(C_n), f(\sigma_1(v)) = f(\sigma_2(v)) = \cdots = f(\sigma_k(v))$.

Let us first show that for every $v \in V(C_n), f(\sigma_1(v)) = f(\sigma_2(v)) = \cdots = f(\sigma_k(v))$. For every $v \in F$, since $\sigma_j(v) = v$, we know that $f(\sigma_j(v)) = f(v) = f(\sigma_{j'}(v))$. Thus, condition (i) is sufficient.

Let us now show that for every $j \neq j', \sigma_j^*(E(C_n)) \cap \sigma_{j'}^*(E(C_n)) = \emptyset$. Assuming that condition (i) holds, we already know that for every labels p, q, if, X being in the previously described partition of $E(C_n)$, then $\sigma_j^*(E_{p,q}) \cap \sigma_{j'}^*(X) = \emptyset$.

Thus, it is enough to show that for every p, q and $(u, v), (u', v') \in E_{p,q}$, $\sigma_j^*(E_{p,q}) \cap \sigma_{j'}^*(E_{p,q}) = \emptyset$ and that $\sigma_j^*(E_F) \cap \sigma_{j'}^*(E_F) = \emptyset$. The first condition is ensured by condition (ii).

To show that $\sigma_j^*(E_F) \cap \sigma_{j'}^*(E_F) = \emptyset$, since each edge of E_F contains one fixed point, we only have to show that its two neighbors u and v verify $\sigma_j(u) \neq \sigma_{j'}(u)$, $\sigma_j(v) \neq \sigma_{j'}(v)$, $\sigma_j(u) \neq \sigma_{j'}(v)$ and $\sigma_j(v) \neq \sigma_{j'}(u)$. The first two conditions are ensured by condition (iii), and the two last conditions are respected if $f(u) \neq f(v)$, assuming that f verifies condition (i). If $f(u) = f(v)$, we get the assertion by using condition (iv). ∎

Theorem 14. *For every cycle C_n of order $n = 2k + x$, with $k \geq 2$ and $1 \leq x \leq 2k - 1$, we have*

$$\lambda^k(C_n) \geq \begin{cases} 2 & \text{if } x = 1 \text{ and } k \text{ is even,} \\ x + 2 & \text{otherwise.} \end{cases}$$

Proof. The proof is divided into five cases, depending on the values of k and x, as follows:

1. k is odd
   (a) $1 \leq x \leq k - 1$
   (b) $k \leq x \leq 2k - 1$
2. k is even and $x = 1$
3. k is even and $x > 1$
   (a) $2 \leq x \leq k - 1$
   (b) $k \leq x \leq 2k - 1$.

For each case, we construct a labeling f with the given number of labels, we give the associated packing $\sigma = \{\sigma_j, 1 \leq j \leq k\}$ and we prove that the four conditions of Lemma 13 hold and that (f, σ) therefore is a labeled packing of k copies of C_n.

Case 1(a). k is odd and $1 \leq x \leq k - 1$. The principle of the construction is the following: For $x = 1$, we label with 1 the vertex v_{2k}, with a every other vertex of even number, and with b every other vertex of odd number. Then, we create the first copy of C_n by joining the vertices in the following order:

$$v_0, v_{n-2}, v_1, v_{n-3}, v_2, v_{n-4}, \ldots, v_{\frac{n-1}{2}-1}, v_{\frac{n-1}{2}}, v_{2k}, v_0.$$

We create each other copy by following the same order as the previous one but with every number but $2k$ added to 2 modulo $n - 1$. Note that, since k is odd, the obtained packing is exactly the construction of the decomposition of K_n into Hamiltonian cycles given by Walecki (see [1]). For $x > 1$, we add fixed points to

the previous construction, on $x - 1$ of the k edges that link a vertex u of label a to a vertex v of label b, cutting every edge into two edges.

To express it in a more formal way, we take the following labeling f, packing $\sigma = \{\sigma_j, 1 \leq j \leq k\}$, and vertices and edges partitions of C_n:

$$f(v_i) = \begin{cases} a & \text{if } 0 \leq i \leq 2k - 1 \text{ and } i \text{ is even,} \\ b & \text{if } 0 \leq i \leq 2k - 1 \text{ and } i \text{ is odd,} \\ i - 2k + 1 & \text{if } i \geq 2k. \end{cases}$$

$$\sigma_j(v_i) = \begin{cases} v_{i + 2(j-1) \, (\text{mod } 2k)} & \text{if } i < 2k, \\ v_i & \text{if } i \geq 2k. \end{cases}$$

$F = \{v_i, 2k \leq i \leq n - 1\}$,
$V_a = \{v_i, 0 \leq i \leq 2k - 1 \text{ and } i \equiv 0 \ (\text{mod } 2)\}$,
$V_b = \{v_i, 0 \leq i \leq 2k - 1 \text{ and } i \equiv 1 \ (\text{mod } 2)\}$,
$E_F = \{(v_{i-1}, v_{2k+i}), (v_{2k-i}, v_{2k+i}), 1 \leq i \leq x - 1\} \cup \{(v_0, v_{2k}), (v_k, v_{2k})\}$,
$E_{a,a} = \{(v_i, v_{2k-i}), 1 \leq i \leq k - 1 \text{ and } i \equiv 0 \ (\text{mod } 2)\}$,
$E_{b,b} = \{(v_i, v_{2k-i}), 1 \leq i \leq k - 1 \text{ and } i \equiv 1 \ (\text{mod } 2)\}$,
$E_{a,b} = \{(v_i, v_{2k-1-i}), x - 1 \leq i \leq k - 1\}$.

The obtained graph G is, as wanted, isomorphic to C_n.

Indeed, for $x = 1$, for H a graph, if $V(H) = \{v_0, v_2, \ldots, v_{n-1}\}$ and $E(H) = \{(v_i, v_{i+1(\text{mod } n)}), 0 \leq i \leq n\}$, we obviously have $H \simeq C_n$, and $G = \tau(H)$, with τ the permutation defined as follows:

$$\tau(v_i) = \begin{cases} v_{\frac{i}{2}} & \text{if } i < 2k \text{ and } i \equiv 0 \ (\text{mod } 2), \\ v_{2k - \frac{i+1}{2}} & \text{if } i < 2k \text{ and } i \equiv 1 \ (\text{mod } 2), \\ v_i & \text{if } i = 2k. \end{cases}$$

For $x > 1$, the extra fixed point are just inserted in some of the edges of C_{2k+1}, so that we always keep $G \simeq C_n$.

We now show that f and σ define an $(x + 2)$-labeled packing of k copies of C_n. As previously stated, for $x = 1$, since k is odd the construction gives us the same decomposition of K_n as the one presented by Walecki (see [1]). The idea of the decomposition is that all the edges that do not involve any fixed point are linking two vertices whose numbers have a unique difference. Thus, those edges do not overlap, and σ defines a packing. Moreover, as $n - 1$ is even, the parity of the indices of a vertex and its image are the same, as well as their labels. For $x > 1$, we know that the added fixed points will not create any superposition of edges, as their neighbors have different labels and are never sent to themselves. More rigorously, we have the following.

(i) In Figure 2, we present the example of the packing of $k = 3$ copies of C_8 with 4 labels. Since $2(j - 1)$ and $2k$ are even, $i + 2(j - 1) \pmod 2 k$ always has the same parity as i, and for any $v \notin F$ and $j \neq j'$, $f(\sigma_j(v)) = f(\sigma_{j'}(v))$.

(ii) For every p, q and every $(u, v), (u', v') \in E_{p,q}$, $\sigma_j^*((u, v)) = \sigma_{j'}^*((u', v'))$ implies $j = j'$.

Indeed, for $1 \leq i, i' \leq k - 1$, if $\sigma_j^*((v_i, v_{2k-1-i})) = \sigma_{j'}^*((v_{i'}, v_{2k-1-i'}))$, then

$$\begin{cases} i + 2(j - 1) = i' + 2(j' - 1) \pmod 2 k, \\ 2k - 1 - i + 2(j - 1) = 2k - 1 - i' + 2(j' - 1) \pmod 2 k. \end{cases}$$

By adding the two equations, since k is odd, and $1 \leq j, j' \leq k$, we get $j = j'$. Similarly, for $1 \leq i, i' \leq k - 1$, if $\sigma_j^*((v_i, v_{2k-i})) = \sigma_{j'}^*((v_{i'}, v_{2k-i'}))$, $j = j'$.

(iii) $i + 2(j - 1) \pmod 2 k = i + 2(j' - 1) \pmod 2 k$ implies $j = j'$, so that for any $v \notin F$ and $j \neq j'$, $\sigma_j(v) \neq \sigma_{j'}(v)$.

(iv) k and 0 have different parity, and so do $2k - i$ and $i - 1$ for every i. Thus, for every pair u, v where there exists $w \in F$ such that $(u, w), (w, v) \in E(C_n)$, $f(u) \neq f(v)$.
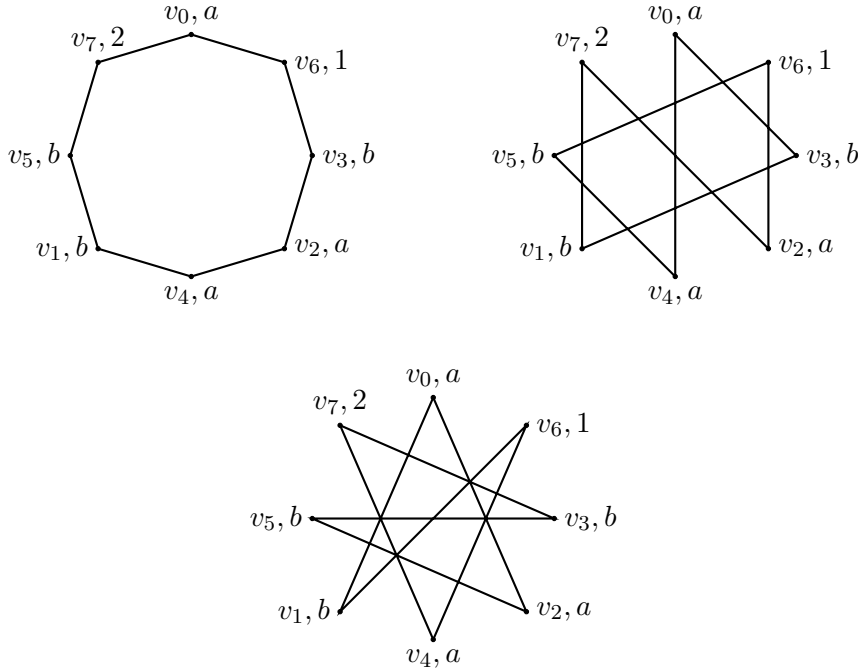

Figure 2. A 4-labeled packing of $k = 3$ copies of C_8.

Case 1(b). k is odd and $k \leq x \leq 2k - 1$. In Figure 3, we give an example of the construction for $k = 3$ and $n = 11$. For $x = k$ and thus $n = 3k$, we construct the first copy of C_n by linking successively a fixed point, a vertex of label a, and a vertex of label b. Then, we construct the $k - 1$ other copies by rotating j times clockwise, $1 \leq j \leq k - 1$, the positions of the vertices of label a, and j times anti-clockwise the positions of the vertices of label b. For $x > k$, we add fixed points to the previous construction on $x - k$ of the k edges that link a vertex of label a to a vertex of label b. We thus take:

$$f(v_i) = \begin{cases} \frac{i}{3} + 1 & \text{if } i \leq 3k \text{ and } i \equiv 0 \ (\mathrm{mod}\ 3), \\ a & \text{if } i \leq 3k \text{ and } i \equiv 1 \ (\mathrm{mod}\ 3), \\ b & \text{if } i \leq 3k \text{ and } i \equiv 2 \ (\mathrm{mod}\ 3), \\ i - 2k + 1 & \text{if } i > 3k. \end{cases}$$

$$\sigma_j(v_i) = \begin{cases} v_i & \text{if } i \equiv 0 \ (\mathrm{mod}\ 3) \text{ or } i > 3k, \\ v_{i+3(j-1)\,(\mathrm{mod}\ 3)\,k} & \text{if } i \leq 3k \text{ and } i \equiv 1 \ (\mathrm{mod}\ 3), \\ v_{i-3(j-1)\,(\mathrm{mod}\ 3)\,k} & \text{if } i \leq 3k \text{ and } i \equiv 2 \ (\mathrm{mod}\ 3). \end{cases}$$

$F = \{v_i, i \equiv 0 \ (\mathrm{mod}\ 3) \text{ or } i > 3k\}$,
$V_a = \{v_i, i \leq 3k \text{ and } i \equiv 1 \ (\mathrm{mod}\ 3)\}$,
$V_b = \{v_i, i \leq 3k \text{ and } i \equiv 2 \ (\mathrm{mod}\ 3)\}$.

$E_F = \big\{(v_{i-1(\mathrm{mod}\ 3)\,k}, v_i), (v_i, v_{i+1(\mathrm{mod}\ 3)\,k}), 0 \leq i < 3k \text{ and } i \equiv 0 \ (\mathrm{mod}\ 3)\big\}$
  $\cup \ \big\{(v_{3i-9k+1}, v_i), (v_i, v_{3i-9k+2}), 3k \leq i < n\big\}$,
$E_{a,a} = E_{b,b} = \emptyset$,
$E_{a,b} = \{(v_{3i+1}, v_{3i+2}), x - k \leq i \leq k - 1\}$.

For $x = k$, we have $E(G) = \{(v_i, v_{i+1\,(\mathrm{mod}\ n)}, 0 \leq i \leq n - 1\}$, so that $G \simeq C_n$. For $x > k$, the extra fixed point are inserted in some of the edges of C_{3k}, and we keep $G \simeq C_n$.

With the way we defined σ and f, we obviously have that all the vertices are always sent to vertices that have the same label. Now, to show that σ defines a packing, the idea is that the edges involving fixed points will never create any overlap since their neighbors have different labels and are never sent to themselves, and the other edges either, precisely because k is odd. The full proof is given next.

(i) Since $3(j-1)$ and $3k$ are multiples of 3, $i + 3(j-1)\,(\mathrm{mod}\ 3)\,k$ always has the same remainder in the euclidean division by 3 as i, and so does $i - 3(j-1)$ $(\mathrm{mod}\ 3)\,k$. Thus, for every $v \notin F$ and $j \neq j'$, $f(\sigma_j(v)) = f(\sigma_{j'}(v))$.

(ii) For any $(u,v), (u',v') \in E_{a,b}$, $\sigma_j^*((u,v)) = \sigma_{j'}^*((u',v'))$ implies $j = j'$. Indeed, for $x - k \leq i \leq k - 1$, if $\sigma_j^*((v_{3i+1}, v_{3i+2})) = \sigma_{j'}^*((v_{3i'+1}, v_{3i'+2}))$, then

$$\begin{cases} 3i + 1 + 3(j-1)\,(\mathrm{mod}\ 3)\,k = 3i' + 1 + 3(j'-1)\,(\mathrm{mod}\ 3)\,k, \\ 3i + 2 - 3(j-1)\,(\mathrm{mod}\ 3)\,k = 3i' + 2 - 3(j'-1)\,(\mathrm{mod}\ 3)\,k. \end{cases}$$

By substracting the two equations, since k is odd, and $1 \leq j, j' \leq k$, we get $j = j'$.

(iii) $i + 3(j - 1) \, (\mathrm{mod} \ 3) \, k = i + 3(j' - 1) \, (\mathrm{mod} \ 3) \, k$ implies $j = j'$, and $i - 3(j - 1) \, (\mathrm{mod} \ 3) \, k = i - 3(j' - 1) \, (\mathrm{mod} \ 3) \, k$ implies $j = j'$.

Thus, for every $v \notin F$ and $j \neq j'$, $\sigma_j(v) \neq \sigma_{j'}(v)$.

(iv) Since for every i, $i-1$ and $i+1$ have a different remainder in the Euclidean division by 3, and so do $3i - 9k + 1$ and $3i - 9k + 2$, we have that for every pair u, v where there exists $w \in F$ such that $(u, w), (w, v) \in E(C_n)$, $f(u) \neq f(v)$.
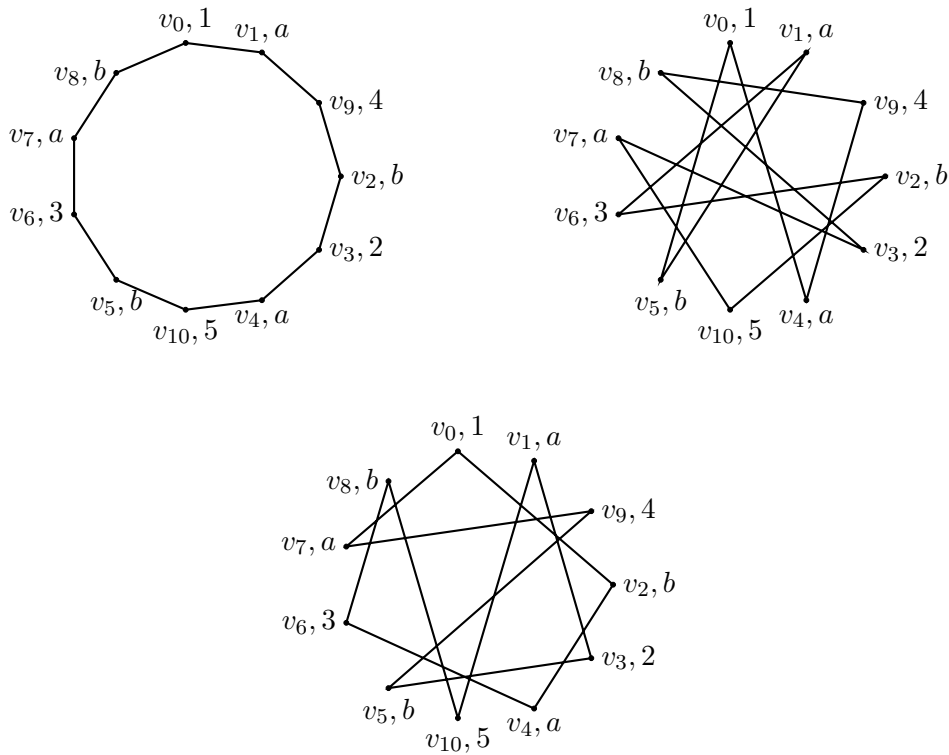

Figure 3. A 7-labeled packing of $k = 3$ copies of C_{11}.

This concludes Case 1 where k is odd.

Case 2. k is even and $x = 1$. We label v_{2k}, the fixed point, with label 1, and every other vertex with label b. Then, we create the first copy of C_n by joining the vertices in the following order

$$v_0, v_{n-2}, v_1, v_{n-3}, v_2, v_{n-4}, \ldots, v_{\frac{n-1}{2}-1}, v_{\frac{n-1}{2}}, v_{2k}, v_0.$$

We construct the other copies by following the same order but with every number added each time to 1 modulo $n - 1$. The corresponding expressions of f, σ, and

the corresponding partitions are

$$f(v_i) = \begin{cases} 1 & \text{if } i = 2k, \\ a & \text{otherwise.} \end{cases}$$

$$\sigma_j(v_i) = \begin{cases} v_i & \text{if } i = 2k, \\ v_{i+j-1\,(\mathrm{mod}\,2)\,k} & \text{otherwise.} \end{cases}$$

$F = \{v_{2k}\}$,
$V_a = \{v_i, 0 \le i \le 2k-1\}$.
$E_F = \{(v_0, v_{2k}), (v_k, v_{2k})\}$,
$E_{a,a} = \{(v_i, v_{2k-i}), 1 \le i \le k-1\} \cup \{(v_i, v_{2k-1-i}), 0 \le i \le k-1\}$.

The obtained graph G is isomorphic to C_n. Indeed, $G = \tau(H)$, with τ the permutation defined as follows.

$$\tau(v_i) = \begin{cases} v_{\frac{i}{2}} & \text{if } i < 2k \text{ and } i \equiv 0\,(\mathrm{mod}\,2), \\ v_{2k-\frac{i+1}{2}} & \text{if } i < 2k \text{ and } i \equiv 1\,(\mathrm{mod}\,2), \\ v_i & \text{if } i = 2k. \end{cases}$$

We now show that (f, σ) define a 2-labeled packing of k copies of C_n.

(i) Since $i + j - 1\,(\mathrm{mod}\,2)\,k < 2k$, we have, for every $v \notin F$, $f(\sigma_j(v)) = a = f(\sigma_{j'}(v))$.

(ii) Similarly to Case 1(a), for every p, q and $(u, v), (u', v') \in E_{p,q}$, $\sigma_j^*((u, v)) = \sigma_{j'}^*((u', v'))$ implies $j = j'$.

(iii) Similarly to Case 1(a), for every $v \notin F$ and $j \ne j'$, $\sigma_j(v) \ne \sigma_{j'}(v)$.

(iv) for every $j \ne j'$, with $1 \le j, j' \le k$, $0 + j - 1\,(\mathrm{mod}\,2)\,k \ne k + j' - 1\,(\mathrm{mod}\,2)\,k$. Thus, for every pair u, v where there exists $w \in F$ such that $(u, w), (w, v) \in E(C_n)$, if $f(u) = f(v)$, then $\sigma_j(u) \ne \sigma_{j'}(v)$ and $\sigma_j(v) \ne \sigma_{j'}(u)$.

The following example, for $k = 4$, gives an idea of the general construction for the case where k is even and $x = 1$.

Case 3(a). k is even and $2 \le x \le k-1$. For $x = 2$, the first copy of C_n follows the following sequence of vertices:

$$C_n = \begin{cases} \Big(v_0, v_2, v_{n-4}, v_4, v_{n-6}, v_6, \dots, v_{\frac{k}{2}}, v_{n-2}, v_1, v_3, v_{n-3}, v_5, v_{n-5}, \dots, \\ \quad v_{\frac{k}{2}+1}, v_{\frac{3k}{2}}, v_{\frac{k}{2}+3}, v_{\frac{3k}{2}-2}, v_{\frac{k}{2}+5}, v_{\frac{3k}{2}-4}, \dots, v_{\frac{3k}{2}+1}, v_{n-1}, v_0 \Big) \text{ if } k \equiv 0\,(\mathrm{mod}\,4) \\ \Big(v_0, v_2, v_{n-4}, v_4, v_{n-6}, v_6, \dots, v_{\frac{3k}{2}+1}, v_{n-2}, v_{n-3}, v_1, v_{n-5}, v_3, \dots, \\ \quad v_{\frac{3k}{2}}, v_{\frac{k}{2}+1}, v_{\frac{3k}{2}-2}, v_{\frac{k}{2}+3}, v_{\frac{3k}{2}-4}, v_{\frac{k}{2}+5}, \dots, v_{\frac{k}{2}}, v_{n-1}, v_0 \Big) \quad \text{if } k \equiv 2\,(\mathrm{mod}\,4). \end{cases}$$

We label v_{2k} and v_{2k+1}, the fixed points, with respective labels $2k$ and $2k+1$, and the other vertices with a if their number is even and b if it is odd. Then,

we create the other copies by adding each time 2 modulo $n - 2$ to every number but $2k$ and $2k + 1$ in this sequence. For $x > 2$, we add extra fixed points in some edges that are linking a vertex of label a and a vertex of label b. Thus, $G \simeq C_n$.
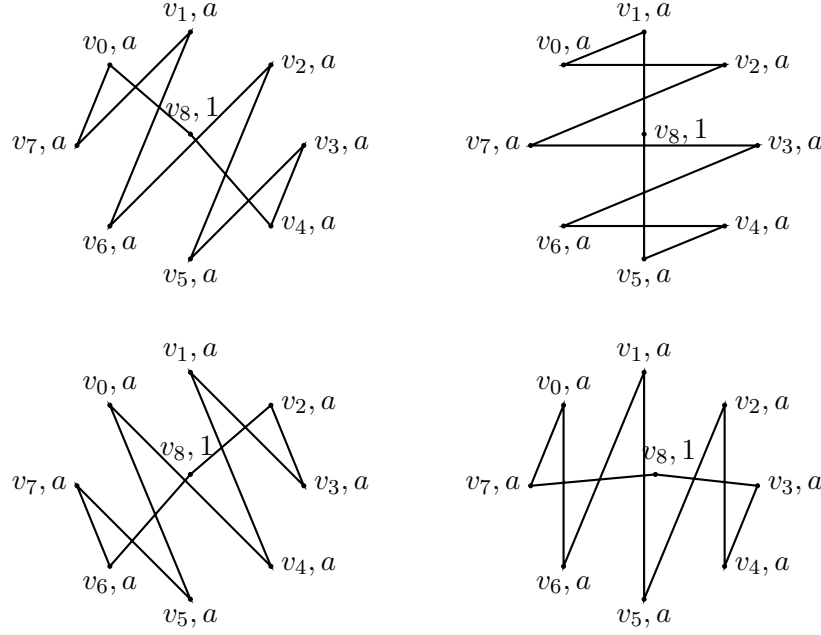

Figure 4. A 2-labeled packing of $k = 4$ copies of C_9.

More formally, we take

$$f(v_i) = \begin{cases} a & \text{if } 0 \leq i \leq 2k - 1 \text{ and } i \text{ is even,} \\ b & \text{if } 0 \leq i \leq 2k - 1 \text{ and } i \text{ is odd,} \\ i - 2k + 1 & \text{if } i \geq 2k. \end{cases}$$

$$\sigma_j(v_i) = \begin{cases} v_{i+2(j-1) \,(\text{mod } 2)\, k} & \text{if } i < 2k, \\ v_i & \text{if } i \geq 2k. \end{cases}$$

$F = \{v_i, 2k \leq i \leq n - 1\}$,
$V_a = \{v_i, 0 \leq i \leq 2k - 1 \text{ and } i \equiv 0 \,(\text{mod } 2)\}$,
$V_b = \{v_i, 0 \leq i \leq 2k - 1 \text{ and } i \equiv 1 \,(\text{mod } 2)\}$.

$E_F = \Big\{ \big(v_{\alpha(k,i)}, v_{k+3+i}\big), \big(v_{k+3+i}, v_{\alpha(k,i+1)}\big), k - 1 \leq i \leq k + x - 4 \Big\}$

$\qquad \cup \Big\{ \big(v_{\beta(k,1)}, v_{2k}\big), \big(v_{(-1)^{\frac{k}{2}} \,(\text{mod } 2)\, k}, v_{2k}\big) \Big\} \cup \Big\{ (v_0, v_{2k+1}), (v_{\beta(k,0)}, v_{2k+1}) \Big\}$,

$E_{a,a} = \Big\{ \big(v_{\gamma(k,i)}, v_{\gamma(k,i+1)}\big), 0 \leq i \leq \frac{k}{2} - 2 \Big\}$,

$$E_{b,b} = \left\{ \left(v_{(-1)^{\frac{k}{2}}+\gamma(k,i) \,(\mathrm{mod}\,2)\,k}, v_{(-1)^{\frac{k}{2}}+\gamma(k,i+1)\,(\mathrm{mod}\,2)\,k} \right) \right\}, 0 \le i \le \tfrac{k}{2} - 2 \right\},$$

$$E_{a,b} = \left\{ \left(v_{\alpha(k,i)}, v_{\alpha(k,i+1)} \right), k + x - 3 \le i \le 2k - 2 \right\}$$

with

$$\alpha(k,i) = (-1)^{\frac{k}{2}} \left(1 + 2 \left\lceil \frac{\frac{k}{2}-1}{2} \right\rceil + (-1)^{\frac{(-1)^i + 1}{2}} \cdot 2 \left\lfloor \frac{i-k+1}{2} \right\rfloor \right)$$
$$+ \frac{(-1)^i + 1}{2} \cdot \left(k + (-1)^{\frac{k}{2}-1} \right) (\mathrm{mod}\,2)\,k,$$

$$\beta(k,i) = \frac{k}{2} + \frac{(-1)^{\frac{k}{2}+i}+1}{2}(k+1)\,(\mathrm{mod}\,2)\,k,$$

$$\gamma(k,i) = 2 \cdot (-1)^i \left\lceil \frac{i+1}{2} \right\rceil (\mathrm{mod}\,2)\,k.$$

The obtained graph G is isomorphic to C_n, as for $x = 2$, $G = \tau(H)$, with τ the permutation defined as follows:

$$\tau(v_i) = \begin{cases} v_{2\cdot(-1)^{i+1}\left\lceil \frac{i}{2} \right\rceil (\mathrm{mod}\,2)\,k} & \text{if } 0 \le i \le \frac{k}{2} - 1, \\ v_{(-1)^{\frac{k}{2}}+2\cdot(-1)^{i+1-\frac{k}{2}}\left\lceil \frac{i-\frac{k}{2}}{2} \right\rceil (\mathrm{mod}\,2)\,k} & \text{if } \frac{k}{2} \le i \le k - 1, \\ v_{\alpha(k,i)} & \text{if } k \le i \le 2k - 1, \\ v_{\frac{k}{2}} & \text{if } i = 2k, \\ v_i & \text{if } i = 2k + 1. \end{cases}$$

For $x > 2$, the extra fixed points are inserted in some edges of C_{2k+2} which are linking a vertex of label a and a vertex of label b. Thus, $G \simeq C_n$.

Since $2k$ is even, all the vertices are always sent to vertices that have the same label. The neighbors of the fixed points always have different labels and are never sent to themselves, so that their edges will not create any problem for the packing. To see that the other edges do not overlap either, the principle is similar to the one of Walecki's construction [1]. Indeed, for every type of edge, or differently said, for every couple of labels an edge can link, we chose the difference of the numbers of the vertices to be unique modulo $2k$ in the sequence, so that no other edge can overlap it. To be more precise, we show that the four previously mentioned properties hold.

(i) Similarly to Case 1(a), for every $v \notin F$ and $j \ne j'$, $f(\sigma_j(v)) = f(\sigma_{j'}(v))$.

(ii) For every p, q and for every $(u, v), (u', v') \in E_{p,q}$, $\sigma_j^*((u, v)) = \sigma_{j'}^*((u', v'))$ implies $j = j'$.

Indeed, for $p = q = a$, for $0 \le i \le \frac{k}{2} - 2$, suppose that

$$\sigma_j^* \left(\left(v_{\gamma(k,i)}, v_{\gamma(k,i+1)} \right) \right) = \sigma_{j'}^* \left(\left(v_{\gamma(k,i')}, v_{\gamma(k,i'+1)} \right) \right).$$

Then, by adding the two equations resulting from the vertices, we get

$$2 \left((-1)^{i+1} \left\lceil \frac{i}{2} \right\rceil + (-1)^i \left\lceil \frac{i+1}{2} \right\rceil \right) + 2j$$
$$\equiv 2 \left((-1)^{i+1} \left\lceil \frac{i}{2} \right\rceil + (-1)^{i'} \left\lceil \frac{i'+1}{2} \right\rceil \right) + 2j' \,(\mathrm{mod}\,2)\,k.$$

Since k is even, this gives $j = j'$. The case where $p = q = b$ is similar.

For $p = a$ and $q = b$, for $k + x - 3 \le i \le 2k - 2$, suppose that

$$\sigma_j^* \left((v_{\alpha(k,i)}, v_{\alpha(k,i+1)}) \right) = \sigma_{j'}^* \left((v_{\alpha(k,i')}, v_{\alpha(k,i'+1)}) \right).$$

Then, by adding the two equations, we get

$$2 \left((-1)^{\frac{(-1)^i+1}{2}} \left\lfloor \frac{i-k+1}{2} \right\rfloor + (-1)^{\frac{(-1)^{i+1}+1}{2}} \left\lfloor \frac{i-k+2}{2} \right\rfloor \right) + 4j$$
$$\equiv 2 \left((-1)^{\frac{(-1)^{i'}+1}{2}} \left\lfloor \frac{i'-k+1}{2} \right\rfloor + (-1)^{\frac{(-1)^{i'+1}+1}{2}} \left\lfloor \frac{i'-k+2}{2} \right\rfloor \right) + 4j' \pmod 2 k.$$

Since k is even, this also gives $j = j'$.

(iii) Similarly to Case 1(a), for every $v \notin F$ and $j \ne j'$, $\sigma_j(v) \ne \sigma_{j'}(v)$.

(iv) Since $k + (-1)^{\frac{k}{2}-1}$ is odd, $\alpha(k,i)$ and $\alpha(k,i+1)$ have different parity for every i such that $k-1 \le i \le k+x-4$. Plus, $\beta(k,1)$ is even while $(-1)^{\frac{k}{2}} \pmod 2 k$ is odd, and 0 is even while $\beta(k,0)$ is odd. Thus, for every pair u, v where there exists $w \in F$ such that $(u,w), (w,v) \in E(C_n)$, $f(u) \ne f(v)$.
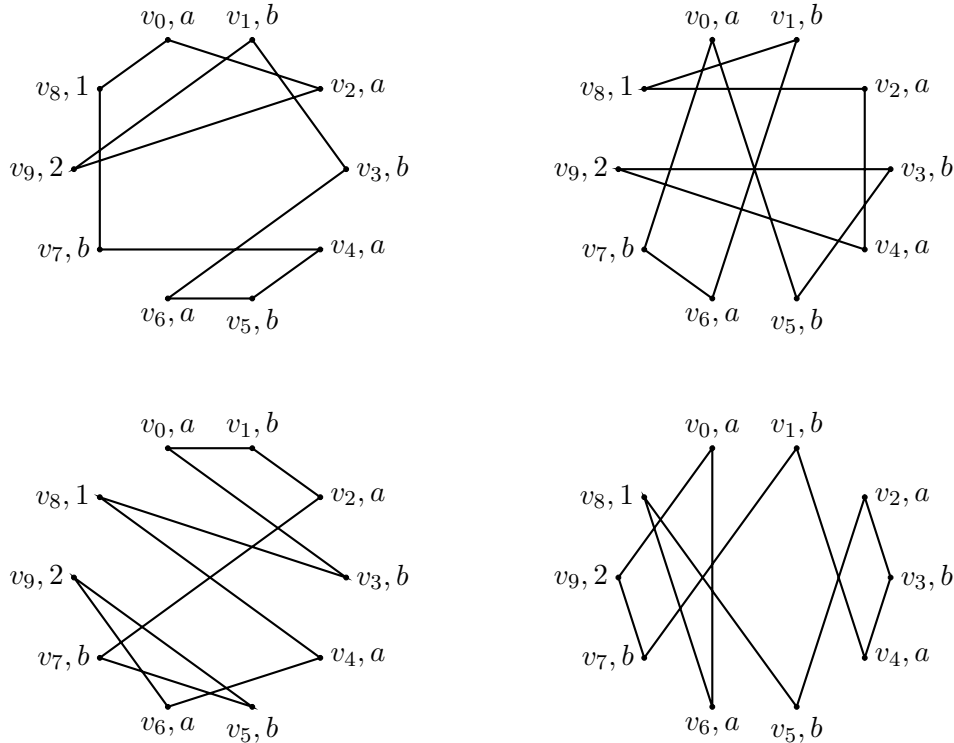

Figure 5. A 4-labeled packing of $k = 4$ copies of C_{10}.

Case 3(b). k is even and $k \leq x \leq 2k - 1$. Figure 6 represents the example of $k = 4$ and $n = 12$. For $x = k$, we build a $(k + 2)$-labeled packing of G in the following way. We label the vertices of number $i \leq 2k$ by their number, and the other vertices with a if their number is even and b if their number is odd. We create the first copy of C_n by joining the vertices in the following order:

$$v_{2k}, v_0, v_{2k+1}, v_1, \ldots, v_{3k-1}, v_{k-1}, v_{2k-2}, v_{k+1}, v_{2k-4}, v_{k+3}, v_{2k-6}, \ldots,$$
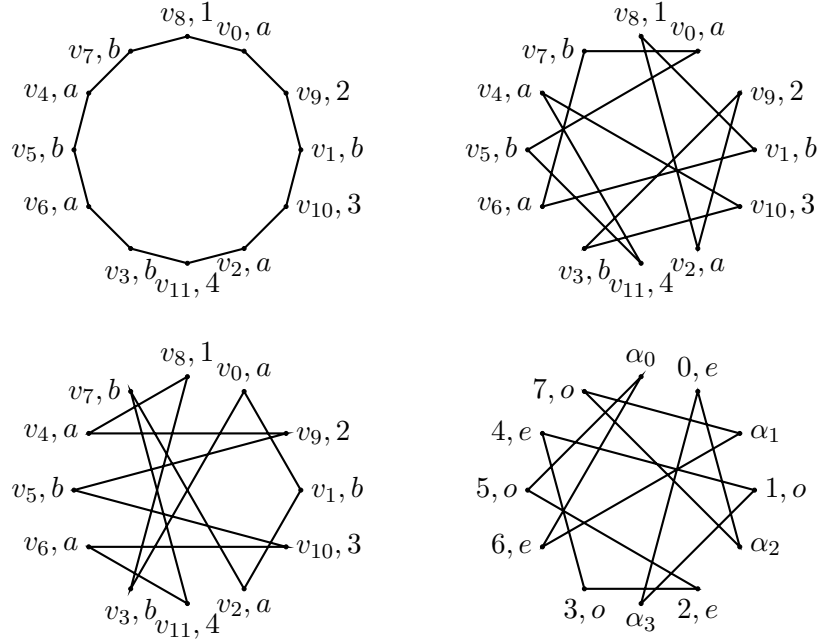
$$v_{2k-k}, v_{k+(k-1)}, v_{2k}.$$

Figure 6. A 6-labeled packing of $k = 4$ copies of C_{12}.

The other copies are created by adding 2 modulo $n - 2$ to the numbers of the a and b label vertices of this sequence. For $x > k$, we add fixed points on the edges linking a vertex of label a to a vertex of label b. We have

$$f(v_i) = \begin{cases} a & \text{if } 0 \leq i \leq 2k - 1 \text{ and } i \text{ is even,} \\ b & \text{if } 0 \leq i \leq 2k - 1 \text{ and } i \text{ is odd,} \\ i - 2k + 1 & \text{if } i \geq 2k. \end{cases}$$

$$\sigma_j(v_i) = \begin{cases} v_{i+2(j-1) \pmod 2) k} & \text{if } i < 2k, \\ v_i & \text{if } i \geq 2k. \end{cases}$$

$$F = \{v_i, 2k \leq i \leq n - 1\},$$

$V_a = \{v_i, 0 \le i \le 2k - 1 \text{ and } i \equiv 0 \,(\mathrm{mod}\,2)\},$

$V_b = \{v_i, 0 \le i \le 2k - 1 \text{ and } i \equiv 1 \,(\mathrm{mod}\,2)\}.$

$E_F = \Big\{ \big(v_{2k+i}, v_{i-1\,(\mathrm{mod}\,2)\,k}\big), (v_{2k+i}, v_i), \ 0 \le i \le k - 1 \Big\}$

$\qquad \cup \Big\{ \big(v_{4k-2\lfloor \frac{i}{2}\rfloor -2}, v_{k+i}\big), \big(v_{k+i}, v_{2\lfloor \frac{i}{2}\rfloor +(-1)^{i+1}-k}\big), 2k \le i \le k + x - 1 \Big\},$

$E_{a,a} = E_{b,b} = \emptyset,$

$E_{a,b} = \Big\{ \big(v_{4k-2\lfloor \frac{i}{2}\rfloor -2}, v_{2\lfloor \frac{i}{2}\rfloor +(-1)^{i+1}-k}\big), k + x \le i \le 3k - 1 \Big\}.$

For $x = k$, $G = \tau(H)$, with τ the permutation defined as follows, so that G is isomorphic to C_n.

$$\tau(v_i) = \begin{cases} v_{2k+\frac{i}{2}} & \text{if } 0 \le i \le 2k - 1,\ i \equiv 0\,(\mathrm{mod}\,2), \\ v_{\frac{i-1}{2}} & \text{if } 0 \le i \le 2k - 1,\ i \equiv 1\,(\mathrm{mod}\,2), \\ v_{4k-i-2} & \text{if } 2k \le i \le 3k - 1,\ i \equiv 0\,(\mathrm{mod}\,2), \\ v_{i-k} & \text{if } 2k \le i \le 3k - 1,\ i \equiv 1\,(\mathrm{mod}\,2). \end{cases}$$

For $x > 2$, the extra fixed points are inserted in some edges of C_{3k}.

In this construction, all the fixed points have one neighbor with label a and one neighbor with label b, that are never sent to themselves or each other. The edges involving them will thus never overlap. The other edges link a vertex of label a with a vertex of label b, and the differences between their numbers are all distinct. Plus, as $2k$ is even, the parity of the vertices remains the same in all the copies, so do their labels. To be more precise, we have the following.

(i) Similarly to Case 1(a), for every $v \notin F$ and $j \ne j'$, $f(\sigma_j(v)) = f(\sigma_{j'}(v))$.

(ii) For every p, q and for every $(u, v), (u', v') \in E_{p,q}$, $\sigma_j^*((u, v)) = \sigma_{j'}^*((u', v'))$ implies $j = j'$.

Indeed, suppose, for a given i, $k + x \le i \le 3k - 1$, that

$$\sigma_j^*\Big(\big(v_{4k-2\lfloor \frac{i}{2}\rfloor -2}, v_{2\lfloor \frac{i}{2}\rfloor +(-1)^{i+1}-k}\big)\Big) = \sigma_{j'}^*\Big(\big(v_{4k-2\lfloor \frac{i'}{2}\rfloor -2}, v_{2\lfloor \frac{i'}{2}\rfloor +(-1)^{i'+1}-k}\big)\Big).$$

Then, by adding the two resulting equations, we get

$$4j + (-1)^{i+1}(\mathrm{mod}\,2)\,k = 4j' + (-1)^{i'+1}(\mathrm{mod}\,2)\,k.$$

Since the parity of both sides of this equality must be the same, this gives $j = j'$.

(iii) Similarly to Case 1(a), for every $v \notin F$ and $j \ne j'$, $\sigma_j(v) \ne \sigma_{j'}(v)$.

(iv) Since for every i with $2k \le i \le k + x - 1$, $4k - 2\lfloor \frac{i}{2}\rfloor - 2$ is even, and $2\lfloor \frac{i}{2}\rfloor + (-1)^{i+1} - k$ is odd, they have different parity, and so do $i - 1 \mod 2k$ and i for every i with $0 \le i \le k - 1$. We therefore have, for every u, v where there exists $w \in F$ such that $(u, w), (w, v) \in E(C_n)$, $f(u) \ne f(v)$.

This concludes Case 3, and thus the entire proof of Theorem 14. ∎

Concerning the upper bound given by Conjecture 7, we have already seen, with a counterexample, that it did not hold in general. That being said, we still found some sufficient conditions for it to hold. The following lemma helped in the process of finding those conditions.

Lemma 15. *Let G be the cycle C_n of order $n = 2k+x$, $k \geq 2$, $1 \leq x \leq 2k-1$. Let f be a p-labeling of k copies of G. Let $q = \min_{i \in \{1,\ldots,p\}} |\{v \in V(C_n) : f(v) = i\}|$ be the minimum, over all p labels, of the number of vertices that have this label. Then, there exists in G a set of at least $2k$ vertices that are associated, together, to at most $2q$ labels.*

__Proof.__ Let j be one of the labels associated to exactly q vertices, and V_j be the set of vertices of G with label j. All the vertices of V_j have two neighbors. Let N_j be the set of neighbors, and $M_j = N_j - (N_j \cap V_j)$ be the set of neighbors that are not in V_j. Let L be the number of different labels $l_1, l_2, \ldots, l_L$ associated to the vertices of $M_j, V(l_1), V(l_2), \ldots, V(l_L)$ be the sets of vertices of G associated to those respective labels, and $U = V(l_1) \cup V(l_2) \cup \cdots \cup V(l_L)$. Let e be the number of edges linking two vertices from V_j in G, and, for every $1 \leq i \leq L$, e_i be the number of edges linking a vertex from V_j to a vertex from $V(l_i)$ in G. To have a valid packing, we must have, for every i with $1 \leq i \leq L$, $ke_i \leq q|V(l_i)|$. Therefore, we have

$$\sum_{i=1}^{L} e_i \leq \frac{q}{k} \sum_{i=1}^{L} |V(l_i)|.$$

But $\sum_{i=1}^{L} e_i = 2q - 2e$, and, since the $V(l_i)$ are pairwise disjoint, $\sum_{i=1}^{L} |V(l_i)| = |U|$. Therefore,

$$|U| \geq \frac{k}{q}(2q - 2e).$$

If $e = 0$, we obtain $|U| \geq 2k$. Moreover, at best, in M_j, we have $2q$ distinct vertices that all have distinct labels, so that $L \leq 2q$. By taking the set U, we obtain a set of at least $2k$ vertices in G associated to at most $2q$ labels. If $e \geq 1$, to have a valid packing, we must have $ke \leq \frac{q(q-1)}{2}$, we thus have $|U| \geq 2k - (q-1)$. Plus, at best, in M_j, we have $2q - 2$ distinct vertices that all have distinct labels, so that $L \leq 2q - 2$. Therefore, by taking the set $U \cup V_j$, we obtain a set of at least $2k + 1$ vertices in G associated to at most $2q - 1$ labels. Those two cases give the result. ∎

Lemma 15 led, in particular, to the following theorem, that gives a sufficient condition for the upper bound given by Conjecture 7 to hold.

Theorem 16. *Let G be the cycle C_n of order $n = 2k+x$, $k \geq 2$, $1 \leq x \leq 2k-1$. Let f be a p-labeling of k copies of G. Let $q = \min_{i \in \{1,\ldots,p\}} |\{v \in V(C_n) : f(v) = $*

$i\}|$ be the minimum, over all p labels, of the number of vertices that have this label. If $q = 1$, meaning that f has fixed points, or $q \leq \frac{x}{2}$, then $p \leq x + 2$.

Proof. We know by Lemma 15 that there exists in G a group of at least $2k$ vertices associated to at most $2q$ labels, and the rest of the labels are represented at least q times. Therefore, we have

$$p \leq 2q + \left\lfloor \frac{n - 2k}{q} \right\rfloor \leq 2q + \frac{x}{q}.$$

But, for every $q \geq 1$, $2q + \frac{x}{q} \leq x + 2$. Indeed, for q=1, we have $2q + \frac{x}{q} = x + 2$. For $q > 1$, $2q + \frac{x}{q} \leq x + 2 \leftrightarrow q \leq \frac{x}{2}$, which is true by hypothesis. Therefore, we have $p \leq x + 2$. ∎

The disadvantage of Theorem 16 is that the conditions it gives are on a preexisting packing, while we want the upper bound to hold for every packings, with conditions on k and x only. Such conditions are thus given in the following theorem.

Theorem 17. Let C_n be the cycle of order $n = 2k + x$, $k \geq 2$, $1 \leq x \leq 2k - 1$. If $x \geq \sqrt{4k - 2}$, then $\lambda^k(C_n) \leq x + 2$.

Proof. Let us suppose that $\lambda^k(C_n) \geq x+3$. Then, there exists an $(x+3)$-labeling f of k copies of G. Let q be the minimum, over all p labels, of the number of vertices that have this label. Then, by Theorem 16, we have $q \geq \frac{x+1}{2}$. We have therefore at least $x + 3$ labels, each represented by at least $\frac{x+1}{2}$ vertices, so that we have

$$\frac{(x + 1)(x + 3)}{2} \leq 2k + x$$

or

$$x^2 + 2x + (3 - 4k) \leq 0.$$

For $x \geq \sqrt{4k - 2}$, this last inequality is false and we obtain a contradiction. ∎

We now have a satisfying sufficient condition for the upper bound of $x + 2$ to be valid, even more satisfying when taking into account the fact that the proportion of cases it covers grows with n. That being said, this bound is not the one given in Conjecture 7 for the particular case where k is even and $x = 1$. The following theorem thus gives a sufficient condition for this case, that depends on the considered packing.

Theorem 18. Let C_n be the cycle of order $n = 2k + 1$, $k \geq 2$, k even. Let f be a p-labeling of k copies of C_n. Let $q = \min_{i \in \{1,...,p\}} |\{v \in V(C_n) : f(v) = i\}|$ be the minimum, over all p labels, of the number of vertices that have this label. If $q = 1$, then $p \leq 2$.

Proof. From Theorem 16, since $q = 1$, we already have $\lambda^k(C_n) \leq 3$. Let us assume that $\lambda^k(C_n) = 3$. Then, there exists a 3-labeled packing of k copies of C_n. By Lemma 15, we know that there exists in this packing a set of $2k$ vertices associated to exactly 2 labels, called a and b, and one fixed point.

We have $k(2k + 1) = \frac{(2k+1)(2k+1-1)}{2}$, so that every edge between any two vertices is going to belong to exactly one of k the copies of G. In particular, every edge between the fixed point and the other vertices is going to belong to a copy, so that the fixed point has to be linked to exactly one vertex with label a, and one vertex with label b. Therefore, there has to be k vertices with label a and k vertices with label b. The number of edges linking two vertices of label a in G is therefore $\frac{k(k-1)}{2k} = \frac{k-1}{2}$, which means that k is odd, and contradicts the hypothesis. Therefore, $\lambda^k(C_n) \neq 3$, and $\lambda^k(C_n) \leq 2$. ∎

For the cases where the previously seen conditions are not verified, the following theorem gives a necessary condition on p for a p-labeled packing of C_{2k+x} to exist and thus provides a different upper bound on $\lambda^k(C_{2k+x})$.

Theorem 19. *Let C_n be the cycle of order $n = 2k+x$. For $p \in \mathbb{N}^*$, if $p \leq \lambda^k(C_n)$, there exists a partition $n_1, n_2, \ldots, n_p$ of n into p parts such that $\sum_{i=1}^{p} \left\lfloor \frac{n_i(n_i-1)}{2k} \right\rfloor + \sum_{i=1}^{p} \sum_{j=i+1}^{p} \left\lfloor \frac{n_i n_j}{k} \right\rfloor \geq n$.*

Proof. Since $p \leq \lambda^k(C_n)$, there exists a p-labeled packing f of k copies of C_n into K_n. Let $1, 2, \ldots, p$ be the name of the labels of f, and $n_1, n_2, \ldots, n_p$ be the number of vertices respectively associated to those labels. Obviously, $n_1, n_2, \ldots, n_p$ is a partition of n into p parts.

k	x	n	maximal possible value of $\lambda^k(C_n)$
9	3	21	$x + 4$
15	6	36	$x + 3$
16	4	36	$x + 5$
16	5	37	$x + 3$
24	7	55	$x + 4$
25	5	55	$x + 6$
25	6	56	$x + 4$
27	3	57	$x + 4$
28	4	60	$x + 3$
33	4	70	$x + 3$
33	5	71	$x + 3$
34	10	78	$x + 3$
35	4	74	$x + 3$
35	8	78	$x + 5$
35	9	79	$x + 3$

Table 1. Values of k and x, until $k = 35$, for which the second inequality of Conjecture 7 is unproven. The upper bound given by Theorem 19 is also expressed, depending on x.

For every $i \in [p]$, $j \in |[i+1, p]|$, let m_i be the number of edges of C_n in f linking two vertices of label i, and let $m_{i,j}$ be the number of edges of C_n in f linking a vertex of label i and a vertex of label j.

We have $\sum_{i=1}^{p} m_i + \sum_{i=1}^{p} \sum_{j=i+1}^{p} m_{i,j} = n$.

For f to be a valid packing, we must have, for every $i \in [p]$, $m_i \leq \left\lfloor \frac{n_i(n_i-1)}{2k} \right\rfloor$, and, for every $j \in |[i+1, p]|$, $m_{i,j} \leq \sum_{j=i+1}^{p} \left\lfloor \frac{n_i n_j}{k} \right\rfloor$.

Thus, $\sum_{i=1}^{p} \left\lfloor \frac{n_i(n_i-1)}{2k} \right\rfloor + \sum_{i=1}^{p} \sum_{j=i+1}^{p} \left\lfloor \frac{n_i n_j}{k} \right\rfloor \geq n$. ∎

By regrouping the sufficient conditions of Theorems 11, 16 and 17, we can already restrict the possible values of k and x for which the second inequality of Conjecture 7 might not hold. Table 1 lists those values and gives the upper bound given by Theorem 19 for each of those, until $k = 35$.

4. LABELED PACKING OF CIRCUITS

Recall that the circuit $\overrightarrow{C_n}$ of order n is the directed graph defined by $V(\overrightarrow{C_n}) = \{v_i, 0 \leq i \leq n-1\}$ and $E(\overrightarrow{C_n}) = \{(v_i, v_{i+1 \mod n}), 0 \leq i \leq n-1\}$. We define a packing of directed graphs $(H_1, H_2, \ldots, H_k)$ in the directed graph G the same way as a packing of graphs, the only difference being that for every i, $E(H_i)$ stands for the arcs of H_i, and $E(G)$ for the arcs of G, so that the induced images of the arcs of the H_i must go into the arcs of G without intersecting. We focus on the packings of k copies of H into the complete digraph $\overleftrightarrow{K_n}$, the digraph of order n with all possible arcs. Given those modifications, the definitions of a labeled packing and of $\lambda^k(G)$ for G a digraph are direct.

The results we present are very similar to the ones presented for cycles, with some adaptations. We first have the following conditions for the p-labeled k-packing to exist.

Theorem 20. $\overrightarrow{C_{k+x}}$ *admits a p-labeled k-packing if and only if $x \geq 1$, and $(x, n) \notin \{(1, 4), (1, 6)\}$.*

Proof. First, for a p-labeled k-packing of $\overrightarrow{C_n}$ to exist, we must have $k|E(\overrightarrow{C_n})| \leq |E(\overleftrightarrow{K_n})|$, so that $n \geq k+1$. Plus, if $x = 1$, the p-labeled k-packing would actually be a decomposition of $\overleftrightarrow{K_n}$, and we know from Theorem 1.1 in [2] that such a decomposition exists if and only if $n \neq 4$, and $n \neq 6$. Now, since the existence of a packing of k copies of $\overrightarrow{C_{k+x}}$ implies the existence of a packing of $k-1$ copies of $\overrightarrow{C_{(k-1)+(x+1)}}$, and since we have the following packings of $k = 2$ copies of $\overrightarrow{C_4}$ and $k = 4$ copies of $\overrightarrow{C_6}$, this gives the result.
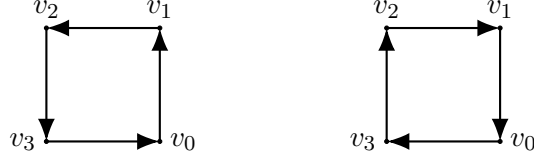
Figure 7. A packing of $k = 2$ copies of $\vec{C}_4$.
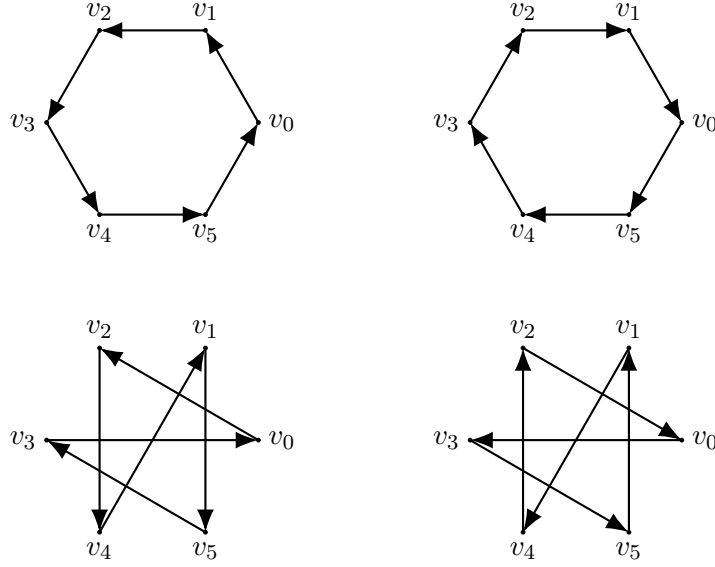
Figure 8. A packing of $k = 4$ copies of $\vec{C}_6$. ∎

When those conditions are satisfied, we can study the value of $\lambda^k(\overrightarrow{C_n})$. We first have the following lemma, that is an extension of Duchêne, Kheddouci, Nowakowski and Tahraoui's Lemma 7 [4] to the case of circuits, following the same proof.

Lemma 21. *For every circuit $\overrightarrow{C_n}$ of order $n > k$, with $(x,n) \notin \{(1,4),(1,6)\}$, we have $\lambda^k(\overrightarrow{C_n}) \leq \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{\lceil \frac{n}{2} \rceil}{k} \right\rfloor$.*

When $n \geq 2k$, the value of $\lambda^k(\overrightarrow{C_n})$ can be exactly found with the following theorem.

Theorem 22. *For every circuit $\overrightarrow{C_n}$ of order $n = 2km + x$, $k, m \geq 1$, $x < 2k$, and $(x,n) \notin \{(1,4),(1,6)\}$ we have $\lambda^k(\overrightarrow{C_n}) = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{\lceil \frac{n}{2} \rceil}{k} \right\rfloor$.*

Proof. From Lemma 21, we get the upper bound. For the lower bound, we give a construction of the corresponding labeled packing of $\overrightarrow{C_n}$. An example of the construction is given in Figure 9, for $k = 3$ and $n = 7$. The idea is to put a fixed point one vertex out of two on the first copy of $\overrightarrow{C_n}$, and to have at least k vertices of any other label so that they will never be sent to themselves and thus never create superpositions of arcs.

Let $V(C_n) = \{v_0, \ldots, v_n\}$ be the set of vertices of $\overrightarrow{C_n}$ and $E(C_n) = \{(v_i, v_{i+1 \,(\mathrm{mod}\,n)}), 0 \leq i \leq n-1\}$ be its set of arcs.

We label the vertices with the labeling f:

$$
f(v_i) = \begin{cases} v_{\frac{i}{2}+1} & \text{if } 0 \leq i \leq 2\left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right) \text{ and } i \text{ is even}, \\ v_{\left\lfloor \frac{i}{2k} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + 1} & \text{if } 1 \leq i \leq 2k\left\lfloor \frac{\left\lceil \frac{n}{2} \right\rceil}{k} \right\rfloor - 1 \text{ and } i \text{ is odd}, \\ v_{\left\lfloor \frac{i}{2k} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor} & \text{otherwise.} \end{cases}
$$

For each label l, we rename the L vertices that have label l, following the increasing order of their number, into $l_0, l_1, \ldots, l_L$, and we associate to the labeling the set of permutations $\sigma = \{\sigma_j, 1 \leq j \leq k\}$:

$$
\begin{cases} \sigma_j(l_i) = l_{i+1 \,(\mathrm{mod}\,L)} & \text{for every } l, \\ \sigma_j(v_i) = v_i & \text{if } i \text{ is even.} \end{cases}
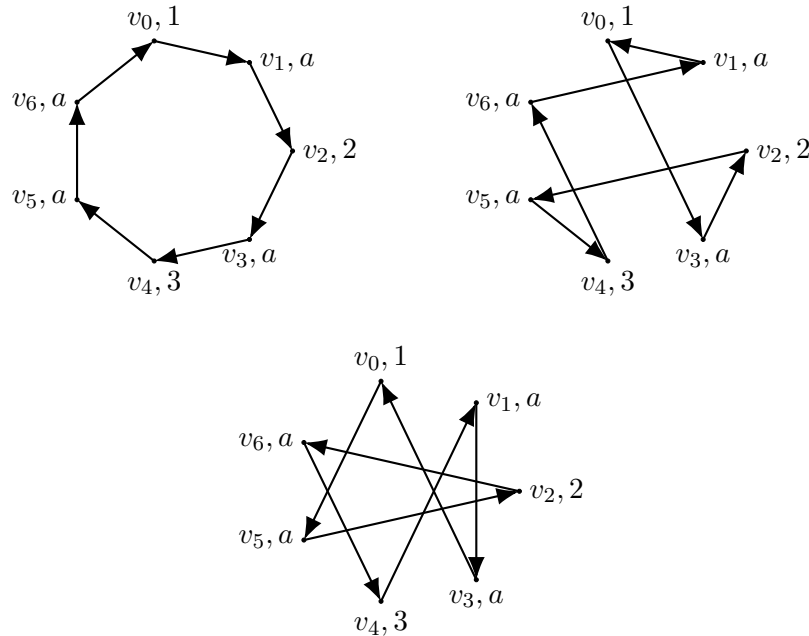$$

Figure 9. A 4-labeled packing of $k = 3$ copies of $\vec{C}_7$.

For any j, a vertex and its image by σ_j obviously have the same label. Plus, since for every i with $0 \leq i \leq n - 1$ and i even, v_i is a fixed points, and for every i odd and, $j \neq j'$, $\sigma_j(v_i) \neq \sigma_{j'}(v_i)$, σ is a packing. ∎

The case where $k + 1 \leq n \leq 2k - 1$ is more complicated. We present here two theorems for this case, one for k even, the second for k odd.

Theorem 23. *For every circuit $\overrightarrow{C_n}$ of order $n = k + x$, $k \geq 2$, $1 \leq x \leq k - 1$, k even, $(x, n) \notin \{(1, 4), (1, 6)\}$, we have $\lambda^k(\overrightarrow{C_n}) \geq x + 1$.*

Proof. We give the associated construction. An example of the construction is given in Figure 10, for $k = 4$ and $x = 3$. Let $\{v_0, v_1, \ldots, v_{n-1}\}$ be the vertices of $\overrightarrow{C_n}$. We label them with f:

$$f(v_i) = \begin{cases} i - k + 1 & \text{if } i \geq k, \\ a & \text{otherwise.} \end{cases}$$

We associate to it the set of permutations $\sigma = \{\sigma_j, 1 \leq j \leq k\}$:

$$\sigma_j(v_i) = \begin{cases} v_i & \text{if } i \geq k, \\ v_{i+j-1 \,(\text{mod } k)} & \text{otherwise.} \end{cases}$$

We partition the set of vertices $V(\overrightarrow{C_n})$ into a set F containing all the fixed points, and a set V_a containing all the vertices of label a. We partition the set of arcs $E(\overrightarrow{C_n})$ into two sets E_F and $E_{a,a}$ defined as follows.

$$E_F = \left\{ (v_{i-1}, v_{k+i}), (v_{k+i}, v_{k-i}), 1 \leq i \leq x - 1 \right\} \cup \left\{ (v_k, v_0), \left(v_{\frac{k}{2}}, v_k \right) \right\},$$

$$E_{a,a} = \left\{ (v_{k-i}, v_i), 1 \leq i \leq \frac{k}{2} - 1 \right\} \cup \left\{ (v_i, v_{k-1-i}), x - 1 \leq i \leq \frac{k}{2} - 1 \right\}.$$

The obtained graph G is isomorphic to $\overrightarrow{C_n}$. Indeed, for $x = 1$, if $H = (V(H) = \{v_0, v_2, \ldots, v_{n-1}\}, E(H) = \{(v_i, v_{i+1 \bmod n}), 0 \leq i \leq n - 1\})$, we have $H \simeq \overrightarrow{C_n}$, and $G = \tau(H)$, with

$$\tau(v_i) = \begin{cases} v_{\frac{i}{2}} & \text{if } i < k, i \equiv 0 \,(\text{mod } 2), \\ v_{k - \frac{i+1}{2}} & \text{if } i < k, i \equiv 1 \,(\text{mod } 2), \\ v_i & \text{if } i = k. \end{cases}$$

For $x > 1$, the extra fixed point are inserted in some of the edges of $\overrightarrow{C_{k+1}}$, so that $G \simeq \overrightarrow{C_n}$.

For every $v \in F$, since $\sigma_j(v) = v$, we know that $f(\sigma_j(v)) = f(v) = f(\sigma_{j'}(v))$. Since $i + j - 1 \bmod k < k$, for any $v \notin F$, $f(\sigma_j(v)) = a = f(\sigma_{j'}(v))$, and f is a valid labeled packing with respect to σ.

Thus, in particular, for every $j, j', \sigma_j^*(E_F) \cap \sigma_{j'}^*(E_{a,a}) = \emptyset$. For every (u, v), $(u', v') \in E_{a,a}$, $\sigma_j^*((u, v)) = \sigma_{j'}^*((u', v'))$ implies $j = j'$.

Indeed, for $1 \le i \le \frac{k}{2} - 1$, if $\sigma_j^*((v_{k-i}, v_i)) = \sigma_{j'}^*((v_{k-i'}, v_{i'}))$, then

$$\begin{cases} i + j - 1 = i' + j' - 1 \mod k, \\ k - i + j - 1 = k - i' + j' - 1 \mod k. \end{cases}$$

By adding the two equations, since k is even, and $1 \le j, j' \le k$, we get $j = j'$, or $j' = j + \frac{k}{2}$. But injecting $j' = j + \frac{k}{2}$ into the first equation gives $i \equiv i' + \frac{k}{2}$ mod k, which is a contradiction, as $1 \le i' \le \frac{k}{2} - 1$.

Similarly, for $x - 1 \le i, i' \le k - 1$, if $\sigma_j^*((v_i, v_{k-1-i})) = \sigma_{j'}^*((v_{i'}, v_{k-1-i'}))$, $j = j'$.

For $1 \le i \le k - 1$ and $x - 1 \le i' \le k - 1$, $\sigma_j^*((v_{k-i}, v_i)) = \sigma_{j'}^*((v_{i'}, v_{k-1-i'}))$ is impossible, as the sum of both equations gives an equality between an odd number and an even number.

Thus, for every $j \ne j'$, $\sigma_j^*(E_{a,a}) \cap \sigma_{j'}^*(E_{a,a}) = \emptyset$.

To show that $\sigma_j^*(E_F) \cap \sigma_{j'}^*(E_F) = \emptyset$, since each edge of E_F contains one fixed point, we only have to show that its ingoing neighbor u and outgoing neighbor v verify $\sigma_j(u) \ne \sigma_{j'}(u)$ and $\sigma_j(v) \ne \sigma_{j'}(v)$. But since $i + j - 1 \mod k = i + j' - 1$ mod k implies $j = j'$, for every $v \notin F$ and $j \ne j'$, $\sigma_j(v) \ne \sigma_{j'}(v)$.
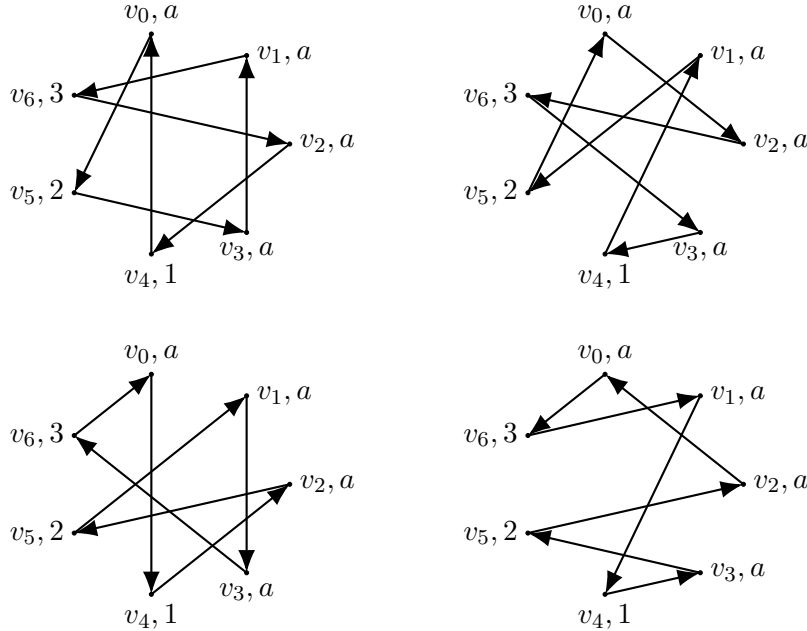

Figure 10. A 4-labeled packing of $k = 4$ copies of $\vec{C}_7$.

This concludes the proof of the theorem. ∎

The following result, for the case where k is odd, can be deduced from the previous one.

Theorem 24. *For every circuit $\overrightarrow{C_n}$ of order $n = k + x$, $k \geq 2$, $2 \leq x \leq k - 1$, k odd, $(x, n) \notin \{(1, 4), (1, 6)\}$, we have $\lambda^k(\overrightarrow{C_n}) \geq x$.*

Proof. We have $n = (k + 1) + (x - 1)$ with $k + 1$ even, and $x - 1 \geq 1$. Thus, by Theorem 22, there exists a x-labeled packing of $k + 1$ copies of $\overrightarrow{C_n}$.   ∎

The next results are interested in finding an upper bound. The proofs are not given as they are similar to the ones for cycles.

The first result is the analogue of Lemma 15 for the case of circuits. The proof is identical, with the exception that we only count the outgoing neighbors and outgoing arcs of the vertices that have one of the labels that are represented by q vertices.

Lemma 25. *Let $\overrightarrow{C_n}$ be the circuit of order $n = k + x$, $k \geq 2$, $1 \leq x \leq k - 1$, $(x, n) \notin \{(1, 4), (1, 6)\}$. Let f be a p-labeling of k copies of $\overrightarrow{C_n}$. Let $q = \min_{i \in \{1, \dots, p\}} |\{v \in V(C_n) \text{ such that } f(v) = i\}|$ be the minimum, over all p labels, of the number of vertices that have this label. Then, there exists in $\overrightarrow{C_n}$ a set of at least k vertices that are associated, together, to at most q labels.*

This lemma leads to the following two theorems.

Theorem 26. *Let $\overrightarrow{C_n}$ be the circuit of order $n = k + x$, $k \geq 2$, $1 \leq x \leq k - 1$, $(x, n) \notin \{(1, 4), (1, 6)\}$. Let f be a p-labeling of k copies of $\overrightarrow{C_n}$. Let $q = \min_{i \in \{1, \dots, p\}} |\{v \in V(\overrightarrow{C_n}) : f(v) = i\}|$ be the minimum, over all p labels, of the number of vertices that have this label. If $q \leq x$, then $p \leq x + 2$.*

As it was the case for cycles, the previous theorem gives conditions on the considered packing, while we want conditions on k and x only. Thus, we give the following result.

Theorem 27. *Let $\overrightarrow{C_n}$ be the circuit of order $n = k + x$, $k \geq 2$, $1 \leq x \leq k - 1$, $(x, n) \notin \{(1, 4), (1, 6)\}$. If $x \geq \sqrt{k - 1}$, then $\lambda^k(\overrightarrow{C_n}) \leq x + 2$.*

Finally, considering necessary conditions on the number of arcs between each pair of labels gives an analogue of Theorem 19 for circuits.

Theorem 28. *Let $\overrightarrow{C_n}$ be the circuit of order $n = k + x$, $(x, n) \notin \{(1, 4), (1, 6)\}$. For $p \in \mathbb{N}^*$, if $p \leq \lambda^k(\overrightarrow{C_n})$, there exists a partition $n_1, n_2, \dots, n_p$ of n into p parts such that $\sum_{i=1}^{p} \left\lfloor \frac{n_i(n_i - 1)}{k} \right\rfloor + \sum_{i=1}^{p} \sum_{j=i+1}^{p} \left\lfloor \frac{2 n_i n_j}{k} \right\rfloor \geq n$.*

References

[1] B. Alspach, *The wonderful Walecki construction*, Bull. Inst. Combin. Appl. **52** (2008) 7–20.

[2] B. Alspach, H. Gavlas, M. Šajna and H. Verrall, *Cycle decompositions* IV*: complete directed graphs and fixed length directed cycles*, J. Combin. Theory Ser. A **103** (2003) 165–208.
https://doi.org/10.1016/S0097-3165(03)00098-0

[3] B. Bollobás and S.E. Eldridge, *Packings of graphs and applications to computational complexity*, J. Combin. Theory Ser. B **25** (1978) 105–124.
https://doi.org/10.1016/0095-8956(78)90030-8

[4] E. Duchene, H. Kheddouci, R.J. Nowakowski and M.A. Tahraoui, *Labeled packing of graphs*, Australas. J. Combin. **57** (2013) 109–126.

[5] A. Kaneko and K. Suzuki, *Packing two copies of a tree into its third power*, Discrete Math. **306** (2006) 2779–2785.
https://doi.org/10.1016/j.disc.2006.04.019

[6] H. Kheddouci, J.-F. Saclé and M. Woźniak, *Packing two copies of a tree into its fourth power*, Discrete Math. **213** (2000) 169–178.
https://doi.org/10.1016/S0012-365X(99)00177-6

[7] H. Kheddouci, S. Marshall, J.-F. Saclé and M. Woźniak, *On the packing of three graphs*, Discrete Math. **236** (2001) 197–225.
https://doi.org/10.1016/S0012-365X(00)00443-X

[8] N. Sauer and J. Spencer, *Edge disjoint placement of graphs*, J. Combin. Theory Ser. B **25** (1978) 295–302.
https://doi.org/10.1016/0095-8956(78)90005-9

[9] M.A. Tahraoui, Coloring, Packing and Embedding of Graphs (Ph.D. Thesis, 2012).

[10] M.A. Tahraoui, E. Duchêne and H. Kheddouci, *Labeled 2-packings of trees*, Discrete Math. **338** (2015) 816–824.
https://doi.org/10.1016/j.disc.2014.12.015

[11] M.A. Tahraoui, E. Duchêne and H. Kheddouci, *Labeled embedding of $(n, n-2)$-graphs in their complements*, Discuss. Math. Graph Theory **37** (2017) 1015–1025.
https://doi.org/10.7151/dmgt.1977

[12] M. Woźniak, *Packing of graphs and permutations—a survey*, Discrete Math. **276** (2004) 379–391.
https://doi.org/10.1016/S0012-365X(03)00296-6

[13] H.P. Yap, *Packing of graphs—a survey*, Ann. Discrete Math. **38** (1988) 395–404.
https://doi.org/10.1016/S0167-5060(08)70809-4