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ARBITRARILY PARTITIONABLE $\{2K_2, C_4\}$-FREE GRAPHS

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Abstract

A graph $G = (V, E)$ of order n is said to be arbitrarily partitionable if for each sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ of positive integers with $\lambda_1 + \cdots + \lambda_p = n$, there exists a partition $(V_1, V_2, \ldots, V_p)$ of the vertex set V such that V_i induces a connected subgraph of order λ_i in G for each $i \in \{1, 2, \ldots, p\}$. In this paper, we show that a threshold graph is arbitrarily partitionable if and only if it admits a perfect matching or a near perfect matching. We also give a necessary and sufficient condition for a $\{2K_2, C_4\}$-free graph being arbitrarily partitionable, as an extension for a result of Broersma, Kratsch and Woeginger [*Fully decomposable split graphs*, European J. Combin. 34 (2013) 567–575] on split graphs.

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1. Introduction

Let $G = (V, E)$ be a simple, undirected graph of order n. A set M of edges of G is called a *matching* of G if any pair of two elements of G have no common end vertex. Furthermore, M is called a *perfect matching* (respectively, a *near perfect matching* if every vertex of G (all but one vertex) is incident with an edge of M.

The matching number of G, denoted by $\alpha'(G)$, is the cardinality of a maximum matching of G. A graph G is called *traceable* if G has a Hamilton path. A subset $S \subseteq V$ is an *independent set* of G if no pair of vertices in S are adjacent in G. The *independence number* of G, denoted by $\alpha(G)$, is the cardinality of a maximum independent set of G.

A sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ of positive integers is called a *partition* of n if $\lambda_1 + \cdots + \lambda_p = n$. The graph G is called λ-*decomposable* (or λ is *realizable*) if there exists a partition $(V_1, V_2, \ldots, V_p)$ of the vertex set V such that $|V_i| = \lambda_i$ and $G[V_i]$ is connected for each $i \in \{1, \ldots, k\}$. In this case, we call such a partition of G a λ-*decomposition* of G, and $G[V_i]$ (or V_i) a λ_i-*component*. Furthermore, G is called *arbitrarily partitionable* (AP, for short) if G is λ-*decomposable* for every partition λ of n. Note that if G is traceable, then it is AP; if G is AP, then it admits a perfect matching or near perfect matching, and $\alpha(G) \leq \lceil \frac{n}{2} \rceil$.

The notion of AP graphs was first introduced by Barth, Baudon and Puech [1], and independently, by Horňák and Woźniak [20]. It is also called arbitrarily vertex decomposable [20] or fully decomposable [12] or decomposable [1]. Similarly, a graph G is called k-*partitionable* if G is λ-decomposable for each partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of n with length k.

A classical theorem of Győri [17] and Lovász [26] is stated as follows.

Theorem 1 (Győri [17] and Lovász [26]). *Every k-connected graph is k-partitionable.*

Structure of AP graphs and minimal AP graphs are investigated in [7, 9]. The problem of deciding whether a given admissible sequence is realizable in a given graph G is NP-complete [2]. Moreover, it is true even if we restrict the problem to the class of trees of degree at most 3 [2]. More results for the algorithmic aspects of AP graphs can be found in [2, 12, 10]. However, it still remains to be an open problem for deciding whether a tree is AP is NP-complete. Barth, Baudon and Puech [1] showed that this problem is polynomial in number of vertices for the class of tripodes. Horňák and Woźniak [20] showed that the maximum degree of a AP tree is at most 6. Later in [2], this bound was dropen to 4. Cichacz, Görlich, Marczyk and Przybyło [15] gave a complete characterization of AP caterpillars with four leaves. They also exhibited two infinite families of AP trees with maximum degree three or four. Ravaux [29] focused on trees with a large diameter. There are also some results on AP star-like trees [21], unicyclic AP graphs [24] and the shape of AP trees [3].

Marczyk [27] showed that if G is connected, $\alpha(G) \leq \lceil \frac{n}{2} \rceil$, and $d_G(x) + d_G(y) \geq n - 2$ for all nonadjacent vertices $x, y \in V(G)$, then G is AP. Later, he [28] further showed that if G is a connected graph on n vertices with independence number at most $\lceil \frac{n}{2} \rceil$ and such that the degree sum of any pair of nonadjacent vertices is at least $n - 3$, then G is AP or is isomorphic to one of two exceptional graphs. Horňák, Marczyk, Schiermeyer and Woźniak [18] showed that if for a connected graph G of order n, the degree sum of any pair of nonadjacent vertices is at least $n-5$, then G is AP. Dense arbitrarily partitionable graphs have been studied in [23].

Various variations of AP graphs, such as on-line arbitrarily partitionable graphs [19, 22, 25], recursively arbitrarily partitionable graphs [4, 8] and AP+k graphs [5, 6] are also investigated.

A graph G is called a *split graph* if its vertex set can be partitioned into two sets I and C, where I is an independent set of G, and C is a clique of G, that is, a set of mutually adjacent vertices in G. For an integer $n \geq 2$, a partition λ of n is called 2-3-*primitive* if it has one of the following forms.

- $\lambda = (1, 3, 3, \ldots, 3)$ consists of threes and a single one;
- $\lambda = (2, \ldots, 2, 3, 3, \ldots, 3)$ only consists of twos and threes.

Broersma, Kratsch and Woeginger [12] characterized AP split graphs as follows.

Theorem 2 (Broersma, Kratsch, Woeginger [12])**.** *A split graph on n vertices is AP if and only if it is λ-decomposable for each* 2-3-*primitive partition λ of n.*

For $n \geq 2$, the *canonical* 2-3-*primitive* partition λ of n is defined as follows.

- If $n = 2k$ is even, then the canonical 2-primitive partition of n consists of k twos. If $n = 2k+1$ is odd, then the canonical 2-primitive partition of n consists of $k-1$ twos and a single three.
- If $n = 3k$, then the canonical 3-primitive partition of n consists of k threes. If $n = 3k+1$, then the canonical 3-primitive partition of n consists of k threes and a single one. If $n = 3k+2$, then the canonical 3-primitive partition of n consists of k threes and a single two.

The canonical 2-3-primitive partitions are a crucial subfamily of the 2-3-primitive partitions.

Theorem 3 (Broersma, Kratsch, Woeginer [12])**.** *A split graph on n vertices is AP if and only if it is λ-decomposable for the canonical* 2-3-*primitive partition λ of n.*

Let $\mathcal{F}$ be a family of graphs. A graph G is called $\mathcal{F}$-free if it contains no induced subgraph isomorphic to a member $F \in \mathcal{F}$. Földes and Hammer[16] proved that a graph is a split graph if and only if it is $\{2K_2, C_4, C_5\}$-free. Hence, split graphs are a subclass of $\{2K_2, C_4\}$-free graphs.

In Section 2, we show that a connected threshold graph is AP if and only if it admits a perfect matching or a near perfect matching (a matching omitting exactly one vertex). In Section 3, we extend the result of Theorem 3 to $\{2K_2, C_4\}$-free graphs, by showing that a $\{2K_2, C_4\}$-free graph is AP if and only if it is λ-decomposable for the canonical 2-3-primitive partition λ of n.

2. Threshold Graphs

Threshold graphs were first introduced and studied by Chvátal and Hammer [14]. Let $a_1, a_2, \ldots, a_n$ be distinct real numbers, and define a simple graph G with vertex set $\{a_1, a_2, \ldots, a_n\}$, in which two vertices a_i and a_j are adjacent if and only if $a_i + a_j > 0$. Without loss of generality, let $a_1 \leq \cdots \leq a_n$. Note that G is connected if and only if $a_1 + a_n > 0$. It is clear that threshold graphs are split graphs. Chvátal and Hammer [14] showed that a graph G is a threshold graph if and only if it is $\{2K_2, P_4, C_4\}$-free.

Theorem 4. *A connected threshold graph G of order n is AP if and only if $\alpha'(G) = \lfloor \frac{n}{2} \rfloor$.*

Proof. To prove the necessity, we consider the admissible sequence $\lambda = \left(2^k 1^{n-2k}\right)$, where $k = \lfloor \frac{n}{2} \rfloor$. Since G is AP, there is λ-decomposition $(V_1, \ldots, V_k)$ of G. Since $G[V_i]$ is connected for each i, we have $\alpha'(G) = \lfloor \frac{n}{2} \rfloor$.

Next we show its sufficiency. Let $V(G) = \{a_1, a_2, \ldots, a_n\}$, and let $V^+(G) = \{a_i | \ a_i \geq 0\}$ and $V^-(G) = \{a_i | \ a_i < 0\}$. By the definition of the threshold graph, $V^+(G)$ is a clique and $V^-(G)$ is an independent set of G. Since $\alpha'(G) = \lfloor \frac{n}{2} \rfloor$, $|V^+(G)| \geq \lfloor \frac{n}{2} \rfloor$.

Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ be a partition of n. We show that G is λ-decomposable. We proceed with induction on n. If $n \leq 2$, then $G \cong K_n$, the result holds trivially. Next let us consider the case when $n \geq 3$. Without loss of generality, let $\lambda_1 \leq \cdots \leq \lambda_l$ and $a_1 < a_2 < \cdots < a_n$. By the definition of threshold graph, $N(a_i) \setminus \{a_j\} \subseteq N(a_j) \setminus \{a_i\}$ for each i, j with $i < j$. Combining this fact with the assumption $\alpha'(G) = \lfloor \frac{n}{2} \rfloor$, it follows that there exists a maximum matching M of G with

$$M = \begin{cases} \{a_i a_{n+1-i} \,|\, 1 \leq i \leq \frac{n}{2}\}, & \text{if } n \text{ is even}, \\ \{a_i a_{n+2-i} \,|\, 2 \leq i \leq \frac{n+1}{2}\}, & \text{if } n \text{ is odd}. \end{cases}$$

Case 1. $\lambda_1 = 1$. Clearly, $G - a_1$ is also a connected threshold graph of order $n-1$ with $\alpha'(G - a_1) = \lfloor \frac{n-1}{2} \rfloor$. By induction hypothesis, $\lambda' = (\lambda_2, \ldots, \lambda_l)$ is realizable for $G - a_1$. Hence, $\lambda = (\lambda_1, \ldots, \lambda_l)$ is realizable for G.

Case 2. $\lambda_1 \geq 2$. Then $\lambda_l \geq 2$. Since G is connected, $a_n a_1 \in E(G)$, i.e., $a_n + a_1 > 0$. It follows that $G[\{a_n, a_1, \ldots, a_{l-1}\}]$ is connected. Let $V_l = \{a_n, a_1, \ldots, a_{l-1}\}$. Note that $\alpha'(G - V_l) \geq \lfloor \frac{n-l}{2} \rfloor$. By the induction hypothesis, $(\lambda_1, \lambda_2, \ldots, \lambda_{l-1})$ is realizable for $G - V_l$. Thus, λ is realizable in G. ■

3. $\{2K_2, C_4\}$-Free Graphs

Blázsik, Hujter, Pluhár and Tuza [11] gave a structural characterization of $\{2K_2, C_4\}$-free graphs.

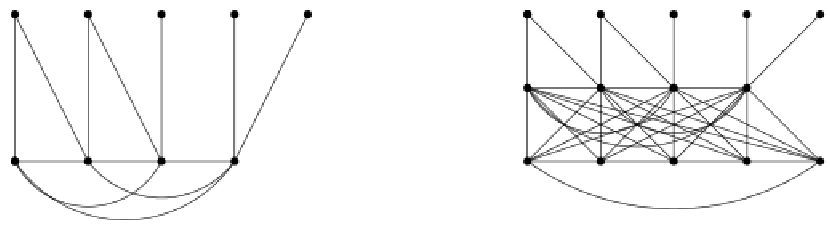
Theorem 5 (Blázsik, Hujter, Pluhár and Tuza [11]). *A graph $G = (V, E)$ is $\{2K_2, C_4\}$-free if and only if there is a partition $V_1 \cup V_2 \cup V_3 = V$ with the following properties.*

(i) *V_1 is an independent set in G.*

(ii) *V_2 is the vertex set of a complete subgraph in G.*

(iii) *$V_3 = \emptyset$ or $|V_3| = 5$, and in the latter case V_3 induces a 5-cycle in G.*

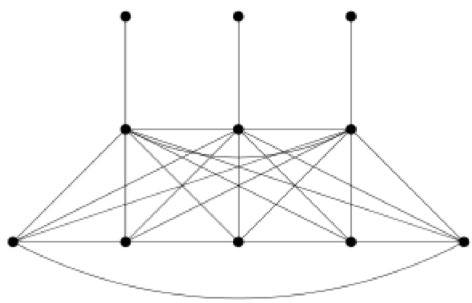
(iv) *If $V_3 \neq \emptyset$, then for all $v_i \in V_i$, $i = 1, 2, 3$, $v_1v_3 \notin E$ and $v_2v_3 \in E$ hold.*



H_1 G_1

Figure 1. A AP split graph H_1 and a $\{2K_2, C_4\}$-free graph G_1 which is not AP.

The graph H_1 in Figure 1 is a split graph. It can be checked that $(2, 2, 2, 3)$ and $(3, 3, 3)$ are realizable in H_1, by Theorem 3, H_1 is AP. Since the admissible sequence $(2, \ldots, 2)$ is not realizable for G_1 in Figure 1, G_1 is not AP.



H_2 G_2

Figure 2. A split graph H_2 that is not AP and a AP $\{2K_2, C_4\}$-free graph G_2.

On the other hand, the graph H_2 in Figure 2 is a $\{2K_2, C_4\}$-free graph, which is not AP, because $(3,3)$ is not realizable in H_2. However, it is easy to check that G_2 is AP.

Theorem 6. *A $2K_2$-free graph G on n vertices is AP if and only if every 2-3-primitive partition λ of n is realizable in G.*

Proof. The necessity is obvious. Next we prove the sufficiency. Let $\lambda = (\lambda, \ldots, \lambda_m)$ be an admissible sequence for G. It is well known that any integer $l \geq 2$ can be expressed as $l = 2a + 3b$, where a and b are two nonnegative integers.

(1) Replace each $\lambda_i \geq 4$ in λ with a_i twos and b_i threes, and denote the resultant partition as λ'.

(2) Let λ_0 denote the number of ones in the vector λ. If $\lambda_0 \geq 2$, then replace the ones in vector λ with a_0 twos and b_0 threes, where $\lambda_0 = 2a_0 + 3b_0$. If $\lambda_0 = 1$ and there is a two in α, then replace the one and a two by a three, otherwise leave the one as it is.

The resultant new partition $\lambda' = (\lambda'_1, \ldots, \lambda'_m)$ of n has the form $(1, 3, \ldots, 3)$ or $(2, \ldots, 2, 3, \ldots, 3)$, and hence is a 2-3-primitive. By the assumption, let $(V'_1, \ldots, V'_m)$ be a realization of λ'. Since G is $2K_2$-free, for any $\lambda'_i \geq 2$ and $\lambda'_j \geq 2$, the union of the λ'_i-component and the λ'_j-component is connected. Therefore, the a_i 2-components and the b_i 3-components are combined into a λ_i-component.

This proves that λ is realizable in G. Thus, G is AP. ∎

Let G be a $\{2K_2, C_4\}$-free graph. In view of Theorem 5, we denote G by (I, C, C_5, E), in which C_5 also denotes $V(C_5)$ in sequel. Assume that T is a connected subgraph of G with $|V(T)| = 3$. We say that T is of type-T_{ijk} if $|V(T) \cap I| = i$, $|V(T) \cap C| = j$ and $|V(T) \cap C_5| = k$. For the special case when $i = 1$, $j = 2$ and $k = 0$, we denote T_{ijk} by $\overline{T_{120}}$ if $T \cong K_3$, otherwise by T_{120}. The types of all connected subgraphs of G with order 3 belong to

$$\{T_{120}, \overline{T_{120}}, T_{111}, T_{030}, T_{021}, T_{012}, T_{003}, T_{210}\}.$$

By Theorem 5, one can see that $T_{120} \cong T_{210} \cong T_{111} \cong T_{003} \cong P_3$, $T_{030} \cong T_{021} \cong K_3$. Moreover, we may assume that $T_{012} \cong K_3$, since by Theorem 5, any two vertices $v_2 \in C$ and $v_3 \in V(C_5)$ are adjacent in G.

We use 2^r3^s to denote an admissible partition of n into r (possibly $r = 0$) twos and s (possibly $s = 0$) threes. A partition of $n = 3k + 1$ into k threes and 1 one is denoted by 3^k1. We say that G is $(3,3)$-*reducible* if and only if 2^r3^s is realizable for some $r \geq 0$ and $s \geq 4$ in G, then $2^{r+3}3^{s-2}$ is also realizable in G. Similarly, we say that G is $(1,3)$-*reducible* if 3^k1 is realizable for some $k \geq 3$ in G, then 2^23^{k-1} is also realizable in G.

Lemma 7. *Let $G = (I, C, C_5, E)$ be a $\{2K_2, C_4\}$-free graph of order n. If a canonical 2-3-primitive partition λ of n is realizable in G, then G is $(3,3)$-reducible.*

Proof. Suppose $\lambda = 2^r 3^s$ is realizable in G for $r \geq 0$ and $s \geq 4$ and Λ be a realization of λ. Assume first that there exist two 3-components in Λ, say T_1 and T_2, of type other than T_{210}, i.e., of type in $\{T_{120}, \overline{T_{120}}, T_{111}, T_{030}, T_{021}, T_{012}, T_{003}\}$. One can see from Figure 3, that $G[T_1 \cup T_2]$ has a perfect matching for all possible cases, except possible the only case when $T_1 \cong T_{111} \cong T_2$. For this case, we may assume that the two vertices of $T_1 \cap C_5$ and $T_2 \cap C_5$ are adjacent in G, because each vertex in C_5 is adjacent to every vertex of C. Thus, by transposing such two 3-components into three 2-components in Λ, we obtain a realization Λ' of $2^{r+3}3^{s-2}$ in G.

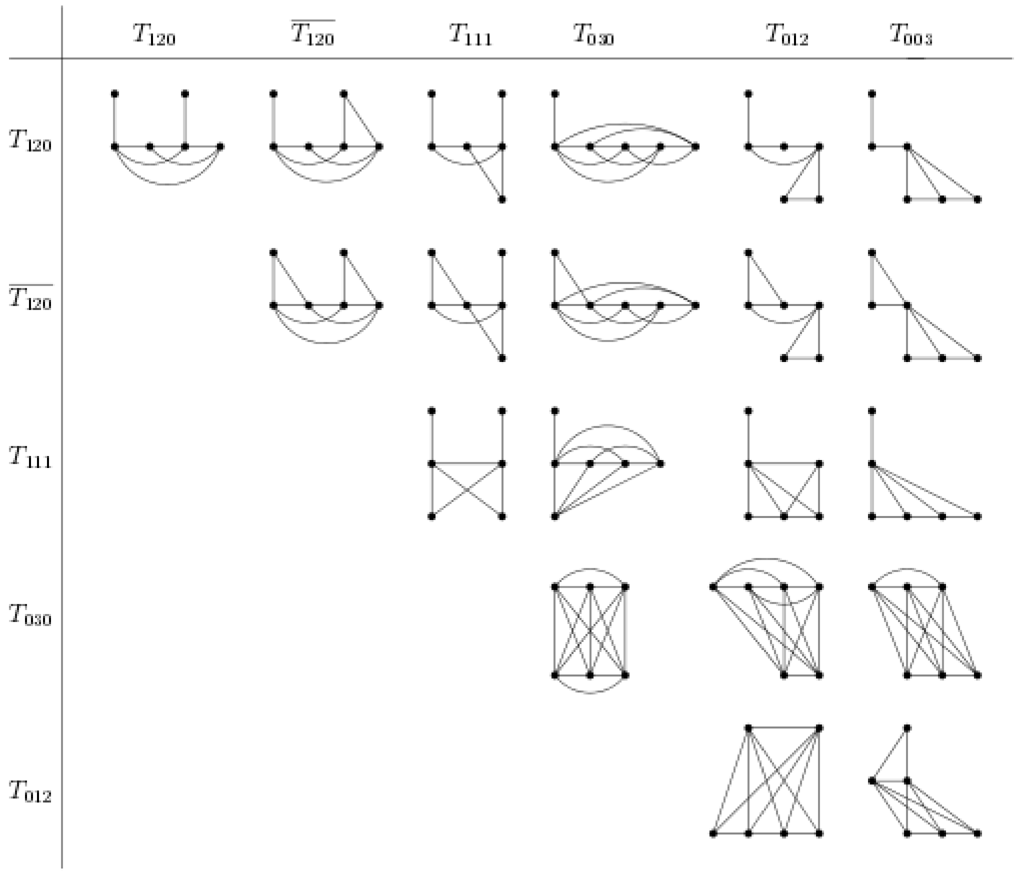


Figure 3. The subgraph of G induced by two 3-components of type in $\{T_{120}, \overline{T_{120}}, T_{111}, T_{030}, T_{021}, T_{012}, T_{003}\}$.

Next assume that there exists at most one 3-components of type in $\{T_{120}, \overline{T_{120}}, T_{111}, T_{030}, T_{021}, T_{012}, T_{003}\}$. Thus, at least $s-1$ 3-components have type T_{210}. Since $s \geq 4$, $|I| \geq 2(s-1) \geq 6$. Moreover, by the assumption that the canonical

2-primitive partition of n is realizable in G, $|C| \geq |I|-1 \geq 5$, and there exists two 3-components T' and T'' of type T_{210}, say $T' = \{u_1, u'_1, v_1\}$, $T'' = \{u_2, u'_2, v_2\}$ with $v_1, v_2 \in C$, and two vertices $w_1, w_2 \in C$ such that $u_1w_1 \in E(G)$ and $u_2w_2 \in E(G)$, and w_1, w_2 are lying in 2-component or 3-component contained in $C \cup C_5$.

First assume that at least one of w_1 and w_2 belongs to a 3-component. Without loss of generality, suppose that $\{w_1, w_0, w'_0\} = T_0$ is a 3-component contained in $C \cup C_5$. Then $T_0 \in \{T_{030}, T_{021}, T_{012}\}$. For the case when $T_0 \cong T_{030}$ or $T_0 \cong T_{021}$, we can decompose the subgraph of G induced by $T' \cup T_0$ into three 2-components. For the case when $T_0 \cong T_{012}$, we may assume that $w_0w'_0 \in E(G)$. So, we can decompose the subgraph of G induced by $T' \cup T_0$ into three 2-components.

If $\{w_1, w_2\}$ is a 2-component, then we can decompose the subgraph of G induced by $T' \cup T'' \cup \{w_1, w_2\}$ into four 2-components. In the following, we assume that w_1 and w_2 belong to different 2-components. Denote $\{w_1, w'_1\}$ and $\{w_2, w'_2\}$ are two 2-components. If $w'_1w'_2 \in E(G)$, we can decompose the subgraph of G induced by $T' \cup T'' \cup \{w_1, w'_1, w_2, w'_2\}$ into five 2-components. If $w'_1w'_2 \notin E(G)$, then $w'_1, w'_2 \in V(C_5)$, there exists at least one 2-component $v_0v'_0$ such that $v_0 \in V(C_5)$, $v_0w'_1 \in E(G)$ and $w'_2v'_0 \in E(G)$, and then we can decompose the subgraph of $G[T' \cup T'' \cup \{w_1, w'_1, w_2, w'_2, v_0, v'_0\}]$ into six 2-components. ■

Lemma 8. *Let $G = (I, C, C_5, E)$ be a $\{2K_2, C_4\}$-free graph of order n. If a canonical 2-3-primitive partition λ of n is realizable in G, then G is $(1,3)$-reducible.*

Proof. Suppose $\lambda = 3^k1$ is realizable in G for some $k \geq 3$. Let $\{v_0\} \cup \Lambda$ be a $\lambda = 3^k1$-decomposition of G, in which $\{v_0\}$ is the 1-component and Λ is the set of 3-components.

Case 1. $v_0 \in C$. By the assumption, every vertex of C_5 belongs to a 3-component, and hence there exists a 3-component $T = \{w, u, v\}$ such that $T \cap C_5 \neq \emptyset$ and $T \cap C \neq \emptyset$. Thus $T \in \{T_{021}, T_{012}, T_{111}\}$. We may assume that $w \in T \cap C_5$ and $u \in T \cap C$. Then $v_0w \in E(G)$ and $uv \in E(G)$, implying that G is $(1,3)$-reducible.

Case 2. $v_0 \in C_5$. Let $w \in C_5$ be a vertex with $v_0w \in E(C_5)$ and $T = \{w, u, v\}$ be a 3-component containing w. Then, $T \in \{T_{021}, T_{012}, T_{111}, T_{003}\}$, and so, $uv \in E(G)$, implying that G is $(1,3)$-reducible.

Case 3. $v_0 \in I$. By the assumption, there exists a vertex $w \in C$ and $T = \{w, u, v\}$. Clearly $T \in \{T_{120}, \overline{T_{120}}, T_{111}, T_{030}, T_{021}, T_{012}, T_{210}\}$.

For the case $T \cong T_{120}$, assume that $u \in C$ and $v \in I$. If $uv \in E(G)$, then $\{v_0, w\}$ and $\{u, v\}$ are two 2-components, and so, G is $(1,3)$-reducible. Otherwise, $uv \notin E(G)$ and $wv \in E(G)$. Then we can choose $\{u\}$ as the new 1-component, and $\{v_0, w, v\}$ as the 3-component. Then, this case is reduced to Case 1.

If $T \cong \overline{T_{120}}$, we may assume that $v \in I$. Without loss of generality, let $uv \in E(G)$. Clearly, the subgraph $G[\{v, w, u, v_0\}]$ can be partitioned into two 2-components $\{v_0, w\}$ and $\{u, v\}$. So, G is $(1,3)$-reducible.

For the case when $T \cong T_{111}$, let $v \in I$ and $u \in C_5$. We can choose $\{u\}$ as the new 1-component, and $\{v_0, w, v\}$ as the 3-component. Then it is reduced to Case 2.

If $T \cong T_{030}$, then $\{v_0\} \cup T$ can be repartitioned into two 2-components $\{v_0, w\}$ and $\{u, v\}$.

If $T \cong T_{021}$, assume that $u \in C_5$ and $v \in C$. Again, $\{v_0\} \cup T$ can be repartitioned into two 2-components $\{v_0, w\}$ and $\{u, v\}$.

If $T \cong T_{012}$, then $\{u, v\} \subseteq C_5$. Actually, we may assume that $uv \in E(G)$. Then $\{v, w, u, v_0\}]$ can be partitioned into two 2-components $\{v_0, w\}$ and $\{u, v\}$, as we desired.

Now we deal with the last case when $T \cong T_{210}$. Since the canonical 2-primitive partition of n is realizable in G, $|C| \geq |I| - 1$. It means that there must exist a 3-component T' with type distinct from T_{210}. Take a sequence of 3-components $T_1, \ldots, T_j$ of $\mathcal{T}$ (Let $T_i = \{u_i, w_i, v_i\}$ with $u_i, v_i \in I$ and $w_i \in C$), such that $v_i w_{i+1} \in E(G)$ and $T_i \cong T_{210}$ for each $i < j$, and $T_j \ncong T_{210}$. Let $T_i' = (T_i \setminus \{v_i\}) \cup \{v_{i-1}\}$ for each $i \in \{1, \ldots, j\}$. Replacing the components $\{v_0\}$, $T_1, \ldots, T_j$ of $\mathcal{T}$ with $\{w_j\}$, $T_1', \ldots, T_j'$, we obtain a new realization $\mathcal{T}'$ of 3^s1 in which 1-component $\{w_j\}$ does not belong to I. By Cases 1 and 2, G is $(1, 3)$-reducible. ■

Theorem 9. *Let $G = (I, C, C_5, E)$ be a connected $\{2K_2, C_4\}$-free graph of order n. If every canonical 2-3-primitive partition of n is realizable in G, then every 2-3-primitive partition of n is realizable in G.*

Proof. Let $\lambda = 2^r 3^s$ be a 2-3-primitive partition of n. Since every canonical 2-3-primitive partition of n is realizable in G, we may assume that $r \geq 2$ and $s \geq 2$.

If $r = 2$, we are able to obtain a λ-decomposition from the λ'-realization of the canonical primitive partition 13^{s+1}, because G is $(1, 3)$-reducible by Lemma 8. If $r \geq 3$, we can obtain a λ-decomposition from the realization of the 2-3-primitive partition $2^{r-3}3^{s+2}$, because G is $(3, 3)$-reducible by Lemma 11. ■

By Theorem 6 and Theorem 9, we obtain the following result.

Theorem 10. *A $\{2K_2, C_4\}$-free graph G on n vertices is AP if and only if every canonical 2-3-primitive partition of n is realizable in G.*

4. $2K_2$-FREE BIPARTITE GRAPHS

Lemma 11. *Let $G = (X, Y)$ be a connected $2K_2$-free bipartite graph. If G has a perfect matching or a near perfect matching, then every 2-3-primitive partition λ of n is realizable in G.*

Proof. Since G has a perfect matching or a near perfect matching, λ^* is realizable in G, where

$$\lambda^* = \begin{cases} 2^{\frac{n}{2}}, & \text{if } n \text{ is even,} \\ 2^{\frac{n-1}{2}}1, & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, since G is connected, $(2, \ldots, 2, 3)$ is realizable in G if n is odd. So, let Λ_0 be a λ_0-decomposition of G, where

$$\lambda_0 = \begin{cases} 2^{\frac{n}{2}}, & \text{if } n \text{ is even,} \\ 2^{\frac{n-3}{2}}3, & \text{if } n \text{ is odd.} \end{cases}$$

To prove every 2-3-primitive partition λ of n is realizable in G, it suffices to show that

(i) the subgraph induced by any three 2-components of Λ_0 can be decomposed into two 3-components; and

(ii) the subgraph induced by any two 2-components of Λ_0 can be decomposed into one 1-component and one 3-component.

We first prove (i). Let x_1y_1, x_2y_2 and x_3y_3 be three 2-components of Λ_0, where $x_i \in X$, $y_i \in Y$, $1 \leq i \leq 3$. Since G is $2K_2$-free, the subgraph induced by any two 2-components is connected. Without loss of generality, we assume that $x_1y_2 \in E(G)$. If $x_2y_3 \in E(G)$, then $\{x_1, y_1, y_2\}$ and $\{x_2, x_3, y_3\}$ are two 3-components. Otherwise, $x_3y_2 \in E(G)$. If $x_1y_3 \in E(G)$, then $\{x_1, y_1, y_3\}$ and $\{x_2, x_3, y_2\}$ are two 3-components. If $x_1y_3 \notin E(G)$, then $x_3y_1 \in E(G)$, and so $\{x_1, x_2, y_2\}$ and $\{x_3, y_1, y_3\}$ are two 3-components. For each case, we can decompose three 2-components into two 3-components. Thus, (i) holds.

Since G is $2K_2$-free, the subgraph induced by any two 2-components is connected. So, it is easy to partition this subgraph into a subgraph of order 1 and a subgraph of order 3. Thus, (ii) holds. ■

By Theorem 6 and Lemma 11, we obtain the following result.

Theorem 12. *Let G be a connected $2K_2$-free bipartite graph. Then G is AP if and only if G has a perfect matching or a near perfect matching.*

5. $2K_2$-Free Nonbipartite Graphs with Clique Number 2

In this section, we consider $2K_2$-free nonbipartite graphs with clique number 2. Recall that $o(H)$ denotes the number of odd components in H. The well-known Tutte's 1-factor theorem says that a graph G has a perfect matching if and only if $o(G - S) \leq |S|$ for all $S \subseteq V(G)$. The following consequence can be derived easily from Tutte's 1-factor theorem.

Proposition 13. *Let G be a graph of odd order. Then G has a near perfect matching if and only if $o(G-S) \leq |S|+1$ for all $S \subset V$.*

For a vertex $v \in V(G)$ and a positive integer n, we say that H is obtained from G by multiplying v by n when H is formed by replacing the vertex v by an independent set of n vertices each having the same neighbors as v.

Theorem 14 (Chung, Gyárfás, Tuza and Trotter [13])**.** *Assume that G is $2K_2$-free, $\omega(G) = 2$ and G is not bipartite. Then G can be obtained from the cycle C_5 by vertex multiplication.*

So let G be a $2K_2$-free, nonbipartite graph with $\omega(G) = 2$. Then, by Theorem 14, we denote $G = (A_1, A_2, A_3, A_4, A_5)$, where the sets A_i are independent sets and form a partition of $V(G)$, and each vertex of A_i is adjacent to all vertices in $A_{i-1} \cup A_{i+1}$ for each $i = 1, 2, \ldots, 5$ where $i-1$ and $i+1$ are taken modulo 5.

Theorem 15. *Assume that G is a $2K_2$-free nonbipartite graph of even order with $\omega(G) = 2$. Then G has a perfect matching if and only if the following conditions are satisfied for each $i \in \{1, 2, \ldots, 5\}$,*

(1) $|A_i| + |A_{i+2}| \leq |A_{i-1}| + |A_{i+1}| + |A_{i-2}|$ *and*

(2) $|A_i| \leq |A_{i-1}| + |A_{i+1}|$, *with equality only if* $|A_{i-2}| = |A_{i+2}|$.

Proof. First assume that G has a perfect matching. The conclusions (1) and (2) can be deduced from Tutte's 1-factor theorem by taking $A_{i-1} \cup A_{i+1} \cup A_{i+3}$ and $A_{i-1} \cup A_{i+1}$ into S, respectively.

Conversely, let G be a $2K_2$-free nonbipartite graph of even order with $\omega(G) = 2$ satisfying conditions (1) and (2). Let $S \subset V(G)$. We shall show that $o(G-S) \leq |S|$. If $G-S$ is connected, then $o(G-S) \leq 1 \leq |S|$ for a nonempty set S, and $o(G-S) = o(G) = 0 = |S|$ for the empty set S, since $|V(G)|$ is even. Now assume that $G-S$ is disconnected. At least two nonadjacent parts of $\{A_1, A_2, A_3, A_4, A_5\}$ are contained in S. Without loss of generality, we assume that $A_2 \cup A_5 \subseteq S$.

Case 1. $S = A_2 \cup A_5$. If $|A_1| = |A_2| + |A_5|$, then by (2) $|A_3| = |A_4|$, and hence

$$o(G-S) = |A_1| = |A_2| + |A_5| = |S|.$$

If $|A_1| \leq |A_2| + |A_5| - 1$, then

$$o(G-S) \leq |A_1| + 1 \leq |A_2| + |A_5| - 1 + 1 = |A_2| + |A_5| = |S|.$$

Case 2. $A_2 \cup A_5 \subset S$ and $S \cap (A_1 \cup A_3 \cup A_4) \neq \emptyset$. If $A_3 \not\subseteq S$ and $A_4 \not\subseteq S$, then $o(G-S) \leq |A_1| + 1 \leq |A_2| + |A_5| + 1 \leq |S|$.

If $A_3 \subset S$, then $o(G-S) \leq |A_1| + |A_4| \leq |A_2| + |A_5| + |A_3| \leq |S|$.

If $A_4 \subset S$, then $o(G-S) \leq |A_1| + |A_3| \leq |A_2| + |A_5| + |A_4| \leq |S|$.

In either case, we obtain $o(G-S) \leq |S|$ for $S \subset V(G)$. By Tutte's 1-factor theorem, G has a perfect matching. ∎

Theorem 16. *Assume that G is a $2K_2$-free nonbipartite graph of odd order with $\omega(G) = 2$. Then G has a near perfect matching if and only if the following conditions are satisfied for each $i \in \{1, 2, \ldots, 5\}$,*

(1) $|A_i| + |A_{i+2}| \leq |A_{i-1}| + |A_{i+1}| + |A_{i-2}| + 1$ *and*

(2) $|A_i| \leq |A_{i-1}| + |A_{i+1}| + 1$, *with equality only if* $|A_{i-2}| = |A_{i+2}|$.

Proof. First assume that G has a near perfect matching. The conclusions (1) and (2) can be deduced from Proposition 13 by taking $A_{i-1} \cup A_{i+1} \cup A_{i+3}$ and $A_{i-1} \cup A_{i+1}$ into S, respectively.

Conversely, let G be a $2K_2$-free nonbipartite graph of odd order with $\omega(G) = 2$ satisfying conditions (1) and (2). Let $S \subset V(G)$. We shall show that $o(G-S) \leq |S|+1$. If $G-S$ is connected, then $o(G-S) \leq 1 \leq |S|$ for a nonempty set S, and $o(G-S) = o(G) = 1 = |S|+1$ for the empty set S, since $|V(G)|$ is odd. If $G-S$ is disconnected, then at least two nonadjacent parts of $\{A_1, A_2, A_3, A_4, A_5\}$ are contained in S. Without loss of generality, we assume that $A_2 \cup A_5 \subseteq S$.

Case 1. $S = A_2 \cup A_5$. If $|A_1| \leq |A_2| + |A_5|$, then

$$o(G-S) = o(G - A_2 - A_5) \leq |A_1| + 1 \leq |A_2| + |A_5| + 1 = |S| + 1.$$

If $|A_1| = |A_2| + |A_5| + 1$, then by the assumption, $|A_3| = |A_4|$. Therefore,

$$o(G-S) = o(G - A_2 - A_5) = |A_1| = |A_2| + |A_5| + 1 = |S| + 1.$$

Case 2. $A_2 \cup A_5 \subset S$ and $S \cap (A_1 \cup A_3 \cup A_4) \neq \emptyset$. If $A_3 \not\subseteq S$ and $A_4 \not\subseteq S$, then $o(G-S) \leq |A_1| + 1 \leq |A_2| + |A_5| + 1 + 1 \leq |S| + 1$.

If $A_3 \subset S$, then $o(G-S) \leq |A_1| + |A_4| \leq |A_2| + |A_5| + |A_3| + 1 \leq |S| + 1$.

If $A_4 \subset S$, then $o(G-S) \leq |A_1| + |A_3| \leq |A_2| + |A_5| + |A_4| + 1 \leq |S| + 1$.

For each case, we obtain $o(G-S) \leq |S| + 1$ for $S \subset V(G)$. By proposition 13, G has a near perfect matching. ∎

Theorem 17. *Let G be a $2K_2$-free nonbipartie graph G with $\omega(G) = 2$. Then G is AP if and only if it has a perfect matching or a near perfect matching.*

Proof. The necessity is obvious. We prove the sufficiency by induction on the order n of G. If $5 \leq n \leq 6$, then G is traceable, and so, G is AP. If $n = 7$, then

$$(|A_1|, |A_2|, |A_3|, |A_4|, |A_5|) \in \{(1, 1, 1, 2, 2), (1, 1, 2, 1, 2)\}.$$

It is easy to check that in the both cases, G is traceable, and thus it is AP. If $n = 8$, then $(|A_1|, |A_2|, |A_3|, |A_4|, |A_5|) = (1, 1, 1, 2, 3)$ or $(1, 1, 2, 2, 2)$ or $(1, 2, 1, 2, 2)$. For each case, it can be checked that G is traceable, and hence G is AP.

Now let $n \geq 9$, and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ be a partition of n with $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_p$. Since G is 2-connected, if $p \leq 2$, then λ is realizable in G by Theorem

1. So, we assume that $p \geq 3$. If there exists $V_1 \subseteq V(G)$ with $|V_1| = \lambda_1$, which come from some (at least two) consecutive parts of G, such that $G_1 = G - V_1$ is $2K_2$-free, $\omega(G_1) = 2$ and G_1 is not bipartite graph with a perfect matching or a near perfect matching, then by induction hypothesis, G_1 is $(\lambda_2, \ldots, \lambda_p)$-realizable, and hence G is λ-realizable. If such a set V_1 does not exist, we have the following result.

Claim 1. *There exist two nonadjacent parts of* $(A_1, A_2, A_3, A_4, A_5)$ *with cardinality* 1*. Moreover,* $\sum_{i=2}^{p} \lambda_i \geq 6$.

Proof. Since $n = \sum_{i=1}^{p} \lambda_i \geq 9$ with $\lambda_1 \leq \cdots \leq \lambda_p$ and $p \geq 3$, if $\sum_{i=2}^{p} \lambda_i \leq 5$, then $\lambda_1 \leq \frac{1}{2}\sum_{i=2}^{p} \lambda_i \leq \frac{1}{2} \times 5 = 2.5$. It follows that $\sum_{i=1}^{p} \lambda_i = \lambda_1 + \sum_{i=2}^{p} \lambda_i \leq 2.5 + 5 = 7.5 < 9$, a contradiction. Thus, $\sum_{i=2}^{p} \lambda_i \geq 6$. □

By Claim 1, suppose that $|A_1| = |A_3| = 1$, without loss of generality. Since G has a perfect matching or a near perfect matching, by Theorem 15 and Theorem 16,

$$|A_2| \leq \begin{cases} 2, & \text{if } n \text{ is even,} \\ 3, & \text{if } n \text{ is odd.} \end{cases}$$

Claim 2. $\min\{|A_4|, |A_5|\} \geq 2$.

Proof. Suppose that $\min\{|A_4|, |A_5|\} = 1$, and without loss of generality, let $|A_4| = 1$. Then by Theorem 16(1), $|A_2| + |A_5| \leq |A_1| + |A_3| + |A_4| + 1 = 4$. Thus, $n = \Sigma_{i=1}^{5} |A_i| \leq 7$, a contradiction. □

Claim 3. $|A_4| + |A_5| - 2 < \lambda_1 \leq \frac{n}{3}$ *and* $9 \leq n \leq 10$.

Proof. Since $p \geq 3$, $\sum_{i=1}^{p} \lambda_i = n$ and $\lambda_1 \leq \cdots \leq \lambda_p$, we have $\lambda_1 \leq \frac{n}{3}$.

If $|A_4| + |A_5| - 2 \geq \lambda_1$, then we can obtain G_1 from G by deleting λ_1 vertices from $A_4 \cup A_5$, a contradiction. Since $|A_4| + |A_5| - 2 < \lambda_1$,

$$n \leq |A_4| + |A_5| + 2 + 3 < \lambda_1 + 2 + 2 + 3 = \lambda_1 + 7 \leq \frac{n}{3} + 7,$$

implying that $n < \frac{21}{2}$, i.e., $9 \leq n \leq 10$. □

If $n = 10$ (n is even), then $|A_2| \leq 2$. It follows that $(|A_1|, |A_2|, |A_3|, |A_4|, |A_5|) = (1, 1, 1, 4, 3)$ or $(1, 2, 1, 3, 3)$. Since $\lambda_1 \leq \frac{10}{3}$, we can obtain G_1 by deleting λ_1 vertices from $|A_4|$ and $|A_5|$, again a contradiction.

If $n = 9$, then $|A_2| \leq 3$. It follows that $(|A_1|, |A_2|, |A_3|, |A_4|, |A_5|) = (1, 1, 1, 3, 3)$ or $(1, 2, 1, 3, 2)$ or $(1, 3, 1, 2, 2)$. Since $\lambda_1 \leq \frac{9}{3}$, for the cases when $(1, 1, 1, 3, 3)$ and $(1, 2, 1, 3, 2)$, we can obtain G_1 from G by deleting λ_1 vertices from $A_4 \cup A_5$. For the case when $(1, 3, 1, 2, 2)$, if $\lambda_1 \leq 2$, we can obtain G_1 from G by deleting λ_1 vertices from $A_4 \cup A_5$. If $\lambda_1 = 3$, then $\lambda = (3, 3, 3)$. Denote $A_2 = \{u_2, v_2, w_2\}$ and $A_4 = \{u_4, v_4\}$. Then we take $V_1 = A_1 \cup \{u_2, v_2\}$, $V_2 = A_3 \cup \{w_2, u_4\}$, $V_3 =$

$A_5 \cup \{v_4\}$. Note that $G[V_i]$ is connected for each $i \in \{1,2,3\}$. That is, $\lambda = (3,3,3)$ is realizable in G. ■

By Theorem 15, Theorem 16 and Theorem 17, we can obtain the following result. Let $\mathcal{G}$ be set of $2K_2$-free graphs G with $\omega(G) = 2$, satisfying the conditions (1) and (2) in Theorem 16 or Theorem 17 (depending whether G has even or odd order).

Theorem 18. *If G is $2K_2$-free, $\omega(G) = 2$ and G is not bipartite, then the following statements are equivalent.*

(i) *G is AP.*

(ii) *$G \in \mathcal{G}$.*

(iii) *G has a perfect matching or a near perfect matching.*

By Theorem 12 and Theorem 18 we obtain the following result.

Theorem 19. *If G is $2K_2$-free and $\omega(G) = 2$, then G is AP if and only if G has a perfect matching or a near perfect matching.*

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