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# ARBITRARILY PARTITIONABLE $\{2K_2, C_4\}$-FREE GRAPHS

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**Abstract**

A graph $G = (V, E)$ of order $n$ is said to be arbitrarily partitionable if for each sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ of positive integers with $\lambda_1 + \cdots + \lambda_p = n$, there exists a partition $(V_1, V_2, \ldots, V_p)$ of the vertex set $V$ such that $V_i$ induces a connected subgraph of order $\lambda_i$ in $G$ for each $i \in \{1, 2, \ldots, p\}$. In this paper, we show that a threshold graph is arbitrarily partitionable if and only if it admits a perfect matching or a near perfect matching. We also give a necessary and sufficient condition for a $\{2K_2, C_4\}$-free graph being arbitrarily partitionable, as an extension for a result of Broersma, Kratsch and Woeginger [*Fully decomposable split graphs*, European J. Combin. 34 (2013) 567–575] on split graphs.

**Keywords:** arbitrarily partitionable graphs, arbitrarily vertex decomposable, threshold graphs, $\{2K_2, C_4\}$-free graphs.

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## 1. Introduction

Let $G = (V, E)$ be a simple, undirected graph of order $n$. A set $M$ of edges of $G$ is called a *matching* of $G$ if any pair of two elements of $G$ have no common end vertex. Furthermore, $M$ is called a *perfect matching* (respectively, a *near perfect matching* if every vertex of $G$ (all but one vertex) is incident with an edge of $M$.

The matching number of $G$, denoted by $\alpha'(G)$, is the cardinality of a maximum matching of $G$. A graph $G$ is called *traceable* if $G$ has a Hamilton path. A subset $S \subseteq V$ is an *independent set* of $G$ if no pair of vertices in $S$ are adjacent in $G$. The *independence number* of $G$, denoted by $\alpha(G)$, is the cardinality of a maximum independent set of $G$.

A sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ of positive integers is called a *partition* of $n$ if $\lambda_1 + \cdots + \lambda_p = n$. The graph $G$ is called $\lambda$*-decomposable* (or $\lambda$ is *realizable*) if there exists a partition $(V_1, V_2, \ldots, V_p)$ of the vertex set $V$ such that $|V_i| = \lambda_i$ and $G[V_i]$ is connected for each $i \in \{1, \ldots, k\}$. In this case, we call such a partition of $G$ a $\lambda$*-decomposition* of $G$, and $G[V_i]$ (or $V_i$) a $\lambda_i$*-component*. Furthermore, $G$ is called *arbitrarily partitionable* (AP, for short) if $G$ is $\lambda$*-decomposable* for every partition $\lambda$ of $n$. Note that if $G$ is traceable, then it is AP; if $G$ is AP, then it admits a perfect matching or near perfect matching, and $\alpha(G) \leq \lceil \frac{n}{2} \rceil$.

The notion of AP graphs was first introduced by Barth, Baudon and Puech [1], and independently, by Horňák and Woźniak [20]. It is also called arbitrarily vertex decomposable [20] or fully decomposable [12] or decomposable [1]. Similarly, a graph $G$ is called $k$*-partitionable* if $G$ is $\lambda$-decomposable for each partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of $n$ with length $k$.

A classical theorem of Győri [17] and Lovász [26] is stated as follows.

**Theorem 1** (Győri [17] and Lovász [26])**.** *Every $k$-connected graph is $k$-partitionable.*

Structure of AP graphs and minimal AP graphs are investigated in [7, 9]. The problem of deciding whether a given admissible sequence is realizable in a given graph $G$ is NP-complete [2]. Moreover, it is true even if we restrict the problem to the class of trees of degree at most 3 [2]. More results for the algorithmic aspects of AP graphs can be found in [2, 12, 10]. However, it still remains to be an open problem for deciding whether a tree is AP is NP-complete. Barth, Baudon and Puech [1] showed that this problem is polynomial in number of vertices for the class of tripodes. Horňák and Woźniak [20] showed that the maximum degree of a AP tree is at most 6. Later in [2], this bound was dropen to 4. Cichacz, Görlich, Marczyk and Przybyło [15] gave a complete characterization of AP caterpillars with four leaves. They also exhibited two infinite families of AP trees with maximum degree three or four. Ravaux [29] focused on trees with a large diameter. There are also some results on AP star-like trees [21], unicyclic AP graphs [24] and the shape of AP trees [3].

Marczyk [27] showed that if $G$ is connected, $\alpha(G) \leq \lceil \frac{n}{2} \rceil$, and $d_G(x) + d_G(y) \geq n - 2$ for all nonadjacent vertices $x, y \in V(G)$, then $G$ is AP. Later, he [28] further showed that if $G$ is a connected graph on $n$ vertices with independence number at most $\lceil \frac{n}{2} \rceil$ and such that the degree sum of any pair of nonadjacent vertices is at least $n - 3$, then $G$ is AP or is isomorphic to one of two exceptional graphs. Horňák, Marczyk, Schiermeyer and Woźniak [18] showed that if for a connected graph $G$ of order $n$, the degree sum of any pair of nonadjacent vertices is at least $n-5$, then $G$ is AP. Dense arbitrarily partitionable graphs have been studied in [23].

Various variations of AP graphs, such as on-line arbitrarily partitionable graphs [19, 22, 25], recursively arbitrarily partitionable graphs [4, 8] and AP+$k$ graphs [5, 6] are also investigated.

A graph $G$ is called a *split graph* if its vertex set can be partitioned into two sets $I$ and $C$, where $I$ is an independent set of $G$, and $C$ is a clique of $G$, that is, a set of mutually adjacent vertices in $G$. For an integer $n \geq 2$, a partition $\lambda$ of $n$ is called 2-3-*primitive* if it has one of the following forms.

- $\lambda = (1, 3, 3, \ldots, 3)$ consists of threes and a single one;
- $\lambda = (2, \ldots, 2, 3, 3, \ldots, 3)$ only consists of twos and threes.

Broersma, Kratsch and Woeginger [12] characterized AP split graphs as follows.

**Theorem 2** (Broersma, Kratsch, Woeginger [12]). *A split graph on $n$ vertices is AP if and only if it is $\lambda$-decomposable for each* 2-3-*primitive partition $\lambda$ of $n$.*

For $n \geq 2$, the *canonical* 2-3-*primitive* partition $\lambda$ of $n$ is defined as follows.

- If $n = 2k$ is even, then the canonical 2-primitive partition of $n$ consists of $k$ twos. If $n = 2k+1$ is odd, then the canonical 2-primitive partition of $n$ consists of $k-1$ twos and a single three.
- If $n = 3k$, then the canonical 3-primitive partition of $n$ consists of $k$ threes. If $n = 3k+1$, then the canonical 3-primitive partition of $n$ consists of $k$ threes and a single one. If $n = 3k+2$, then the canonical 3-primitive partition of $n$ consists of $k$ threes and a single two.

The canonical 2-3-primitive partitions are a crucial subfamily of the 2-3-primitive partitions.

**Theorem 3** (Broersma, Kratsch, Woeginer [12]). *A split graph on $n$ vertices is AP if and only if it is $\lambda$-decomposable for the canonical* 2-3-*primitive partition $\lambda$ of $n$.*

Let $\mathcal{F}$ be a family of graphs. A graph $G$ is called $\mathcal{F}$-free if it contains no induced subgraph isomorphic to a member $F \in \mathcal{F}$. Földes and Hammer[16] proved that a graph is a split graph if and only if it is $\{2K_2, C_4, C_5\}$-free. Hence, split graphs are a subclass of $\{2K_2, C_4\}$-free graphs.

In Section 2, we show that a connected threshold graph is AP if and only if it admits a perfect matching or a near perfect matching (a matching omitting exactly one vertex). In Section 3, we extend the result of Theorem 3 to $\{2K_2, C_4\}$-free graphs, by showing that a $\{2K_2, C_4\}$-free graph is $AP$ if and only if it is $\lambda$-decomposable for the canonical 2-3-primitive partition $\lambda$ of $n$.

## 2. Threshold Graphs

Threshold graphs were first introduced and studied by Chvátal and Hammer [14]. Let $a_1, a_2, \ldots, a_n$ be distinct real numbers, and define a simple graph $G$ with vertex set $\{a_1, a_2, \ldots, a_n\}$, in which two vertices $a_i$ and $a_j$ are adjacent if and only if $a_i + a_j > 0$. Without loss of generality, let $a_1 \leq \cdots \leq a_n$. Note that $G$ is connected if and only if $a_1 + a_n > 0$. It is clear that threshold graphs are split graphs. Chvátal and Hammer [14] showed that a graph $G$ is a threshold graph if and only if it is $\{2K_2, P_4, C_4\}$-free.

**Theorem 4.** *A connected threshold graph $G$ of order $n$ is $AP$ if and only if $\alpha'(G) = \lfloor \frac{n}{2} \rfloor$.*

***Proof.*** To prove the necessity, we consider the admissible sequence $\lambda = \left(2^k 1^{n-2k}\right)$, where $k = \lfloor \frac{n}{2} \rfloor$. Since $G$ is AP, there is $\lambda$-decomposition $(V_1, \ldots, V_k)$ of $G$. Since $G[V_i]$ is connected for each $i$, we have $\alpha'(G) = \lfloor \frac{n}{2} \rfloor$.

Next we show its sufficiency. Let $V(G) = \{a_1, a_2, \ldots, a_n\}$, and let $V^+(G) = \{a_i | \ a_i \geq 0\}$ and $V^-(G) = \{a_i | \ a_i < 0\}$. By the definition of the threshold graph, $V^+(G)$ is a clique and $V^-(G)$ is an independent set of $G$. Since $\alpha'(G) = \lfloor \frac{n}{2} \rfloor$, $|V^+(G)| \geq \lfloor \frac{n}{2} \rfloor$.

Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ be a partition of $n$. We show that $G$ is $\lambda$-decomposable. We proceed with induction on $n$. If $n \leq 2$, then $G \cong K_n$, the result holds trivially. Next let us consider the case when $n \geq 3$. Without loss of generality, let $\lambda_1 \leq \cdots \leq \lambda_l$ and $a_1 < a_2 < \cdots < a_n$. By the definition of threshold graph, $N(a_i) \setminus \{a_j\} \subseteq N(a_j) \setminus \{a_i\}$ for each $i, j$ with $i < j$. Combining this fact with the assumption $\alpha'(G) = \lfloor \frac{n}{2} \rfloor$, it follows that there exists a maximum matching $M$ of $G$ with

$$M = \begin{cases} \{a_i a_{n+1-i} \,|\, 1 \leq i \leq \frac{n}{2}\}, & \text{if } n \text{ is even,} \\ \{a_i a_{n+2-i} \,|\, 2 \leq i \leq \frac{n+1}{2}\}, & \text{if } n \text{ is odd.} \end{cases}$$

*Case* 1. $\lambda_1 = 1$. Clearly, $G - a_1$ is also a connected threshold graph of order $n-1$ with $\alpha'(G - a_1) = \lfloor \frac{n-1}{2} \rfloor$. By induction hypothesis, $\lambda' = (\lambda_2, \ldots, \lambda_l)$ is realizable for $G - a_1$. Hence, $\lambda = (\lambda_1, \ldots, \lambda_l)$ is realizable for $G$.

*Case* 2. $\lambda_1 \geq 2$. Then $\lambda_l \geq 2$. Since $G$ is connected, $a_n a_1 \in E(G)$, i.e., $a_n + a_1 > 0$. It follows that $G[\{a_n, a_1, \ldots, a_{l-1}\}]$ is connected. Let $V_l = \{a_n, a_1, \ldots, a_{l-1}\}$. Note that $\alpha'(G - V_l) \geq \lfloor \frac{n-l}{2} \rfloor$. By the induction hypothesis, $(\lambda_1, \lambda_2, \ldots, \lambda_{l-1})$ is realizable for $G - V_l$. Thus, $\lambda$ is realizable in $G$. ∎

## 3. $\{2K_2, C_4\}$-Free Graphs

Blázsik, Hujter, Pluhár and Tuza [11] gave a structural characterization of $\{2K_2, C_4\}$-free graphs.

**Theorem 5** (Blázsik, Hujter, Pluhár and Tuza [11])**.** *A graph $G = (V, E)$ is $\{2K_2, C_4\}$-free if and only if there is a partition $V_1 \cup V_2 \cup V_3 = V$ with the following properties.*

(i) *$V_1$ is an independent set in $G$.*

(ii) *$V_2$ is the vertex set of a complete subgraph in $G$.*

(iii) *$V_3 = \emptyset$ or $|V_3| = 5$, and in the latter case $V_3$ induces a 5-cycle in $G$.*

(iv) *If $V_3 \neq \emptyset$, then for all $v_i \in V_i$, $i = 1, 2, 3$, $v_1v_3 \notin E$ and $v_2v_3 \in E$ hold.*

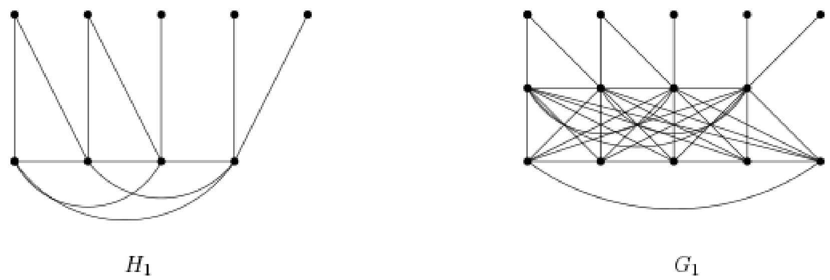


Figure 1. A AP split graph $H_1$ and a $\{2K_2, C_4\}$-free graph $G_1$ which is not AP.

The graph $H_1$ in Figure 1 is a split graph. It can be checked that $(2, 2, 2, 3)$ and $(3, 3, 3)$ are realizable in $H_1$, by Theorem 3, $H_1$ is AP. Since the admissible sequence $(2, \ldots, 2)$ is not realizable for $G_1$ in Figure 1, $G_1$ is not AP.

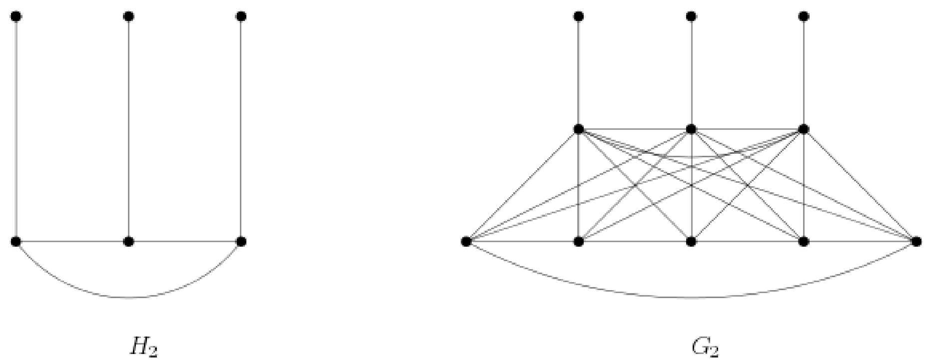


Figure 2. A split graph $H_2$ that is not AP and a AP $\{2K_2, C_4\}$-free graph $G_2$.

On the other hand, the graph $H_2$ in Figure 2 is a $\{2K_2, C_4\}$-free graph, which is not AP, because $(3,3)$ is not realizable in $H_2$. However, it is easy to check that $G_2$ is AP.

**Theorem 6.** *A $2K_2$-free graph $G$ on $n$ vertices is AP if and only if every 2-3-primitive partition $\lambda$ of $n$ is realizable in $G$.*

***Proof.*** The necessity is obvious. Next we prove the sufficiency. Let $\lambda = (\lambda, \ldots, \lambda_m)$ be an admissible sequence for $G$. It is well known that any integer $l \geq 2$ can be expressed as $l = 2a + 3b$, where $a$ and $b$ are two nonnegative integers.

(1) Replace each $\lambda_i \geq 4$ in $\lambda$ with $a_i$ twos and $b_i$ threes, and denote the resultant partition as $\lambda'$.

(2) Let $\lambda_0$ denote the number of ones in the vector $\lambda$. If $\lambda_0 \geq 2$, then replace the ones in vector $\lambda$ with $a_0$ twos and $b_0$ threes, where $\lambda_0 = 2a_0 + 3b_0$. If $\lambda_0 = 1$ and there is a two in $\alpha$, then replace the one and a two by a three, otherwise leave the one as it is.

The resultant new partition $\lambda' = (\lambda'_1, \ldots, \lambda'_m)$ of $n$ has the form $(1, 3, \ldots, 3)$ or $(2, \ldots, 2, 3, \ldots, 3)$, and hence is a 2-3-primitive. By the assumption, let $(V'_1, \ldots, V'_m)$ be a realization of $\lambda'$. Since $G$ is $2K_2$-free, for any $\lambda'_i \geq 2$ and $\lambda'_j \geq 2$, the union of the $\lambda'_i$-component and the $\lambda'_j$-component is connected. Therefore, the $a_i$ 2-components and the $b_i$ 3-components are combined into a $\lambda_i$-component.

This proves that $\lambda$ is realizable in $G$. Thus, $G$ is AP. ■

Let $G$ be a $\{2K_2, C_4\}$-free graph. In view of Theorem 5, we denote $G$ by $(I, C, C_5, E)$, in which $C_5$ also denotes $V(C_5)$ in sequel. Assume that $T$ is a connected subgraph of $G$ with $|V(T)| = 3$. We say that $T$ is of type-$T_{ijk}$ if $|V(T) \cap I| = i$, $|V(T) \cap C| = j$ and $|V(T) \cap C_5| = k$. For the special case when $i = 1$, $j = 2$ and $k = 0$, we denote $T_{ijk}$ by $\overline{T_{120}}$ if $T \cong K_3$, otherwise by $T_{120}$. The types of all connected subgraphs of $G$ with order 3 belong to

$$\{T_{120}, \overline{T_{120}}, T_{111}, T_{030}, T_{021}, T_{012}, T_{003}, T_{210}\}.$$

By Theorem 5, one can see that $T_{120} \cong T_{210} \cong T_{111} \cong T_{003} \cong P_3$, $T_{030} \cong T_{021} \cong K_3$. Moreover, we may assume that $T_{012} \cong K_3$, since by Theorem 5, any two vertices $v_2 \in C$ and $v_3 \in V(C_5)$ are adjacent in $G$.

We use $2^r 3^s$ to denote an admissible partition of $n$ into $r$ (possibly $r = 0$) twos and $s$ (possibly $s = 0$) threes. A partition of $n = 3k + 1$ into $k$ threes and 1 one is denoted by $3^k 1$. We say that $G$ is $(3,3)$-*reducible* if and only if $2^r 3^s$ is realizable for some $r \geq 0$ and $s \geq 4$ in $G$, then $2^{r+3} 3^{s-2}$ is also realizable in $G$. Similarly, we say that $G$ is $(1,3)$-*reducible* if $3^k 1$ is realizable for some $k \geq 3$ in $G$, then $2^2 3^{k-1}$ is also realizable in $G$.

**Lemma 7.** *Let $G = (I, C, C_5, E)$ be a $\{2K_2, C_4\}$-free graph of order $n$. If a canonical 2-3-primitive partition $\lambda$ of $n$ is realizable in $G$, then $G$ is $(3,3)$-reducible.*

***Proof.*** Suppose $\lambda = 2^r 3^s$ is realizable in $G$ for $r \geq 0$ and $s \geq 4$ and $\Lambda$ be a realization of $\lambda$. Assume first that there exist two 3-components in $\Lambda$, say $T_1$ and $T_2$, of type other than $T_{210}$, i.e., of type in $\{T_{120}, \overline{T_{120}}, T_{111}, T_{030}, T_{021}, T_{012}, T_{003}\}$. One can see from Figure 3, that $G[T_1 \cup T_2]$ has a perfect matching for all possible cases, except possible the only case when $T_1 \cong T_{111} \cong T_2$. For this case, we may assume that the two vertices of $T_1 \cap C_5$ and $T_2 \cap C_5$ are adjacent in $G$, because each vertex in $C_5$ is adjacent to every vertex of $C$. Thus, by transposing such two 3-components into three 2-components in $\Lambda$, we obtain a realization $\Lambda'$ of $2^{r+3}3^{s-2}$ in $G$.

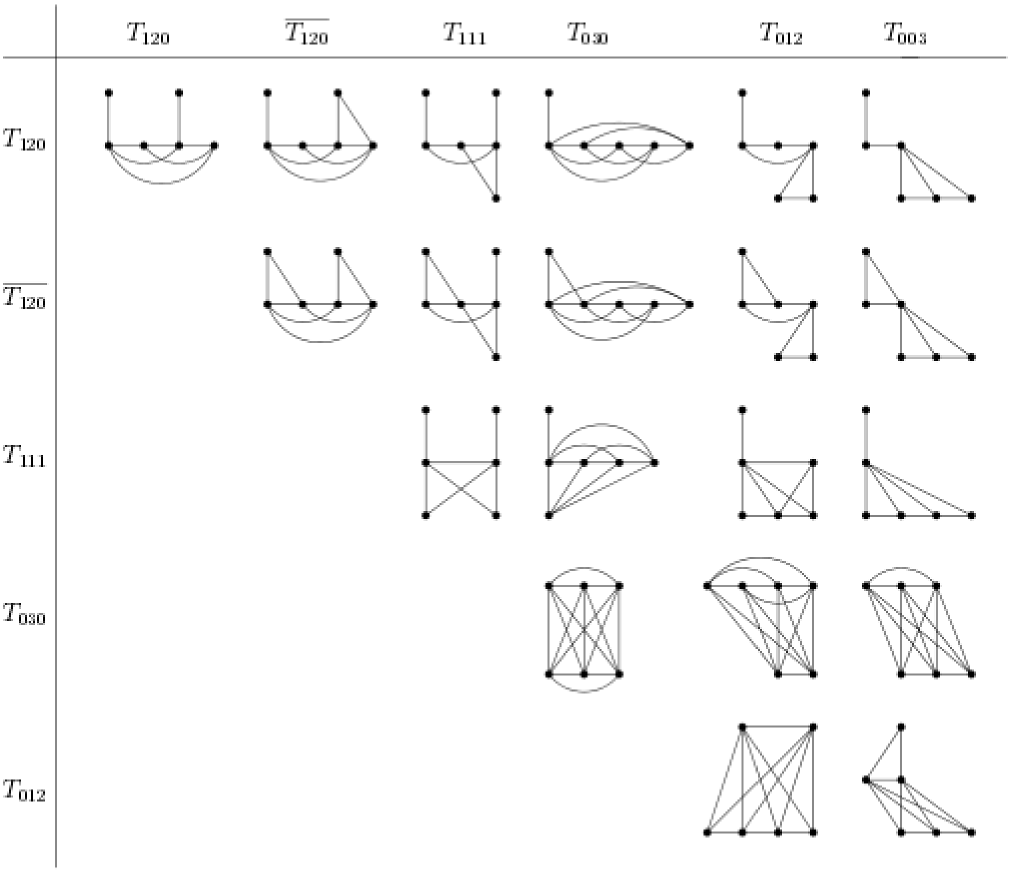


Figure 3. The subgraph of $G$ induced by two 3-components of type in $\{T_{120}, \overline{T_{120}}, T_{111}, T_{030}, T_{021}, T_{012}, T_{003}\}$.

Next assume that there exists at most one 3-components of type in $\{T_{120}, \overline{T_{120}}, T_{111}, T_{030}, T_{021}, T_{012}, T_{003}\}$. Thus, at least $s-1$ 3-components have type $T_{210}$. Since $s \geq 4$, $|I| \geq 2(s-1) \geq 6$. Moreover, by the assumption that the canonical

2-primitive partition of $n$ is realizable in $G$, $|C| \geq |I|-1 \geq 5$, and there exists two 3-components $T'$ and $T''$ of type $T_{210}$, say $T' = \{u_1, u_1', v_1\}$, $T'' = \{u_2, u_2', v_2\}$ with $v_1, v_2 \in C$, and two vertices $w_1, w_2 \in C$ such that $u_1w_1 \in E(G)$ and $u_2w_2 \in E(G)$, and $w_1, w_2$ are lying in 2-component or 3-component contained in $C \cup C_5$.

First assume that at least one of $w_1$ and $w_2$ belongs to a 3-component. Without loss of generality, suppose that $\{w_1, w_0, w_0'\} = T_0$ is a 3-component contained in $C \cup C_5$. Then $T_0 \in \{T_{030}, T_{021}, T_{012}\}$. For the case when $T_0 \cong T_{030}$ or $T_0 \cong T_{021}$, we can decompose the subgraph of $G$ induced by $T' \cup T_0$ into three 2-components. For the case when $T_0 \cong T_{012}$, we may assume that $w_0w_0' \in E(G)$. So, we can decompose the subgraph of $G$ induced by $T' \cup T_0$ into three 2-components.

If $\{w_1, w_2\}$ is a 2-component, then we can decompose the subgraph of $G$ induced by $T' \cup T'' \cup \{w_1, w_2\}$ into four 2-components. In the following, we assume that $w_1$ and $w_2$ belong to different 2-components. Denote $\{w_1, w_1'\}$ and $\{w_2, w_2'\}$ are two 2-components. If $w_1'w_2' \in E(G)$, we can decompose the subgraph of $G$ induced by $T' \cup T'' \cup \{w_1, w_1', w_2, w_2'\}$ into five 2-components. If $w_1'w_2' \notin E(G)$, then $w_1', w_2' \in V(C_5)$, there exists at least one 2-component $v_0v_0'$ such that $v_0 \in V(C_5)$, $v_0w_1' \in E(G)$ and $w_2'v_0' \in E(G)$, and then we can decompose the subgraph of $G[T' \cup T'' \cup \{w_1, w_1', w_2, w_2', v_0, v_0'\}]$ into six 2-components. ∎

**Lemma 8.** *Let $G = (I, C, C_5, E)$ be a $\{2K_2, C_4\}$-free graph of order $n$. If a canonical 2-3-primitive partition $\lambda$ of $n$ is realizable in $G$, then $G$ is $(1,3)$-reducible.*

***Proof.*** Suppose $\lambda = 3^k1$ is realizable in $G$ for some $k \geq 3$. Let $\{v_0\} \cup \Lambda$ be a $\lambda = 3^k1$-decomposition of $G$, in which $\{v_0\}$ is the 1-component and $\Lambda$ is the set of 3-components.

*Case* 1. $v_0 \in C$. By the assumption, every vertex of $C_5$ belongs to a 3-component, and hence there exists a 3-component $T = \{w, u, v\}$ such that $T \cap C_5 \neq \emptyset$ and $T \cap C \neq \emptyset$. Thus $T \in \{T_{021}, T_{012}, T_{111}\}$. We may assume that $w \in T \cap C_5$ and $u \in T \cap C$. Then $v_0w \in E(G)$ and $uv \in E(G)$, implying that $G$ is $(1,3)$-reducible.

*Case* 2. $v_0 \in C_5$. Let $w \in C_5$ be a vertex with $v_0w \in E(C_5)$ and $T = \{w, u, v\}$ be a 3-component containing $w$. Then, $T \in \{T_{021}, T_{012}, T_{111}, T_{003}\}$, and so, $uv \in E(G)$, implying that $G$ is $(1,3)$-reducible.

*Case* 3. $v_0 \in I$. By the assumption, there exists a vertex $w \in C$ and $T = \{w, u, v\}$. Clearly $T \in \{T_{120}, \overline{T_{120}}, T_{111}, T_{030}, T_{021}, T_{012}, T_{210}\}$.

For the case $T \cong T_{120}$, assume that $u \in C$ and $v \in I$. If $uv \in E(G)$, then $\{v_0, w\}$ and $\{u, v\}$ are two 2-components, and so, $G$ is $(1,3)$-reducible. Otherwise, $uv \notin E(G)$ and $wv \in E(G)$. Then we can choose $\{u\}$ as the new 1-component, and $\{v_0, w, v\}$ as the 3-component. Then, this case is reduced to Case 1.

If $T \cong \overline{T_{120}}$, we may assume that $v \in I$. Without loss of generality, let $uv \in E(G)$. Clearly, the subgraph $G[\{v, w, u, v_0\}]$ can be partitioned into two 2-components $\{v_0, w\}$ and $\{u, v\}$. So, $G$ is $(1,3)$-reducible.

For the case when $T \cong T_{111}$, let $v \in I$ and $u \in C_5$. We can choose $\{u\}$ as the new 1-component, and $\{v_0, w, v\}$ as the 3-component. Then it is reduced to Case 2.

If $T \cong T_{030}$, then $\{v_0\} \cup T$ can be repartitioned into two 2-components $\{v_0, w\}$ and $\{u, v\}$.

If $T \cong T_{021}$, assume that $u \in C_5$ and $v \in C$. Again, $\{v_0\} \cup T$ can be repartitioned into two 2-components $\{v_0, w\}$ and $\{u, v\}$.

If $T \cong T_{012}$, then $\{u, v\} \subseteq C_5$. Actually, we may assume that $uv \in E(G)$. Then $\{v, w, u, v_0\}]$ can be partitioned into two 2-components $\{v_0, w\}$ and $\{u, v\}$, as we desired.

Now we deal with the last case when $T \cong T_{210}$. Since the canonical 2-primitive partition of $n$ is realizable in $G$, $|C| \geq |I| - 1$. It means that there must exist a 3-component $T'$ with type distinct from $T_{210}$. Take a sequence of 3-components $T_1, \ldots, T_j$ of $\mathcal{T}$ (Let $T_i = \{u_i, w_i, v_i\}$ with $u_i, v_i \in I$ and $w_i \in C$), such that $v_i w_{i+1} \in E(G)$ and $T_i \cong T_{210}$ for each $i < j$, and $T_j \ncong T_{210}$. Let $T_i' = (T_i \setminus \{v_i\}) \cup \{v_{i-1}\}$ for each $i \in \{1, \ldots, j\}$. Replacing the components $\{v_0\}$, $T_1, \ldots, T_j$ of $\mathcal{T}$ with $\{w_j\}$, $T_1', \ldots, T_j'$, we obtain a new realization $\mathcal{T}'$ of $3^s1$ in which 1-component $\{w_j\}$ does not belong to $I$. By Cases 1 and 2, $G$ is $(1, 3)$-reducible. ∎

**Theorem 9.** *Let $G = (I, C, C_5, E)$ be a connected $\{2K_2, C_4\}$-free graph of order $n$. If every canonical 2-3-primitive partition of $n$ is realizable in $G$, then every 2-3-primitive partition of $n$ is realizable in $G$.*

***Proof.*** Let $\lambda = 2^r3^s$ be a 2-3-primitive partition of $n$. Since every canonical 2-3-primitive partition of $n$ is realizable in $G$, we may assume that $r \geq 2$ and $s \geq 2$.

If $r = 2$, we are able to obtain a $\lambda$-decomposition from the $\lambda'$-realization of the canonical primitive partition $13^{s+1}$, because $G$ is $(1, 3)$-reducible by Lemma 8. If $r \geq 3$, we can obtain a $\lambda$-decomposition from the realization of the 2-3-primitive partition $2^{r-3}3^{s+2}$, because $G$ is $(3, 3)$-reducible by Lemma 11. ∎

By Theorem 6 and Theorem 9, we obtain the following result.

**Theorem 10.** *A $\{2K_2, C_4\}$-free graph $G$ on $n$ vertices is AP if and only if every canonical 2-3-primitive partition of $n$ is realizable in $G$.*

## 4. $2K_2$-Free Bipartite Graphs

**Lemma 11.** *Let $G = (X, Y)$ be a connected $2K_2$-free bipartite graph. If $G$ has a perfect matching or a near perfect matching, then every 2-3-primitive partition $\lambda$ of $n$ is realizable in $G$.*

***Proof.*** Since $G$ has a perfect matching or a near perfect matching, $\lambda^*$ is realizable in $G$, where

$$\lambda^* = \begin{cases} 2^{\frac{n}{2}}, & \text{if } n \text{ is even,} \\ 2^{\frac{n-1}{2}}1, & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, since $G$ is connected, $(2, \ldots, 2, 3)$ is realizable in $G$ if $n$ is odd. So, let $\Lambda_0$ be a $\lambda_0$-decomposition of $G$, where

$$\lambda_0 = \begin{cases} 2^{\frac{n}{2}}, & \text{if } n \text{ is even,} \\ 2^{\frac{n-3}{2}}3, & \text{if } n \text{ is odd.} \end{cases}$$

To prove every 2-3-primitive partition $\lambda$ of $n$ is realizable in $G$, it suffices to show that

(i) the subgraph induced by any three 2-components of $\Lambda_0$ can be decomposed into two 3-components; and

(ii) the subgraph induced by any two 2-components of $\Lambda_0$ can be decomposed into one 1-component and one 3-component.

We first prove (i). Let $x_1y_1$, $x_2y_2$ and $x_3y_3$ be three 2-components of $\Lambda_0$, where $x_i \in X$, $y_i \in Y$, $1 \le i \le 3$. Since $G$ is $2K_2$-free, the subgraph induced by any two 2-components is connected. Without loss of generality, we assume that $x_1y_2 \in E(G)$. If $x_2y_3 \in E(G)$, then $\{x_1, y_1, y_2\}$ and $\{x_2, x_3, y_3\}$ are two 3-components. Otherwise, $x_3y_2 \in E(G)$. If $x_1y_3 \in E(G)$, then $\{x_1, y_1, y_3\}$ and $\{x_2, x_3, y_2\}$ are two 3-components. If $x_1y_3 \notin E(G)$, then $x_3y_1 \in E(G)$, and so $\{x_1, x_2, y_2\}$ and $\{x_3, y_1, y_3\}$ are two 3-components. For each case, we can decompose three 2-components into two 3-components. Thus, (i) holds.

Since $G$ is $2K_2$-free, the subgraph induced by any two 2-components is connected. So, it is easy to partition this subgraph into a subgraph of order 1 and a subgraph of order 3. Thus, (ii) holds. ■

By Theorem 6 and Lemma 11, we obtain the following result.

**Theorem 12.** *Let $G$ be a connected $2K_2$-free bipartite graph. Then $G$ is AP if and only if $G$ has a perfect matching or a near perfect matching.*

## 5. $2K_2$-Free Nonbipartite Graphs with Clique Number 2

In this section, we consider $2K_2$-free nonbipartite graphs with clique number 2. Recall that $o(H)$ denotes the number of odd components in $H$. The well-known Tutte's 1-factor theorem says that a graph $G$ has a perfect matching if and only if $o(G - S) \le |S|$ for all $S \subseteq V(G)$. The following consequence can be derived easily from Tutte's 1-factor theorem.

**Proposition 13.** *Let $G$ be a graph of odd order. Then $G$ has a near perfect matching if and only if $o(G-S) \leq |S|+1$ for all $S \subset V$.*

For a vertex $v \in V(G)$ and a positive integer $n$, we say that $H$ is obtained from $G$ by multiplying $v$ by $n$ when $H$ is formed by replacing the vertex $v$ by an independent set of $n$ vertices each having the same neighbors as $v$.

**Theorem 14** (Chung, Gyárfás, Tuza and Trotter [13]). *Assume that $G$ is $2K_2$-free, $\omega(G)=2$ and $G$ is not bipartite. Then $G$ can be obtained from the cycle $C_5$ by vertex multiplication.*

So let $G$ be a $2K_2$-free, nonbipartite graph with $\omega(G)=2$. Then, by Theorem 14, we denote $G=(A_1, A_2, A_3, A_4, A_5)$, where the sets $A_i$ are independent sets and form a partition of $V(G)$, and each vertex of $A_i$ is adjacent to all vertices in $A_{i-1} \cup A_{i+1}$ for each $i=1,2,\ldots,5$ where $i-1$ and $i+1$ are taken modulo 5.

**Theorem 15.** *Assume that $G$ is a $2K_2$-free nonbipartite graph of even order with $\omega(G)=2$. Then $G$ has a perfect matching if and only if the following conditions are satisfied for each $i \in \{1,2,\ldots,5\}$,*

(1) $|A_i|+|A_{i+2}| \leq |A_{i-1}|+|A_{i+1}|+|A_{i-2}|$ *and*

(2) $|A_i| \leq |A_{i-1}|+|A_{i+1}|$, *with equality only if $|A_{i-2}|=|A_{i+2}|$.*

***Proof.*** First assume that $G$ has a perfect matching. The conclusions (1) and (2) can be deduced from Tutte's 1-factor theorem by taking $A_{i-1} \cup A_{i+1} \cup A_{i+3}$ and $A_{i-1} \cup A_{i+1}$ into $S$, respectively.

Conversely, let $G$ be a $2K_2$-free nonbipartite graph of even order with $\omega(G)=2$ satisfying conditions (1) and (2). Let $S \subset V(G)$. We shall show that $o(G-S) \leq |S|$. If $G-S$ is connected, then $o(G-S) \leq 1 \leq |S|$ for a nonempty set $S$, and $o(G-S)=o(G)=0=|S|$ for the empty set $S$, since $|V(G)|$ is even. Now assume that $G-S$ is disconnected. At least two nonadjacent parts of $\{A_1, A_2, A_3, A_4, A_5\}$ are contained in $S$. Without loss of generality, we assume that $A_2 \cup A_5 \subseteq S$.

*Case* 1. $S=A_2 \cup A_5$. If $|A_1|=|A_2|+|A_5|$, then by (2) $|A_3|=|A_4|$, and hence

$$o(G-S)=|A_1|=|A_2|+|A_5|=|S|.$$

If $|A_1| \leq |A_2|+|A_5|-1$, then

$$o(G-S) \leq |A_1|+1 \leq |A_2|+|A_5|-1+1=|A_2|+|A_5|=|S|.$$

*Case* 2. $A_2 \cup A_5 \subset S$ and $S \cap (A_1 \cup A_3 \cup A_4) \neq \emptyset$. If $A_3 \not\subseteq S$ and $A_4 \not\subseteq S$, then $o(G-S) \leq |A_1|+1 \leq |A_2|+|A_5|+1 \leq |S|$.

If $A_3 \subset S$, then $o(G-S) \leq |A_1|+|A_4| \leq |A_2|+|A_5|+|A_3| \leq |S|$.

If $A_4 \subset S$, then $o(G-S) \leq |A_1|+|A_3| \leq |A_2|+|A_5|+|A_4| \leq |S|$.

In either case, we obtain $o(G-S) \leq |S|$ for $S \subset V(G)$. By Tutte's 1-factor theorem, $G$ has a perfect matching. ∎

**Theorem 16.** *Assume that $G$ is a $2K_2$-free nonbipartite graph of odd order with $\omega(G) = 2$. Then $G$ has a near perfect matching if and only if the following conditions are satisfied for each $i \in \{1, 2, \ldots, 5\}$,*

(1) $|A_i| + |A_{i+2}| \leq |A_{i-1}| + |A_{i+1}| + |A_{i-2}| + 1$ *and*

(2) $|A_i| \leq |A_{i-1}| + |A_{i+1}| + 1$, *with equality only if* $|A_{i-2}| = |A_{i+2}|$.

***Proof.*** First assume that $G$ has a near perfect matching. The conclusions (1) and (2) can be deduced from Proposition 13 by taking $A_{i-1} \cup A_{i+1} \cup A_{i+3}$ and $A_{i-1} \cup A_{i+1}$ into $S$, respectively.

Conversely, let $G$ be a $2K_2$-free nonbipartite graph of odd order with $\omega(G) = 2$ satisfying conditions (1) and (2). Let $S \subset V(G)$. We shall show that $o(G-S) \leq |S|+1$. If $G-S$ is connected, then $o(G-S) \leq 1 \leq |S|$ for a nonempty set $S$, and $o(G-S) = o(G) = 1 = |S|+1$ for the empty set $S$, since $|V(G)|$ is odd. If $G-S$ is disconnected, then at least two nonadjacent parts of $\{A_1, A_2, A_3, A_4, A_5\}$ are contained in $S$. Without loss of generality, we assume that $A_2 \cup A_5 \subseteq S$.

*Case* 1. $S = A_2 \cup A_5$. If $|A_1| \leq |A_2| + |A_5|$, then

$$o(G-S) = o(G - A_2 - A_5) \leq |A_1| + 1 \leq |A_2| + |A_5| + 1 = |S| + 1.$$

If $|A_1| = |A_2| + |A_5| + 1$, then by the assumption, $|A_3| = |A_4|$. Therefore,

$$o(G-S) = o(G - A_2 - A_5) = |A_1| = |A_2| + |A_5| + 1 = |S| + 1.$$

*Case* 2. $A_2 \cup A_5 \subset S$ and $S \cap (A_1 \cup A_3 \cup A_4) \neq \emptyset$. If $A_3 \not\subseteq S$ and $A_4 \not\subseteq S$, then $o(G-S) \leq |A_1| + 1 \leq |A_2| + |A_5| + 1 + 1 \leq |S| + 1$.

If $A_3 \subset S$, then $o(G-S) \leq |A_1| + |A_4| \leq |A_2| + |A_5| + |A_3| + 1 \leq |S| + 1$.

If $A_4 \subset S$, then $o(G-S) \leq |A_1| + |A_3| \leq |A_2| + |A_5| + |A_4| + 1 \leq |S| + 1$.

For each case, we obtain $o(G-S) \leq |S|+1$ for $S \subset V(G)$. By proposition 13, $G$ has a near perfect matching. ■

**Theorem 17.** *Let $G$ be a $2K_2$-free nonbipartie graph $G$ with $\omega(G) = 2$. Then $G$ is AP if and only if it has a perfect matching or a near perfect matching.*

***Proof.*** The necessity is obvious. We prove the sufficiency by induction on the order $n$ of $G$. If $5 \leq n \leq 6$, then $G$ is traceable, and so, $G$ is AP. If $n = 7$, then

$$(|A_1|, |A_2|, |A_3|, |A_4|, |A_5|) \in \{(1,1,1,2,2), (1,1,2,1,2)\}.$$

It is easy to check that in the both cases, $G$ is traceable, and thus it is AP. If $n = 8$, then $(|A_1|, |A_2|, |A_3|, |A_4|, |A_5|) = (1,1,1,2,3)$ or $(1,1,2,2,2)$ or $(1,2,1,2,2)$. For each case, it can be checked that $G$ is traceable, and hence $G$ is AP.

Now let $n \geq 9$, and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ be a partition of $n$ with $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_p$. Since $G$ is 2-connected, if $p \leq 2$, then $\lambda$ is realizable in $G$ by Theorem

1. So, we assume that $p \geq 3$. If there exists $V_1 \subseteq V(G)$ with $|V_1| = \lambda_1$, which come from some (at least two) consecutive parts of $G$, such that $G_1 = G - V_1$ is $2K_2$-free, $\omega(G_1) = 2$ and $G_1$ is not bipartite graph with a perfect matching or a near perfect matching, then by induction hypothesis, $G_1$ is $(\lambda_2, \ldots, \lambda_p)$-realizable, and hence $G$ is $\lambda$-realizable. If such a set $V_1$ does not exist, we have the following result.

**Claim 1.** *There exist two nonadjacent parts of* $(A_1, A_2, A_3, A_4, A_5)$ *with cardinality* 1. *Moreover,* $\sum_{i=2}^{p} \lambda_i \geq 6$.

***Proof.*** Since $n = \sum_{i=1}^{p} \lambda_i \geq 9$ with $\lambda_1 \leq \cdots \leq \lambda_p$ and $p \geq 3$, if $\sum_{i=2}^{p} \lambda_i \leq 5$, then $\lambda_1 \leq \frac{1}{2} \sum_{i=2}^{p} \lambda_i \leq \frac{1}{2} \times 5 = 2.5$. It follows that $\sum_{i=1}^{p} \lambda_i = \lambda_1 + \sum_{i=2}^{p} \lambda_i \leq 2.5 + 5 = 7.5 < 9$, a contradiction. Thus, $\sum_{i=2}^{p} \lambda_i \geq 6$. □

By Claim 1, suppose that $|A_1| = |A_3| = 1$, without loss of generality. Since $G$ has a perfect matching or a near perfect matching, by Theorem 15 and Theorem 16,

$$|A_2| \leq \begin{cases} 2, & \text{if } n \text{ is even,} \\ 3, & \text{if } n \text{ is odd.} \end{cases}$$

**Claim 2.** $\min\{|A_4|, |A_5|\} \geq 2$.

***Proof.*** Suppose that $\min\{|A_4|, |A_5|\} = 1$, and without loss of generality, let $|A_4| = 1$. Then by Theorem 16(1), $|A_2| + |A_5| \leq |A_1| + |A_3| + |A_4| + 1 = 4$. Thus, $n = \Sigma_{i=1}^{5} |A_i| \leq 7$, a contradiction. □

**Claim 3.** $|A_4| + |A_5| - 2 < \lambda_1 \leq \frac{n}{3}$ *and* $9 \leq n \leq 10$.

***Proof.*** Since $p \geq 3$, $\sum_{i=1}^{p} \lambda_i = n$ and $\lambda_1 \leq \cdots \leq \lambda_p$, we have $\lambda_1 \leq \frac{n}{3}$.

If $|A_4| + |A_5| - 2 \geq \lambda_1$, then we can obtain $G_1$ from $G$ by deleting $\lambda_1$ vertices from $A_4 \cup A_5$, a contradiction. Since $|A_4| + |A_5| - 2 < \lambda_1$,

$$n \leq |A_4| + |A_5| + 2 + 3 < \lambda_1 + 2 + 2 + 3 = \lambda_1 + 7 \leq \frac{n}{3} + 7,$$

implying that $n < \frac{21}{2}$, i.e., $9 \leq n \leq 10$. □

If $n = 10$ ($n$ is even), then $|A_2| \leq 2$. It follows that $(|A_1|, |A_2|, |A_3|, |A_4|, |A_5|) = (1, 1, 1, 4, 3)$ or $(1, 2, 1, 3, 3)$. Since $\lambda_1 \leq \frac{10}{3}$, we can obtain $G_1$ by deleting $\lambda_1$ vertices from $|A_4|$ and $|A_5|$, again a contradiction.

If $n = 9$, then $|A_2| \leq 3$. It follows that $(|A_1|, |A_2|, |A_3|, |A_4|, |A_5|) = (1, 1, 1, 3, 3)$ or $(1, 2, 1, 3, 2)$ or $(1, 3, 1, 2, 2)$. Since $\lambda_1 \leq \frac{9}{3}$, for the cases when $(1, 1, 1, 3, 3)$ and $(1, 2, 1, 3, 2)$, we can obtain $G_1$ from $G$ by deleting $\lambda_1$ vertices from $A_4 \cup A_5$. For the case when $(1, 3, 1, 2, 2)$, if $\lambda_1 \leq 2$, we can obtain $G_1$ from $G$ by deleting $\lambda_1$ vertices from $A_4 \cup A_5$. If $\lambda_1 = 3$, then $\lambda = (3, 3, 3)$. Denote $A_2 = \{u_2, v_2, w_2\}$ and $A_4 = \{u_4, v_4\}$. Then we take $V_1 = A_1 \cup \{u_2, v_2\}$, $V_2 = A_3 \cup \{w_2, u_4\}$, $V_3 =$

$A_5 \cup \{v_4\}$. Note that $G[V_i]$ is connected for each $i \in \{1, 2, 3\}$. That is, $\lambda = (3, 3, 3)$ is realizable in $G$. ■

By Theorem 15, Theorem 16 and Theorem 17, we can obtain the following result. Let $\mathcal{G}$ be set of $2K_2$-free graphs $G$ with $\omega(G) = 2$, satisfying the conditions (1) and (2) in Theorem 16 or Theorem 17 (depending whether $G$ has even or odd order).

**Theorem 18.** *If $G$ is $2K_2$-free, $\omega(G) = 2$ and $G$ is not bipartite, then the following statements are equivalent.*

(i) *$G$ is AP.*

(ii) *$G \in \mathcal{G}$.*

(iii) *$G$ has a perfect matching or a near perfect matching.*

By Theorem 12 and Theorem 18 we obtain the following result.

**Theorem 19.** *If $G$ is $2K_2$-free and $\omega(G) = 2$, then $G$ is AP if and only if $G$ has a perfect matching or a near perfect matching.*

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