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GRAPHS WHOSE A_α-SPECTRAL RADIUS DOES NOT EXCEED 2

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This paper is dedicated to the memory of our excellent colleague Slobodan K. Simić who recently passed away.

Abstract

Let $A(G)$ and $D(G)$ be the adjacency matrix and the degree matrix of a graph G, respectively. For any real $\alpha \in [0,1]$, we consider $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$ as a graph matrix, whose largest eigenvalue is called the A_α-spectral radius of G. We first show that the smallest limit point for the A_α-spectral radius of graphs is 2, and then we characterize the connected graphs whose A_α-spectral radius is at most 2. Finally, we show that all such graphs, with four exceptions, are determined by their A_α-spectra.

Keywords: A_α-matrix, Smith graphs, limit point, spectral radius, index.

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1. Introduction

All graphs considered here are simple, undirected and connected. Let $G = (V(G), E(G))$ be such a graph with the order $|V(G)| = n$ and size $|E(G)| = m$. Let $M = M(G)$ be a corresponding *graph matrix* defined in some prescribed way. The *M-polynomial* of G is defined as $\phi_M(G, \lambda) = \det(\lambda I - M)$, where I is the identity matrix. The *M-eigenvalues* of G are those of its graph matrix M, and constitute the *M-spectrum* of G. The *M-index* of G is its largest M-eigenvalue, which often is also the *M-spectral radius*, denoted by $\rho_M(G)$. In the literature, M takes the role of several matrices, as the adjacency matrix A, the Laplacian matrix L, the signless Laplacian matrix Q, the distance matrix $\mathcal{D}$, among others.

Usually, the M-index of a graph increases with the complexity of the graph structure. Therefore, graphs showing a simple structure get a relatively small M-index. The first results in this direction were obtained for $M = A$, the adjacency matrix. It was Smith in [19], who detected all connected graphs whose A-index is equal to 2 (see also [8, 15] for a generalization), such graphs are known as the *Smith graphs*. In [9] Hoffman proved that 2 is the smallest limit point for the spectral radius of sequences of vertex-increasing graphs, and he found all limit values up to $\sqrt{2+\sqrt{5}}$. Finally, Hoffman and Smith in [10] proved that adding infinitely many vertices of degree 2 (by subdividing all edges) in any graph whose maximum degree is Δ, then the corresponding A-index converges to $\frac{\Delta}{\sqrt{\Delta-1}}$.

The seminal papers of Hoffman and Smith inspired similar investigations, as the complete characterization of connected graphs whose A-index does not exceed $\sqrt{2+\sqrt{5}}$ [2, 4], and of connected graphs whose A-index does not exceed $\frac{3}{\sqrt{2}}$ [3, 23, 24]. Analogous investigations were conducted for the Laplacian and signless Laplacian matrices [20–22], as well.

Let $D(G) = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ be the diagonal matrix of G, where $d_i = d(v_i)$ is the degree of vertex $v_i \in V(G)$. For any $\alpha \in [0, 1]$ and for any graph G, the A_α matrix of G is defined as

$$A_\alpha(G) = \alpha D(G) + (1 - \alpha)A(G).$$

Here, we study $M = A_\alpha$ (recall, $\alpha \in [0, 1]$), and our aim is to characterize the graphs G getting a small A_α-index (in fact, the A_α-spectral radius) for the graph matrix A_α. For the sake of readability, we shall use A_α-index instead of A_α-spectral radius, and we denote the characteristic polynomial $|\lambda I - A_\alpha(G)|$ by $\phi_\alpha(G, \lambda)$. The matrix A_α was first defined in [16] by Nikiforov as an unifying approach to study the graph matrices $A = A_0$, $D = A_1$, $Q = 2A_{1/2}$ and $L = \frac{A_\alpha - A_\beta}{\alpha - \beta}$. This matrix has attracted the attention of several scholars, and there is already an interesting literature covering this graph matrix [11, 13, 14, 17, 18].

As usual, let P_n, C_n, $K_{1,n-1}$ and W_n be the *path*, the *cycle*, the *star* and the *double-snake* of order n, respectively. Let $T_{a,b,c}$ stand for a T-shaped tree defined as a tree with a single vertex u of degree 3 such that $T_{a,b,c} - u = P_a \cup P_b \cup P_c$ $(a \leq b \leq c)$; when $a = b = 1$, the tree $T_{1,1,n-3}$ is also called a *snake*. Both snake $T_{1,1,n-3}$ and double-snake W_n are depicted in Figure 1.

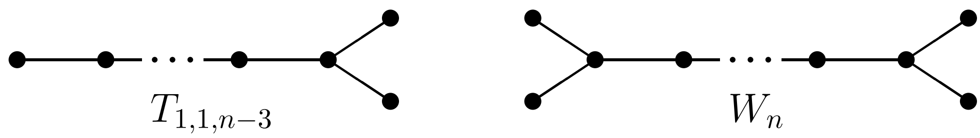

Figure 1. The snake and the double-snake.

The main results of this paper are stated as follows. The first one is about the smallest limit point for the A_α-index of graphs.

Theorem 1.1. *The smallest limit point of the A_α-index $\rho_{A_\alpha}(G)$ of graphs is* 2.

Next, in Theorem 1.2 we characterize the connected graphs with A_α-index at most 2.

Theorem 1.2. *Let G be a connected graph with order n. Then, for $\alpha \in [0,1]$,*

(i) $\rho_{A_\alpha}(G) < 2$ *if and only if G is one of the following graphs.*
   (a) P_n $(n \geq 1)$ *for* $\alpha \in [0,1)$,
   (b) $T_{1,1,n-3}$ $(n \geq 4)$ *for* $\alpha \in [0, s_1)$,
   (c) $T_{1,2,2}$ *for* $\alpha \in [0, s_2)$, $T_{1,2,3}$ *for* $\alpha \in [0, s_3)$ *and* $T_{1,2,4}$ *for* $\alpha \in [0, s_4)$.

(ii) $\rho_{A_\alpha}(G) = 2$ *if and only if G is one of the following graphs.*
   (a) C_n, $n \geq 3$,
   (b) P_n $(n \geq 3)$ *for* $\alpha = 1$,
   (c) W_n $(n \geq 6)$ *for* $\alpha = 0$,
   (d) $T_{1,1,n-3}$ *for* $\alpha = s_1$, *where* $s_1 = \frac{4}{n+1+\sqrt{(n+1)^2-16}}$,
   (e) $T_{1,2,2}$ *for* $\alpha = s_2$, $T_{1,2,3}$ *for* $\alpha = s_3$, $T_{1,2,4}$ *for* $\alpha = s_4$,
   (f) $T_{1,3,3}$ *for* $\alpha = 0$, $T_{1,2,5}$ *for* $\alpha = 0$, $K_{1,4}$ *for* $\alpha = 0$ *and* $T_{2,2,2}$ *for* $\alpha = 0$,

where $s_2 = 0.2192+$ *is the solution of* $2\alpha^3 - 11\alpha^2 + 16\alpha - 3 = 0$, $s_3 = 0.1206+$ *is the solution of* $\alpha^3 - 6\alpha^2 + 9\alpha - 1 = 0$ *and* $s_4 = 0.0517+$ *is the solution of* $2\alpha^3 - 13\alpha^2 + 20\alpha - 1 = 0$.

Finally, we study the spectral determination of graphs with A_α-index at most 2. We mention some basic notions. Two non-isomorphic graphs G and H with the same M-spectrum are called *M-cospectral graphs*. A graph G is said to be *determined by its M-spectrum* if there is no other non-isomorphic graph with the same M-spectrum. There are dozens of papers on this problem, especially for the adjacency matrix; a good starting point are the excellent surveys [5, 6]. For $M = A_\alpha$ there is already some literature. In [12], it is proved that the path P_n and the cycle C_n are determined by their A_α-spectra. Our result reads as follows.

Theorem 1.3. *Let $\alpha \in [0, 1]$ and G be a connected graph with A_α-index at most 2. Then*

(i) *P_n $(n \geq 1)$ and C_n $(n \geq 3)$ are determined by their A_α-spectra for $\alpha \in [0, 1]$,*
(ii) *$T_{1,1,n-3}$ $(n \geq 4)$ for $\alpha \in [0, s_1]$ and $T_{1,2,c}$ $(c \in \{2, 3, 4\})$ for $\alpha \in [0, s_c]$ are determined by their A_α-spectra,*
(iii) *$T_{1,3,3}$ and $T_{1,2,5}$ are determined by the A_0-spectrum,*
(iv) *$T_{1,1,1}$ and $K_1 \cup C_3$ are $A_{\frac{1}{2}}$-cospectral,*
(v) *W_n and $C_4 \cup P_{n-4}$ $(n \geq 6)$, $T_{2,2,2}$ and $K_1 \cup C_6$, $K_{1,4}$ and $K_1 \cup C_4$ are A_0-cospectral.*

The rest of the paper is organized as follows. In Section 2, the smallest limit point of A_α-index of graphs is determined. In Section 3, all the connected graphs with the A_α-index at most 2 are characterized, and we also study their spectral determination. In Section 4, some concluding remarks about this topic are given.

2. The Limit Points of A_α-Index of Graphs

In this section, we investigate the smallest limit point of A_α-index of graphs.

Lemma 2.1 [16]**.** *Let $\alpha \in [0, 1]$. Then the A_α-index of $K_{1,n-1}$ is*

$$\rho_{A_\alpha}(K_{1,n-1}) = \frac{1}{2}\left(\alpha n + \sqrt{\alpha^2 n^2 + 4(n-1)(1-2\alpha)}\right).$$

Lemma 2.2 [17]**.** *The spectral radius of $A_\alpha(P_n)$ satisfies*

$$\rho_{A_\alpha}(P_n) \leq \begin{cases} 2\alpha + 2(1-\alpha)\cos\left(\frac{\pi}{n+1}\right), & \alpha \in [0, 1/2), \\ 2\alpha + 2(1-\alpha)\cos\left(\frac{\pi}{n}\right), & \alpha \in [1/2, 1]. \end{cases}$$

Equality holds if and only if $\alpha = 0$, $\alpha = 1/2$, $\alpha = 1$.

Lemma 2.3 [17]**.** *The spectral radius of $A_\alpha(P_n)$ satisfies*

$$\rho_{A_\alpha}(P_n) \geq \begin{cases} 2\alpha + 2(1-\alpha)\cos\left(\frac{\pi}{n}\right), & \alpha \in [0, 1/2), \\ 2\alpha + 2\alpha\cos\left(\frac{\pi}{n}\right) - 2(2\alpha-1)\cos\left(\frac{\pi}{n+1}\right), & \alpha \in [1/2, 1]. \end{cases}$$

Equality holds if and only if $\alpha = 1/2$.

In fact, the proof of Hoffman [9] given for $\alpha = 0$ can be re-used for any $\alpha \in [0, 1]$, so that Theorem 1.1 can be deduced. To keep the paper self-contained, we provide the proof adapted to the general case.

***Proof of Theorem* 1.1.** Let $G_1, G_2, \ldots$ be a sequence of graphs such that $\rho_{A_\alpha}(G_i) \neq \rho_{A_\alpha}(G_j)$ for $i \neq j$, and $\rho_{A_\alpha}(G_n) \to \lambda < 2$. Suppose that G is a connected graph on at least three vertices, the maximum degree of the vertices of G is $\Delta(G)$, and the diameter of G is $d(G)$. Then $|V(G)| \leq \Delta^{d(G)} + 1$. Therefore, $\max(\Delta(G), d(G)) \geq (\log |V(G)| - 1)^{1/2}$. But since the graphs G_i are different, $|V(G)| \to \infty$. Hence, for sufficiently large n, G_n contains as a subgraph an arbitrarily long path P_k or arbitrarily large star $K_{1,t}$. But from Lemma 2.1 we know $\rho_{A_\alpha}(K_{1,t}) \to \infty$; while from Lemmas 2.2 and 2.3 it follows that $\rho_{A_\alpha}(P_k) \to 2$. ■

3. Graphs with A_α-Index at Most 2

In this section we characterize all the connected graph whose A_α-index does not exceed 2. The graphs mentioned in Lemma 3.1(ii) are depicted in Figure 2.

Lemma 3.1 [19]**.** *Let G be a connected graph with A-index $\rho_A(G)$. Then*

(i) $\rho_A(G) < 2$ *if and only if*

$$G \in \mathcal{G}_1 = \{P_n (n \geq 1), T_{1,1,n-3} (n \geq 4), T_{1,2,2}, T_{1,2,3}, T_{1,2,4}\},$$

(ii) $\rho_A(G) = 2$ *if and only if*

$$G \in \mathcal{G}_2 = \{C_n (n \geq 3), W_n (n \geq 6), K_{1,4}, T_{2,2,2}, T_{1,2,5}, T_{1,3,3}\}.$$

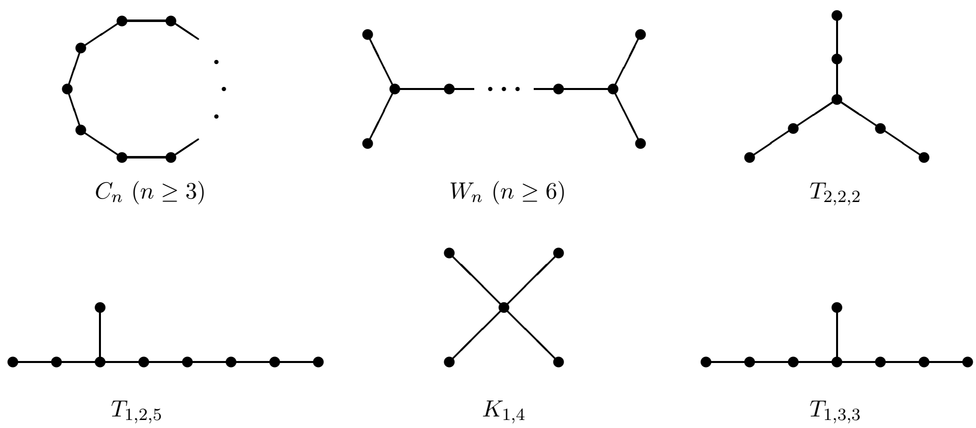

Figure 2. The Smith graphs.

Some useful properties of A_α-matrix are summarized in the following lemma.

Lemma 3.2. *Let G be a connected graph of order n and maximum degree $\Delta(G)$.*

(i) [16] *Then $\rho_{A_\alpha}(G) \geq \rho_A(G)$ for $\alpha \in [0,1]$,*

(ii) [16] *Then $\rho_{A_\alpha}(G) \geq \rho_{A_\beta}(G)$ for $0 \leq \beta < \alpha \leq 1$, where inequality is strict, unless G is regular.*

(iii) [17] *For $\alpha \in [0,1]$,*

  (a) *if $\alpha = 1$, then $\rho_{A_\alpha}(G) = \Delta(G)$,*

  (b) *if $\alpha \in [0,1)$, then $\rho_{A_\alpha}(G) \geq \rho_{A_\alpha}(P_n)$, where the equality holds if and only if $G = P_n$.*

(iv) [16] *If H is a proper subgraph of G, then $\rho_{A_\alpha}(H) < \rho_{A_\alpha}(G)$.*

(v) [16] *If λ is an eigenvalue of $A_\alpha(G)$ with a nonnegative eigenvector, then $\lambda = \rho_{A_\alpha}(G)$.*

Let $\alpha \in [0,1]$. We now pick up the connected graphs G with $\rho_{A_\alpha}(G) \leq 2$. Assume that $\rho_A(G) > 2$. Then by Lemma 3.2(i) we get $\rho_{A_\alpha}(G) \geq \rho_A(G) > 2$. Hence, $\rho_A(G) \leq 2$, and thus $G \in \mathcal{G}_1 \cup \mathcal{G}_2$ by Lemma 3.1. It is easy to check that if G is k-regular, then $A_\alpha(G)$ has constant row sum equal to k, and we get $\rho_{A_\alpha}(G) = k$ (cf. also [16]). Consequently, we have $\rho_{A_\alpha}(C_n) = 2$, and then $\rho_{A_\alpha}(P_n) < 2$ for $\alpha \in [0,1)$ and $\rho_{A_\alpha}(P_n) = 2$ ($n \geq 3$) for $\alpha = 1$ by Lemma 3.2(iii). Thereby, the left work is to discuss the graphs in the set $(\mathcal{G}_1 \cup \mathcal{G}_2)\backslash\{P_n, C_n\}$. If $G \in \mathcal{G}_2\backslash\{C_n \,|\, n \geq 3\}$, by Lemmas 3.1 and 3.2(ii) we obtain $\rho_{A_\alpha}(G) \geq \rho_{A_0}(G) = 2$ with equality if and only if $\alpha = 0$. Thus, we have shown the following proposition.

Proposition 3.3. *Let G be a connected graph with order n. For $\alpha \in [0,1]$,*

(i) *if $\rho_{A_\alpha}(G) \leq 2$, then $G \in \mathcal{G}_1 \cup \mathcal{G}_2$,*

(ii) *$\rho_{A_\alpha}(P_n) = 2$ ($n \geq 3$) for $\alpha = 1$, and $\rho_{A_\alpha}(P_n) < 2$ ($n \geq 1$) for $\alpha \in [0,1)$;*

(iii) *$\rho_{A_\alpha}(C_n) = 2$ ($n \geq 3$),*

(iv) *if $G \in \mathcal{G}_2\backslash\{C_n \,|\, n \geq 3\}$, then $\rho_{A_\alpha}(G) = 2$ for $\alpha = 0$.*

At this stage, we need only discuss the graphs in $\mathcal{G}_1\backslash\{P_n \,|\, n \geq 1\}$. Recall that $\phi_\alpha(G,\lambda) = \det(\lambda I - A_\alpha(G))$ is the A_α-polynomial of a graph G. Let

$$D_n = \begin{vmatrix} 2-\alpha & 0 & \alpha-1 & & & & \\ 0 & 2-\alpha & \alpha-1 & & & & \\ \alpha-1 & \alpha-1 & 2-3\alpha & \alpha-1 & & & \\ & & \alpha-1 & 2-2\alpha & \alpha-1 & & \\ & & & & \ddots & & \\ & & & & \alpha-1 & 2-2\alpha & \alpha-1 \\ & & & & & \alpha-1 & 2-2\alpha \end{vmatrix}.$$

Lemma 3.4. *For $n \geq 4$, $D_n = (-1)^n(\alpha-2)(\alpha-1)^{n-3}(\alpha^2-(n+1)\alpha+2)$.*

Proof. We prove the lemma by induction on the order n. For $n = 4$, a direct calculation shows that the lemma holds. Suppose that the lemma is true when $n \leq k-1$. For $n = k$, by expanding the determinant obtained by the last column we arrive at

$$\begin{aligned} D_k &= (2-2\alpha)D_{k-1} - (\alpha-1)^2 D_{k-2} \\ &= (2-2\alpha)((-1)^{k-1}(\alpha-2)(\alpha-1)^{k-4}(\alpha^2-k\alpha+2)) \\ &- (\alpha-1)^2((-1)^{k-2}(\alpha-2)(\alpha-1)^{k-5}(\alpha^2-(k-1)\alpha+2)) \\ &= (-1)^{k-2}(\alpha-2)(\alpha-1)^{k-5}[(2\alpha-2)(\alpha-1)(\alpha^2-k\alpha+2) \\ &- (\alpha-1)^2(\alpha^2-(k-1)\alpha+2)] \\ &= (-1)^{k-2}(\alpha-2)(\alpha-1)^{k-5}((\alpha-1)^2(\alpha^2-(k+1)\alpha+2)) \\ &= (-1)^k(\alpha-2)(\alpha-1)^{k-3}(\alpha^2-(k+1)\alpha+2). \end{aligned}$$

Hence, the result follows. ∎

Lemma 3.5. *The A_α-polynomial of $T_{1,1,n-3}$, for $n \geq 4$, computed at 2 is*

$$\phi_{A_\alpha}(T_{1,1,n-3}, 2) = (-1)^{n+1}(\alpha-2)(\alpha-1)^{n-4}(2\alpha^2-(n+1)\alpha+2).$$

Proof. Clearly, the A_α-polynomial of $T_{1,1,n-3}$ when $\lambda = 2$ is

$$\phi_{A_\alpha}(T_{1,1,n-3}, 2) = \begin{vmatrix} 2-\alpha & 0 & \alpha-1 & & & & \\ 0 & 2-\alpha & \alpha-1 & & & & \\ \alpha-1 & \alpha-1 & 2-3\alpha & \alpha-1 & & & \\ & & \alpha-1 & 2-2\alpha & \alpha-1 & & \\ & & & & \ddots & & \\ & & & & \alpha-1 & 2-2\alpha & \alpha-1 \\ & & & & & \alpha-1 & 2-\alpha \end{vmatrix}.$$

From Lemma 3.4, expanding the above determinant obtained by the last column we obtain that

$$\begin{aligned} \phi_{A_\alpha}(T_{1,1,n-3}, 2) &= D_n + \alpha D_{n-1} \\ &= (-1)^n(\alpha-2)(\alpha-1)^{n-3}(\alpha^2-(n+1)\alpha+2) \\ &+ \alpha((-1)^{n-1}(\alpha-2)(\alpha-1)^{n-4}(\alpha^2-n\alpha+2)) \\ &= (-1)^{n-1}(\alpha-2)(\alpha-1)^{n-4}(\alpha(\alpha^2-n\alpha+2) \\ &- (\alpha-1)(\alpha^2-(n+1)\alpha+2)) \\ &= (-1)^{n+1}(\alpha-2)(\alpha-1)^{n-4}(2\alpha^2-(n+1)\alpha+2). \end{aligned}$$

This completes the proof. ∎

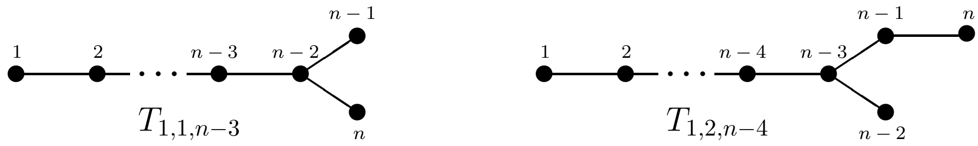

Figure 3. The labeled trees $T_{1,1,n-3}$ and $T_{1,2,n-4}$.

Proposition 3.6. *Let $T_{1,1,n-3}$ be a tree ($n \geq 4$) and $\alpha \in [0,1]$. Then*

$$\rho_{A_\alpha}(T_{1,1,n-3}) \begin{cases} < 2, & \alpha \in [0, s_1), \\ = 2, & \alpha = s_1, \\ > 2, & \alpha \in (s_1, 1], \end{cases} \quad \textit{where } s_1 = \frac{4}{n+1+\sqrt{(n+1)^2-16}}.$$

Proof. To obtain the results, in view of Lemma 3.2(ii) we only need to show $\rho_{A_\alpha}(T_{1,1,n-3}) = 2$ if and only if $\alpha = s_1$. From Lemma 3.5 and $\alpha \in [0,1]$, it follows that 2 is an A_α-eigenvalue of $T_{1,1,n-3}$ if and only if $\phi_\alpha(T_{1,1,n-3}, 2) = 0$, that is

$$\alpha \in \left\{1, \frac{n+1-\sqrt{(n+1)^2-16}}{4}\right\}.$$

If $\alpha = 1$, then $A_1(T_{1,1,n-3}) = D(T_{1,1,n-3}) = \mathrm{diag}(1,1,1,2,\ldots,2,3)$ with $\rho_{A_1}(T_{1,1,n-3}) = 3 > 2$. Hence, the left work is to show $\rho_{A_{s_1}}(T_{1,1,n-3}) = 2$ when

$$s_1 = \alpha = \frac{n+1-\sqrt{(n+1)^2-16}}{4} = \frac{4}{n+1+\sqrt{(n+1)^2-16}}.$$

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be the eigenvector associated to the eigenvalue 2. Without loss of generality, set $x_1 = 1$. By $A_{s_1}(T_{1,1,n-3})\mathbf{x} = 2\mathbf{x}$ we get

$$s_1 d_i x_i + (1-s_1)\sum_{j \backsim i} x_j = 2x_i.$$

As labelled in Figure 3, we get $x_2 = \frac{2-s_1}{1-s_1} > 0$ and $2x_i = x_{i-1} + x_{i+1}$ ($i = 2, \ldots, n-3$). Thereby,

$$x_{i+1} - x_i = x_i - x_{i-1} = \cdots = x_2 - x_1 = \frac{1}{1-s_1},$$

which leads to

$$x_{i+1} = x_i + \frac{1}{1-s_1} > 0 \ \ (i = 2, 3, \ldots, n-3)$$

and

$$x_{n-3} = x_{n-4} + \frac{1}{1-s_1} = x_{n-5} + \frac{2}{1-s_1} = \cdots = x_2 + \frac{n-5}{1-s_1} = \frac{n-3-s_1}{1-s_1} > 0.$$

For x_{n-2}, x_{n-1} and x_n, solving the following equations

$$\begin{cases} 3s_1x_{n-2} + (1-s_1)(x_{n-3} + x_{n-1} + x_n) = 2x_{n-2}, \\ s_1x_{n-1} + (1-s_1)\,x_{n-2} = 2x_{n-1}, \\ s_1x_n + (1-s_1)\,x_{n-2} = 2x_n, \end{cases}$$

we get $x_{n-2} = \frac{(n-s_1-3)(2-s_1)}{s_1^2-4s_1+2}$ and $x_{n-1} = x_n = \frac{(n-s_1-3)(1-s_1)}{s_1^2-4s_1+2}$. Note, it is not difficult to obtain $s_1 \le 0.5 < 2-\sqrt{2}$ (which is the least root of $s^2-4s+2=0$). Thus, for $n \ge 4$ we get $n-s_1-3>0$, $2-s_1>0$, $1-s_1>0$ and $s_1^2-4s_1+2 > 0$. Hence, $x_{n-2}, x_{n-1}, x_n > 0$, which indicates $\mathbf{x}$ is a non-negative eigenvector. Thereby, $\rho_{A_\alpha}(T_{1,1,n-3}) = 2$ by Lemma 3.2(v).

This completes the proof. ■

With similar methods as above, we next check up the A_α-index of three small trees.

Proposition 3.7. *Let $G \in \{T_{1,2,2}, T_{1,2,3}, T_{1,2,4}\}$. For $\alpha \in [0,1]$,*

(i) $$\rho_{A_\alpha}(T_{1,2,2}) \begin{cases} < 2, & \alpha \in [0, s_2), \\ = 2, & \alpha = s_2, \\ > 2, & \alpha \in (s_2, 1], \end{cases}$$

where $s_2 \in [0,1/2]$ is the solution of the equation $2\alpha^3 - 11\alpha^2 + 16\alpha - 3 = 0$,

(ii) $$\rho_{A_\alpha}(T_{1,2,3}) \begin{cases} < 2, & \alpha \in [0, s_3), \\ = 2, & \alpha = s_3, \\ > 2, & \alpha \in (s_3, 1], \end{cases}$$

where $s_3 \in [0,1/2]$ is the solution of the equation $\alpha^3 - 6\alpha^2 + 9\alpha - 1 = 0$,

(iii) $$\rho_{A_\alpha}(T_{1,2,4}) \begin{cases} < 2, & \alpha \in [0, s_4), \\ = 2, & \alpha = s_4, \\ > 2, & \alpha \in (s_4, 1], \end{cases}$$

where $s_4 \in [0,1/2]$ is the solution of the equation $2\alpha^3 - 13\alpha^2 + 20\alpha - 1 = 0$.

Proof. If $\alpha \in (1/2, 1]$, from Lemma 3.2(iv) we obtain for $n = 6, 7, 8$ that

$$\rho_{A_\alpha}(T_{1,2,n-4}) > \rho_{A_\alpha}(T_{1,1,1}) = 2\alpha + \sqrt{3 - 6\alpha + 4\alpha^2} > 2.$$

Therefore, $\alpha \in [0, 1/2]$. By calculations we arrive at

$$\phi_\alpha(T_{1,2,2}, 2) = -(\alpha - 1)^2(2\alpha^3 - 11\alpha^2 + 16\alpha - 3), \tag{1}$$
$$\phi_\alpha(T_{1,2,3}, 2) = 2(\alpha - 1)^3(\alpha^3 - 6\alpha^2 + 9\alpha - 1), \tag{2}$$
$$\phi_\alpha(T_{1,2,4}, 2) = -(\alpha - 1)^4(2\alpha^3 - 13\alpha^2 + 20\alpha - 1), \tag{3}$$

which implies that 2 is, respectively, an A_α-eigenvalue of $T_{1,2,2}$, $T_{1,2,3}$ and $T_{1,2,4}$ if and only if $s_2 = \alpha = 0.2192+$ in (1), $s_3 = \alpha = 0.1206+$ in (2) and $s_4 = \alpha = 0.0517+$ in (3). Let $\mathbf{x} = (x_1, x_2, \dots, x_n)$ $(n = 6, 7, 8)$ be the eigenvector associated to the eigenvalue 2. Without loss of generality, set $x_1 = 1$. By $A_{s_{n-4}}(T_{1,2,n-4})\mathbf{x} = 2\mathbf{x}$ we get

$$s_{n-4} d_i x_i + (1 - s_{n-4}) \sum_{j \backsim i} x_j = 2x_i.$$

As labelled in Figure 3, we get $x_2 = \frac{2 - s_{n-4}}{1 - s_{n-4}} > 0$ and $2x_i = x_{i-1} + x_{i+1}$ $(i = 2, \dots, n-4)$. Thereby,

$$x_{i+1} - x_i = x_i - x_{i-1} = \cdots = x_2 - x_1 = \frac{1}{1 - s_{n-4}},$$

which results in

$$x_{i+1} = x_i + \frac{1}{1 - s_{n-4}} > 0 \ \ (i = 2, 3, \dots, n-4)$$

and

$$\begin{aligned} x_{n-4} &= x_{n-5} + \frac{1}{1 - s_{n-4}} = x_{n-6} + \frac{2}{1 - s_{n-4}} \\ &= \cdots = x_2 + \frac{n-6}{1 - s_{n-4}} = \frac{n - 4 - s_{n-4}}{1 - s_{n-4}} > 0. \end{aligned}$$

For x_{n-3}, x_{n-2}, x_{n-1} and x_n, solving the next equations

$$\begin{cases} 3s_{n-4}x_{n-3} + (1 - s_{n-4})(x_{n-4} + x_{n-2} + x_{n-1}) = 2x_{n-3}, \\ s_{n-4}x_{n-2} + (1 - s_{n-4})\, x_{n-3} = 2x_{n-2}, \\ 2s_{n-4}x_{n-1} + (1 - s_{n-4})(x_{n-3} + x_n) = 2x_{n-1}, \\ s_{n-4}x_n + (1 - s_{n-4})\, x_{n-1} = 2x_n, \end{cases}$$

we get

$$x_{n-3} = \frac{(s_{n-4}+4-n)(s_{n-4}-2)(s_{n-4}-3)}{s_{n-4}^3-7s_{n-4}^2+13s_{n-4}-5} > 0,$$
$$x_{n-2} = \frac{(s_{n-4}+4-n)(s_{n-4}-1)(s_{n-4}-3)}{s_{n-4}^3-7s_{n-4}^2+13s_{n-4}-5} > 0,$$
$$x_{n-1} = \frac{(s_{n-4}+4-n)(s_{n-4}-2)^2}{s_{n-4}^3-7s_{n-4}^2+13s_{n-4}-5} > 0,$$
$$x_{n} = \frac{(s_{n-4}+4-n)(s_{n-4}-1)(s_{n-4}-2)}{s_{n-4}^3-7s_{n-4}^2+13s_{n-4}-5} > 0.$$

So, $\mathbf{x}$ is a positive eigenvector associated to the eigenvalue 2. By Lemma 3.2(v) we obtain $\rho_{A_\alpha}(T_{1,2,n-4}) = 2$ ($n = 6, 7, 8$). Consequently, the desired result follows from Lemma 3.2(ii).

This completes the proof. ■

***Proof of Theorem* 1.2.** The proof comes as a consequence of Propositions 3.3, 3.6 and 3.7. ■

To conclude this section, we study the A_α-spectral determination of the graphs so far considered.

***Proof of Theorem* 1.3.** Let G be one of graphs in Theorem 1.2. Let H be any graph such that H and G are A_α-cospectral.

If $\rho_{A_\alpha}(G) < 2$, then $\rho_{A_\alpha}(H) < 2$, and thus H is one of graphs in Theorem 1.2(i). Recall that the path P_n ($0 \le \alpha < 1$) is determined by the A_α-spectrum (see [12]). By Lemma 3.5 and Proposition 3.7, it follows that the graphs $T_{1,1,n-3}, T_{1,2,2}, T_{1,2,3}$ and $T_{1,2,4}$ are not pairwise A_α-cospectral. Hence, if $G = T_{1,1,n-3}$, then $H = T_{1,1,t-3}$. By $|V(G)| = |V(H)|$ we get $n = t$ and thus $H \cong G$. Clearly, $T_{1,2,a}$ ($a = 2, 3, 4$) is determined by the A_α-spectrum.

If $\rho_{A_\alpha}(G) = 2$, then $\rho_{A_\alpha}(H) = 2$, and thus H is one of graphs in Theorem 1.2(ii). Recall that the cycle C_n is determined by the A_α-spectrum (see [12]), and Ghareghani [7] proved for $\alpha = 0$ that $T_{1,3,3}$ and $T_{1,2,5}$ are determined by A_0-spectra, but $W_n, T_{2,2,2}$ and $K_{1,4}$ are not. Moreover, $K_{1,3}$ is $A_{\frac{1}{2}}$-cospectral with $K_1 \cup C_3$ [20].

So, the left work is to check the remaining graphs. Using the same method as above, we can prove that P_n for $\alpha = 1$, $T_{1,1,n-3}$ ($n \ge 4$) for $\alpha = s_1$, and $T_{1,2,c}$ ($c = 2, 3, 4$) for $\alpha = s_c$ are determined by their A_α-spectra, as well. ■

4. CONCLUDING REMARKS

Fifty years after the publication of the paper of Smith, his research is still inspiring investigations in Spectral Graph Theory. The work presented in this paper is a generalization of Smith's results, but it can be seen as a first step towards a more general problem, which is known in the literature as the *Hoffman Program*. The Hoffman Program consists in the classification and identification of graphs with *small* index, where the term 'small' means that the index does not exceed the *Hoffman limit value*, that is, the limit value of indices for the sequence of cycles with a pendant vertex and increasing girth. Such value in the adjacency theory is the well-studied number $\sqrt{2+\sqrt{5}} = \frac{\tau^{1/2}+\tau^{-1/2}}{2}$, where τ is the golden mean. For details about the Hoffmann Program for the (signless) Laplacian matrix we refer the reader to [1].

Next step of this research is the study of the Hoffman limit value and the corresponding graphs in the context of the A_α-matrix of graphs.

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