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TURÁN'S THEOREM IMPLIES STANLEY'S BOUND

V. Nikiforov

Department of Mathematical Sciences
University of Memphis, Memphis TN 38152, USA

e-mail: vnikifrv@memphis.edu

Dedicated to the memory of Slobodan K. Simić

Abstract

Let G be a graph with m edges and let ρ be the largest eigenvalue of its adjacency matrix. It is shown that

$$\rho \leq \sqrt{2\left(1-\left\lfloor 1/2+\sqrt{2m+1/4}\right\rfloor^{-1}\right)m},$$

improving the well-known bound of Stanley. Moreover, writing ω for the clique number of G and W_k for the number of its walks on k vertices, it is shown that the sequence

$$\left\{\left((1-1/\omega)\,W_{2^k}\right)^{1/2^k}\right\}_{k=1}^{\infty}$$

is nonincreasing and converges to ρ.

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How large can be the spectral radius ρ of a graph with m edges? This problem was raised by Brualdi and Hoffman in 1976, who published their ground-breaking results a few years later in [1]. Particular cases of this problem were resolved by Friedland [4, 5], and the complete solution was given by Rowlinson in [10]: *The maximum spectral radius ρ of a graph G with m edges is attained if and only if G has a single nontrivial component of order* $\left\lceil 1/2+\sqrt{2m+1/4}\right\rceil$. It is not hard to see that such G is unique up to isomorphism and its nontrivial component is either a complete graph or a complete graph plus an additional vertex.

Independently, Stanley [12] proved the bound

$$\rho \leq -1/2 + \sqrt{2m + 1/4}, \tag{1}$$

which is exact for complete graphs with possibly some isolated vertices, and is never too far from the best possible value proved by Rowlinson. In fact, it would be interesting to determine how far bound (1) can be from the best possible value.

In this note, we deduce Stanley's bound from Turán's theorem [13] in extremal graph theory, which we state for convenience next.

Theorem A (Turán [13]). *If G is a graph of order n and clique number ω, then*

$$e(G) \leq e(T_\omega(n)). \tag{2}$$

Equality holds if and only if $G = T_\omega(n)$.

Here $e(G)$ is the number of edges of G, and $T_r(n)$ is the complete r-partite graph of order n, with color classes of size either $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$. Some counting yields the following expression for $e(T_r(n))$: if s is the remainder of n mod r, then

$$e(T_r(n)) = \left(1 - \frac{1}{r}\right)\frac{n^2}{2} - \frac{s(r-s)}{2r}.$$

As an obvious conclusion from this formula, one gets a neat corollary.

Corollary B (Turán [13]). *If G is a graph of order n and clique number ω, then*

$$e(G) \leq \left(1 - \frac{1}{\omega}\right)\frac{n^2}{2}. \tag{3}$$

Equality holds if and only if ω divides n and $G = T_\omega(n)$.

In the literature, sometimes Corollary B is called *Turán's theorem*, but this is not correct. Indeed, using the fact that $s(r-s)/2r \leq r/8$, bound (3) implies bound (2) for $\omega \leq 7$, yet for $\omega \geq 8$ the implication is not at all clear, unless one essentially reproves Turán's theorem.

Another landmark result, essentially equivalent to Corollary B, is the inequality of Motzkin and Straus [6].

Theorem C (Motzkin and Straus [6]). *Let G be a graph of order n and clique number ω. If $x_1, \ldots, x_n$ are nonnegative real numbers such that $x_1 + \cdots + x_n = 1$, then*

$$\sum_{\{u,v\} \in E(G)} x_u x_v \leq (1 - 1/\omega)/2. \tag{4}$$

Theorem C has multiple applications, hence in honor of Motzkin and Straus we call the value

$$\mu(G) = \max\left\{ \sum_{\{u,v\}\in E(G)} x_u x_v : |x_1| + \cdots + |x_n| = 1 \right\}$$

the *MS-index* of G. Note that $\mu(G)$ can be naturally extended to hypergraphs, where its determination is a major open problem, in contrast to the equality $\mu(G) = (1 - 1/\omega)/2$, which always holds for graphs.

Taking $x_1 = \cdots = x_n = 1/n$, it turns out that (4) implies (3). Conversely, Sidorenko [11] showed by a fairly general argument that (3) implies (4). That is to say, Corollary B and Theorem C are essentially equivalent statements. Nonetheless, Theorem C seems better suited for applications, particularly in spectral graph theory, as pioneered by Wilf [14].

For an illustration, let $(x_1, \ldots, x_n)$ be a unit eigenvector to ρ. Applying the AM–QM inequality and using the fact that $x_1^2 + \cdots + x_n^2 = 1$, we get an upper bound on ρ

(5)
$$\rho = 2 \sum_{\{u,v\}\in E(G)} x_u x_v \le 2\sqrt{m \sum_{\{u,v\}\in E(G)} x_u^2 x_v^2} \le \sqrt{4m\mu(G)} = \sqrt{2(1-1/\omega)m}.$$

Note that for $\omega = 2$ inequality (5) was proved earlier by Nosal [9] using a different method. The general inequality $\rho \le \sqrt{2(1-1/\omega)m}$ was conjectured by Edwards and Elphick in [3], and first proved in [7].

In view of the inequalities

$$2m/n \le \rho \le \sqrt{2(1-1/\omega)m},$$

we see that bound (5) implies bound (3). Hence, bounds (3), (4), and (5) are essentially equivalent.

Now we are ready to show that bound (5) implies Stanley's bound (1). Indeed, clearly

$$m \ge \binom{\omega}{2},$$

as G contains a complete graph of order ω. Solving this quadratic inequality, we get

$$\omega \le 1/2 + \sqrt{2m + 1/4},$$

and bound (1) follows by

$$\rho \le \sqrt{2(1-1/\omega)m} \le \sqrt{2m \cdot \frac{-1/2 + \sqrt{2m+1/4}}{1/2 + \sqrt{2m+1/4}}} = -1/2 + \sqrt{2m+1/4}.$$

Nevertheless, one cannot claim that bound (5) is better than bound (1), because of the extra parameter ω in (5). Yet the above argument can be refined further by noting that ω is an integer and therefore,

$$\omega \leq \left\lfloor 1/2 + \sqrt{2m + 1/4} \right\rfloor, \tag{6}$$

yielding the following clear improvement over (1)

$$\rho \leq \sqrt{2\left(1 - \left\lfloor 1/2 + \sqrt{2m + 1/4} \right\rfloor^{-1}\right) m}. \tag{7}$$

In fact, Theorem C provides more than that. We can see a broader picture, in which (5) and (7) are just the first terms of decreasing sequences of upper bounds on ρ that converge to ρ.

Indeed, let G be a graph with m edges, clique number ω, and spectral radius ρ. Write W_k for the number of walks on k vertices in G. Using Theorem C, it is shown in [8] that for every $k \geq 1$, we have

$$\rho \leq \left((1 - 1/\omega)\, W_k\right)^{1/k}$$

and

$$W_{2^{k+1}} \leq (1 - 1/\omega)\, W_{2^k}^2.$$

Hence, we see that

$$\left\{ \left((1 - 1/\omega)\, W_{2^k} \right)^{1/2^k} \right\}_{k=1}^{\infty}$$

is a nonincreasing sequence of upper bounds on ρ and its first term is $\sqrt{2\,(1 - 1/\omega)\, m}$. The result of Cvetković [2] $\lim_{n\to\infty} W_n^{1/n} = \rho$ implies that the sequence converges to ρ.

Further, using (6), one comes up with the sequence

$$\left\{ \left(\left(1 - \left\lfloor 1/2 + \sqrt{2m + 1/4} \right\rfloor^{-1}\right) W_{2^k} \right)^{1/2^k} \right\}_{k=1}^{\infty},$$

which is nonincreasing, converges to ρ, and its first element is the right side of bound (7).

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