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GRAPHS WITH CLUSTERS PERTURBED BY REGULAR GRAPHS—A_α-SPECTRUM AND APPLICATIONS

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Abstract

Given a graph G, its adjacency matrix $A(G)$ and its diagonal matrix of vertex degrees $D(G)$, consider the matrix $A_\alpha(G) = \alpha D(G) + (1 - \alpha)A(G)$, where $\alpha \in [0, 1)$. The A_α-spectrum of G is the multiset of eigenvalues of $A_\alpha(G)$ and these eigenvalues are the α-eigenvalues of G. A cluster in G is a pair of vertex subsets (C, S), where C is a set of cardinality $|C| \geq 2$ of pairwise co-neighbor vertices sharing the same set S of $|S|$ neighbors. Assuming that G is connected and it has a cluster (C, S), $G(H)$ is obtained from G and an r-regular graph H of order $|C|$ by identifying its vertices with the vertices in C, eigenvalues of $A_\alpha(G)$ and $A_\alpha(G(H))$ are deduced and if $A_\alpha(H)$ is positive semidefinite, then the i-th eigenvalue of $A_\alpha(G(H))$ is greater than or equal to i-th eigenvalue of $A_\alpha(G)$. These results are extended to graphs with several pairwise disjoint clusters $(C_1, S_1), \ldots, (C_k, S_k)$. As an application, the effect on the energy, α-Estrada index and α-index of a graph G with clusters when the edges of regular graphs are added to G are analyzed. Finally, the A_α-spectrum of the corona product $G \circ H$ of a connected graph G and a regular graph H is determined.

Keywords: cluster, convex combination of matrices, A_α-spectrum, corona product of graphs.

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1. Introduction and Preliminaries

We deal with simple undirected graphs $G = (V(G), E(G))$ on n vertices with vertex set $V(G)$ and edge set $E(G)$. The complement of G is the graph $\overline{G}$ with the same vertex set as G in which any two distinct vertices are adjacent if and only if they are non-adjacent in G. The complete graph on n vertices is denoted by K_n (therefore, $\overline{K_n}$ has no edges, that is, all its vertices are isolated). The complete bipartite graph on $p+q$ vertices is denoted by $K_{p,q}$ (in particular, $K_{1,s}$ is a star on $s+1$ vertices).

Throughout the text, N_k denotes the set of positive integers not greater than k, the identity matrix of order m and the transpose of a matrix A are denoted by I_m and A^T, respectively. Furthermore, 0 is the zero matrix of appropriate order, $\mathbf{1}_n$ is the all-one column vector of size n and $J_{p,q}$ is the all-one matrix of order $p \times q$. The remainder notation is standard. However for the reader's convenience, as it follows, the fundamental concepts and their notation is briefly recalled.

Let $D(G)$ be the diagonal matrix of order n whose (i, i)-entry is the degree of the i-th vertex of G and let $A\,(G)$ be the adjacency matrix of G. The matrices $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ are the Laplacian and signless Laplacian matrix of G, respectively. The matrices $L(G)$ and $Q(G)$ are both positive semidefinite and $(0, \mathbf{1})$ is an eigenpair of $L\,(G)$. Fiedler [7] proved that G is a connected graph if and only if the second smallest eigenvalue of $L(G)$ is positive. This eigenvalue is called the algebraic connectivity of G. Moreover, it is known that for any bipartite graph G, the characteristic polynomials of $L(G)$ and $Q(G)$ coincide [6, Prop. 2.3]. For a connected graph G, the least eigenvalue of $Q(G)$ is positive if and only if G is non-bipartite [6, Proposition 2.1].

In [13] Nikiforov introduced the family of matrices

$$A_\alpha\,(G) = \alpha D\,(G) + (1-\alpha)A\,(G)$$

where $\alpha \in [0, 1]$. We see that $A_\alpha(G)$ is a convex combination of the matrices $A(G)$ and $D(G)$. The multiset of eigenvalues of $A_\alpha(G)$ is called the A_α-spectrum of G.

Since $A_\alpha(G)$ is a real symmetric matrix, its eigenvalues are real numbers. Observe that $A_0\,(G) = A\,(G)$ and $2A_{1/2}\,(G) = Q\,(G)$. Thus, the family $A_\alpha(G)$ extends both $A(G)$ and $Q\,(G)$. Since $A_1(G) = D(G)$, from now on, we take $\alpha \in [0, 1)$.

If G is a graph of order n, we denote by

$$\nu_1\,(G) \leq \nu_2\,(G) \leq \cdots \leq \nu_n\,(G)$$

the eigenvalues of $A_\alpha(G)$. If necessary, these eigenvalues are also denoted by $\nu_1(A_\alpha(G)), \nu_2(A_\alpha(G)), \ldots, \nu_n(A_\alpha(G))$.

In particular, $\nu_n(G)$ is called the α-index of G. From the Perron-Frobenius Theory for nonnegative matrices, it follows that

- for a connected graph G, the α-index of G (Perron root) is a simple eigenvalue of $A_\alpha(G)$ that has a positive eigenvector (Perron vector),
- for a connected graph G, the α-index of G increases if any entry of $A_\alpha(G)$ increases,
- if G is a proper subgraph of a connected graph H, then $\nu_n(G) < \nu_n(H)$, and
- if G is an r-regular graph of order n, then $A_\alpha(G) = r\alpha I_n + (1-\alpha)A(G)$ and $\nu_n(G) = r$ with eigenvector $\mathbf{1}_n$.

Now, we recall the concept of cluster which appears first in [11] and more recently in [5].

Definition 1.1. A cluster of order c and degree s in a graph G is a pair of vertex subsets (C, S), where C is a set of cardinality $|C| = c \geq 2$ of pairwise co-neighbor vertices sharing the same set S of s neighbors.

A pendent vertex is a vertex of degree 1 and a quasi-pendent vertex is a vertex adjacent to at least one pendent vertex. For the star $K_{1,s}$, C is the set of the pendent vertices and $S = \{v\}$ where v is the root vertex and a complete bipartite graph $K_{p,q}$ has the clusters $(\overline{K}_p, \overline{K}_q)$ and $(\overline{K}_q, \overline{K}_p)$. Also, note that each quasi-pendent vertex adjacent with more than one pendent vertex define a cluster (C, S) in which $|S| = 1$. In [4], among other results, it was proved that α is an eigenvalue of $A_\alpha(G)$ with multiplicity at least $p(G) - q(G)$, when G has $p(G) > 0$ pendent vertices and $q(G)$ quasi-pendent vertices. It is easy to prove that any set of pairwise co-neighbor vertices is an independent set.

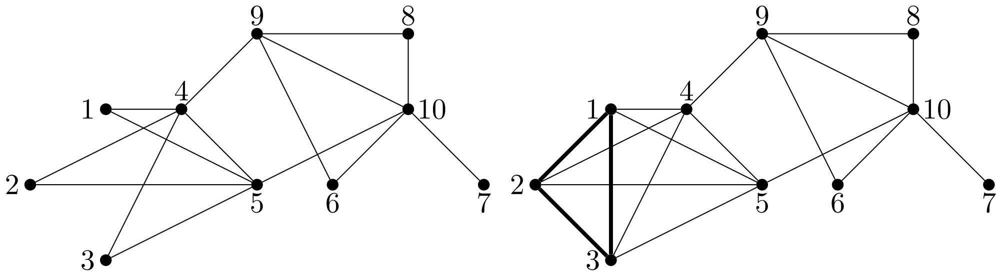
Definition 1.2. Let G be a connected graph of order n with a cluster (C, S) and let H be a graph of order $|C|$. Assuming that $V(H) = C$, then $G(H)$ is the graph with vertex set $V(G(H)) = V(G)$ and edge set $E(G(H)) = E(G) \cup E(H)$.

From Definition 1.2, $G(H)$ is the graph obtained from G and H adding the edges of H to the edges of G by identifying the vertices of H with the vertices in C.

Example 1.3. Let G be the graph below depicted which has the cluster (C, S), where $C = \{1, 2, 3\}$ and $S = \{4, 5\}$. Let H be the cycle on 3 vertices, $V(H) = \{1, 2, 3\}$. Then the graphs G and $G(H)$ are displayed, respectively, below.

Definition 1.4. Let (C_1, S_1) and (C_2, S_2) be clusters in a graph G. We say that (C_1, S_1) and (C_2, S_2) are disjoint if $C_1 \cap C_2 = \emptyset$ and $S_1 \cap S_2 = \emptyset$.

The Laplacian and signless Laplacian spectra of a graph G with a cluster (C, S) are studied in [1]. The effects on the Laplacian spectral radius and algebraic connectivity of a graph perturbed by adding edges between its pendent vertices are considered in [9] and [17], respectively. Moreover, the effects on others spectral invariants are determined in [15] and [16].



Definition 1.5. Let G be a connected graph with pairwise disjoint clusters $(C_1, S_1), \ldots, (C_k, S_k)$. For $i = 1, \ldots, k$, let H_i be a graph of order $|C_i|$. Let $G(H_i : i \in N_k)$ be the graph obtained from G and the graphs H_i when the edges of H_i are added to the edges of G by identifying the vertices of H_i with the vertices in C_i for $i = 1, \ldots, k$.

From this definition, we have $V(H_i) = C_i$, for $i = 1, \ldots, k$,

$$V(G(H_i : i \in N_k)) = V(G)$$

and

$$E(G(H_i : i \in N_k)) = E(G) \cup E(H_1) \cup \cdots \cup E(H_k).$$

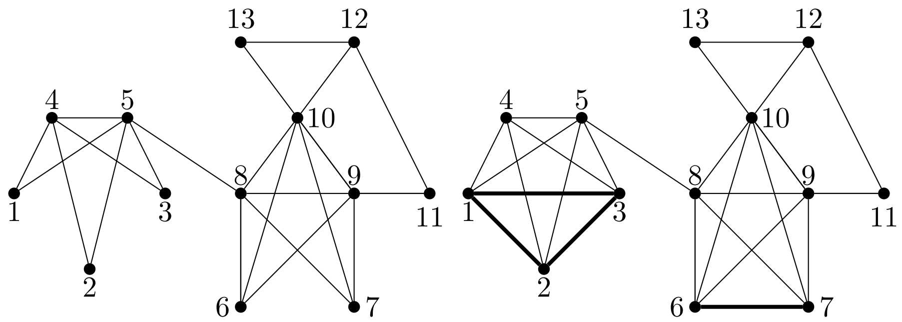
Observe that the graph $G(H_i : i \in N_k)$ can be constructed as follows.

- The graph $G_1 = G(H_1)$ is obtained from G and H_1 identifying the vertices of H_1 with C_1, and
- for $i = 2, \ldots, k$, the graph $G_i = G(H_1, \ldots, H_i)$ is obtained from $G_{i-1} = G(H_1, \ldots, H_{i-1})$ and H_i identifying the vertices of H_i with C_i.

Example 1.6. Let G be the graph below depicted which has two disjoint clusters (C_1, S_1) and (C_2, S_2) where $C_1 = \{1, 2, 3\}$, $S_1 = \{4, 5\}$ and $C_2 = \{6, 7\}$, $S_2 = \{8, 9, 10\}$. Let H_1 be the cycle on 3 vertices, $V(H_1) = \{1, 2, 3\}$, and H_2 be the path on 2 vertices, $V(H_2) = \{6, 7\}$. Then the graphs G and $G(H_1, H_2)$ are displayed, respectively, below.

A unified approach to the determination of the spectra of adjacency, Laplacian and signless Laplacian matrices of graphs with edge perturbation on their clusters was presented in [5]. Moreover, the invariance of algebraic connectivity and Laplacian index under those perturbation was proved.

In this article, using a methodology similar to the one followed in [5], new results about the spectra of $A_\alpha(G)$ and $A_\alpha(G(H))$ are deduced. Namely, in



Section 2, assuming that G is a connected graph of order n with a cluster (C, S) and $G(H)$ is obtained according to Definition 1.2, the following results about the spectra of $A_\alpha(G)$ and $A_\alpha(G(H))$ are proven.

1. $|S|\alpha + \nu_j(H)$, $1 \leq j \leq |C| - 1$, are eigenvalues of $A_\alpha(G(H))$, where

$$\nu_1(H) \leq \cdots \leq \nu_{|C|-1}(H) \leq \nu_{|C|}(H) = r$$

are the eigenvalues of $A_\alpha(H)$. As direct consequence, $|S|\alpha$ is an eigenvalue of $A_\alpha(G)$ with multiplicity at least $|C|-1$. In both cases, the remaining eigenvalues can be computed from a special matrix, (5) and (8), respectively (Theorem 2.1 and Corollary 2.2).

2. If $A_\alpha(H)$ is a positive semidefinite matrix, then

$$\nu_i(G) \leq \nu_i(G(H)),$$

for $i = 1, \ldots, n$, where $\{\nu_i(G) : 1 \leq i \leq n\}$ and $\{\nu_i(G(H)) : 1 \leq i \leq n\}$ are the A_α-spectra of G and $G(H)$, respectively (Theorem 2.6).

3. Assuming that G has $k \geq 2$ pairwise disjoint clusters $(C_1, S_1), \ldots, (C_k, S_k)$, the above results are extended to the graph $G(H_i : i = 1, \ldots, k)$ (Theorem 2.7).

Finally, in Section 3, the obtained results are applied to study the effect on the energy (Theorems 3.1 and 3.2), α-Estrada index (Theorems 3.3 and 3.4) and α-index (Theorem 3.5) of a graph G with clusters when the edges of regular graphs are added to G. Additionally, the A_α-spectrum of the corona product $G \circ H$ of a connected graph G and a regular graph H is determined (Theorem 3.7).

2. Effects by Adding the Edges of a Regular Graph

Consider $G(H)$ as in Definition 1.2. Let $|C| = c$ and $|S| = s$. We assume that H is a connected r-regular graph of order $|C| = c$ and that

$$\nu_1(H) \leq \cdots \leq \nu_{c-1}(H) < \nu_c(H) = r$$

are the eigenvalues of $A_\alpha(H)$ with an orthogonal basis of eigenvectors

$$\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{c-1}, \mathbf{x}_c = \frac{1}{\sqrt{c}}\mathbf{1}_c$$

in which, for $1 \leq i \leq c$, $A_\alpha(H)\mathbf{x}_i = \nu_i(H)\mathbf{x}_i$. In particular

$$A_\alpha(H)\mathbf{1}_c = r\mathbf{1}_c. \tag{1}$$

Let

$$X = \begin{bmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_{c-1} & \frac{1}{\sqrt{c}}\mathbf{1}_c \end{bmatrix}$$

and

$$U = \begin{bmatrix} X & \\ & I_{n-c} \end{bmatrix}. \tag{2}$$

Clearly X and U are both orthonormal matrices.

Through this paper $\beta = 1 - \alpha$ and d_i is the degree of the vertex i of the graph G.

We recall that G is a graph that has a cluster (C, S). The graphs G and $G(H)$ have the same set of vertices. We label the vertices of G as follows. The labels $1, 2, \ldots, c$ are for the vertices of C, the labels $c+1, c+2, \ldots, c+s$ are for the vertices in S and the labels $c+s+1, \ldots, n$ are for the remaining vertices of G. This labeling is illustrated in Example 1.3. For this labeling, $A_\alpha(G)$ and $A_\alpha(G(H))$ become as follows

$$A_\alpha(G) = \begin{bmatrix} s\alpha I_c & \begin{bmatrix}\beta\mathbf{1}_c\mathbf{1}_s^T & 0\end{bmatrix} \\ \begin{bmatrix}\beta\mathbf{1}_s\mathbf{1}_c^T \\ 0\end{bmatrix} & R(\alpha) \end{bmatrix} \tag{3}$$

and

$$A_\alpha(G(H)) = \begin{bmatrix} s\alpha I_c + A_\alpha(H) & \begin{bmatrix}\beta\mathbf{1}_c\mathbf{1}_s^T & 0\end{bmatrix} \\ \begin{bmatrix}\beta\mathbf{1}_s\mathbf{1}_c^T \\ 0\end{bmatrix} & R(\alpha) \end{bmatrix} \tag{4}$$

where $R(\alpha) = \begin{bmatrix} A & B \\ B^T & Z \end{bmatrix}$ with submatrices A, B and Z of size $s \times s$, $s \times (n-c-s)$ and $(n-c-s) \times (n-c-s)$, respectively. The diagonal entries of the matrices A and Z are αd_i, $c+1 \leq i \leq n$ and the off-diagonal entries of A and Z as well as the entries of B are β if the corresponding vertices of G are adjacent and 0 otherwise.

Theorem 2.1. *Let G be a graph with a cluster (C, S) of order $|C| = c$ and degree $|S| = s$. If H is an r-regular graph of order c and $G(H)$ is obtained according to*

Definition 1.2, *then*

$$s\alpha + \nu_j(H), \qquad 1 \le j \le c-1,$$

are eigenvalues of $A_\alpha(G(H))$, *where* $\nu_1(H) \le \cdots \le \nu_{c-1}(H) \le \nu_c(H) = r$ *are the eigenvalues of* $A_\alpha(H)$ *and the remaining eigenvalues of* $A_\alpha(G(H))$ *are the eigenvalues of the matrix*

$$X = \begin{bmatrix} s\alpha + r & [\beta\sqrt{c}\mathbf{1}_s^T \;\; 0] \\ \begin{bmatrix} \beta\sqrt{c}\mathbf{1}_s \\ 0 \end{bmatrix} & R(\alpha) \end{bmatrix}. \tag{5}$$

Proof. We use (4) and the orthogonal matrix U defined in (2) obtaining

$$\begin{aligned}
&U^T A_\alpha(G(H))U \\
&= \begin{bmatrix} X^T & \\ & I_{n-c} \end{bmatrix} \begin{bmatrix} s\alpha I_c + A_\alpha(H) & [\beta \mathbf{1}_c \mathbf{1}_s^T \;\; 0] \\ \begin{bmatrix} \beta \mathbf{1}_s \mathbf{1}_c^T \\ 0 \end{bmatrix} & R(\alpha) \end{bmatrix} \begin{bmatrix} X & \\ & I_{n-c} \end{bmatrix} \\
&= \begin{bmatrix} s\alpha I_c + X^T A_\alpha(H) X & [\beta X^T \mathbf{1}_c \mathbf{1}_s^T \;\; 0] \\ \begin{bmatrix} \beta \mathbf{1}_s \mathbf{1}_c^T X \\ 0 \end{bmatrix} & R(\alpha) \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} s\alpha + \nu_1(H) & & & \\ & \ddots & & \\ & & s\alpha + \nu_{c-1}(H) & \\ & & & s\alpha + r \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ \beta\sqrt{c}\mathbf{1}_s^T & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & \cdots & 0 & \beta\sqrt{c}\mathbf{1}_s \\ & \cdots & 0 & 0 \end{bmatrix} & R(\alpha) \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} s\alpha + \nu_1(H) & & \\ & \ddots & \\ & & s\alpha + \nu_{c-1}(H) \end{bmatrix} & \\ & \begin{bmatrix} s\alpha + r & [\beta\sqrt{c}\mathbf{1}_s^T \;\; 0] \\ \begin{bmatrix} \beta\sqrt{c}\mathbf{1}_s \\ 0 \end{bmatrix} & R(\alpha) \end{bmatrix} \end{bmatrix}.
\end{aligned}$$

Then $U^T A_\alpha(G(H))U =$

$$\begin{bmatrix} s\alpha + \nu_1(H) & & \\ & \ddots & \\ & & s\alpha + \nu_{c-1}(H) \end{bmatrix} \oplus \begin{bmatrix} s\alpha + r & [\beta\sqrt{c}\mathbf{1}_s^T \;\; 0] \\ \begin{bmatrix} \beta\sqrt{c}\mathbf{1}_s \\ 0 \end{bmatrix} & R(\alpha) \end{bmatrix}. \tag{6}$$

Therefore, the conclusion follows from (6). ■

Applying Theorem 2.1 to the particular case of $H = \overline{K_c}$, it follows that $G(H) = G$ and

$$U^T A_\alpha(G)U = \begin{bmatrix} s\alpha & & \\ & \ddots & \\ & & s\alpha \end{bmatrix} \oplus \begin{bmatrix} s\alpha & [\beta\sqrt{c}\mathbf{1}_s^T \;\; 0] \\ \begin{bmatrix} \beta\sqrt{c}\mathbf{1}_s \\ 0 \end{bmatrix} & R(\alpha) \end{bmatrix}. \tag{7}$$

Thus the next corollary is immediate.

Corollary 2.2. *Let G be a graph with a cluster (C, S) of order $|C| = c$ and degree $|S| = s$. Then $s\alpha$ is an eigenvalue of $A_\alpha(G)$ with multiplicity at least $c-1$ and the remaining eigenvalues are the eigenvalues of the matrix*

$$Y = \begin{bmatrix} s\alpha & [\beta\sqrt{c}\mathbf{1}_s^T \;\; 0] \\ \begin{bmatrix} \beta\sqrt{c}\mathbf{1}_s \\ 0 \end{bmatrix} & R(\alpha) \end{bmatrix}. \tag{8}$$

Taking into account that $A_0(G) = A(G)$ and $2A_{\frac{1}{2}}(G) = Q(G)$, another immediate corollary is the following.

Corollary 2.3. *Let G be a graph with a cluster (C, S) of order $|C| = c$ and degree $|S| = s$. If H is an r-regular graph of order c and $G(H)$ is obtained according to Definition* 1.2, *then*

(i) 0 *is an eigenvalue of $A(G)$ with multiplicity at least $c-1$,*

(ii) *if $\lambda_j(H) \neq r$ is an eigenvalue of $A(H)$, then it is also an eigenvalue of $A(G(H))$,*

(iii) *s is an eigenvalue of $Q(G)$ with multiplicity at least $c-1$, and*

(iv) *if $q_j(H) \neq 2r$ is an eigenvalue of $Q(H)$, then $q_j(H) + s$ is an eigenvalue of $Q(G(H))$.*

2.1. The nonnegative A_α-spectrum case

In this subsection we study the A_α-spectrum of $G(H)$ when $A_\alpha(H)$ is a positive semidefinite matrix.

Among the basic results on $A_\alpha(G)$ obtained in [13] we recall the following theorem.

Theorem 2.4 [13, Proposition 4]**.** *Let $1 \geq \alpha > \beta \geq 0$. Then*

$$\nu_j(A_\alpha(G)) \geq \nu_j(A_\beta(G)) \tag{9}$$

for $j = 1, 2, \ldots, n$. If G is connected, then inequality (9) *is strict, unless $j = n$ and G is regular.*

The function $f_G(\alpha) = \nu_1(A_\alpha(G))$ is continuous and, from (9) with $j = 1$, it is nondecreasing in α. Moreover, $f_G(0) = \nu_1(A_0(G)) < 0$. Therefore, there is a smallest value $\alpha \in \left(0, \frac{1}{2}\right]$ such that $\nu_1(A_\alpha(G)) = 0$. Hence, denoting this value by $\alpha_0(G)$, $A_\alpha(G)$ is a positive semidefinite matrix if and only if $\alpha_0(G) \leq \alpha \leq 1$.

Now, we restate a problem proposed in [13, Problem 8] as follows: *given a graph G, find $\alpha_0(G)$.*

Some advances on this problem obtained in [14] are presented in the next proposition.

Proposition 2.5 [14, Proposition 5]. *If H is an r-regular graph, then*

$$\alpha_0(H) = \frac{-\nu_{\min}(A(H))}{r - \nu_{\min}(A(H))} \tag{10}$$

where $\nu_{\min}(A(H))$ is the least eigenvalue of $A(H)$.

Theorem 2.6. *Let G be a graph with a cluster (C, S) of order $|C| = c$ and degree $|S| = s$. If H is an r-regular graph of order c, $\alpha \geq \alpha_0(H)$, where $\alpha_0(H)$ is given by* (10), *and $G(H)$ is obtained according to Definition* 1.2, *then*

$$\nu_i(G) \leq \nu_i(G(H)),$$

for $i = 1, \ldots, n$, where $\{\nu_i(G) : 1 \leq i \leq n\}$ and $\{\nu_i(G(H)) : 1 \leq i \leq n\}$ are the A_α-spectra of G and $G(H)$, respectively.

Proof. Since $\alpha \geq \alpha_0(H)$ with $\alpha_0(H)$ given by (10), $A_\alpha(H)$ is a positive semidefinite matrix and then its eigenvalues are nonnegative. Thus the result follows from (6) and (7) applying the Weyl's inequalities for eigenvalues of Hermitian matrices ([10], p. 181). ∎

2.2. The multiple pairwise disjoint clusters case

In this subsection the graphs with more than one cluster are analyzed.

Theorem 2.7. *Let G be a graph with a set of pairwise disjoint clusters $\{(C_i, S_i) : i \in N_k\}$, with $k \geq 2$, and let $|C_i| = c_i$ and $|S_i| = s_i$, for $i \in N_k$. Assuming that each H_i is an r_i-regular graph of order c_i and $G(H_i : i \in N_k)$ is obtained according to Definition* 1.5, *it follows, for each $p \in N_k$, that*

(i) *$s_p\alpha$ is an eigenvalue of $A_\alpha(G)$ with multiplicity at least $c_p - 1$,*

(ii) *$s_p\alpha + \nu_j(H_p)$, $1 \leq j \leq c_p - 1$, is an eigenvalue of $A_\alpha(G(H_i : i \in N_k))$, where*

$$\nu_1(H_p) \leq \cdots \leq \nu_{c_p-1}(H_p) \leq \nu_{c_p}(H_p) = r_p$$

are the eigenvalues of $A_\alpha(H_p)$,

(iii) *if*

$$\alpha \geq \frac{-\alpha_{\min}(A(H_p))}{r_p - \alpha_{\min}(A(H_p))}$$

where $\alpha_{\min}(A(H_p))$ is the least eigenvalue of $A(H_p)$, then the j-th eigenvalue of $A_\alpha(G(H_i : i \in N_k))$ is greater or equal to the j-th eigenvalue of $A_\alpha(G)$.

Proof. Considering $p \in N_k$, since

$$G(H_i : i \in N_k \setminus \{p\})(H_p) = G(H_i : i \in N_k),$$

the results are immediate from Theorems 2.1 and 2.6. ∎

As a consequence, we have the following corollary.

Corollary 2.8. *Let G be a graph with a set of pairwise disjoint clusters $\{(C_i, S_i) : i \in N_k\}$, with $k \geq 2$, and let $|C_i| = c_i$ and $|S_i| = s_i$, for $i \in N_k$. Assuming that each H_i is an r_i-regular graph of order c_i and $G(H_i : i \in N_k)$ is obtained according to Definition* 1.5, *then* 0 *is an eigenvalue of $A(G)$ with multiplicity at least $\sum_{i=1}^{k} c_i - k$. Moreover, for each $p \in N_k$,*

(i) *if $\lambda_j(H_p) \neq r_p$ is an eigenvalue of $A(H_p)$, then it is also an eigenvalue of $A(G(H_i : i \in N_k))$,*

(ii) *s_p is an eigenvalue of $Q(G)$ with multiplicity at least $c_p - 1$,*

(iii) *if $q_j(H_p) \neq 2r_p$ is an eigenvalue of $Q(H_p)$, then $q_j(H_p) + s_p$ is an eigenvalue of $Q(G(H_i : i \in N_k))$.*

3. Some Applications

In this section, the energy, α-Estrada index, and α-index of graphs with clusters are considered, and the A_α-spectrum of the corona of a connected graph G and a regular graph H is determined.

We recall that the energy of a graph G is $\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i(G)|$ and the Estrada index of G is $E\mathcal{E}(G) = \sum_{i=1}^{n} e^{\lambda_i(G)}$, where

$$\lambda_1 (G) \leq \lambda_2 (G) \leq \cdots \leq \lambda_{n-1} (G) \leq \lambda_n (G)$$

are the eigenvalues of $A(G)$. Similarly, the signless Laplacian Estrada index of G is defined as $SLE\mathcal{E}(G) = \sum_{i=1}^{n} e^{q_i(G)}$, where

$$q_1 (G) \leq q_2 (G) \leq \cdots \leq q_{n-1} (G) \leq q_n (G)$$

are the eigenvalues of $Q(G)$.

The corona $G \circ H$ of two graphs G and H (where $|V(G)| = n$ and $|V(H)| = m$) introduced by Frucht and Harary [8] is defined as the graph obtained by taking one copy of G and n copies of H and then joining by an edge the i-th vertex of G to every vertex of the i-th copy of H. It is immediate that the corona graph operation is not commutative, that is, in general $G \circ H \neq H \circ G$.

3.1. The energy of graphs with clusters

Let M be an $m \times n$ complex matrix, $q = \min\{m, n\}$ and

$$\sigma_1 (M) \geq \sigma_2 (M) \geq \cdots \geq \sigma_q (M)$$

be the singular values of M. Nikiforov [12] defines the energy of M as $\mathcal{E}(M) = \sum_{j=1}^{q} \sigma_j (M)$. Since $A(G)$ is symmetric, its singular values are the modulus of its eigenvalues. Then $\mathcal{E}(G) = \mathcal{E}(A(G))$.

Given a natural number k such that $1 \leq k \leq n$, the Ky Fan k-norm of a matrix M of order $n \times n$ is the sum of the k largest singular values of M, that is, assuming that $\sigma_1(M), \ldots, \sigma_k(M)$ are the k largest singular values of M, $\|M\|_k = \sum_{i=1}^{k} \sigma_i(M)$. In particular, $\|M\|_n = \mathcal{E}(M)$.

Theorem 3.1. *Let G be a graph with a cluster (C, S) of order $|C| = c$ and degree $|S| = s$. Let H be an r-regular graph of order c. Let $G(H)$ as in Definition* 1.2. *Then*

$$\mathcal{E}(G(H)) - \mathcal{E}(G) \leq \mathcal{E}(H).$$

Proof. We apply Theorem 2.1 with $\alpha = 0$. From (6) and (7), using the fact that the singular values are invariant under unitary transformations, we have

$$\mathcal{E}(G(H)) = \mathcal{E}(A(G(H))) = \sum_{i=1}^{c-1} |\nu_i(H)| + \mathcal{E}(C), \tag{11}$$

where $C = \begin{bmatrix} r & [\sqrt{c}\mathbf{1}_s^T \;\; 0] \\ \begin{bmatrix} \sqrt{c}\mathbf{1}_s \\ 0 \end{bmatrix} & R(0) \end{bmatrix}$ and $\mathcal{E}(G) = \mathcal{E}(A(G)) = \mathcal{E}(D)$, where $D = \begin{bmatrix} 0 & [\sqrt{c}\mathbf{1}_s^T \;\; 0] \\ \begin{bmatrix} \sqrt{c}\mathbf{1}_s \\ 0 \end{bmatrix} & R(0) \end{bmatrix}$. Then $C = D + F$, where

$$F = \begin{bmatrix} r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{12}$$

Hence $\mathcal{E}(C) = \|C\|_{n-c+1} \leq \|D\|_{n-c+1} + \|F\|_{n-c+1} = \mathcal{E}(D) + r = \mathcal{E}(G) + r$. Using this inequality in (11), we obtain

$$\mathcal{E}(G(H)) - \mathcal{E}(G) \leq \sum_{i=1}^{c-1} |\nu_i(H)| + r = \sum_{i=1}^{c} |\nu_i(H)| = \mathcal{E}(H).$$

■

Theorem 3.2. *Let G be a graph with a set of clusters $\{(C_i, S_i) : i \in N_k\}$, $k \geq 2$. For $i \in N_k$, let $|C_i| = c_i$, $|S_i| = s_i$ and H_i be an r_i-regular graph of order c_i. Let $G(H_i : i \in N_k)$ as in Definition* 1.5. *Then*

$$\mathcal{E}(G(H_i : i \in N_k)) - \mathcal{E}(G) \leq \sum_{i=1}^{k} \mathcal{E}(H_i).$$

Proof. The result follows easily by a repeated application of Theorem 3.1. ■

3.2. The α-Estrada index of graphs with clusters

In [16], for a graph with pendent vertices, the effects on the energy, Estrada index ($\alpha = 0$) and signless Laplacian Estrada index (essentially, $\alpha = 0.5$) are obtained when the edges of regular graphs are added among the pendent vertices. In this subsection, we extend these results to a graph with clusters, for all $\alpha \in [0, 1)$.

Since $A_0(G) = A(G)$, it seems natural to define the α-Estrada index of G, denoted by $E\mathcal{E}_\alpha(G)$, as

$$E\mathcal{E}_\alpha(G) = \sum_{i=1}^{n} e^{\nu_i(G)}$$

where

$$\nu_1(G) \leq \nu_2(G) \leq \cdots \leq \nu_{n-1}(G) \leq \nu_n(G)$$

are the eigenvalues of $A_\alpha(G)$. Hence $E\mathcal{E}_\alpha(G) = trace(e^{A_\alpha(G)})$

Next, we study the effect on the α-Estrada index.

Theorem 3.3. *Let G be a graph with a cluster (C, S) of order $|C| = c$ and degree $|S| = s$. Let H be an r-regular graph of order c. Let $G(H)$ be as in Definition 1.2. Then*

$$E\mathcal{E}_\alpha(G(H)) - E\mathcal{E}_\alpha(G) \geq e^{s\alpha} E\mathcal{E}_\alpha(H) - \left[(c-1)e^{s\alpha} + e^r\left(e^{s\alpha} - 1\right)\right].$$

Proof. We use again the fact that the singular values under unitary transformations to obtain, from (6) and (7), that

$$E\mathcal{E}_\alpha(G(H)) = \sum_{i=1}^{c-1} e^{(s\alpha + \nu_i(H))} + trace\left(e^X\right) \tag{13}$$

and

$$E\mathcal{E}_\alpha(G) = \sum_{i=1}^{c-1} e^{s\alpha} + trace\left(e^Y\right) \tag{14}$$

where X and Y are as in Theorem 2.1. From the series-expansion of e^N, we have

$$e^X = \sum_{j=0}^{\infty} \frac{1}{j!} X^j = \sum_{j=0}^{\infty} \frac{1}{j!} (Y + F)^j = \sum_{j=0}^{\infty} \frac{1}{j!} \left(Y^j + \cdots + F^j\right),$$

where F is given in (12). Since Y and F are nonnegative matrices, it follows that

$$trace\left(e^X\right) \geq trace\left(\sum_{j=0}^{\infty} \frac{1}{j!} Y^j\right) + trace\left(\sum_{j=0}^{\infty} \frac{1}{j!} F^j\right).$$

Hence,

$$trace\big(e^X\big) \geq trace\big(e^Y\big) + \sum_{j=0}^{\infty} \frac{1}{j!} r^j = trace\big(e^Y\big) + e^r.$$

Using this inequality in (13), we get

$$\begin{aligned} E\mathcal{E}_\alpha(G(H)) &\geq \sum_{i=1}^{c-1} e^{(s\alpha+\nu_i(H))} + trace\big(e^Y\big) + e^r \\ &= e^{s\alpha} \sum_{i=1}^{c-1} e^{\nu_i(H)} + E\mathcal{E}_\alpha(G) - \sum_{i=1}^{c-1} e^{s\alpha} + e^r. \end{aligned}$$

Finally,

$$\begin{aligned} E\mathcal{E}_\alpha(G(H)) - E\mathcal{E}_\alpha(G) &\geq e^{s\alpha} \sum_{i=1}^{c-1} e^{\nu_i(H)} - \sum_{i=1}^{c-1} e^{s\alpha} + e^r \\ &= e^{s\alpha} \sum_{i=1}^{c-1} e^{\nu_i(H)} - (c-1)e^{s\alpha} + e^r + e^{s\alpha}e^r - e^{s\alpha}e^r \\ &= e^{s\alpha} E\mathcal{E}_\alpha(H) - (c-1)e^{s\alpha} - e^r\big(e^{s\alpha} - 1\big). \end{aligned}$$

Therefore,

$$E\mathcal{E}_\alpha(G(H)) - E\mathcal{E}_\alpha(G) \geq e^{s\alpha} E\mathcal{E}_\alpha(H) - \big[(c-1)e^{s\alpha} + e^r\big(e^{s\alpha} - 1\big)\big]. \qquad \blacksquare$$

A repeated application of Theorem 3.3 yields to the following result.

Theorem 3.4. *Let G be a graph with a set of clusters $\{(C_i, S_i) : i \in N_k\}$, $k \geq 2$. For $i \in N_k$, let $|C_i| = c_i$, $|S_i| = s_i$ and H_i be an r_i-regular graph of order c_i. Let $G(H_i : i \in N_k)$ as in Definition* 1.5. *Then*

$$E\mathcal{E}_\alpha(G(H_i : i \in N_k)) - E\mathcal{E}_\alpha(G) \geq \sum_{i=1}^{k} \big(e^{s_i\alpha} E\mathcal{E}_\alpha(H_i) - (c_i - 1)e^{s_i\alpha} - e^{r_i}\big(e^{s_i\alpha} - 1\big)\big).$$

3.3. The α-index of graphs with a cluster

Now, we study the effect on the α-index. We remember that $\nu_n(G)$ and $\nu_n(G(H))$ denote the α-index of G and $G(H)$, respectively. We denote by $\rho(X)$ and $\rho(Y)$ the spectral radius of the matrices X and Y given in (5) and (8), respectively.

Theorem 3.5. *Let G be a graph with a cluster (C, S) of order $|C| = c$ and degree $|S| = s$. Let H be an r-regular graph of order c. Let $G(H)$ be as in Definition* 1.2. *Then*

$$0 < \nu_n(G(H)) - \nu_n(G) < r.$$

Proof. Clearly, from Theorem 2.1, $\nu_n(G(H)) = \rho(X)$ and $\nu_n(G) = \rho(Y)$. We have $X = Y + F$ with F as in (12). Since $X - Y \geq 0$ with strict inequality in the entry (1,1), we get that $0 < \rho(X) - \rho(Y)$. Moreover, applying the Weyl's inequalities for eigenvalues of Hermitian matrices ([10], p. 181) and the conditions for the equality [18], we obtain that $\rho(X) - \rho(Y) < r$. ■

3.4. The corona product

In [2, Theorem 3.1] the authors compute the entire spectrum of the adjacency matrix of $G \circ H$ ($\alpha = 0$), when H is regular. In this subsection we extend this result to all $\alpha \in [0, 1)$, when H is regular. Before that, it is worth mention the following lemma which is an immediate consequence of Lemma 2.3.1 in [3].

Lemma 3.6. *If $\{X_1, X_2, \ldots, X_m\}$ is a partition of $X = \{1, 2, \ldots, n\}$ which is equitable for the square matrix A whose rows and columns are indexed by the elements of X, then each eigenvalue of the corresponding quotient matrix is an eigenvalue of A.*

Let $V(G) = \{v_1, \ldots, v_n\}$. Observe that $G \circ H = (G \circ \overline{K_m})(H_i : 1 \leq i \leq n)$ where $H_i = H$. Each pair of vertex subsets (C_i, S_i), with $C_i = V(\overline{K_m})$ and $S_i = \{v_i\}$ is a cluster, for $i = 1, \ldots, n$.

Theorem 3.7. *If G is a connected graph of order n and H is a r-regular graph of order m, then $G \circ H$ is a graph of order $n(m+1)$ and its A_α-spectrum includes the eigenvalues*

$$\alpha + \nu_j(H) \text{ for } 1 \leq j \leq m-1, \tag{15}$$

each one with multiplicity n.

The remaining $2n$ eigenvalues of $A_\alpha(G \circ H)$ are the eigenvalues of the matrix

$$B = \begin{bmatrix} A_\alpha(G) + m\alpha I_n & m\beta I_n \\ \beta I_n & (\alpha + r) I_n \end{bmatrix}. \tag{16}$$

Proof. Let $V(G) = \{v_1, \ldots, v_n\}$. We recall that $G \circ H = (G \circ \overline{K_m})(H_i : 1 \leq i \leq n)$ with $H_i = H$ for all i. Applying Theorem 2.7(ii) to $(G \circ \overline{K_m})(H_i : 1 \leq i \leq n)$, it follows that, for $1 \leq i \leq n$ and $1 \leq j \leq m-1$, $\alpha + \nu_j(H)$ is an eigenvalue of $A_\alpha(G \circ H)$ with multiplicity n. Therefore, the expression (15) follows. We label the vertices of $G \circ H$ as follows: $1, \ldots, n$ for the vertices of G and, for $1 \leq i \leq n$, the labels $n+(i-1)m+1, \ldots, n+im$ for the vertices of H_i. Let $X = \{1, \ldots, n, n+1, \ldots, n+mn\}$. Consider the partition $\{X_1, \ldots, X_n, X_{n+1}, \ldots, X_{2n}\}$ of X where $X_1 = \{1\}, \ldots, X_n = \{n\}$ and, for $1 \leq i \leq n$, $X_{n+i} = \{n+(i-1)m+1, \ldots, n+im\}$. For this partition $A_\alpha(G \circ H)$ becomes a $2n \times 2n$ - block matrix such that the row sum of each of the blocks is constant. Hence $\{X_1, \ldots, X_{2n}\}$ is an equitable partition. The corresponding quotient matrix is the matrix B given in (16). Therefore, by Lemma 3.6, the eigenvalues of B are eigenvalues of $A_\alpha(G \circ H)$. ■

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