Discussiones Mathematicae
Graph Theory 40 (2020) 451–466
doi:10.7151/dmgt.2284

GRAPHS WITH CLUSTERS PERTURBED BY REGULAR GRAPHS—A_α-SPECTRUM AND APPLICATIONS

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Abstract

Given a graph G, its adjacency matrix $A(G)$ and its diagonal matrix of vertex degrees $D(G)$, consider the matrix $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$, where $\alpha \in [0, 1)$. The A_α-spectrum of G is the multiset of eigenvalues of $A_\alpha(G)$ and these eigenvalues are the α-eigenvalues of G. A cluster in G is a pair of vertex subsets (C, S), where C is a set of cardinality $|C| \geq 2$ of pairwise co-neighbor vertices sharing the same set S of $|S|$ neighbors. Assuming that G is connected and it has a cluster (C, S), $G(H)$ is obtained from G and an r-regular graph H of order $|C|$ by identifying its vertices with the vertices in C, eigenvalues of $A_\alpha(G)$ and $A_\alpha(G(H))$ are deduced and if $A_\alpha(H)$ is positive semidefinite, then the i-th eigenvalue of $A_\alpha(G(H))$ is greater than or equal to i-th eigenvalue of $A_\alpha(G)$. These results are extended to graphs with several pairwise disjoint clusters $(C_1, S_1), \ldots, (C_k, S_k)$. As an application, the effect on the energy, α-Estrada index and α-index of a graph G with clusters when the edges of regular graphs are added to G are analyzed. Finally, the A_α-spectrum of the corona product $G \circ H$ of a connected graph G and a regular graph H is determined.

Keywords: cluster, convex combination of matrices, A_α-spectrum, corona product of graphs.

2010 Mathematics Subject Classification: 05E30, 15A18, 05C50.

1. Introduction and Preliminaries

We deal with simple undirected graphs $G = (V(G), E(G))$ on n vertices with vertex set $V(G)$ and edge set $E(G)$. The complement of G is the graph $\overline{G}$ with the same vertex set as G in which any two distinct vertices are adjacent if and only if they are non-adjacent in G. The complete graph on n vertices is denoted by K_n (therefore, $\overline{K_n}$ has no edges, that is, all its vertices are isolated). The complete bipartite graph on $p+q$ vertices is denoted by $K_{p,q}$ (in particular, $K_{1,s}$ is a star on $s+1$ vertices).

Throughout the text, N_k denotes the set of positive integers not greater than k, the identity matrix of order m and the transpose of a matrix A are denoted by I_m and A^T, respectively. Furthermore, 0 is the zero matrix of appropriate order, $\mathbf{1}_n$ is the all-one column vector of size n and $J_{p,q}$ is the all-one matrix of order $p \times q$. The remainder notation is standard. However for the reader's convenience, as it follows, the fundamental concepts and their notation is briefly recalled.

Let $D(G)$ be the diagonal matrix of order n whose (i,i)-entry is the degree of the i-th vertex of G and let $A\,(G)$ be the adjacency matrix of G. The matrices $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ are the Laplacian and signless Laplacian matrix of G, respectively. The matrices $L(G)$ and $Q(G)$ are both positive semidefinite and $(0, \mathbf{1})$ is an eigenpair of $L\,(G)$. Fiedler [7] proved that G is a connected graph if and only if the second smallest eigenvalue of $L(G)$ is positive. This eigenvalue is called the algebraic connectivity of G. Moreover, it is known that for any bipartite graph G, the characteristic polynomials of $L(G)$ and $Q(G)$ coincide [6, Prop. 2.3]. For a connected graph G, the least eigenvalue of $Q(G)$ is positive if and only if G is non-bipartite [6, Proposition 2.1].

In [13] Nikiforov introduced the family of matrices

$$A_\alpha\,(G) = \alpha D\,(G) + (1-\alpha) A\,(G)$$

where $\alpha \in [0,1]$. We see that $A_\alpha(G)$ is a convex combination of the matrices $A(G)$ and $D(G)$. The multiset of eigenvalues of $A_\alpha(G)$ is called the A_α-spectrum of G.

Since $A_\alpha(G)$ is a real symmetric matrix, its eigenvalues are real numbers. Observe that $A_0\,(G) = A\,(G)$ and $2A_{1/2}\,(G) = Q\,(G)$. Thus, the family $A_\alpha(G)$ extends both $A(G)$ and $Q\,(G)$. Since $A_1(G) = D(G)$, from now on, we take $\alpha \in [0,1)$.

If G is a graph of order n, we denote by

$$\nu_1\,(G) \leq \nu_2\,(G) \leq \cdots \leq \nu_n\,(G)$$

the eigenvalues of $A_\alpha(G)$. If necessary, these eigenvalues are also denoted by $\nu_1(A_\alpha(G)), \nu_2(A_\alpha(G)), \ldots, \nu_n(A_\alpha(G))$.

In particular, $\nu_n(G)$ is called the α-index of G. From the Perron-Frobenius Theory for nonnegative matrices, it follows that

- for a connected graph G, the α-index of G (Perron root) is a simple eigenvalue of $A_\alpha(G)$ that has a positive eigenvector (Perron vector),
- for a connected graph G, the α-index of G increases if any entry of $A_\alpha(G)$ increases,
- if G is a proper subgraph of a connected graph H, then $\nu_n(G) < \nu_n(H)$, and
- if G is an r-regular graph of order n, then $A_\alpha(G) = r\alpha I_n + (1-\alpha)A(G)$ and $\nu_n(G) = r$ with eigenvector $\mathbf{1}_n$.

Now, we recall the concept of cluster which appears first in [11] and more recently in [5].

Definition 1.1. A cluster of order c and degree s in a graph G is a pair of vertex subsets (C, S), where C is a set of cardinality $|C| = c \geq 2$ of pairwise co-neighbor vertices sharing the same set S of s neighbors.

A pendent vertex is a vertex of degree 1 and a quasi-pendent vertex is a vertex adjacent to at least one pendent vertex. For the star $K_{1,s}$, C is the set of the pendent vertices and $S = \{v\}$ where v is the root vertex and a complete bipartite graph $K_{p,q}$ has the clusters $(\overline{K}_p, \overline{K}_q)$ and $(\overline{K}_q, \overline{K}_p)$. Also, note that each quasi-pendent vertex adjacent with more than one pendent vertex define a cluster (C, S) in which $|S| = 1$. In [4], among other results, it was proved that α is an eigenvalue of $A_\alpha(G)$ with multiplicity at least $p(G) - q(G)$, when G has $p(G) > 0$ pendent vertices and $q(G)$ quasi-pendent vertices. It is easy to prove that any set of pairwise co-neighbor vertices is an independent set.

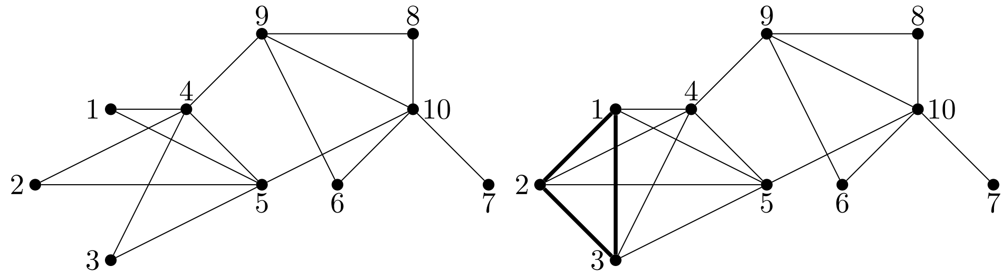
Definition 1.2. Let G be a connected graph of order n with a cluster (C, S) and let H be a graph of order $|C|$. Assuming that $V(H) = C$, then $G(H)$ is the graph with vertex set $V(G(H)) = V(G)$ and edge set $E(G(H)) = E(G) \cup E(H)$.

From Definition 1.2, $G(H)$ is the graph obtained from G and H adding the edges of H to the edges of G by identifying the vertices of H with the vertices in C.

Example 1.3. Let G be the graph below depicted which has the cluster (C, S), where $C = \{1, 2, 3\}$ and $S = \{4, 5\}$. Let H be the cycle on 3 vertices, $V(H) = \{1, 2, 3\}$. Then the graphs G and $G(H)$ are displayed, respectively, below.

Definition 1.4. Let (C_1, S_1) and (C_2, S_2) be clusters in a graph G. We say that (C_1, S_1) and (C_2, S_2) are disjoint if $C_1 \cap C_2 = \emptyset$ and $S_1 \cap S_2 = \emptyset$.

The Laplacian and signless Laplacian spectra of a graph G with a cluster (C, S) are studied in [1]. The effects on the Laplacian spectral radius and algebraic connectivity of a graph perturbed by adding edges between its pendent vertices are considered in [9] and [17], respectively. Moreover, the effects on others spectral invariants are determined in [15] and [16].



Definition 1.5. Let G be a connected graph with pairwise disjoint clusters $(C_1, S_1), \ldots, (C_k, S_k)$. For $i = 1, \ldots, k$, let H_i be a graph of order $|C_i|$. Let $G(H_i : i \in N_k)$ be the graph obtained from G and the graphs H_i when the edges of H_i are added to the edges of G by identifying the vertices of H_i with the vertices in C_i for $i = 1, \ldots, k$.

From this definition, we have $V(H_i) = C_i$, for $i = 1, \ldots, k$,

$$V(G(H_i : i \in N_k)) = V(G)$$

and

$$E(G(H_i : i \in N_k)) = E(G) \cup E(H_1) \cup \cdots \cup E(H_k).$$

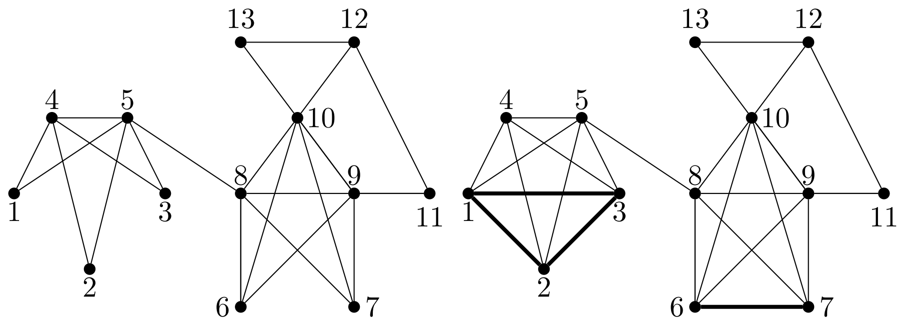
Observe that the graph $G(H_i : i \in N_k)$ can be constructed as follows.

- The graph $G_1 = G(H_1)$ is obtained from G and H_1 identifying the vertices of H_1 with C_1, and
- for $i = 2, \ldots, k$, the graph $G_i = G(H_1, \ldots, H_i)$ is obtained from $G_{i-1} = G(H_1, \ldots, H_{i-1})$ and H_i identifying the vertices of H_i with C_i.

Example 1.6. Let G be the graph below depicted which has two disjoint clusters (C_1, S_1) and (C_2, S_2) where $C_1 = \{1, 2, 3\}$, $S_1 = \{4, 5\}$ and $C_2 = \{6, 7\}$, $S_2 = \{8, 9, 10\}$. Let H_1 be the cycle on 3 vertices, $V(H_1) = \{1, 2, 3\}$, and H_2 be the path on 2 vertices, $V(H_2) = \{6, 7\}$. Then the graphs G and $G(H_1, H_2)$ are displayed, respectively, below.

A unified approach to the determination of the spectra of adjacency, Laplacian and signless Laplacian matrices of graphs with edge perturbation on their clusters was presented in [5]. Moreover, the invariance of algebraic connectivity and Laplacian index under those perturbation was proved.

In this article, using a methodology similar to the one followed in [5], new results about the spectra of $A_\alpha(G)$ and $A_\alpha(G(H))$ are deduced. Namely, in



Section 2, assuming that G is a connected graph of order n with a cluster (C, S) and $G(H)$ is obtained according to Definition 1.2, the following results about the spectra of $A_\alpha(G)$ and $A_\alpha(G(H))$ are proven.

1. $|S|\alpha + \nu_j(H)$, $1 \leq j \leq |C| - 1$, are eigenvalues of $A_\alpha(G(H))$, where

$$\nu_1(H) \leq \cdots \leq \nu_{|C|-1}(H) \leq \nu_{|C|}(H) = r$$

are the eigenvalues of $A_\alpha(H)$. As direct consequence, $|S|\alpha$ is an eigenvalue of $A_\alpha(G)$ with multiplicity at least $|C|-1$. In both cases, the remaining eigenvalues can be computed from a special matrix, (5) and (8), respectively (Theorem 2.1 and Corollary 2.2).

2. If $A_\alpha(H)$ is a positive semidefinite matrix, then

$$\nu_i(G) \leq \nu_i(G(H)),$$

for $i = 1, \ldots, n$, where $\{\nu_i(G) : 1 \leq i \leq n\}$ and $\{\nu_i(G(H)) : 1 \leq i \leq n\}$ are the A_α-spectra of G and $G(H)$, respectively (Theorem 2.6).

3. Assuming that G has $k \geq 2$ pairwise disjoint clusters $(C_1, S_1), \ldots, (C_k, S_k)$, the above results are extended to the graph $G(H_i : i = 1, \ldots, k)$ (Theorem 2.7).

Finally, in Section 3, the obtained results are applied to study the effect on the energy (Theorems 3.1 and 3.2), α-Estrada index (Theorems 3.3 and 3.4) and α-index (Theorem 3.5) of a graph G with clusters when the edges of regular graphs are added to G. Additionally, the A_α-spectrum of the corona product $G \circ H$ of a connected graph G and a regular graph H is determined (Theorem 3.7).

2. Effects by Adding the Edges of a Regular Graph

Consider $G(H)$ as in Definition 1.2. Let $|C| = c$ and $|S| = s$. We assume that H is a connected r-regular graph of order $|C| = c$ and that

$$\nu_1(H) \leq \cdots \leq \nu_{c-1}(H) < \nu_c(H) = r$$

are the eigenvalues of $A_\alpha(H)$ with an orthogonal basis of eigenvectors

$$\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{c-1}, \mathbf{x}_c = \frac{1}{\sqrt{c}}\mathbf{1}_c$$

in which, for $1 \le i \le c$, $A_\alpha(H)\mathbf{x}_i = \nu_i(H)\mathbf{x}_i$. In particular

$$A_\alpha(H)\mathbf{1}_c = r\mathbf{1}_c. \tag{1}$$

Let

$$X = \begin{bmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_{c-1} & \frac{1}{\sqrt{c}}\mathbf{1}_c \end{bmatrix}$$

and

$$U = \begin{bmatrix} X & \\ & I_{n-c} \end{bmatrix}. \tag{2}$$

Clearly X and U are both orthonormal matrices.

Through this paper $\beta = 1 - \alpha$ and d_i is the degree of the vertex i of the graph G.

We recall that G is a graph that has a cluster (C, S). The graphs G and $G(H)$ have the same set of vertices. We label the vertices of G as follows. The labels $1, 2, \ldots, c$ are for the vertices of C, the labels $c+1, c+2, \ldots, c+s$ are for the vertices in S and the labels $c+s+1, \ldots, n$ are for the remaining vertices of G. This labeling is illustrated in Example 1.3. For this labeling, $A_\alpha(G)$ and $A_\alpha(G(H))$ become as follows

$$A_\alpha(G) = \begin{bmatrix} s\alpha I_c & \begin{bmatrix}\beta\mathbf{1}_c\mathbf{1}_s^T & 0\end{bmatrix} \\ \begin{bmatrix}\beta\mathbf{1}_s\mathbf{1}_c^T \\ 0\end{bmatrix} & R(\alpha) \end{bmatrix} \tag{3}$$

and

$$A_\alpha(G(H)) = \begin{bmatrix} s\alpha I_c + A_\alpha(H) & \begin{bmatrix}\beta\mathbf{1}_c\mathbf{1}_s^T & 0\end{bmatrix} \\ \begin{bmatrix}\beta\mathbf{1}_s\mathbf{1}_c^T \\ 0\end{bmatrix} & R(\alpha) \end{bmatrix} \tag{4}$$

where $R(\alpha) = \begin{bmatrix} A & B \\ B^T & Z \end{bmatrix}$ with submatrices A, B and Z of size $s\times s$, $s\times(n-c-s)$ and $(n-c-s)\times(n-c-s)$, respectively. The diagonal entries of the matrices A and Z are αd_i, $c+1 \le i \le n$ and the off-diagonal entries of A and Z as well as the entries of B are β if the corresponding vertices of G are adjacent and 0 otherwise.

Theorem 2.1. *Let G be a graph with a cluster (C, S) of order $|C| = c$ and degree $|S| = s$. If H is an r-regular graph of order c and $G(H)$ is obtained according to*

Definition 1.2, *then*

$$s\alpha + \nu_j(H), \qquad 1 \le j \le c-1,$$

are eigenvalues of $A_\alpha(G(H))$, *where* $\nu_1(H) \le \cdots \le \nu_{c-1}(H) \le \nu_c(H) = r$ *are the eigenvalues of* $A_\alpha(H)$ *and the remaining eigenvalues of* $A_\alpha(G(H))$ *are the eigenvalues of the matrix*

$$X = \begin{bmatrix} s\alpha + r & [\beta\sqrt{c}\mathbf{1}_s^T \ \ 0] \\ \begin{bmatrix} \beta\sqrt{c}\mathbf{1}_s \\ 0 \end{bmatrix} & R(\alpha) \end{bmatrix}. \tag{5}$$

Proof. We use (4) and the orthogonal matrix U defined in (2) obtaining

$$\begin{aligned}
&U^T A_\alpha(G(H))U \\
&= \begin{bmatrix} X^T & \\ & I_{n-c} \end{bmatrix} \begin{bmatrix} s\alpha I_c + A_\alpha(H) & [\beta\mathbf{1}_c\mathbf{1}_s^T \ \ 0] \\ \begin{bmatrix} \beta\mathbf{1}_s\mathbf{1}_c^T \\ 0 \end{bmatrix} & R(\alpha) \end{bmatrix} \begin{bmatrix} X & \\ & I_{n-c} \end{bmatrix} \\
&= \begin{bmatrix} s\alpha I_c + X^T A_\alpha(H) X & [\beta X^T\mathbf{1}_c\mathbf{1}_s^T \ \ 0] \\ \begin{bmatrix} \beta\mathbf{1}_s\mathbf{1}_c^T X \\ 0 \end{bmatrix} & R(\alpha) \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} s\alpha + \nu_1(H) & & & \\ & \ddots & & \\ & & s\alpha + \nu_{c-1}(H) & \\ & & & s\alpha + r \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ \beta\sqrt{c}\mathbf{1}_s^T & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & \cdots & 0 & \beta\sqrt{c}\mathbf{1}_s \\ & \cdots & 0 & 0 \end{bmatrix} & R(\alpha) \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} s\alpha + \nu_1(H) & & \\ & \ddots & \\ & & s\alpha + \nu_{c-1}(H) \end{bmatrix} & \\ & \begin{bmatrix} s\alpha + r & [\beta\sqrt{c}\mathbf{1}_s^T \ \ 0] \\ \begin{bmatrix} \beta\sqrt{c}\mathbf{1}_s \\ 0 \end{bmatrix} & R(\alpha) \end{bmatrix} \end{bmatrix}.
\end{aligned}$$

Then $U^T A_\alpha(G(H))U =$

$$\begin{bmatrix} s\alpha + \nu_1(H) & & \\ & \ddots & \\ & & s\alpha + \nu_{c-1}(H) \end{bmatrix} \oplus \begin{bmatrix} s\alpha + r & [\beta\sqrt{c}\mathbf{1}_s^T \ \ 0] \\ \begin{bmatrix} \beta\sqrt{c}\mathbf{1}_s \\ 0 \end{bmatrix} & R(\alpha) \end{bmatrix}. \tag{6}$$

Therefore, the conclusion follows from (6). ■

Applying Theorem 2.1 to the particular case of $H = \overline{K_c}$, it follows that $G(H) = G$ and

$$U^T A_\alpha(G)U = \begin{bmatrix} s\alpha & & \\ & \ddots & \\ & & s\alpha \end{bmatrix} \oplus \begin{bmatrix} s\alpha & [\beta\sqrt{c}\mathbf{1}_s^T \ \ 0] \\ \begin{bmatrix} \beta\sqrt{c}\mathbf{1}_s \\ 0 \end{bmatrix} & R(\alpha) \end{bmatrix}. \tag{7}$$

Thus the next corollary is immediate.

Corollary 2.2. *Let G be a graph with a cluster (C, S) of order $|C| = c$ and degree $|S| = s$. Then $s\alpha$ is an eigenvalue of $A_\alpha(G)$ with multiplicity at least $c - 1$ and the remaining eigenvalues are the eigenvalues of the matrix*

$$Y = \begin{bmatrix} s\alpha & [\beta\sqrt{c}\mathbf{1}_s^T \ \ 0] \\ \begin{bmatrix} \beta\sqrt{c}\mathbf{1}_s \\ 0 \end{bmatrix} & R(\alpha) \end{bmatrix}. \tag{8}$$

Taking into account that $A_0(G) = A(G)$ and $2A_{\frac{1}{2}}(G) = Q(G)$, another immediate corollary is the following.

Corollary 2.3. *Let G be a graph with a cluster (C, S) of order $|C| = c$ and degree $|S| = s$. If H is an r-regular graph of order c and $G(H)$ is obtained according to Definition* 1.2, *then*

(i) 0 *is an eigenvalue of $A(G)$ with multiplicity at least $c - 1$,*

(ii) *if $\lambda_j(H) \neq r$ is an eigenvalue of $A(H)$, then it is also an eigenvalue of $A(G(H))$,*

(iii) *s is an eigenvalue of $Q(G)$ with multiplicity at least $c - 1$, and*

(iv) *if $q_j(H) \neq 2r$ is an eigenvalue of $Q(H)$, then $q_j(H) + s$ is an eigenvalue of $Q(G(H))$.*

2.1. The nonnegative A_α-spectrum case

In this subsection we study the A_α-spectrum of $G(H)$ when $A_\alpha(H)$ is a positive semidefinite matrix.

Among the basic results on $A_\alpha(G)$ obtained in [13] we recall the following theorem.

Theorem 2.4 [13, Proposition 4]**.** *Let $1 \geq \alpha > \beta \geq 0$. Then*

$$\nu_j(A_\alpha(G)) \geq \nu_j(A_\beta(G)) \tag{9}$$

for $j = 1, 2, \ldots, n$. If G is connected, then inequality (9) *is strict, unless $j = n$ and G is regular.*

The function $f_G(\alpha) = \nu_1(A_\alpha(G))$ is continuous and, from (9) with $j = 1$, it is nondecreasing in α. Moreover, $f_G(0) = \nu_1(A_0(G)) < 0$. Therefore, there is a smallest value $\alpha \in \left(0, \frac{1}{2}\right]$ such that $\nu_1(A_\alpha(G)) = 0$. Hence, denoting this value by $\alpha_0(G)$, $A_\alpha(G)$ is a positive semidefinite matrix if and only if $\alpha_0(G) \leq \alpha \leq 1$.

Now, we restate a problem proposed in [13, Problem 8] as follows: *given a graph G, find $\alpha_0(G)$.*

Some advances on this problem obtained in [14] are presented in the next proposition.

Proposition 2.5 [14, Proposition 5]. *If H is an r-regular graph, then*

$$\alpha_0(H) = \frac{-\nu_{\min}(A(H))}{r - \nu_{\min}(A(H))} \tag{10}$$

where $\nu_{\min}(A(H))$ is the least eigenvalue of $A(H)$.

Theorem 2.6. *Let G be a graph with a cluster (C, S) of order $|C| = c$ and degree $|S| = s$. If H is an r-regular graph of order c, $\alpha \geq \alpha_0(H)$, where $\alpha_0(H)$ is given by* (10), *and $G(H)$ is obtained according to Definition* 1.2, *then*

$$\nu_i(G) \leq \nu_i(G(H)),$$

for $i = 1, \ldots, n$, where $\{\nu_i(G) : 1 \leq i \leq n\}$ and $\{\nu_i(G(H)) : 1 \leq i \leq n\}$ are the A_α-spectra of G and $G(H)$, respectively.

Proof. Since $\alpha \geq \alpha_0(H)$ with $\alpha_0(H)$ given by (10), $A_\alpha(H)$ is a positive semidefinite matrix and then its eigenvalues are nonnegative. Thus the result follows from (6) and (7) applying the Weyl's inequalities for eigenvalues of Hermitian matrices ([10], p. 181). ∎

2.2. The multiple pairwise disjoint clusters case

In this subsection the graphs with more than one cluster are analyzed.

Theorem 2.7. *Let G be a graph with a set of pairwise disjoint clusters $\{(C_i, S_i) : i \in N_k\}$, with $k \geq 2$, and let $|C_i| = c_i$ and $|S_i| = s_i$, for $i \in N_k$. Assuming that each H_i is an r_i-regular graph of order c_i and $G(H_i : i \in N_k)$ is obtained according to Definition* 1.5, *it follows, for each $p \in N_k$, that*

(i) *$s_p\alpha$ is an eigenvalue of $A_\alpha(G)$ with multiplicity at least $c_p - 1$,*

(ii) *$s_p\alpha + \nu_j(H_p)$, $1 \leq j \leq c_p - 1$, is an eigenvalue of $A_\alpha(G(H_i : i \in N_k))$, where*

$$\nu_1(H_p) \leq \cdots \leq \nu_{c_p - 1}(H_p) \leq \nu_{c_p}(H_p) = r_p$$

are the eigenvalues of $A_\alpha(H_p)$,

(iii) *if*

$$\alpha \geq \frac{-\alpha_{\min}(A(H_p))}{r_p - \alpha_{\min}(A(H_p))}$$

where $\alpha_{\min}(A(H_p))$ is the least eigenvalue of $A(H_p)$, then the j-th eigenvalue of $A_\alpha(G(H_i : i \in N_k))$ is greater or equal to the j-th eigenvalue of $A_\alpha(G)$.

Proof. Considering $p \in N_k$, since

$$G(H_i : i \in N_k \setminus \{p\})(H_p) = G(H_i : i \in N_k),$$

the results are immediate from Theorems 2.1 and 2.6. ∎

As a consequence, we have the following corollary.

Corollary 2.8. *Let G be a graph with a set of pairwise disjoint clusters $\{(C_i, S_i) : i \in N_k\}$, with $k \geq 2$, and let $|C_i| = c_i$ and $|S_i| = s_i$, for $i \in N_k$. Assuming that each H_i is an r_i-regular graph of order c_i and $G(H_i : i \in N_k)$ is obtained according to Definition* 1.5, *then* 0 *is an eigenvalue of $A(G)$ with multiplicity at least $\sum_{i=1}^{k} c_i - k$. Moreover, for each $p \in N_k$,*

- (i) *if $\lambda_j(H_p) \neq r_p$ is an eigenvalue of $A(H_p)$, then it is also an eigenvalue of $A(G(H_i : i \in N_k))$,*
- (ii) *s_p is an eigenvalue of $Q(G)$ with multiplicity at least $c_p - 1$,*
- (iii) *if $q_j(H_p) \neq 2r_p$ is an eigenvalue of $Q(H_p)$, then $q_j(H_p) + s_p$ is an eigenvalue of $Q(G(H_i : i \in N_k))$.*

3. Some Applications

In this section, the energy, α-Estrada index, and α-index of graphs with clusters are considered, and the A_α-spectrum of the corona of a connected graph G and a regular graph H is determined.

We recall that the energy of a graph G is $\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i(G)|$ and the Estrada index of G is $E\mathcal{E}(G) = \sum_{i=1}^{n} e^{\lambda_i(G)}$, where

$$\lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_{n-1}(G) \leq \lambda_n(G)$$

are the eigenvalues of $A(G)$. Similarly, the signless Laplacian Estrada index of G is defined as $SLE\mathcal{E}(G) = \sum_{i=1}^{n} e^{q_i(G)}$, where

$$q_1(G) \leq q_2(G) \leq \cdots \leq q_{n-1}(G) \leq q_n(G)$$

are the eigenvalues of $Q(G)$.

The corona $G \circ H$ of two graphs G and H (where $|V(G)| = n$ and $|V(H)| = m$) introduced by Frucht and Harary [8] is defined as the graph obtained by taking one copy of G and n copies of H and then joining by an edge the i-th vertex of G to every vertex of the i-th copy of H. It is immediate that the corona graph operation is not commutative, that is, in general $G \circ H \neq H \circ G$.

3.1. The energy of graphs with clusters

Let M be an $m \times n$ complex matrix, $q = \min\{m, n\}$ and

$$\sigma_1(M) \geq \sigma_2(M) \geq \cdots \geq \sigma_q(M)$$

be the singular values of M. Nikiforov [12] defines the energy of M as $\mathcal{E}(M) = \sum_{j=1}^{q} \sigma_j(M)$. Since $A(G)$ is symmetric, its singular values are the modulus of its eigenvalues. Then $\mathcal{E}(G) = \mathcal{E}(A(G))$.

Given a natural number k such that $1 \le k \le n$, the Ky Fan k-norm of a matrix M of order $n \times n$ is the sum of the k largest singular values of M, that is, assuming that $\sigma_1(M), \dots, \sigma_k(M)$ are the k largest singular values of M, $\|M\|_k = \sum_{i=1}^{k} \sigma_i(M)$. In particular, $\|M\|_n = \mathcal{E}(M)$.

Theorem 3.1. *Let G be a graph with a cluster (C, S) of order $|C| = c$ and degree $|S| = s$. Let H be an r-regular graph of order c. Let $G(H)$ as in Definition* 1.2. *Then*

$$\mathcal{E}(G(H)) - \mathcal{E}(G) \le \mathcal{E}(H).$$

Proof. We apply Theorem 2.1 with $\alpha = 0$. From (6) and (7), using the fact that the singular values are invariant under unitary transformations, we have

$$\mathcal{E}(G(H)) = \mathcal{E}(A(G(H))) = \sum_{i=1}^{c-1} |\nu_i(H)| + \mathcal{E}(C), \tag{11}$$

where $C = \begin{bmatrix} r & [\sqrt{c}\mathbf{1}_s^T \;\; 0] \\ \begin{bmatrix} \sqrt{c}\mathbf{1}_s \\ 0 \end{bmatrix} & R(0) \end{bmatrix}$ and $\mathcal{E}(G) = \mathcal{E}(A(G)) = \mathcal{E}(D)$, where $D = \begin{bmatrix} 0 & [\sqrt{c}\mathbf{1}_s^T \;\; 0] \\ \begin{bmatrix} \sqrt{c}\mathbf{1}_s \\ 0 \end{bmatrix} & R(0) \end{bmatrix}$. Then $C = D + F$, where

$$F = \begin{bmatrix} r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{12}$$

Hence $\mathcal{E}(C) = \|C\|_{n-c+1} \le \|D\|_{n-c+1} + \|F\|_{n-c+1} = \mathcal{E}(D) + r = \mathcal{E}(G) + r$. Using this inequality in (11), we obtain

$$\mathcal{E}(G(H)) - \mathcal{E}(G) \le \sum_{i=1}^{c-1} |\nu_i(H)| + r = \sum_{i=1}^{c} |\nu_i(H)| = \mathcal{E}(H).$$

■

Theorem 3.2. *Let G be a graph with a set of clusters $\{(C_i, S_i) : i \in N_k\}$, $k \ge 2$. For $i \in N_k$, let $|C_i| = c_i$, $|S_i| = s_i$ and H_i be an r_i-regular graph of order c_i. Let $G(H_i : i \in N_k)$ as in Definition* 1.5. *Then*

$$\mathcal{E}(G(H_i : i \in N_k)) - \mathcal{E}(G) \le \sum_{i=1}^{k} \mathcal{E}(H_i).$$

Proof. The result follows easily by a repeated application of Theorem 3.1. ■

3.2. The α-Estrada index of graphs with clusters

In [16], for a graph with pendent vertices, the effects on the energy, Estrada index ($\alpha = 0$) and signless Laplacian Estrada index (essentially, $\alpha = 0.5$) are obtained when the edges of regular graphs are added among the pendent vertices. In this subsection, we extend these results to a graph with clusters, for all $\alpha \in [0, 1)$.

Since $A_0(G) = A(G)$, it seems natural to define the α-Estrada index of G, denoted by $E\mathcal{E}_\alpha(G)$, as

$$E\mathcal{E}_\alpha(G) = \sum_{i=1}^{n} e^{\nu_i(G)}$$

where

$$\nu_1(G) \le \nu_2(G) \le \cdots \le \nu_{n-1}(G) \le \nu_n(G)$$

are the eigenvalues of $A_\alpha(G)$. Hence $E\mathcal{E}_\alpha(G) = trace(e^{A_\alpha(G)})$

Next, we study the effect on the α-Estrada index.

Theorem 3.3. *Let G be a graph with a cluster (C, S) of order $|C| = c$ and degree $|S| = s$. Let H be an r-regular graph of order c. Let $G(H)$ be as in Definition 1.2. Then*

$$E\mathcal{E}_\alpha(G(H)) - E\mathcal{E}_\alpha(G) \ge e^{s\alpha} E\mathcal{E}_\alpha(H) - \left[(c-1)e^{s\alpha} + e^r\left(e^{s\alpha} - 1\right)\right].$$

Proof. We use again the fact that the singular values under unitary transformations to obtain, from (6) and (7), that

$$E\mathcal{E}_\alpha(G(H)) = \sum_{i=1}^{c-1} e^{(s\alpha+\nu_i(H))} + trace\left(e^X\right) \tag{13}$$

and

$$E\mathcal{E}_\alpha(G) = \sum_{i=1}^{c-1} e^{s\alpha} + trace\left(e^Y\right) \tag{14}$$

where X and Y are as in Theorem 2.1. From the series-expansion of e^N, we have

$$e^X = \sum_{j=0}^{\infty} \frac{1}{j!} X^j = \sum_{j=0}^{\infty} \frac{1}{j!} (Y + F)^j = \sum_{j=0}^{\infty} \frac{1}{j!} \left(Y^j + \cdots + F^j\right),$$

where F is given in (12). Since Y and F are nonnegative matrices, it follows that

$$trace\left(e^X\right) \ge trace\left(\sum_{j=0}^{\infty} \frac{1}{j!} Y^j\right) + trace\left(\sum_{j=0}^{\infty} \frac{1}{j!} F^j\right).$$

Hence,

$$trace\big(e^X\big) \geq trace\big(e^Y\big) + \sum_{j=0}^{\infty} \frac{1}{j!} r^j = trace\big(e^Y\big) + e^r.$$

Using this inequality in (13), we get

$$\begin{aligned} E\mathcal{E}_\alpha(G(H)) &\geq \sum_{i=1}^{c-1} e^{(s\alpha+\nu_i(H))} + trace\big(e^Y\big) + e^r \\ &= e^{s\alpha} \sum_{i=1}^{c-1} e^{\nu_i(H)} + E\mathcal{E}_\alpha(G) - \sum_{i=1}^{c-1} e^{s\alpha} + e^r. \end{aligned}$$

Finally,

$$\begin{aligned} E\mathcal{E}_\alpha(G(H)) - E\mathcal{E}_\alpha(G) &\geq e^{s\alpha} \sum_{i=1}^{c-1} e^{\nu_i(H)} - \sum_{i=1}^{c-1} e^{s\alpha} + e^r \\ &= e^{s\alpha} \sum_{i=1}^{c-1} e^{\nu_i(H)} - (c-1)e^{s\alpha} + e^r + e^{s\alpha}e^r - e^{s\alpha}e^r \\ &= e^{s\alpha} E\mathcal{E}_\alpha(H) - (c-1)e^{s\alpha} - e^r\big(e^{s\alpha} - 1\big). \end{aligned}$$

Therefore,

$$E\mathcal{E}_\alpha(G(H)) - E\mathcal{E}_\alpha(G) \geq e^{s\alpha} E\mathcal{E}_\alpha(H) - \big[(c-1)e^{s\alpha} + e^r\big(e^{s\alpha} - 1\big)\big]. \qquad \blacksquare$$

A repeated application of Theorem 3.3 yields to the following result.

Theorem 3.4. *Let G be a graph with a set of clusters $\{(C_i, S_i) : i \in N_k\}$, $k \geq 2$. For $i \in N_k$, let $|C_i| = c_i$, $|S_i| = s_i$ and H_i be an r_i-regular graph of order c_i. Let $G(H_i : i \in N_k)$ as in Definition* 1.5. *Then*

$$E\mathcal{E}_\alpha(G(H_i : i \in N_k)) - E\mathcal{E}_\alpha(G) \geq \sum_{i=1}^{k} \big(e^{s_i\alpha} E\mathcal{E}_\alpha(H_i) - (c_i - 1)e^{s_i\alpha} - e^{r_i}\big(e^{s_i\alpha} - 1\big)\big).$$

3.3. The α-index of graphs with a cluster

Now, we study the effect on the α-index. We remember that $\nu_n(G)$ and $\nu_n(G(H))$ denote the α-index of G and $G(H)$, respectively. We denote by $\rho(X)$ and $\rho(Y)$ the spectral radius of the matrices X and Y given in (5) and (8), respectively.

Theorem 3.5. *Let G be a graph with a cluster (C, S) of order $|C| = c$ and degree $|S| = s$. Let H be an r-regular graph of order c. Let $G(H)$ be as in Definition* 1.2. *Then*

$$0 < \nu_n(G(H)) - \nu_n(G) < r.$$

Proof. Clearly, from Theorem 2.1, $\nu_n(G(H)) = \rho(X)$ and $\nu_n(G) = \rho(Y)$. We have $X = Y + F$ with F as in (12). Since $X - Y \geq 0$ with strict inequality in the entry (1,1), we get that $0 < \rho(X) - \rho(Y)$. Moreover, applying the Weyl's inequalities for eigenvalues of Hermitian matrices ([10], p. 181) and the conditions for the equality [18], we obtain that $\rho(X) - \rho(Y) < r$. ∎

3.4. The corona product

In [2, Theorem 3.1] the authors compute the entire spectrum of the adjacency matrix of $G \circ H$ ($\alpha = 0$), when H is regular. In this subsection we extend this result to all $\alpha \in [0, 1)$, when H is regular. Before that, it is worth mention the following lemma which is an immediate consequence of Lemma 2.3.1 in [3].

Lemma 3.6. *If $\{X_1, X_2, \ldots, X_m\}$ is a partition of $X = \{1, 2, \ldots, n\}$ which is equitable for the square matrix A whose rows and columns are indexed by the elements of X, then each eigenvalue of the corresponding quotient matrix is an eigenvalue of A.*

Let $V(G) = \{v_1, \ldots, v_n\}$. Observe that $G \circ H = (G \circ \overline{K_m})(H_i : 1 \leq i \leq n)$ where $H_i = H$. Each pair of vertex subsets (C_i, S_i), with $C_i = V(\overline{K_m})$ and $S_i = \{v_i\}$ is a cluster, for $i = 1, \ldots, n$.

Theorem 3.7. *If G is a connected graph of order n and H is a r-regular graph of order m, then $G \circ H$ is a graph of order $n(m+1)$ and its A_α-spectrum includes the eigenvalues*

$$\alpha + \nu_j(H) \text{ for } 1 \leq j \leq m-1, \tag{15}$$

each one with multiplicity n.

The remaining $2n$ eigenvalues of $A_\alpha(G \circ H)$ are the eigenvalues of the matrix

$$B = \begin{bmatrix} A_\alpha(G) + m\alpha I_n & m\beta I_n \\ \beta I_n & (\alpha + r) I_n \end{bmatrix}. \tag{16}$$

Proof. Let $V(G) = \{v_1, \ldots, v_n\}$. We recall that $G \circ H = (G \circ \overline{K_m})(H_i : 1 \leq i \leq n)$ with $H_i = H$ for all i. Applying Theorem 2.7(ii) to $(G \circ \overline{K_m})(H_i : 1 \leq i \leq n)$, it follows that, for $1 \leq i \leq n$ and $1 \leq j \leq m-1$, $\alpha + \nu_j(H)$ is an eigenvalue of $A_\alpha(G \circ H)$ with multiplicity n. Therefore, the expression (15) follows. We label the vertices of $G \circ H$ as follows: $1, \ldots, n$ for the vertices of G and, for $1 \leq i \leq n$, the labels $n+(i-1)m+1, \ldots, n+im$ for the vertices of H_i. Let $X = \{1, \ldots, n, n+1, \ldots, n+mn\}$. Consider the partition $\{X_1, \ldots, X_n, X_{n+1}, \ldots, X_{2n}\}$ of X where $X_1 = \{1\}, \ldots, X_n = \{n\}$ and, for $1 \leq i \leq n$, $X_{n+i} = \{n+(i-1)m+1, \ldots, n+im\}$. For this partition $A_\alpha(G \circ H)$ becomes a $2n \times 2n$ - block matrix such that the row sum of each of the blocks is constant. Hence $\{X_1, \ldots, X_{2n}\}$ is an equitable partition. The corresponding quotient matrix is the matrix B given in (16). Therefore, by Lemma 3.6, the eigenvalues of B are eigenvalues of $A_\alpha(G \circ H)$. ■

Acknowledgments

The authors are indebted to the anonymous reviewers for their suggestions which helped us to make this article more concise.

D.M. Cardoso is partially supported by the Portuguese Foundation for Science and Technology ("FCT-Fundação para a Ciência e a Tecnologia"), through CIDMA — Center for Research and Development in Mathematics and Applications, within project UID/MAT/04106/2019. G. Pastén is supported by CONICYT-PFCHA/Doctorado Nacional/2017-21170391, Chile and O. Rojo is supported by Project Fondecyt Regular 1170313, Chile.

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Received 30 March 2019
Revised 25 October 2019
Accepted 12 November 2019